ISSN 1997-1397 (Print)
ISSN 2313-6022 (Online)

Журнал Сибирского федерального университета

Математика и физика

Journal of Siberian Federal University

Mathematics & Physics

2020 13 (1)

ISSN 1997-1997-1397 (Print)

ISSN 2313-6022 (Online)

ЖУРНАЛ СИБИРСКОГО ФЕДЕРАЛЬНОГО УНИВЕРСИТЕТА
Математика и Физика

2020 13 (1)

JOURNAL OF SIBERIAN FEDERAL UNIVERSITY
Mathematics & Physics

Издание индексируется Scopus (Elsevier), Emerging Sources Citation Index (WoS, Clarivate Analytics), Российским индексом научного цитирования (НЭБ), представлено в международных и российских информационных базах: Ulrich's periodicals directiory, ProQuest, EBSCO (США), Google Scholar, MathNet.ru, КиберЛенинке.

Включено в список Высшей аттестационной комиссии «Рецензируемые научные издания, входящие в международные реферативные базы данных и системы цитирования».

Все статьи представлены в открытом доступе http://journal.sfu-kras.ru/en/series/mathematics_physics.

Журнал Сибирского федерального университета. Математика и физика.
Journal of Siberian Federal University. Mathematics & Physics.

Учредитель: Федеральное государственное автономное образовательное учреждение высшего образования "Сибирский федеральный университет"(СФУ)
Главный редактор: А.М. Кытманов. Редакторы: В.Е. Зализняк, А.В. Щуплев.
Компьютерная верстка: Г.В. Хрусталева
№ 1. 26.02.2020. Индекс: 42327. Тираж: 1000 экз. Свободная цена
Адрес редакции и издательства: 660041 г. Красноярск, пр. Свободный, 79, оф. 32-03.
Отпечатано в типографии Издательства БИК СФУ
660041 г. Красноярск, пр. Свободный, 82а.
Свидетельство о регистрации СМИ ПИ № ФС 77-28724 от 27.06.2007 г., выданное Федеральной службой по надзору в сфере массовых коммуникаций, связи и охраны культурного наследия
http://journal.sfu-kras.ru
Подписано в печать 15.02.20. Формат 84×108/16. Усл.печ. л. 11,9.
Уч.-изд. л. 11,7. Бумага тип. Печать офсетная.
Тираж 1000 экз. Заказ 10776
Возрастная маркировка в соответствии с Федеральным законом № 436-ФЗ:16+

CONTENTS

СОДЕРЖАНИЕ

Journal of Siberian Federal University. Mathematics & Physics 2020, 13(1), 5–25

DOI: 10.17516/1997-1397-2020-13-1-5-25
УДК 517.55

A Degree Theory for Lagrangian Boundary Value Problems

Ammar Alsaedy*
Alnahrain University
Baghdad, Iraq
Nikolai Tarkhanov†
University of Potsdam
Potsdam, Germany

Received 08.05.2019, received in revised form 06.09.2019, accepted 06.11.2019

Abstract. We study those nonlinear partial differential equations which appear as Euler-Lagrange equations of variational problems. On defining weak boundary values of solutions to such equations we initiate the theory of Lagrangian boundary value problems in spaces of appropriate smoothness. We also analyse if the concept of mapping degree of current importance applies to Lagrangian problems.

Keywords: nonlinear equations, Lagrangian system, weak boundary values, quasilinear Fredholm operators, mapping degree.

Citation: A.Alsaedy, N.Tarkhanov, A Degree Theory for Lagrangian Boundary Value Problems, J. Sib. Fed. Univ. Math. Phys., 2020, 13(1), 5–25. DOI: 10.17516/1997-1397-2020-13-1-5-25.

Introduction

Distribution theory steams from weak solutions of linear differential equations and it is hardly efficient for nonlinear equations. The use of distributions is actually difficult in linear boundary value problems, for no canonical duality theory is available for manifolds with boundary $\mathcal{X}$. The scale of Sobolev-Slobodetskij spaces $W^{s,p}(\mathcal{X})$ makes it possible to consider the restrictions of functions to the boundary surface, however, these latter are defined only if $s - 1/p > 0$. To go beyond this range, one applies integral equalities obtained by manipulation of the Green formula. The study of general boundary value problems for differential equations in Sobolev-Slobodetskij spaces of negative smoothness goes back at least as far as [22].

For a boundary value problem, the Green formula is determined uniquely up to the counterpart of boundary data within the entire Cauchy data, see [26, 9.2.2]. This allows one to avoid much ambiguity in the choice of formal adjoint boundary value problem and to set up duality. As a result one is in a position to introduce weak solutions of the boundary value problem, see for instance Section 9.3.1 *ibid.* and elsewhere. The Cauchy data of a weak solution to an overdetermined elliptic system in the interior of $\mathcal{X}$ are proved to possess weak boundary values at $\partial\mathcal{X}$ if and only if the solution is of finite order of growth near the boundary surface, see [26, 9.3.6].

When considering a boundary value problem for a nonlinear equation, one has no good guide to an appropriate concept of weak solution. Perhaps one has to pass to the linearised problem. In any case the definition of a weak solution is implicitly contained in the variational setting of the boundary value problem. If the problem itself fails to be Lagrangian, it can be relaxed to variational one. It is just the task of experienced researcher to recover the concept of weak solution in the variational formulation, see [2].

*ammar.alsaedy@ymail.com
†tarkhanov@math.uni-potsdam.de

As but one tool of this work we introduce the concept of weak boundary values for solutions of nonlinear differential equations. We restrict the discussion to those equations which appear as Euler-Lagrange equations for a variational problem of minimasing the discrepancy $Au-f$ in the problem of finding a function u in $\mathcal{X}$, such that $Au = f(x,u)$ in $\mathcal{X}$ and $Bu = u_0$ at $\partial\mathcal{X}$. Here, A is an overdetermined elliptic operator of order one and B is a matrix of functions at $\partial\mathcal{X}$. The direct approach of variational calculus of [17] applies well to search for a solution in the Sobolev spaces $W^{1,p}(\mathcal{X})$ with non-extreme $1 < p < \infty$. However, the Euler-Lagrange equations include the boundary condition $B^*|Au-f|^{p-2}(Au-f) = 0$ at $\partial\mathcal{X}$. The function $|Au-f|^{p-2}(Au-f)$ is of class $L^{p'}(\mathcal{X})$, where $1/p+1/p' = 1$, and hence $B^*|Au-f|^{p-2}(Au-f)$ has no clear meaning at the boundary. We give this expression a weak meaning using the variational setting and an appropriate Green formula.

On specifying the spaces of weak boundary values one is in a position to consider the nonlinear mapping of Banach spaces or, more generally, Banach manifolds corresponding to the Lagrangian problem. The tangent mapping is a morphism of tangent (Banach) bundles and it is given by the linearisation of the nonlinear mapping at the points of $\mathcal{X}$. The nonlinear mapping is called elliptic if its tangent mapping is elliptic at each tangent space, cf. [20]. In this sense the Lagrangian boundary value problems are never elliptic but for $p = 2$, for they degenerate at each boundary point where $Au = f(x,u)$. By a Hodge theory for a nonlinear mapping is meant the Hodge theory for the corresponding morphism of tangent (Banach) bundles. This bundle is Hilbert, if $p = 2$, in which case the problem arises if the Hodge decompositions depend continuously on the point of the underlying Hilbert manifold. To treat this problem of differential geometry on Hilbert manifold we exploit the results of [27].

Any Lagrangian boundary value problem proves to be a quasilinear Fredholm mapping. To the best of our knowledge, this class of nonlinear mappings was first introduced in [24]. The quasilinear Fredholm mappings admit a reasonable degree theory elaborated in [9]. As but one consequence of our results we show that the degree theory of [9] applies to the Lagrangian boundary value problems.

1. Lagrangian boundary value problems

By Lagrangian boundary value problems are meant those arising as the Euler-Lagrange equations for functionals minimising discrepancy in overdetermined problems.

Let $\mathcal{X}$ be a bounded closed domain with C^∞ boundary in $\mathbb{R}^n$. Consider the boundary value problem

$$\left\{ \begin{array}{rclll} Au & = & f(x,u) & \text{in} & \mathcal{X}, \\ Bu & = & u_0 & \text{at} & \partial\mathcal{X}, \end{array} \right. \tag{1.1}$$

where A is a (possibly, overdetermined) elliptic linear partial differential operator of the first order near $\mathcal{X}$, f a function of its numerical variables $(x,u) \in \mathcal{X} \times \mathbb{R}^\ell$ with values in $\mathbb{R}^m$, and B an $(\ell' \times \ell)$-matrix of smooth functions on the boundary of $\mathcal{X}$ whose rank is ℓ' for all $x \in \partial\mathcal{X}$.

The operator A is given by an $(m \times \ell)$-matrix of scalar differential operators in a neighbourhood U of $\mathcal{X}$, and the principal symbol of A has rank ℓ for all $(x,\xi) \in U \times (\mathbb{R}^n \setminus \{0\})$. Our standing requirement on f is that $u \mapsto f(x,u)$ be a continuous mapping of $W^{1,p}(\mathcal{X},\mathbb{R}^\ell)$ into $L^p(\mathcal{X},\mathbb{R}^m)$.

Remark 1.1. Classical elliptic boundary value problems correspond to the case $m = \ell$ and $\ell' = \ell/2$.

The most conventional Banach space setting of this problem is $W^{1,p}$, where $1 < p < \infty$. Hence, we pick u_0 in $W^{1-1/p,p}(\partial\mathcal{X},\mathbb{R}^{\ell'})$ and look for a $u \in W^{1,p}(\mathcal{X},\mathbb{R}^\ell)$ satisfying (1.1).

If the operator

$$\mathcal{A} = \begin{pmatrix} A \\ B \end{pmatrix} : W^{1,p}(\mathcal{X}, \mathbb{R}^\ell) \to L^p(\mathcal{X}, \mathbb{R}^m) \times W^{1-1/p,p}(\partial\mathcal{X}, \mathbb{R}^{\ell'})$$

has a left parametrix $\mathcal{P} = (G, P)$, then on applying $\mathcal{P}$ to (1.1) from the left we obtain

$$u = G\, f(\cdot, u) + P u_0 + (\mathcal{P}\mathcal{A} - I)u \tag{1.2}$$

in $\mathcal{X}$ for all $u \in W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ satisfying (1.1). (Note that $\mathcal{A}$ possesses a left parametrix if and only if its null space is finite dimensional and its range is complemented, see [18]. In this case $\mathcal{P}\mathcal{A} - I$ can be thought of as projection onto the null space.) The operator $u \mapsto G \circ f(\cdot, u)$ is known as the Hammerstein operator. If $u \mapsto f(\cdot, u)$ maps $W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ compactly into $L^p(\mathcal{X}, \mathbb{R}^m)$, then the Leray-Schauder theory applies to equation (1.2). However, the solutions of the latter equation need not satisfy (1.1).

Moreover, if A is overdetermined (i.e. $m > \ell$) then there is a nonzero differential operator A^1, such that $A^1 A = 0$. Then, for the equation $Au = f(\cdot, u)$ to be solvable, it is necessary that $A^1 f(\cdot, u) = 0$ in $\mathcal{X}$ for some function $u \in W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$. Another obstacle to the existence of solutions of problem (1.1) is possible overdeterminacy of boundary conditions. This is the case, e.g., if $\ell' = \ell$, i.e. Bu represents the whole Cauchy data of u with respect to $A - f(x, \cdot)$ at the boundary surface $\partial\mathcal{X}$. This gives evidence of replacing the exact equation $Au = f(\cdot, u)$ in $\mathcal{X}$ by minimising the discrepancy $Au - f(\cdot, u)$ in the norm of $L^p(\mathcal{X}, \mathbb{R}^m)$. For this purpose, we introduce the functional

$$I(u) = \int_{\mathcal{X}} |Au - f(x, u)|^p\, dx \tag{1.3}$$

whose domain is the affine subspace $\mathcal{D}_I$ of $W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ consisting of all u, such that $Bu = u_0$ at $\partial\mathcal{X}$. Obviously, every solution of (1.1) minimises (1.3). The converse assertion is not true.

Write m for the infimum of $I(u)$ over $u \in \mathcal{D}_I$. In order that $u \in \mathcal{D}_I$ may satisfy $I(u) = m$ it is necessary that u would fulfill the so-called Euler-Lagrange equations. We now describe these.

Lemma 1.2. *Let C be an $((\ell - \ell') \times \ell)$-matrix C of smooth functions on $\partial\mathcal{X}$, such that*

$$\operatorname{rank} \begin{pmatrix} B(x) \\ C(x) \end{pmatrix} = \ell$$

for all $x \in \partial\mathcal{X}$. Then there are unique matrices B^* and C^* of continuous functions on $\partial\mathcal{X}$ with the property that

$$\int_{\partial\mathcal{X}} \big((Bu, C^* g)_x - (Cu, B^* g)_x\big) ds = \int_{\mathcal{X}} \big((Au, g)_x - (u, A^* g)_x\big) dx \tag{1.4}$$

for all $u \in W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ and $g \in W^{1,p'}(\mathcal{X}, \mathbb{R}^m)$, where ds is the surface measure on the boundary.

As usual, A^* stands for the formal adjoint of the differential operator A in a neighbourhood of $\mathcal{X}$.

Proof. For an explicit construction of matrices B^* and C^* we refer the reader to [2]. □

Formula (1.4) is usually referred to as the Green formula. On arguing as in Section 3 of [2] one sees that if functional (1.3) has a local extremum at a function $u \in \mathcal{D}_I$ then

$$\int_{\mathcal{X}} \big((A - f'_u)v, |Au - f|^{p-2}(Au - f)\big)_x\, dx = 0 \tag{1.5}$$

for all $v \in W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ such that $Bv = 0$ at $\partial\mathcal{X}$. Here, f'_u is the Jacobi matrix of $f(x,u)$ with respect to $u = (u_1, \ldots, u_\ell)$, i.e., the $(m \times \ell)$-matrix whose entries are f'_{i,u_j}.

If $g = |Au - f|^{p-2}(Au - f)$ is of class $W^{1,p'}(\mathcal{X}, \mathbb{R}^m)$, then we can apply formula (1.4) on the left-hand side and move $A - f'_u$ from v to $|Au - f|^{p-2}(Au - f)$, thus obtaining

$$\int_{\partial\mathcal{X}} (Cv, B^* g)_x \, ds + \int_{\mathcal{X}} (v, (A - f'_u)^* g)_x \, dx = 0$$

for all $v \in W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ satisfying $Bv = 0$ at the boundary. We first choose v to be arbitrary with compact support in the interior of $\mathcal{X}$ and so we conclude by the main lemma of variational calculus that $(A - f'_u)^* g$ vanishes almost everywhere in $\mathcal{X}$. Hence, the boundary integral is equal to zero for all $v \in W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$, such that $Bv = 0$ on $\partial\mathcal{X}$. It is a simple matter to see that the boundary integral actually vanishes for all functions $v \in W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$. Hence it follows immediately that $B^* g = 0$ on $\partial\mathcal{X}$.

Lemma 1.3. *For the variational problem $I(u) \to \min$ over $u \in \mathcal{D}_I$, Euler-Lagrange's equations just amount to*

$$\left\{ \begin{array}{rcll} (A - f'_u)^* \big(|Au - f|^{p-2}(Au - f)\big) & = & 0 & \mathit{in} \ \ \mathcal{X}, \\ Bu & = & u_0 & \mathit{at} \ \ \partial\mathcal{X}, \\ B^* \big(|Au - f|^{p-2}(Au - f)\big) & = & 0 & \mathit{at} \ \ \partial\mathcal{X}. \end{array} \right. \tag{1.6}$$

Proof. If $u \in \mathcal{D}_I$ and $|Au - f|^{p-2}(Au - f)$ is of class $W^{1,p'}(\mathcal{X}, \mathbb{R}^m)$ then this is precisely what has been proved above. For general $u \in \mathcal{D}_I$ equalities (1.6) are understood in the weak sense suggested by (1.5). To wit, the differential equation is satisfied in the sense of distributions in the interior of $\mathcal{X}$. The interpretation of the second boundary condition in (1.6) is more sophisticated. This will be discussed in detail in Section 2. □

The differential equation of (1.6) represents a system of ℓ second order partial differential equations for ℓ unknown functions. The number of boundary conditions just amounts to ℓ.

Example 1.4. *The variational problem of minimising the functional*

$$I(u) := \int_{\mathcal{X}} \big(|du|^p + |d^* u|^p\big) dx$$

over the set of all i-forms u of class $W^{1,p}(\mathcal{X})$ with normal part $\nu(u) = u_0$ at the boundary leads to the L^p-setting of the Neumann problem for the de Rham complex in $\mathcal{X}$. To wit,

$$\left\{ \begin{array}{rcll} d^*(|du|^{p-2}du) + d(|d^*u|^{p-2}d^*u) & = & 0 & \mathit{in} \ \ \mathcal{X}, \\ \nu(u) & = & u_0 & \mathit{at} \ \ \partial\mathcal{X}, \\ \nu(|du|^{p-2}du) & = & 0 & \mathit{at} \ \ \partial\mathcal{X}, \end{array} \right.$$

cf. [16].

2. Weak boundary values

In (1.6), u is an element of $W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$, and so $g = |Au - f|^{p-2}(Au - f)$ belongs to $L^{p'}(\mathcal{X}, \mathbb{R}^m)$, where $p' = p/(p-1)$ is the dual exponent for p. Hence, the differential equation $(A - f'_u)^* g = 0$ is readily interpreted in the sense of distributions in the interior of $\mathcal{X}$, just as it comes from (1.5) into consideration. One encounters difficulties in interpreting the equality $B^* g = 0$ at the boundary surface $\partial\mathcal{X}$, for g is defined almost everywhere in $\mathcal{X}$. To give a meaning to $B^* g$ at $\partial\mathcal{X}$, we strongly invoke the fact that g satisfies $(A - f'_u)^* g = 0$ weakly in the interior of $\mathcal{X}$. Namely, if $g \in W^{1,p'}(\mathcal{X}, \mathbb{R}^m)$, then

$$\int_{\partial\mathcal{X}} \big((Bv, C^* g)_x - (Cv, B^* g)_x\big) ds = \int_{\mathcal{X}} \big(((A - f'_u)v, g)_x - (v, (A - f'_u)^* g)_x\big) dx \qquad (2.1)$$

holds for all $v \in W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$, which is due to Green formula (1.4). Since $(A - f'_u)^* g$ vanishes in the interior of $\mathcal{X}$, we may neglect the second term on the right-hand side and use (2.1) to specify both $C^* g$ and $B^* g$ at the boundary in the general case $g \in L^{p'}(\mathcal{X}, \mathbb{R}^m)$.

Definition 2.1. *Let $g \in L^{p'}(\mathcal{X}, \mathbb{R}^m)$ satisfy $(A - f'_u)^* g = 0$ weakly in the interior of $\mathcal{X}$. Then we define*

$$\int_{\partial\mathcal{X}} \big((v_0, C^* g)_x - (v_1, B^* g)_x\big) ds = \int_{\mathcal{X}} \big((A - f'_u)v, g\big)_x \, dx$$

for all $v_0 \in W^{1/p',p}(\partial\mathcal{X}, \mathbb{R}^{\ell'})$ and $v_1 \in W^{1/p',p}(\partial\mathcal{X}, \mathbb{R}^{\ell-\ell'})$, where $v \in W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ is an arbitrary function satisfying $Bv = v_0$ and $Cv = v_1$ at $\partial\mathcal{X}$.

Note that the equalities $Bv = v_0$ and $Cv = v_1$ at the boundary surface just amount to

$$v = \begin{pmatrix} B \\ C \end{pmatrix}^{-1} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$$

at $\partial\mathcal{X}$, where the right-hand side belongs to $W^{1/p',p}(\partial\mathcal{X}, \mathbb{R}^\ell)$. Hence, the existence of a function $v \in W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ with the property that $Bv = v_0$ and $Cv = v_1$ at $\partial\mathcal{X}$ and

$$\|v\|_{W^{1,p}(\mathcal{X},\mathbb{R}^\ell)} \leqslant C \left(\|v_0\|_{W^{1/p',p}(\partial\mathcal{X},\mathbb{R}^{\ell'})} + \|v_1\|_{W^{1/p',p}(\partial\mathcal{X},\mathbb{R}^{\ell-\ell'})} \right) \qquad (2.2)$$

follows from the Sobolev trace theorem.

Theorem 2.2. *Definition 2.1 is correct and specifies the boundary values $C^* g$ and $B^* g$ in the dual spaces $W^{-1/p',p'}(\partial\mathcal{X}, \mathbb{R}^{\ell'})$ and $W^{-1/p',p'}(\partial\mathcal{X}, \mathbb{R}^{\ell-\ell'})$, respectively.*

Proof. Suppose v and w are two functions in $W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ satisfying $Bv = Bw$ and $Cv = Cw$ at $\partial\mathcal{X}$. Set $z = v - w$. Then $z \in W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ satisfies $Bz = 0$ and $Cz = 0$ at the boundary. By the spectral synthesis theorem for Sobolev spaces, there is a sequence

$$z_\nu \in C^\infty_{\mathrm{comp}}(\overset{o}{\mathcal{X}}, \mathbb{R}^\ell)$$

which approximates z in the $W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$-norm. Hence it follows that

$$\begin{aligned} \int_{\mathcal{X}} \big((A - f'_u)v, g\big)_x \, dx &= \int_{\mathcal{X}} \big((A - f'_u)w, g\big)_x \, dx + \int_{\mathcal{X}} \big((A - f'_u)z, g\big)_x \, dx = \\ &= \int_{\mathcal{X}} \big((A - f'_u)w, g\big)_x \, dx + \lim_{\nu\to\infty} \int_{\mathcal{X}} \big((A - f'_u)z_\nu, g\big)_x \, dx, \end{aligned}$$

where the last integral on the right-hand side vanishes, for g satisfies $(A - f'_u)^* g = 0$ weakly in the interior of $\mathcal{X}$. We have thus proved that Definition 2.1 is correct, i.e. it does not depend on the choice of v. Finally, combining Definition 2.1 and estimate (2.2) yields

$$\begin{aligned} \Big| \int_{\partial\mathcal{X}} ((v_0, C^* g)_x - (v_1, B^* g)_x) \, ds \Big| &\leqslant \|(A - f'_u)v\|_{L^p(\mathcal{X},\mathbb{R}^m)} \|g\|_{L^{p'}(\mathcal{X},\mathbb{R}^m)} \\ &\leqslant C \left(\|v_0\|_{W^{1/p',p}(\partial\mathcal{X},\mathbb{R}^{\ell'})} + \|v_1\|_{W^{1/p',p}(\partial\mathcal{X},\mathbb{R}^{\ell-\ell'})} \right) \end{aligned}$$

for all $v_0 \in W^{1/p',p}(\partial\mathcal{X}, \mathbb{R}^{\ell'})$ and $v_1 \in W^{1/p',p}(\partial\mathcal{X}, \mathbb{R}^{\ell-\ell'})$, the constant C being independent of v_0 and v_1. Hence it follows that $C^* g \in W^{-1/p',p'}(\partial\mathcal{X}, \mathbb{R}^{\ell'})$ and $B^* g \in W^{-1/p',p'}(\partial\mathcal{X}, \mathbb{R}^{\ell-\ell'})$, as desired. □

Thus, for each $u \in W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ satisfying $(A - f'_u)^*(|Au - f|^{p-2}(Au - f)) = 0$ weakly in the interior of $\mathcal{X}$, both $C^*(|Au - f|^{p-2}(Au - f))$ and $B^*(|Au - f|^{p-2}(Au - f))$ have weak values at the boundary surface $\partial\mathcal{X}$ which belong to $W^{-1/p',p'}(\partial\mathcal{X}, \mathbb{R}^{\ell'})$ and $W^{-1/p',p'}(\partial\mathcal{X}, \mathbb{R}^{\ell-\ell'})$, respectively. This completes, in particular, the result of [23].

For a thorough treatment of weak boundary values of solutions to linear overdetermined elliptic equations we refer the reader to [26, 9.4].

3. Variational boundary value problems after Browder

By the very nature, the function $(A - f'_u)^*(|Au - f|^{p-2}(Au - f))$ appears as distribution in the interior of $\mathcal{X}$, i.e. as element of

$$\big(\overset{o}{W}{}^{1,p}(\mathcal{X}, \mathbb{R}^\ell)\big)'.$$

Since $\overset{o}{W}{}^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ is not dense in $W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$, the continuous extension of this functional to all of $W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ is not uniquely determined. In fact, any continuous extension of $(A - f'_u)^*(|Au - f|^{p-2}(Au - f))$ to a closed subspace V of $W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ containing C^∞ functions of compact support in the interior of $\mathcal{X}$ with values in $\mathbb{R}^\ell$ defines a variational boundary value problem in the sense of [7]. We confine the discussion to (1.5).

Corresponding to the representation (1.5) for the critical points of functional (1.3), we have the nonlinear Dirichlet form $a(u, v)$ defined for all u and v in $W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ by

$$a(u, v) = \big(|Au - f|^{p-2}(Au - f), (A - f'_u)v\big),$$

where (g, h) stands for the natural sesquilinear pairing between g in $L^{p'}(\mathcal{X}, \mathbb{R}^m)$ and h in $L^p(\mathcal{X}, \mathbb{R}^m)$. By assumption, $a(u, v)$ is well defined for all u and v in $W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ and

$$|a(u, v)| \leqslant c\big(\|u\|_{W^{1,p}(\mathcal{X},\mathbb{R}^\ell)}\big)\, \|v\|_{W^{1,p}(\mathcal{X},\mathbb{R}^\ell)}$$

by Hölder's inequality, where $c(r)$ is a continuous function of the real variable r depending on A and f.

Let V be the closed subspace of $W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ that consists of all v satisfying $Bv = 0$ at the boundary $\partial\mathcal{X}$, and V^* be the conjugate space of V, i.e. the space of all bounded conjugate linear functionals on V. For $w \in V^*$ and $v \in V$, the value of w at v is denoted by (w, v). In particular, if $w \in L^{p'}(\mathcal{X}, \mathbb{R}^\ell)$, the bounded conjugate linear functional (w, v) on V yields an element of V^* which we may again denote by w.

We are now in a position to define the variational boundary problem corresponding to (a, V). Denote by F the mapping $V \to V^*$ given by $(Fu, v) := a(u, v)$ for all $v \in V$. In particular, we get

$$Fu = (A - f'_u)^* \left(|Au - f|^{p-2}(Au - f)\right) \tag{3.1}$$

in the sense of distributions in the interior of $\mathcal{X}$. Given $w \in V^*$, the variational boundary problem corresponding to (a, V) consists in finding $u \in V$ such that $Fu = w$. Hence it follows that $Fu = w$ holds weakly in the interior of $\mathcal{X}$ and $Bu = 0$ at the boundary. As usual, in order to include also inhomogeneous conditions $Bu = u_0$ at $\partial\mathcal{X}$, one solves these first in functions $u \in W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ which need not satisfy $Fu = w$.

If $u \in V$ satisfies $Fu = w$ with $w \in V^*$, then w is a relevant extension of the distribution $(A - f'_u)^*(|Au - f|^{p-2}(Au - f))$ in the interior of $\mathcal{X}$ to a continuous linear functional on V. Then Definition 2.1 for the weak value of B^*g at $\partial\mathcal{X}$ transforms to

$$\begin{aligned} -\int_{\partial\mathcal{X}} (B^*g, v_1)_x\, ds &= \int_{\mathcal{X}} \big(g, (A - f'_u)v\big)_x\, dx - (w, v) = \\ &= a(u, v) - (w, v) \end{aligned}$$

for all $v_1 \in W^{1/p',p}(\partial\mathcal{X}, \mathbb{R}^{\ell-\ell'})$, where $v \in W^{1,p}(\mathcal{X}, \mathbb{R}^{\ell})$ is an arbitrary function satisfying $Bv = 0$ and $Cv = v_1$ at $\partial\mathcal{X}$. Since $a(u,v) = (w,v)$ for all $v \in V$, it follows that $B^*g = 0$ at the boundary. Thus, the study of Euler-Lagrange's equations (1.6) can be carried out within the framework of mapping properties of $F : V \to V^*$.

To formulate the hypothesis of our existence theorem, we need an additional concept. Namely, by an admissible lower order operator is meant $u \to \Delta f(x,u)$, where Δf is a continuous function of its numerical arguments satisfying an inequality of the form

$$|\Delta f(x,u)| \leqslant c\Big(\|u\|_{W^{1,p}(\mathcal{X},\mathbb{R}^{\ell})}\Big)\,\Big(|u(x)|^{(p-1)+Q}+1\Big)$$

where $0 \leqslant Q < \dfrac{p^2}{n-p}$, if $p \leqslant n$, and $Q = 0$, if $p > n$.

Theorem 3.1. *Suppose that there exists an admissible lower order operator Δf and a continuous function $c(r)$ of the real variable r with $c(r) \to +\infty$ as $r \to \infty$, such that*

1) If $\Delta a(u,v) := (\Delta f(x,u), v)$ is the nonlinear Dirichlet form corresponding to Δf, then

$$\Re\Big(a(u,u-v) - a(v,u-v) + \Delta a(u,u-v) - \Delta a(v,u-v)\Big) \geqslant 0$$

for all u and v of V.

2) For all u in V,

$$\Re a(u,u) \geqslant c\Big(\|u\|_{W^{1,p}(\mathcal{X},\mathbb{R}^{\ell})}\Big)\,\|u\|_{W^{1,p}(\mathcal{X},\mathbb{R}^{\ell})}.$$

Then, for every w in V^*, the variational boundary problem for $Fu = w$ with null V-boundary conditions has at least one solution u.

Proof. The proof is along the lines of Theorem 1 of [7]. □

Note that in the case $f \equiv 0$ and $\Delta f = 0$ the condition 1) is fulfilled. Indeed, we get

$$\begin{aligned}
&\Re\Big(a(u,u-v) - a(v,u-v)\Big) = \\
&= \int_{\mathcal{X}} \Big(|Au|^p - |Au|^{p-2}\Re(Au,Av)_x - |Av|^{p-2}\Re(Av,Au)_x + |Av|^p\Big)dx \geqslant \\
&\geqslant \int_{\mathcal{X}} \Big(|Au|^p - |Au|^{p-1}|Av| - |Av|^{p-1}|Au| + |Av|^p\Big)dx \geqslant \\
&\geqslant \int_{\mathcal{X}} \Big(|Au|^{p-1} - |Av|^{p-1}\Big)\Big(|Au| - |Av|\Big)\,dx
\end{aligned}$$

which is obviously nonnegative for all $u, v \in V$. Furthermore, the condition 2) reduces to

$$\|Au\|^p_{L^p(\mathcal{X},\mathbb{R}^m)} \geqslant c\Big(\|u\|_{W^{1,p}(\mathcal{X},\mathbb{R}^{\ell})}\Big)\,\|u\|_{W^{1,p}(\mathcal{X},\mathbb{R}^{\ell})}$$

for all $u \in V$.

4. Hodge theory for nonlinear mappings

Let $\mathcal{V}$ and $\mathcal{W}$ be Banach manifolds and F a differentiable mapping of $\mathcal{V}$ to $\mathcal{W}$, i.e. we have a short complex

$$0 \to \mathcal{V} \xrightarrow{F} \mathcal{W} \to 0. \tag{4.1}$$

Given an arbitrary point $v \in \mathcal{V}$, the tangent mapping $F'(v) : T_v\mathcal{V} \to T_w\mathcal{W}$ is a bounded linear mapping of tangent spaces to $\mathcal{V}$ and $\mathcal{W}$ at v and $w = F(v)$, respectively. These mappings are gathered together to form the Banach bundle morphism

$$0 \to T\mathcal{V} \stackrel{F'}{\to} T\mathcal{W} \to 0,$$

see [27].

Definition 4.1. *A differentiable mapping $F : \mathcal{V} \to \mathcal{W}$ is said to be Fredholm if the linear mappings $F'(v) : T_v\mathcal{V} \to T_{F(w)}\mathcal{W}$ are Fredholm for all $v \in \mathcal{V}$.*

By the Hodge theory for the nonlinear mapping F we mean the Hodge theory for the tangent bundle morphism. According to the properties of Fredholm mappings, there are bounded linear projections $P(v)$ and $Q(v)$ in $T_v\mathcal{V}$ and $T_w\mathcal{W}$, respectively, such that

$$\begin{array}{rcccl} T_v\mathcal{V} & = & N(F'(v)) & \oplus & R(I-P(v)), \\ T_w\mathcal{W} & = & R(Q(v)) & \oplus & R(F'(v)), \end{array} \tag{4.2}$$

$P(v)$ being a projection onto the finite-dimensional null-space of $F'(v)$ and $Q(v)$ being a projection onto a finite-dimensional direct complement of the range of $F'(v)$ in $T_w\mathcal{W}$.

Using the inverse mapping theorem of Banach we conclude that the restriction of $F'(v)$ to $R(I-P(v))$ is an isomorphism of this Banach space onto $R(F'(v))$. The mapping

$$\Pi(v) = \Big(F'(v) \restriction_{R(I-P(v))} \Big)^{-1} \big(I - Q(v) \big)$$

is therefore a bounded linear operator from $T_w\mathcal{W}$ to $T_v\mathcal{V}$ satisfying

$$\begin{array}{rcl} \Pi(v)F'(v) & = & I - P(v), \\ F'(v)\Pi(v) & = & I - Q(v), \end{array}$$

i.e. $\Pi(v)$ is a parametrix of $F'(v)$ for each $v \in \mathcal{V}$. Note that if $\mathcal{V}$ is contractible then the parametrix $\Pi(v)$ can be chosen to depend continuously on the point $v \in \mathcal{V}$, see [9, 27].

If $\mathcal{V}$ and $\mathcal{W}$ are Hilbert manifolds, there is a canonical way for the choice of $P(v)$ and $Q(v)$. Namely, $P(v)$ is the orthogonal projection onto $N(F'(v))$ and $I-Q(v)$ is the orthogonal projection onto $R(F'(v))$. By the lemma on the annihilator of the kernel of operator,

$$\begin{array}{rcl} R(I-P(v)) & = & R(F'(v)^*), \\ R(Q(v)) & = & N(F'(v)^*), \end{array}$$

where $F'(v)^*$ is the Hilbert space adjoint for $F'(v) : T_v\mathcal{V} \to T_w\mathcal{W}$. We have thus proved

Theorem 4.2. *If $F : \mathcal{V} \to \mathcal{W}$ is a Fredholm mapping of Hilbert manifolds, then the tangent bundles of $\mathcal{V}$ and $\mathcal{W}$ split as*

$$\begin{array}{rcccl} T\mathcal{V} & = & N(F') & \oplus & R(F'^*), \\ T\mathcal{W} & = & N(F'^*) & \oplus & R(F'). \end{array}$$

These decompositions are scarcely useful to characterise the range of the global nonlinear mapping (4.1).

Example 4.3. *Let F be a differentiable selfmapping of $\mathbb{R}^n$, such that $\det F' \equiv 1$ in all of $\mathbb{R}^n$. Then the decompositions of Theorem 4.2 reduce to $T\mathbb{R}^n = R(F'^*)$ and $T\mathbb{R}^n = R(F')$, however, F need not be surjective in general. This is related to Jacobian problems, cf. [15].*

5. Quasilinear Fredholm mappings

Let V and W be real Banach spaces. Throughout we assume that V is compactly embedded into another Banach space V^-. When we refer to topological properties of a set $U \subset V$, we will mean the topology induced by V, unless we explicitly refer to the topology induced by V^-.

A mapping $F : V \to W$ is called quasilinear Fredholm if it can be written in the form

$$F(v) = L(v)v + C(v) \tag{5.1}$$

for $v \in V$, where L is the restriction to V of a continuous mapping L^- of V^- into the subset of $\mathcal{L}(V, W)$ consisting of Fredholm operators of index zero, and $C : V \to W$ is compact. Of course, quasilinear Fredholm mappings need not be differentiable.

Quasilinear Fredholm mappings were introduced in [24] in the study of the nonlinear Riemann-Hilbert problem. Another typical situation in which quasilinear Fredholm mappings arise quite naturally is the study of the Dirichlet problem for quasilinear elliptic equations. By [3], fully nonlinear elliptic equations with general nonlinear Shapiro-Lopatinskii boundary conditions induce quasilinear Fredholm mappings between appropriate function spaces, provided that the "coefficients" are sufficiently smooth.

If $F : V \to W$ is any C^1 mapping, we may write F as $F(v) = L(v)v + F(0)$ for $v \in V$, where $L(v) \in \mathcal{L}(V, W)$ is defined by

$$L(v) = \int_0^1 F'(tv)dt,$$

which is a curve integral in the space of bounded linear operators from V to W. Thus, the algebraic representation of (5.1) is not very restrictive. The crucial point is that each $L(v)$ is a Fredholm operator of index zero and that the family $L(v)$ is defined and depends continuously on v for v belonging to a larger space V^- in which V is compactly embedded. The latter property implies that $v \mapsto L(v)$ factors through a compact embedding $V \hookrightarrow V^-$, and so it is a compact mapping from V to $\mathcal{L}(V, W)$.

We now establish several general properties of quasilinear Fredholm mappings, following [9]. The mapping L is usually referred to as a principal part of f. Note that if $L : V \to \mathcal{L}(V, W)$ is continuous at $v_0 \in V$ then the mapping of V to W given by $v \mapsto L(v)(v - v_0)$ is Fréchet differentiable at v_0 and its Fréchet derivative at v_0 just amounts to $L(v_0)$.

Lemma 5.1. *Two principal parts of a quasilinear Fredholm mapping $F : V \to W$ differ by a family of compact operators.*

Proof. Suppose that $F : V \to W$ is represented by $F(v) = L_j(v)v + C_j(v)$, for $j = 1, 2$. Fix $v_0 \in V$ and set $G_j(v) = L_j(v)(v - v_0)$ for $v \in V$. As mentioned, we get $G_j'(v_0) = L_j(v_0)$, for $j = 1, 2$. From the equality of both representations it follows that the difference

$$G_1(v) - G_2(v) = -\big(C_1(v) - C_2(v)\big) - \big(L_1(v) - L_2(v)\big)v_0$$

is a compact mapping of V to W. But the Fréchet derivative of a compact mapping is compact, so that $G_1'(v_0) - G_2'(v_0) = L_1(v_0) - L_2(v_0)$ is compact. □

Lemma 5.2. *Let $F : V \to W$ be quasilinear Fredholm and be represented by $F(v) = L(v)v+C(v)$ for $v \in V$. If $F : V \to W$ is Fréchet differentiable at $v_0 \in V$, then $F'(v_0) - L(v_0)$ is compact.*

Proof. Write

$$R(v) = F(v) - L(v)(v - v_0)$$

for $v \in V$. The differentiability of F at v_0 implies that $R'(v_0) = F'(v_0)-L(v_0)$. Since $R : V \to W$ is compact, it follows that $F'(v_0) - L(v_0)$ is compact, too, as desired. □

So far we have not used the property of $L : V \to \mathcal{L}(V, W)$ to take on its values in Fredholm operators of index zero. Our next lemma makes use of this property. The Fredholm operators of index zero possess parametrices which are invertible mappings of W onto V. We confine ourselves to formulation of this result, referring the reader to [27] and [9] for a proof. Recall that an operator $A \in \mathcal{L}(V, W)$ is Fredholm of index zero if and only if there exists $P \in GL(W, V)$ with $PA - I \in \mathcal{K}(V)$. Let $A(\lambda)$ be a family of Fredholm operators of index zero acting from V to W and continuously depending on a parameter $\lambda \in \Lambda$, Λ being a topological space. By a strong parametrix for $A(\lambda)$ is meant any continuous family $P : \Lambda \to GL(W, V)$ satisfying $P(\lambda)A(\lambda) - I \in \mathcal{K}(V)$ for all $\lambda \in \Lambda$. In general, a family $A(\lambda)$ has no strong parametrix. For instance, when Λ is the unit circle in the plane, the non-existence of strong parametrices for certain continuous families $A(\lambda)$ of Fredholm operators of index zero just amounts to the nontriviality of the Poincaré group of the Fredholm operators of index zero in $\mathcal{L}(V, W)$. However, if Λ is a contractible paracompact Hausdorff space, then any continuous family $A(\lambda)$ of $\lambda \in \Lambda$ with values in Fredholm operators of index zero in $\mathcal{L}(V, W)$ possesses a strong parametrix, see Theorem 2.1 of [9] which is referred to as a fundamental result.

Lemma 5.3. *Suppose $F : V \to W$ is a quasilinear Fredholm mapping represented by $F(v) = L(v)v + C(v)$ for $v \in V$. Let $\Pi^- : V^- \to GL(W, V)$ be a continuous mapping with the property that $\Pi^-(v)L^-(v) - I \in \mathcal{K}(V)$ for all $v \in V^-$. Then $\Pi^-(v)F(v) = v - K(v)$ holds valid for all $v \in V$, where $K : V \to V$ is a compact mapping.*

Proof. We get $\Pi^-(v)L^-(v) = I - R^-(v)$ for $v \in V^-$, where $R^- : V^- \to \mathcal{K}(V)$ is continuous. Hence,

$$\begin{aligned} \Pi^-(v)F(v) &= \Pi^-(v)\,(L(v)v + C(v)) = \\ &= \big(I - R^-(v)\big)\,v + \Pi^-(v)C(v) = \\ &= v - K(v) \end{aligned}$$

for all $v \in V$, where $K(v) = R^-(v)v - \Pi^-(v)C(v)$. Since V is compactly embedded into V^- and both

$$\begin{array}{rcl} R^- : & V^- & \to \mathcal{L}(V, W), \\ \Pi^- : & V^- & \to \mathcal{L}(V, W) \end{array}$$

are continuous, the compactness of $K : V \to V$ follows from the compactness of $C : V \to W$ and of each $R^-(v)$ for $v \in V^-$. □

Theorem 5.4. *Let $F : V \to W$ be a quasilinear Fredholm mapping. Then F can be represented as*

$$F(v) = T^-(v)\,(v - K(v)) \tag{5.2}$$

for $v \in V$, where $T^- : V^- \to GL(V, W)$ is a continuous family of isomorphisms and K is a compact mapping of V.

Proof. Write F in the form $F(v) = L(v)v + C(v)$ for $v \in V$. On applying Theorem 2.1 of [9] we choose $\Pi^- : V^- \to GL(W, V)$ to be any strong parametrix for the family L^-. Set

$$T^-(v) := \big(\Pi^-(v)\big)^{-1}$$

for $v \in V^-$ and use Lemma 5.3 to get (5.2), as desired. □

If $A \in \mathcal{L}(V, W)$ is a Fredholm operator of index zero, then the restriction of A to any bounded closed subset of V is proper. The following lemma is a generalisation of this assertion to nonlinear mappings, which is of independent interest as a quite general criterion for establishing properness.

Lemma 5.5. *Assume that* $F : V \to W$ *is a quasilinear Fredholm mapping. If* $\Sigma \subset V$ *is closed and bounded, then* $F : \Sigma \to W$ *is proper.*

Proof. Let $F : V \to W$ be represented by (5.1). Then the properness of $F : \Sigma \to W$ follows from the compactness of the embedding of V into V^-, the compactness of $C : V \to W$ and the continuity of $L^- : V^- \to \mathcal{L}(V, W)$, together with the properness of $L^-(v) : \Sigma \to W$ for each $v \in V^-$. □

We now turn to the boundary value problem composed in Lemma 1.3. The advantage of using quasilinear Fredholm mappings lies in the fact that they require no linearisation of the problem, which may be cumbersome. To illustrate the results explicitly, we restrict our attention to the case $p = 2$, for the theory for $p \neq 2$ does not fit immediately the framework of quasilinear Fredholm operators. If $p = 2$ then (1.6) transforms to

$$\left\{ \begin{array}{rcll} (A - f'_u)^*(Au - f) & = & 0 & \text{in } \overset{\circ}{\mathcal{X}}, \\ Bu & = & u_0 & \text{at } \partial\mathcal{X}, \\ B^*(A - f) & = & 0 & \text{at } \partial\mathcal{X}, \end{array} \right. \tag{5.3}$$

cf. [2]. The differential equation of (5.3) is understood in the sense of distributions in the interior of $\mathcal{X}$. While the direct methods of variational calculus apply to look for a solution $u \in H^1(\mathcal{X}, \mathbb{R}^\ell)$, direct constructions along more classical lines deal with solutions in $H^{2+s}(\mathcal{X}, \mathbb{R}^\ell)$, where $s = 0, 1, \ldots$. Under obvious assumption on f, the problem corresponds to

$$F : H^{s+2}(\mathcal{X}, \mathbb{R}^\ell) \to \begin{array}{c} H^s(\mathcal{X}, \mathbb{R}^\ell) \\ \oplus \\ H^{s+3/2}(\partial\mathcal{X}, \mathbb{R}^{\ell'}) \\ \oplus \\ H^{s+1/2}(\partial\mathcal{X}, \mathbb{R}^{\ell-\ell'}) \end{array}$$

given by $F(u) = L(u)u + C(u)$, where

$$L(v)u = \begin{pmatrix} A^*Au \\ Bu \\ B^*Au \end{pmatrix}, \quad C(u) = \begin{pmatrix} -A^*f - (f'_u)^*(Au - f) \\ 0 \\ -B^*f \end{pmatrix}$$

for $v \in H^1(\mathcal{X}, \mathbb{R}^\ell)$.

Denote by $H^{s+2}_{B,B^*A}(\mathcal{X}, \mathbb{R}^\ell)$ the subspace of $H^{s+2}(\mathcal{X}, \mathbb{R}^\ell)$ that consists of all functions $u \in H^{s+2}(\mathcal{X}, \mathbb{R}^\ell)$ satisfying $Bu = 0$ and $B^*(Au) = 0$ at $\partial\mathcal{X}$. Applying Theorem 5 of [1] we conclude that the boundary value problem $L(v)$ is formally selfadjoint relative to the Green formula for the Laplacian $\Delta := A^*A$. Hence it follows that the operator $\Delta : H^{s+2}_{B,B^*A}(\mathcal{X}, \mathbb{R}^\ell) \to H^s(\mathcal{X}, \mathbb{R}^\ell)$ has index zero. We may select a compact operator $K : H^{s+2}_{B,B^*A}(\mathcal{X}, \mathbb{R}^\ell) \to H^s(\mathcal{X}, \mathbb{R}^\ell)$ such that $\Delta + K : H^{s+2}_{B,B^*A}(\mathcal{X}, \mathbb{R}^\ell) \to H^s(\mathcal{X}, \mathbb{R}^\ell)$ is a bijection. The surjectivity of the boundary operators $\{B, B^*A\}$ then implies that the perturbation of $L(v)$ by $\{KP, 0, 0\}$ is bijective, where P is the projection of $H^{s+2}(\mathcal{X}, \mathbb{R}^\ell)$ onto the kernel of $\{B, B^*A\}$. Since the Fredholm index is invariant under compact perturbation, we deduce that $L(v)$ is Fredholm of index zero, cf. Lemma 10.11 of [9]. Hence, F is a quasilinear Fredholm mapping.

6. Mapping degree of Lagrangian problems

In [9], an additive integer-valued degree theory for quasilinear Fredholm mappings is constructed. The theory is based upon a modification of the well-known techniques of [14] for formulating the solutions of the Dirichlet problem for a quasilinear second order elliptic equation as the zeroes of a compact perturbation of the identity, i.e., fixed points of a compact mapping. Following an idea of [3], it is shown in [9] that general elliptic boundary value problems with sufficiently smooth "coefficients", induce quasilinear Fredholm mappings both in Sobolev and Hölder spaces.

The definition of degree in [9] turns upon first assigning a degree to each linear isomorphism and then extending the degree to general quasilinear Fredholm mappings.

If V and W are finite dimensional of the same dimension, the choice of orientation of V and W defines the determinant $\det T$ for all $T \in GL(V, W)$. Then $\varepsilon : GL(V, W) \to \{\pm 1\}$, defined by $\varepsilon(T) = \operatorname{sgn} \det T$, distinguishes the two connected components of $GL(V, W)$. Of course, $\varepsilon(T)$ is the Brower degree of T with respect to the choice of orientations.

If $V = W$ is infinite dimensional, then the group of compact perturbations of the identity in $GL(V, V)$ also has two components, which are distinguished by the function $\varepsilon(T) = (-1)^N$ where N is the number of the negative eigenvalues of T counted with their algebraic multiplicities. Obviously, $\varepsilon(T)$ just amounts to the Leray-Schauder degree of T.

For general spaces V and W the "group" $GL(V, W)$ may be connected. If we divide $GL(V, W)$ into equivalence classes under the Calkin equivalence relation, to wit $T \sim S$ if $T - S$ is compact, then each equivalence class has two connected components. In fact, if $T - S = K$ then $I - T^{-1}S = T^{-1}K$, and so $T^{-1}S$ is a compact perturbation of the identity. The Leray-Schauder degree of $T^{-1}S$ distinguishes two connected components of the equivalence class indeed. It is reasonable to define the degree so that it would distinguish the components of each Calkin equivalence class. If T and S in $GL(V, W)$ are equivalent, then they lie in the same component of their equivalence class if and only if the Leray-Schauder degree of $T^{-1}S$ is equal to 1. Accordingly, [9] defines a function $\varepsilon : GL(V, W) \to \{\pm 1\}$ to be an orientation provided that $\varepsilon(T)\varepsilon(S)$ just amounts to the Leray-Schauder degree of $T^{-1}S$, if $T, S \in GL(V, W)$ are equivalent. An orientation of $GL(V, V)$ is always required to assign 1 to the identity.

Once an orientation ε is chosen, the degree of F on an open set $U \subset V$ is defined by

$$\deg(F, U) = \varepsilon(T^{-}(0)) \deg(I - K, U, 0), \tag{6.1}$$

where T^{-} and K are as in (5.2) and $\deg(I - K, U, 0)$ is the Leray-Schauder degree of $I - K$ in U with respect to the value 0. The right-hand side of (6.1) is independent of representation (5.2).

The degree defined by (6.1) has the usual additivity, existence and Borsuk-Ulam properties, see [9]. If $V = W$ and $GL(V, V)$ is connected, then any integer-valued degree theory on a class of mappings which includes all linear isomorphisms and which coincides with the Leray-Schauder degree on the class of compact perturbation of the identity can neither be homotopy invariant nor can the classical regular value formula hold.

In [10] a rather different construction of mapping degree is given which uses a stronger notion of orientation than the one used in [9]. If $F : V \to W$ is a C^2 quasilinear Fredholm map which has 0 as a regular point, then the function o defined by $o(x) = \varepsilon(F'(x))\, \sigma(F' \circ \gamma)$, where γ is any path between 0 and the regular point x and $\sigma(F' \circ \gamma)$ the parity of the family F' along γ, is an orientation of the map F in the sense of [10]. Moreover, for any admissible set U in V, the degree of F with respect to o is

$$\deg_o(F, U, w) := \sum_{x \in F^{-1}(w) \cap U} o(x)$$

provided that $w \notin F(\partial U)$ is a regular value of $F : U \to W$. We write it $\deg_o(F, U)$ for short, if $w = 0$.

A major breakthrough came with the paper [11] which remedied the shortcomings of [10]. Indeed, the theory of [10] has required C^2 mappings whereas C^1 mappings would be more natural. The paper [4] is inspired by the approach of [10] though the details are different. The authors define the orientation of a linear Fredholm operator $T : V \to W$ of index zero between Banach spaces as the choice of either of the connected components of the set of all finite rank operators K such that $T + K$ is invertible. They succeed in defining the degree $\deg(F, U, w)$ whenever $F : U \to W$ is a C^1 oriented Fredholm map of index zero between Banach manifolds and $f^{-1}(w)$ is compact, and this degree satisfies the expected properties including invariance under oriented homotopies. For a further progress we refer the reader to [5, 6].

We now turn to the Euler-Lagrange equations of Lemma 1.3. In the initial setting the operator

$$u \mapsto (A - f'_u)^* (|Au - f|^{p-2} (Au - f))$$

is given the domain $W^{1,p}(\mathcal{X}, \mathbb{R}^\ell)$ and maps it to $(\overset{o}{W}{}^{1,p}(\mathcal{X}, \mathbb{R}^\ell))'$. Our objective is to single out the principal part of the operator containing all second order derivatives of u. For this reason our computations will be modulo terms which include the derivatives up to the first order of u. Under obvious conditions on f they can be comprehended as nonlinear compact operators in the relevant Banach spaces. We first write

$$Au = \sum_{j=1}^{n} A^j \, \partial_j u + A^0 u,$$

where A^j and A^0 are $(m \times \ell)$-matrices of smooth functions on $\mathcal{X}$. On using this formula we get

$$\begin{aligned} & (A - f'_u)^* \big(|Au - f|^{p-2} (Au - f)\big) = \\ = \ & |Au - f|^{p-2} A^* A u - \sum_{j=1}^{n} A^{j*} \, (Au - f) \, \partial_j |Au - f|^{p-2} \end{aligned} \qquad (6.2)$$

modulo first order terms. The function Au takes on its values in $\mathbb{R}^m$, and we think of Au as an m-column with entries $A_1 u, \ldots, A_m u$. By the definition, each A_k is an ℓ-row of scalar partial differential operators of the first order on $\mathcal{X}$. More precisely, we obtain

$$A_k u = \sum_{i=1}^{n} A_k^i \, \partial_i u + A_k^0 u$$

for $k = 1, \ldots, m$, where A_k^i and A_k^0 are the kth rows of the matrices A^i and A^0, respectively. Now a trivial verification shows that

$$\begin{aligned} \partial_j |Au - f|^{p-2} &= \partial_j \Big(\sum_{k=1}^{m} (A_k u - f_k)^2 \Big)^{\frac{p-2}{2}} = \\ &= \frac{p-2}{2} \, |Au - f|^{p-4} \Big(\sum_{k=1}^{m} 2 \, (A_k u - f_k) \, \partial_j (A_k u - f_k) \Big) = \\ &= (p-2) \, |Au - f|^{p-4} \Big(\sum_{k=1}^{m} (A_k u - f_k) \sum_{i=1}^{n} A_k^i \partial_j \partial_i u \Big) \end{aligned}$$

modulo nonlinear terms which include the derivatives of u of order not exceeding one. On the other hand, we have

$$A^{j*} = \begin{pmatrix} A_1^{j*} & \ldots & A_m^{j*} \end{pmatrix}$$

for all $j = 1, \ldots, n$, whence

$$A^{j*} \, (Au - f) = \sum_{l=1}^{m} A_l^{j*} \, (A_l u - f_l).$$

Substituting these equalities into (6.2) yields

$$
\begin{aligned}
&(A-f'_u)^*\big(|Au-f|^{p-2}(Au-f)\big) = \\
&= \ |Au-f|^{p-2}\Big(A^*Au-(p-2)\sum_{k,l=1}^{m}\frac{A_l u-f_l}{|Au-f|}\,\frac{A_k u-f_k}{|Au-f|}\sum_{i,j=1}^{n}A_l^{j\,*}A_k^i\partial_j\partial_i u\Big)
\end{aligned}
$$

modulo nonlinear terms containing the derivatives of u of order $\leqslant 1$. It is easily seen that

$$
-\sum_{i,j=1}^{n}A_l^{j\,*}A_k^i\partial_j\partial_i u = A_l^* A_k u
$$

up to terms containing the derivatives of u of order at most one. This gives the final formula

$$
\begin{aligned}
&(A-f'_u)^*\big(|Au-f|^{p-2}(Au-f)\big) = \\
&= \ |Au-f|^{p-2}\Big(A^*Au+(p-2)\sum_{k,l=1}^{m}\frac{A_l u-f_l}{|Au-f|}\,\frac{A_k u-f_k}{|Au-f|}\,A_l^*A_k u\Big)
\end{aligned}
\tag{6.3}
$$

up to terms containing the derivatives of u of order $\leqslant 1$. Formula (6.3) gains in interest if we observe that

$$
A^*A=\sum_{k=1}^{n}A_k^*A_k.
$$

Remark 6.1. For the classical p-Laplace operator in $\mathbb{R}^n$ equality (6.3) takes the form

$$
\Delta_p u = |\nabla u|^{p-2}\Big(-\Delta u-(p-2)\sum_{k,l=1}^{n}\frac{\partial_l u}{|\nabla u|}\,\frac{\partial_k u}{|\nabla u|}\,\partial_l\partial_k u\Big)
$$

modulo terms containing the derivatives of u up to order one.

Summarising we conclude that the operator corresponding to the Euler-Lagrange equations (1.6)

$$
F:\,W^{1,p}(\mathcal{X},\mathbb{R}^\ell)\to
\begin{array}{c}
W^{-1,p'}(\mathcal{X},\mathbb{R}^\ell)\\
\oplus\\
W^{1/p',p}(\partial\mathcal{X},\mathbb{R}^{\ell'})\\
\oplus\\
W^{-1/p',p'}(\partial\mathcal{X},\mathbb{R}^{\ell-\ell'})
\end{array}
$$

can be written in the form $F(u)=L(u)u+C(u)$, where

$$
L(v)u=\left(
\begin{array}{c}
|Av-f|^{p-2}\Big(A^*Au+(p-2)\displaystyle\sum_{k,l=1}^{m}\frac{A_l v-f_l}{|Av-f|}\,\frac{A_k v-f_k}{|Av-f|}\,A_l^*A_k u\Big)\\
Bu\\
|Av-f|^{p-2}B^*Au
\end{array}
\right)
$$

for $v\in W^{1,p}(\mathcal{X},\mathbb{R}^\ell)$, and C is a nonlinear compact operator. One sees readily that, if $Av-f(\cdot,v)$ vanishes at some point of $\mathcal{X}$, then the boundary value problem $L(v)$ is degenerate.

Theorem 6.2. *Let $Av(x)-f(x,v)\neq 0$ for all $x\in\mathcal{X}$. Then the differential equation of $L(v)$ is elliptic in $\mathcal{X}$.*

Proof. The theorem just amounts to saying that the second order partial differential operator

$$L = A^*A + (p-2) \sum_{k,l=1}^{m} \bar{a}_l a_k \, A_l^* A_k$$

is elliptic in $\mathcal{X}$, where $a_k = \dfrac{A_k v - f_k}{|Av - f|}$ for $k = 1, \ldots, m$.

Fix $x \in \mathcal{X}$ and denote by $\sigma(L) = \sigma^2(L)(x,\xi)$ the principal symbol of L at a point $(x,\xi) \in T^*\mathcal{X}$, where $\xi \in T_x^*\mathcal{X}$ is different from zero. An easy computation shows that

$$\sigma(L) = \sum_{k=1}^{m} (\sigma(A_k))^* \sigma(A_k) + (p-2) \Big(\sum_{l=1}^{m} a_l \, \sigma(A_l) \Big)^* \Big(\sum_{k=1}^{m} a_k \, \sigma(A_k) \Big),$$

where $\sigma(A_k) = \sigma^1(A_k)(x,\xi)$ is the principal symbol of A_k at (x,ξ). The invertibility of $\sigma(L) : \mathbb{R}^\ell \to \mathbb{R}^\ell$ will be established once we prove that $(\sigma(L)u, u) > 0$ for each nonzero vector $u \in \mathbb{R}^\ell$.

We get

$$(\sigma(L)u, u) = \sum_{k=1}^{m} |\sigma(A_k)u|^2 + (p-2) \Big| \sum_{k=1}^{m} a_k \, \sigma(A_k) u \Big|^2,$$

which is obviously nonnegative if $p \geqslant 2$. Furthermore, if $1 < p < 2$, then using the Cauchy inequality yields

$$\begin{aligned} (\sigma(L)u, u) &\geqslant \sum_{k=1}^{m} |\sigma(A_k)u|^2 + (p-2) \sum_{k=1}^{m} |\sigma(A_k)u|^2 \geqslant \\ &\geqslant 0, \end{aligned} \tag{6.4}$$

for $1 + (p-2) > 0$.

It remains to show that $(\sigma(L)u, u) = 0$ for $u \in \mathbb{R}^\ell$ implies $u = 0$. If $p \geqslant 2$, then from $(\sigma(L)u, u) = 0$ it follows that $\sigma(A_k)u = 0$ for all $k = 1, \ldots, m$. Since the principal symbol mapping of A is injective, we conclude that $u = 0$, as desired. The same proof remains valid for $1 < p < 2$, for if $\sigma(A)u \neq 0$, then $(\sigma(L)u, u) > 0$, which is due to (6.4). $\square$

Thus, if the system of boundary operators $\{B, B^*A\}$ satisfies the Shapiro-Lopatinskii condition, then $L(v)$ is actually an elliptic boundary value problem. To get rid of degeneracy it suffices to cancel the scalar factor $|Av - f|^{p-2}$, thus obtaining a problem essentially selfadjoint with respect to the Green formula, see Theorem 5 of [1]. Therefore, the theory of [4, 11] still applies to Lagrangian boundary value problems.

7. Perturbed Dirichlet problem

In this section we consider the Dirichlet problem for the perturbed Laplace equation and prove criteria which are needed to apply the degree.

Let $\mathcal{X}$ be a bounded closed domain with smooth boundary in $\mathbb{R}^n$. Consider the problem

$$\left\{ \begin{array}{rcll} \Delta u &=& f(x, u, u') & \text{in } \ \mathcal{X}, \\ u &=& 0 & \text{at } \ \partial\mathcal{X}, \end{array} \right. \tag{7.1}$$

where f is a nonlinear C^1 function of its numerical arguments $(x, u, p) \in \mathcal{X} \times \mathbb{R} \times \mathbb{R}^n$ satisfying

$$|f| \leqslant C \, \langle p \rangle^\gamma, \quad |f'_u| \leqslant C \, \langle p \rangle^\gamma, \quad |f'_p| \leqslant C, \tag{7.2}$$

with $\gamma < 1$ and C a constant independent of x, u and p. Here, we use the designation $\langle p \rangle = (1+|p|^r)^{1/r}$ with $r=2$ or with any other $r>0$, for all the expressions are equivalent.

Choose $V := \overset{o}{H}{}^1(\mathcal{X})$ and $W := H^{-1}(\mathcal{X})$ with norms

$$\begin{aligned} \|u\|_V &= \Big(\int_{\mathcal{X}} |u'|^2 dx \Big)^{1/2}, \\ \|f\|_W &= \sup_{\|u\|_V=1} \Big| \int_{\mathcal{X}} f \bar{u}\, dx \Big|. \end{aligned}$$

Then, $F(u) := \Delta u - f(x,u,u')$ maps V continuously into W and it is an elliptic operator.

Lemma 7.1. *The Laplace operator $\Delta : V \to W$ is an isomorphism and C^1, and so a C^1 Fredholm operator of index* 0.

Proof. To show that $\Delta : V \to W$ is an isomorphism, note that if $u \in V$ and $\Delta u = 0$ then $u = 0$, for u is a harmonic function vanishing at the boundary. Thus, $\Delta : V \to W$ is one to one. We now assume that $f \in H^{-1}(\mathcal{X})$. The equation $\Delta u = f$ for $u \in V$ is understood in the weak sense, i.e., $a(u,v) = f(\bar{v})$ for every $v \in V$, where

$$a(u,v) = \int_{\mathcal{X}} (u', v')_x dx$$

stands for the inner product in V. By the Riesz representation theorem there is a unique $u \in V$ satisfying $a(u,v) = f(\bar{v})$ for all $v \in V$. Hence it follows that $\Delta : V \to W$ is onto. Moreover, Δ is a linear operator and hence C^1. Thus, $\Delta : V \to W$ is a C^1 isomorphism. □

Lemma 7.2. *Under assumptions (7.2) the Nemytskii map $u \mapsto f(x,u,u')$ is a C^1 compact operator.*

Proof. We first observe that, for a fixed $u \in V$, the function $x \mapsto f(x,u(x),u'(x))$ belongs to $L^p(\mathcal{X})$ with any $p \geqslant 1$. Consider the map

$$\overset{o}{H}{}^1(\mathcal{X}) \to L^2(\mathcal{X}) \times L^2(\mathcal{X})^n \overset{N_f}{\to} L^2(\mathcal{X}) \hookrightarrow H^{-1}(\mathcal{X}), \tag{7.3}$$

where by the first arrow is meant the map $u \mapsto (u,u')$ and by the second arrow the map $(u,u') \mapsto f(x,u,u')$. The first map is linear and bounded, hence it is continuous and C^1. On the other hand, from Theorem 10.58 of [21] and the first inequality of (7.2) it follows that N_f is a continuous map from $L^2(\mathcal{X}) \times L^2(\mathcal{X})^n$ to $L^2(\mathcal{X})$. And finally the embedding of $L^2(\mathcal{X})$ into $H^{-1}(\mathcal{X})$ is also continuous and C^1. Therefore, (7.3) is a composition of continuous maps and thus is continuous. Moreover, since the last embedding is compact, (7.3) is a compact map from V to W. On the other hand, the remaining estimates of (7.2) together with Theorem 10.58 of [21] imply that N_f is C^1, and so (7.3) is C^1 as composition of C^1 maps. □

We conclude that the map $F : V \to W$ is of the form $L + C$, where $Lu := \Delta u$ is a linear Fredholm operator of index zero and $Cu := -f(x,u,u')$ is a compact operator. If u is a smooth function with compact support in the interior of the closed domain $\mathcal{X}$, then

$$\|\Delta u\|_W = \sup_{\|v\|_V=1} \int_{\mathcal{X}} \Delta u\, \bar{v} dx.$$

On integrating by parts we get

$$\|\Delta u\|_W = \sup_{\|v\|_V=1} \Big| \int_{\mathcal{X}} (u', v')_x dx \Big| = \sup_{\|v\|_V=1} \big| (u,v)_V \big|$$

and choosing

$$v = \frac{u}{\|u\|_V}$$

yields $\|\Delta u\|_W \geqslant \|u\|_V$. On the other hand, $\|\Delta u\|_W \leqslant \|u\|_V$, which is clear from the Cauchy-Schwarz inequality. Thus,

$$\|\Delta u\|_W = \|u\|_V$$

which extends by continuity to all functions $u \in \overset{o}{H}{}^1(\mathcal{X})$.

If $u \in V$ is a solution of (7.1) then $\Delta u = f(x, u, u')$, hence

$$\begin{array}{rcl} \|u\|_V &=& \|f(x,u,u')\|_W \leqslant \\ &\leqslant& c\,\|f(x,u,u')\|_{L^2(\mathcal{X})} \end{array}$$

with c a constant independent of u. Furthermore, applying the first estimate of (7.2) on f we get

$$\begin{array}{rcl} \|f(x,u,u')\|^2_{L^2(\mathcal{X})} &\leqslant& C^2 \displaystyle\int_{\mathcal{X}} \langle u' \rangle^{2\gamma}\, dx \leqslant \\ &\leqslant& C^2 \Big(\displaystyle\int_{\mathcal{X}} dx \Big)^{1-\gamma} \Big(\displaystyle\int_{\mathcal{X}} \langle u' \rangle^2 dx \Big)^{\gamma} \leqslant \\ &\leqslant& C \left(1 + \|u\|^2_V \right)^{\gamma}, \end{array}$$

where C is a constant independent of u which may be different in diverse applications. Thus,

$$\|u\|_V \leqslant C \left(1 + \|u\|^2_V \right)^{\gamma/2}$$

for all $u \in V$ satisfying (7.1). Since the right hand side is a sublinear function of $\|u\|_V$, such an a priori estimate occurs only if $\|u\|_V$ is bounded, i.e. $\|u\|_V \leqslant R$ for some constant $R > 0$ independent of u.

We may now appeal to the concept of mapping degree to show the existence of a solution to problem (7.1). The specific concept we use here is that of regular point degree clarified in [11, 7.1].

Let U be the ball of radius $2R$ with centre at the origin in V. By Lemmata 7.1 and 7.2, F is a C^1 map from $\overline{U}$ to W. By the above a priori estimate, $F^{-1}(0)$ belongs to the ball $U/2$, and hence F does not vanish at ∂U. It follows that the mapping degree $\deg(F, U)$ is well defined. To compute this degree, we consider the homotopy

$$F_t(u) = \Delta u - t\, f(x, u, u')$$

for $t \in [0, 1]$. Obviously, F_t is a C^1 map, for each $t \in [0, 1]$, and the same a priori estimate shows that $F_t^{-1}(0) \subset U/2$. Therefore, F_t does not vanish at ∂U for all $t \in [0, 1]$. Then, the homotopy invariance of the mapping degree implies that $\deg(F, U) = \deg(\Delta, U)$.

By Lemma 7.1, $\Delta : V \to W$ is a (linear) isomorphism, and so the mapping degree $\deg(\Delta, U)$ is different from zero. This implies immediately that $\deg(F, U) \neq 0$. On using the normalisation property of mapping degree [11] we conclude that the set $F^{-1}(0)$ is nonempty, i.e., problem (7.1) has at least one solution $u \in V$, as desired.

This result extends in an obvious way to the Dirichlet problem for perturbations of the Laplace operator $\Delta = A^* A$, where A is a first order overdetermined elliptic differential operator satisfying the uniqueness condition for the local Cauchy problem $(U)_s$, see [26].

8. The Dirichlet problem for the p-Laplace equation

In this section we consider the Dirichlet problem for the perturbed p-Laplace equation. Let $\mathcal{X}$ be a bounded closed domain with smooth boundary in $\mathbb{R}^n$. Consider the problem

$$\left\{ \begin{array}{rcll} \Delta_p u & = & f(x,u,u') & \text{in } \ \mathcal{X}, \\ u & = & 0 & \text{at } \ \partial\mathcal{X}, \end{array} \right. \tag{8.1}$$

where $\Delta_p u := \nabla^* (|\nabla u|^{p-2} \nabla u)$. The right hand side f is assumed to be a nonlinear C^1 function of its numerical arguments $(x,u,p) \in \mathcal{X} \times \mathbb{R} \times \mathbb{R}^n$ satisfying inequalities (7.2) with some $\gamma < p-1$.

Choose $V := \overset{o}{W}{}^{1,p}(\mathcal{X})$ and $W := W^{-1,p'}(\mathcal{X})$ with norms

$$\begin{array}{rcl} \|u\|_V & = & \displaystyle \Big(\int_{\mathcal{X}} |u'|^p dx \Big)^{1/2}, \\ \|f\|_W & = & \displaystyle \sup_{\|u\|_V = 1} \Big| \int_{\mathcal{X}} f \bar{u} \, dx \Big|, \end{array}$$

where $1/p + 1/p' = 1$. Then, $F(u) := \Delta_p u - f(x,u,u')$ maps V continuously into W and it is a degenerate elliptic operator.

Lemma 8.1. *The map $F : V \to W$ is C^1 and it admits a regular point u_0 in V, i.e., $F'(u_0) \in \mathrm{GL}(V,W)$.*

Proof. Using the chain rule we see that the Fréchet derivative of the p-Laplace operator at a point $u_0 \in V$ is given by

$$\begin{array}{rcl} \Delta_p'(u_0) u & = & \displaystyle \nabla^* \Big(|\nabla u_0|^{p-2} \Big(E_n + (p-2) \frac{\nabla u_0}{|\nabla u_0|} \Big(\frac{\nabla u_0}{|\nabla u_0|} \Big)^* \Big) \nabla u \Big) = \\ & = & \nabla^* \left(a(x) \nabla u \right) \end{array}$$

for $u \in V$. Note that $a(x)$ is a symmetric $(n \times n)$-matrix with entries in $L^{\frac{p}{p-2}}(\mathcal{X})$. By Theorem 6.2, $\Delta_p'(u_0)$ is a second order elliptic operator away from the critical points of u_0 in $\mathcal{X}$.

On the other hand, the Fréchet derivative of the map $\hat{f} : V \to W$ given by $u \mapsto f(x,u,u')$ is

$$\hat{f}'(u_0) u = f_u'(x, u_0, \nabla u_0) u + f_p'(x, u_0, \nabla u_0) \nabla u$$

for $u \in V$. The inhomogeneous equation $F'(u_0)u = w$ with $w \in W$ just amounts to finding a $u \in V$ which satisfies

$$\nabla^* (a(x) \nabla u) - f_u'(x, u_0, \nabla u_0) u - f_p'(x, u_0, \nabla u_0) \nabla u = w$$

weakly in $\mathcal{X}$.

We now refer to [12] to see that in any ball around the origin in V there is a function u_0, such that $F'(u_0)u = w$ has a unique solution $u \in V$ for each right hand side $w \in W$. In other words, $F'(u_0) \in \mathrm{GL}(V,W)$, i.e., u_0 is a regular point of F, as desired. $\square$

If u is a smooth function with compact support in the interior of the closed domain $\mathcal{X}$, then

$$\|\Delta_p u\|_W = \sup_{\|v\|_V = 1} \int_{\mathcal{X}} \Delta_p u \, \bar{v} dx.$$

On integrating by parts we get

$$\|\Delta_p u\|_W = \sup_{\|v\|_V = 1} \Big| \int_{\mathcal{X}} (\Delta_p u, v)_x dx \Big| = \sup_{\|v\|_V = 1} \Big| \int_{\mathcal{X}} |\nabla u|^{p-2} \, (\nabla u, \nabla v)_x dx \Big|.$$

Let

$$v = \frac{u}{\|u\|_V},$$

then

$$\begin{aligned}\int_{\mathcal{X}} |\nabla u|^{p-2} (\nabla u, \nabla v)_x dx &= \frac{1}{\|u\|_V} \int_{\mathcal{X}} |\nabla u|^{p-2} (\nabla u, \nabla u)_x dx = \\ &= \frac{1}{\|u\|_V} \int_{\mathcal{X}} |\nabla u|^p \, dx = \\ &= \|u\|_V^{p-1}\end{aligned}$$

whence $\|\Delta_p u\|_W \geqslant \|u\|_V^{p-1}$. On the other hand, if $v \in \overset{o}{W}{}^{1,p}(\mathcal{X})$ and $\|v\|_V = 1$, then

$$\begin{aligned}\Big| \int_{\mathcal{X}} |\nabla u|^{p-2} (\nabla u, \nabla v)_x dx \Big| &\leqslant \|\nabla v\|_{L^p(\mathcal{X})} \Big(\int_{\mathcal{X}} |\nabla u|^{(p-1)p'} (\nabla u, \nabla v)_x dx \Big)^{1/p'} = \\ &= \|u\|_V^{p-1},\end{aligned}$$

the first estimate being due to the Hölder inequality. Thus,

$$\|\Delta_p u\|_W = \|u\|_V^{p-1}$$

which extends by continuity to all functions $u \in \overset{o}{W}{}^{1,p}(\mathcal{X})$.

If $u \in V$ is a solution of (8.1) then $\Delta_p u = f(x, u, u')$, hence

$$\begin{aligned}\|u\|_V^{p-1} &= \|f(x, u, u')\|_W \leqslant \\ &\leqslant c \, \|f(x, u, u')\|_{L^p(\mathcal{X})}\end{aligned}$$

with c a constant independent of u. Furthermore, on applying the first estimate of (7.2) on f we obtain

$$\begin{aligned}\|f(x, u, u')\|_{L^p(\mathcal{X})}^p &\leqslant C^p \int_{\mathcal{X}} \langle u' \rangle^{p\gamma} \, dx \leqslant \\ &\leqslant C \Big(\int_{\mathcal{X}} \langle u' \rangle^p dx \Big)^{\gamma} \leqslant \\ &\leqslant C \, (1 + \|u\|_V^p)^{\gamma},\end{aligned}$$

where C is a constant independent of u which may be different in diverse applications. Thus,

$$\|u\|_V \leqslant C \, (1 + \|u\|_V^p)^{\gamma/p(p-1)}$$

for all $u \in V$ satisfying (8.1). Since $\gamma < p-1$, the right hand side of this inequality is a sublinear function of $\|u\|_V$. On arguing as in Section 7. we see that there is a constant $R > 0$ with the property that $\|u\|_V \leqslant R$ is fulfilled for all $u \in V$ satisfying (8.1).

Let U be the ball of radius $2R$ with centre at the origin in V. By Lemma 8.1, F is a C^1 map from $\overline{U}$ to W and it has a regular point $u_0 \in U$. By the above a priori estimate, $F^{-1}(0)$ belongs to the ball $U/2$, and hence $F(u) \neq 0$ for all $u \in \partial U$. It follows that the mapping degree $\deg_{u_0}(F, U)$ is well defined, see [11, 7.1]. To compute this degree, we consider the homotopy

$$F_t(u) = \Delta_p u - t \, f(x, u, u')$$

for $t \in [0, 1]$. Obviously, F_t is a C^1 map, for each $t \in [0, 1]$, and the same a priori estimate shows that $F_t^{-1}(0) \subset U/2$. Therefore, F_t does not vanish at ∂U for all $t \in [0, 1]$. Then, the homotopy invariance of the mapping degree implies that $\deg\,(F, U) = \deg\,(\Delta_p, U)$.

The mapping $\Delta_p : V \to W$ is well known to be an isomorphism, see for instance [23] and elsewhere. This allows one to conclude that the mapping degree $\deg(\Delta_p, U)$ is different from zero. Hence it follows that $\deg(F, U) \neq 0$, which implies immediately that $F^{-1}(0) \neq \emptyset$. Therefore, problem (8.1) has at least one solution $u \in V$, as desired.

For a deeper discussion of the Dirichlet problem for compact perturbations of the p-Laplace equation along more classical lines with $f : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function we refer the reader to [8].

No attempt has been made here to generalise this result to the Dirichlet problem for the p-Laplace operator $u \mapsto A^*(|Au|^{p-2} Au)$ related to a first order overdetermined elliptic differential operator A satisfying the uniqueness condition for the local Cauchy problem $(U)_s$.

Conclusion

As a byproduct of our study of Lagrangian boundary value problems in $\mathcal{X}$ we derived a linearisation of the nonlinear Laplace operator in general outline up to first order terms. It looks like

$$\Delta(v)u = A^*Au + \lambda \sum_{k,l=1}^{m} \frac{A_l v - f_l(\cdot, v)}{|Av - f(\cdot, v)|} \frac{A_k v - f_k(\cdot, v)}{|Av - f(\cdot, v)|} A_l^* A_k u,$$

where A is an $(m \times \ell)$-matrix of first order partial differential operators on $\mathcal{X}$ and $A_1, \dots, A_m$ the rows of A. If the principal symbol mapping of A is injective away from the zero section of $T^*\mathcal{X}$ and $\lambda > -1$, then $\Delta(v)$ is elliptic. This operator is supplied with two boundary operators B and B^*A and the problem of solvability of the corresponding boundary value problem in $\mathcal{X}$ is of central interest in the present paper.

Remark 8.2. The operator $\Delta(v)$ is elliptic for all real $\lambda > -1$ and it coincides with A^*A for $\lambda = 0$. Hence, the index of the boundary value problem $\{\Delta(v), B, B^*A\}$ amounts to that of $\{A^*A, B, B^*A\}$ if the boundary operators satisfy the Shapiro-Lopatinskii condition.

The first author gratefully acknowledges the financial support of the Deutscher Akademischer Austauschdienst.

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Теория степени для лагранжевых краевых задач

Аммар Аль-Саеди
Университет Анахрайна
Багдад, Ирак
Николай Тарханов
Потсдамский университет
Потсдам, Германия

Аннотация. Мы изучаем те нелинейные уравнения с частными производными, которые возникают как уравнения Эйлера-Лагранжа вариационных задач. Определяя слабые граничные значения решений таких уравнений, мы инициируем теорию лагранжевых краевых задач в функциональных пространствах подходящей гладкости. Мы также анализируем, применяется ли современная концепция степени отображения к лагранжевым проблемам.

Ключевые слова: нелинейные уравнения, лагранжева система, слабые граничные значения, квазилинейные операторы Фредхольма, степень отображения.

Journal of Siberian Federal University. Mathematics & Physics 2020, 13(1), 26–36

DOI: 10.17516/1997-1397-2020-13-1-26-36
УДК 532.5.013.4

On the Asymptotic Behavior of the Conjugate Problem Describing a Creeping Axisymmetric Thermocapillary Motion

Victor K. Andreev*
Evgeniy P. Magdenko†
Institute of Computational Modelling SB RAS
Krasnoyarsk, Russian Federation
Siberian Federal University
Krasnoyarsk, Russian Federation

Received 04.03.2019, received in revised form 10.11.2019, accepted 08.12.2019

Abstract. In this paper the conditions for the law of temperature behavior on a solid cylinder wall describes, under which the solution of a linear conjugate inverse initial-boundary value problem describing a two-layer axisymmetric creeping motion of viscous heat-conducting fluids tends to zero exponentially with increases of time.

Keywords:the conjugate nonlinear inverse problem, interface, a crawling motion.

Citation: V.K.Andreev, E.P.Magdenko, On the Asymptotic Behavior of the Conjugate Problem Describing a Creeping Axisymmetric Thermocapillary Motion, J. Sib. Fed. Univ. Math. Phys., 2020, 13(1), 26–36. DOI: 10.17516/1997-1397-2020-13-1-26-36.

1. Introduction and preliminaries

In work [1], the linear conjugate inverse initial boundary value problem describing a two-layer creeping motion of viscous heat-conducting fluids in a cylinder with a solid side surface $r = R_2 = \text{const}$ and interface $r = h(t)$, $0 < h(t) < R_2$ was considered

$$v_{1t} = \nu_1 \left(v_{1rr} + \frac{1}{r} v_{1r} \right) + f_1(t), \quad 0 < r < R_1, \tag{1}$$

$$v_{2t} = \nu_2 \left(v_{2rr} + \frac{1}{r} v_{2r} \right) + f_2(t), \quad R_1 < r < R_2, \tag{2}$$

$$v_1(R_1, t) = v_2(R_1, t), \quad \int_0^{R_1} r v_1(r, t) dr + \int_{R_1}^{R_2} r v_2(r, t) dr = 0, \tag{3}$$

$$\mu_1 v_{1r}(R_1, t) - \mu_2 v_{2r}(R_1, t) = -2æ a_1(R_1, t), \tag{4}$$

$$|v_1(0, t)| < \infty, \quad v_2(R_2, t) = 0, \tag{5}$$

$$v_1(r, 0) = 0, \quad v_2(r, 0) = 0, \tag{6}$$

$$\rho_1 f_1(t) = \rho_2 f_2(t) - \frac{2æ a_1(R_1, t)}{R_1} \tag{7}$$

*andr@icm.krasn.ru
†magdenko_evgeniy@icm.krasn.ru https://orcid.org/0000-0003-2354-1152

and the closed conjugate problem for functions $a_j(r,t)$ is described the following equations:

$$a_{jt} = \chi_j \left(a_{jrr} + \frac{1}{r} a_{jr} \right), \tag{8}$$

$$a_j(r,0) = a_j^0(r), \quad |a_1(0,t)| < \infty, \tag{9}$$

$$a_2(R_2,t) = \alpha(t), \tag{10}$$

$$a_1(R_1,t) = a_2(R_1,t), \quad k_1 a_{1r}(R_1,t) = k_2 a_{2r}(R_2,t). \tag{11}$$

The interface is described by the formula

$$h(t) = R_1[1 + \mathrm{M}\, h_1(t)], \quad h_1(t) = -\frac{1}{R_1} \int_0^t r v_1(R_1,t) dt. \tag{12}$$

Here $\mathrm{M} = æ a^1 R_1^3/\mu_1\chi_1$ is Marangoni number, $a^1 = \max\limits_{t\in[0,T]} |\alpha(t)|$. Note that $\mathrm{M} \to 0$ since the creeping motion considers in this paper.

In paper [1] the priori estimates were obtained for the functions $v_j(r,t)$, $a_j(r,t)$, $f_j(t)$. In this paper, it will be proved that under certain conditions which set for the temperature on the cylinder surface, the solution of the problem (1)–(11) tends to zero exponentially with increasing time.

2. The behavior of the solution under $t \to \infty$

A priori estimates for the function $a_j(r,t)$ satisfying the problem (8)–(11) have form [1]

$$|a_1(r,t)| \leqslant 2\left[\max_{t\in[0,T]} |\alpha(t)| + \frac{1}{(R_1^2 k_2 \rho_2 c_{\rho_2})^{1/4}} \max_{t\in[0,T]} (A(t)A_1(t))^{1/4}\right] + \max_{r\in[0,R_1]} |a_1^0(r)|, \tag{13}$$

$$|a_2(r,t)| \leqslant |\alpha(t)| + 2\left(\frac{1}{R_1^2 k_2 \rho_2 c_{\rho_2}} A(t) A_1(t)\right)^{1/4}, \tag{14}$$

where

$$A(t) \leqslant \left(\sqrt{A_0} + \frac{1}{2}\int_0^t G(\tau) e^{\eta\tau} d\tau\right)^2 e^{-2\eta t}, \tag{15}$$

$$A_1(t) = k_1 \int_0^{R_1} r (a_{1r}^0)^2\, dr + k_2 \int_{R_1}^{R_2} r(\bar{a}_{2r}^0)^2\, dr \; + \rho_2 c_{p_2} \int_0^t \int_{R_1}^{R_2} r g_2(r,t)\, dr\, dt. \tag{16}$$

Here A_0 is value of function $A(t)$ at $t = 0$ and

$$G(t) = \max_j \left(\frac{2}{\rho_j c_{p_j}}\right)^{1/2} \left(\int_{R_1}^{R_2} r g_2^2\, dr\right)^{1/2}, \tag{17}$$

$$\bar{a}_2(r,t) = a_2(r,t) - \frac{\alpha(t)(r-R_1)^2}{(R_2-R_1)^2}, \tag{18}$$

$$g_2(r,t) = \frac{2\chi_2 \alpha(t)}{(R_2-R_1)^2}\left(2 - \frac{R_1}{r}\right) - \frac{\alpha'(r-R_1)^2}{(R_2-R_1)^2}. \tag{19}$$

If the function $\alpha(t)$ and its derivatives $\alpha'(t)$, $\alpha''(t)$, $\alpha'''(t)$ are defined for all $t \geqslant 0$, there is a question about the behavior of the problems solutions (1)–(11) at $t \to \infty$. From the definition of (19) the inequality is valid for the functions $g_2(r,t)$

$$\int_{R_1}^{R_2} r g_2^2 dr \leqslant \frac{2}{(R_2-R_1)^4}\int_{R_1}^{R_2}\left[4\chi_2^2\left(2-\frac{R_1}{r}\right)^2\alpha^2(t) + \right.$$
$$\left. + (r-R_1)^4(\alpha'(t))^2\right] r dr \leqslant 2R_2(R_2-R_1)(\alpha'(t))^2 + \frac{32\chi_2^2\alpha^2(t)}{(R_2-R_1)^3}$$

(for integrals over r, an upper estimate is given but not their exact value, which can be quite cumbersome), so from (17) we have

$$G(t) \leqslant \left[\max_j\left(\frac{2}{\rho_j c_{\rho_j}}\right)\right]^{1/2}\left[2R_2(R_2-R_1)(\alpha'(t))^2 + \frac{32\chi_2^2\alpha^2(t)}{(R_2-R_1)^3}\right]^{1/2} \leqslant$$
$$\leqslant 2\left[\max_j\left(\frac{1}{\rho_j c_{\rho_j}}\right)\right]^{1/2}\left[\frac{4\chi_2}{(R_2-R_1)^{3/2}}|\alpha(t)| + \sqrt{R_2(R_2-R_1)}|\alpha'(t)|\right]. \qquad (20)$$

So from (15) we obtain

$$A(t) \leqslant \left\{\sqrt{A_0} + \left[\max_j\left(\frac{1}{\rho_j c_{\rho_j}}\right)\right]^{1/2}\left[\frac{4\chi_2}{(R_2-R_1)^{3/2}}\int_0^t |\alpha(\tau)|e^{\eta\tau}d\tau + \right.\right.$$
$$\left.\left. + \sqrt{R_2(R_2-R_1)}\int_0^t |\alpha'(\tau)|e^{\eta\tau}d\tau\right]\right\}^2 e^{-2\eta t}. \qquad (21)$$

From (16) and (19) the estimate is valid

$$|A_1(t)| \leqslant k_1\int_0^{R_1} r(a_{1r}^0)^2 dr + k_2\int_{R_1}^{R_2} r(\bar{a}_{2r}^0)^2 dr +$$
$$+ \rho_2 c_{\rho_2} R_2\left[\frac{4\chi_2}{R_2-R_1}\int_0^t |\alpha(\tau)|d\tau + (R_2-R_1)\int_0^t |\alpha'(\tau)|d\tau\right]. \qquad (22)$$

We suppose that the following integrals converge

$$\int_0^\infty |\alpha(\tau)|e^{\eta\tau}d\tau, \quad \int_0^\infty |\alpha'(\tau)|e^{\eta\tau}d\tau, \qquad (23)$$

then the expression for function modules $|\alpha(\tau)|$ and $|\alpha'(\tau)|$ have the form

$$|\alpha(\tau)| = \alpha_1(t)e^{-\eta\tau}, \quad |\alpha'(\tau)| = \alpha_2(t)e^{-\eta\tau} \qquad (24)$$

with non-negative functions $\alpha_1(t)$, $\alpha_2(t)$, at that $\alpha_1(t) \to 0$, $\alpha_2(t) \to 0$ at $t \to \infty$ and the following estimate is valid

$$\int_0^\infty \alpha_k(\tau)d\tau < \infty, \quad k = 1, 2. \qquad (25)$$

The convergence of integrals

$$\int_0^\infty |\alpha(\tau)|d\tau, \quad \int_0^\infty |\alpha'(\tau)|d\tau,$$

follows from (24), (25), so from (14), (21), (22) we obtain exponential convergence to zero of the function $a_2(r,t)$ $\forall r \in [R_1, R_2]$:

$$|a_2(r,t)| \leqslant \alpha_1(t)e^{-\eta t} + 2\left(\frac{A_1(\infty)D^2}{R_1^2 k_2 \rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}, \tag{26}$$

where in the quality D we have designed the value of the expression in curly brackets (21) at $t = \infty$.

For $a_1(r,t)$ from the estimate (13)we find

$$|a_1(r,t)| \leqslant 2\left[\alpha_1(t)e^{-\eta t} + \left(\frac{A_1(\infty)D^2}{R_1^2 k_2 \rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}\right] + \max_{r\in[0,R_1]} |a_1^0(r)| \exp\left(-\frac{\chi_1 \xi_1 t}{R_1}\right), \tag{27}$$

where $\xi_1 \approx 2.4048$ is the first roots of equation $J_0(\xi) = 0$ [2]. So there is

Lemma 2.1. *If the functions $\alpha(\tau)$, $\alpha'(\tau)$ satisfy conditions (23)–(25), then for the solutions of the initial-boundary value problems (8)–(11) $a_j(r,t)$ the following estimates are valid: (26), (27), from which it follows that these functions tend exponentially to zero with increasing time.*

The priori estimates for functions $v_j(r,t)$ and $f_j(t)$ have form [1]

$$|v_2(r,t)| \leqslant \frac{2æ}{\mu_2}|a_1(R_1,t)| \max_{r\in[R_1,R_2]} |P_4(r)| + \sqrt{\frac{2}{R_1}}\left(\frac{2}{\rho_2\mu_2}H_2(t)E(t)\right)^{1/4}. \tag{28}$$

$$\begin{aligned} |f_1(t)| \leqslant 2\nu_1 &\left[\left(\frac{1}{7}R_1^4 + \sum_{n=1}^{\infty}|h_n^2|\right) + 2R_1^2\sum_{n=1}^{\infty}\left(\frac{|h_n^1|}{\zeta_n^2} + \frac{|h_n^2|}{R_1^2}\right)\right]\max_{t\in[0,T]}|g(t)| + \\ &+ \frac{R_2^2 - R_1^2}{R_1^2}\left[\frac{2æ}{\mu_2}\max_{t\in[0,T]}|a_{1t}(R_1,t)| \max_{r\in[R_1,R_2]}|P_4(r)| + \right. \\ &\left. + \sqrt{\frac{2}{R_1}}\max_{t\in[0,T]}\left(\frac{2}{\rho_2\mu_2}H_3(t)E_1(t)\right)^{1/4}\right]. \end{aligned} \tag{29}$$

$$|v_1(r,t)| \leqslant R_1 \max_{t\in[0,T]}|v_2(R_1,t)| + \frac{2R_1}{\nu_1}\max_{t\in[0,T]}|f_1(t)| \sum_{n=1}^{\infty}\frac{1}{\xi_n^3|J_1(\xi_n)|}, \tag{30}$$

$$|f_2(t)| \leqslant \rho|f_1(t)| + \frac{2æ}{\rho_2 R_1}\max_{t\in[0,T]}|a_1(\rho_2 R_1, t)|. \tag{31}$$

Here $\rho = \rho_1/\rho_2$, ξ_n are the roots of the Bessel function $J_0(\xi_n)=0$, ζ_n are the positive roots of equation $J_2(\zeta) = 0$ [3], $h_n^1 = \beta_n^1/\zeta_n$ and $h_n^2 = \beta_n^2/\zeta_n$ (where β_n^1, β_n^2 are coefficients of Fourier series of functions $-15R_1 r$ and $3R_1(r^3 - 4R_1 r^2/7)$ when they are decomposed by function $J_2(R_1^{-1}\zeta_n r)$ [1]). Further we have

$$P_4(r) = \frac{1}{R_1^2(R_1 - R_2)}(r^2 - (R_1 + R_2)r + R_1 R_2)(r^2 + C_1 r + C_2) \tag{32}$$

with constants

$$C_1 = -\frac{(R_1 + R_2)(2R_1^2 + 2R_2^2 + R_1 R_2)}{(R_2 - R_1)(3R_2 + 2R_1)}, \quad C_2 = -R_1 C_1 \tag{33}$$

and

$$E(t) \leqslant \left[\sqrt{E(0)} + \int_0^t H_1(\tau)e^{\delta\tau}d\tau\right]^2 e^{-2\delta t}, \tag{34}$$

$$E(0) = \frac{2æ^2\rho_2}{\mu_2^2}(a_1^0(R_1))^2\int_{R_1}^{R_2} rP_4^2(r)dr. \tag{35}$$

$$H_1(t) = \left[\sqrt{\frac{\rho_2}{2}}\left(\int_{R_1}^{R_2} rQ_2^2 dr\right)^{1/2} + \frac{æ}{\sqrt{\rho_1}}\,|a_1(R_1,t)|\right], \tag{36}$$

$$Q_2(r,t) = \frac{2æ}{\mu_2}\left[a_{2t}(R_1,t)P_4(r) - \nu_2\left(P_{4rr} + \frac{1}{r}\,P_{4r}\right)a_2(R_1,t)\right]. \tag{37}$$

$$H_2(t) = \mu_2\int_{R_1}^{R_2} r(\bar v_{2r}^0)^2 dr + \frac{\rho_2}{2}\int_0^t\int_{R_1}^{R_2} rQ_2^2(r,t)drdt + \frac{æ^2}{\rho_1}\int_0^t a_1^2(R_1,t)dt, \tag{38}$$

Below, in order to determine the behavior of $v_1(r,t)$ and $f_j(t)$ for large t, we need the estimate $|a_{2t}(r,t)|$. It was obtained in [1], that

$$|a_{2t}(r,t)| \leqslant |\alpha'(t)| + 2\left(\frac{1}{R_1^2 k_2\rho_2 c_{\rho_2}}A_2(t)A_3(t)\right)^{1/4}, \tag{39}$$

where

$$A_2(t) = \frac{\rho_1 c_{\rho_1}}{2}\int_0^{R_1} ra_{1t}^2(r,t)dr + \frac{\rho_2 c_{\rho_2}}{2}\int_{R_1}^{R_2} r\bar a_{2t}^2(r,t)dr,$$

$$A_{20} = A_2(0) = \frac{\chi_1^2\rho_1 c_{\rho_1}}{2}\int_0^{R_1} r\left(a_{1rr}^0 + \frac{1}{r}\,a_{1r}^0\right)^2 dr +$$

$$+\frac{\rho_2 c_{\rho_2}}{2}\int_{R_1}^{R_2} r\left[\chi_2\left(\bar a_{2rr}^0 + \frac{1}{r}\,\bar a_{2r}^0\right) + \frac{2\chi_2\alpha(0)}{(R_2-R_1)^2}\left(2 - \frac{R_1}{r}\right) - \frac{\alpha'(0)(r-R_1)^2}{(R_2-R_1)^2}\right]^2 dr,$$

$$\bar a_2^0(r) = a_2^0(r) - \frac{\alpha(0)(r-R_1)^2}{(R_2-R_1)^2};$$

$$\begin{gathered} A_3(t) = k_1\chi_1^2\int_0^{R_1} r\left(a_{1rr}^0 + \frac{1}{r}\,a_{1r}^0\right)^2 dr + \\ +k_2\int_{R_1}^{R_2} r\left[\chi_2\left(a_{2rr}^0 + \frac{1}{r}\,a_{2r}^0\right) - \frac{\alpha'(0)(r-R_1)^2}{(R_2-R_1)^2}\right]^2 dr \,+\, \rho_2 c_{\rho_2}\int_0^t\int_{R_1}^{R_2} rg_3(r,t)drdt, \\ g_3(r,t) = \frac{1}{(R_2-R_1)^2}\left[2\chi_2\alpha'(t)\left(2 - \frac{R_1}{r}\right) - \alpha''(t)(r-R_1)^2\right]. \end{gathered} \tag{40}$$

Therefore, for $A_2(t)$ we obtain inequality (21) with replacement A_0 by A_{10}, $\alpha(\tau)$ by $\alpha'(\tau)$ and $\alpha'(\tau)$ by $\alpha''(\tau)$. For the function $A_3(t)$ inequality form (22) is satisfied with the replacement

$$\int_0^{R_1} r\left(a_{1r}^0\right)^2 dr \quad \text{by} \quad \chi_1^2\int_0^{R_1} r\left(a_{1rr}^0 + \frac{1}{r}\,a_{1r}^0\right)^2 dr \equiv d_1,$$

$$\int_{R_1}^{R_2} r\left(\bar a_{2r}^0\right)^2 dr \quad \text{by} \quad \int_{R_1}^{R_2} r\left[\chi_2\left(a_{2rr}^0 + \frac{1}{r}\,a_{2r}^0\right) - \frac{\alpha'(0)(r-R_1)^2}{(R_2-R_1)^2}\right]^2 dr \equiv d_2$$

and $\alpha(\tau)$ by $\alpha'(\tau)$, $\alpha'(\tau)$ by $\alpha''(\tau)$.

In addition to (23)–(25) we assume the convergence of the integral

$$\int_0^\infty |\alpha''(\tau)|e^{\eta\tau}d\tau < \infty, \tag{41}$$

so that there is valid

$$|\alpha''(t)| = \alpha_3(t)e^{-\eta t}, \quad \int_0^\infty \alpha_3(\tau)d\tau < \infty, \quad \alpha_3(t) \to 0 \ \text{ at } \ t \to \infty. \tag{42}$$

Taking into account the above, we find from (14) that

$$|a_{2t}(r,t)| \leqslant \alpha_2(t)e^{-\eta t} + 2\left(\frac{A_3(\infty)D_1^2}{R_1^2 k_2 \rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}, \tag{43}$$

where

$$\begin{gathered} A_3(\infty) = k_1 d_1 + k_2 d_2 + \frac{R_2\rho_2 c_{p_2}}{R_2 - R_1}\left[4\chi_2 \int_0^\infty |\alpha'(\tau)| + (R_2-R_1)^2 \int_0^\infty |\alpha''(\tau)|d\tau\right], \\ D_1 = \sqrt{A_{10}} + \max_j \left(\frac{1}{\rho_j c_{p_j}}\right)^{1/2} \left[\frac{4\chi_2}{(R_2-R_1)^{3/2}} \int_0^\infty |\alpha'(\tau)|e^{\eta\tau}d\tau + \right. \\ \left. + \sqrt{R_2(R_2-R_1)} \int_0^\infty |\alpha''(\tau)|e^{\eta\tau}d\tau\right]. \end{gathered} \tag{44}$$

We turn to inequality for $|a_{2tt}(r,t)|$ [1]. We have

$$|a_{2tt}(r,t)| \leqslant |\alpha''(t)| + 2\left(\frac{1}{R_1^2 k_2 \rho_2 c_{\rho_2}} A_4(t) A_5(t)\right)^{1/4}, \tag{45}$$

where

$$\begin{gathered} A_4(t) = \frac{\rho_1 c_{p_1}}{2} \int_0^{R_1} r a_{1tt}^2 dr + \frac{\rho_2 c_{p_2}}{2} \int_{R_1}^{R_2} r \bar{a}_{2tt}^2 dr, \\ A_{40} = \frac{\rho_1 c_{p_1}}{2} \int_0^{R_1} r(a_{1tt}^0(r))^2 dr + \frac{\rho_2 c_{p_2}}{2} \int_{R_1}^{R_2} r(\bar{a}_{2tt}^0(r))^2 dr. \end{gathered} \tag{46}$$

The initial data are found from equations (9) and replacement of (18):

$$\begin{gathered} a_{1tt}^0(r) = \chi_1 \left[\left(a_{1rr}^0 + \frac{1}{r}\, a_{1r}^0\right)_{rr} + \frac{1}{r}\left(a_{1rr}^0 + \frac{1}{r}\, a_{1r}^0\right)_r\right], \\ \bar{a}_{2tt}^0(r) = \chi_2 \left[\left(a_{2rr}^0 + \frac{1}{r}\, a_{2r}^0\right)_{rr} + \frac{1}{r}\left(a_{2rr}^0 + \frac{1}{r}\, a_{2r}^0\right)_r\right] - \frac{\alpha''(0)(r-R_1)^2}{(R_2-R_1)^2}. \end{gathered} \tag{47}$$

Further we have

$$\begin{gathered} A_5(t) = k_1 \int_0^{R_1} r(a_{1tt}^0)^2 dr + k_2 \int_{R_1}^{R_2} r(\bar{a}_{2tt}^0)^2 dr + \\ + \frac{\rho_2 c_{\rho_2}}{(R_2-R_1)^2} \int_0^t \int_{R_1}^{R_2} r\left[2\chi_2 \alpha''(\tau)\left(2 - \frac{R_1}{r}\right) - \alpha'''(\tau)\,(r-R_1)^2\right] dr. \end{gathered} \tag{48}$$

Similarly to function $A(t)$ the function $A_4(t)$ satisfies an estimate of type (15), and hence (21) with the replacement A_0 by A_{40}, $\alpha(t)$ by $\alpha''(\tau)$ and $\alpha'(\tau)$ by $\alpha'''(\tau)$.

If we require convergence of the integral

$$\int_0^\infty |\alpha'''(\tau)|e^{\eta t}d\tau < \infty, \tag{49}$$

$$|\alpha'''(t)| = \alpha_4(t)e^{-\eta t}, \quad \int_0^\infty \alpha_4(\tau)d\tau < \infty, \tag{50}$$

we obtain an estimate of the function $A_5(t)$ (we use the formula (22)

$$|A_5(t)| \leqslant k_1 \int_0^{R_1} r(a^0_{1tt})^2 dr + k_2 \int_{R_1}^{R_2} r(\bar{a}^0_{2tt})^2 dr + \\ + \rho_2 c_{\rho_2} R_2 \left[\frac{4\chi_2}{R_2 - R_1} \int_0^t |\alpha''(\tau)| d\tau + (R_2 - R_1) \int_0^t |\alpha'''(\tau)| d\tau \right], \tag{51}$$

where $a^0_{jtt}(r)$ are defined by formulas (24). By virtue of (41), (49) $|A_5(t)| \leqslant A_5(\infty)$ and, similarly to estimate (21), we obtain from (45)

$$|a_{2tt}(r,t)| \leqslant \alpha_4(t) e^{-\eta t} + 2 \left(\frac{A_5(\infty) D_2^2}{R_1^2 k_2 \rho_2 c_{\rho_2}} \right)^{1/4} e^{-\eta t/2}, \tag{52}$$

$$D_2 = \sqrt{A_{40}} + \left[\max_j \left(\frac{1}{\rho_j c_{\rho_j}} \right) \right]^{1/2} \left[\frac{4\chi_2}{(R_2 - R_1)^{3/2}} \int\limits_0^\infty |\alpha''(\tau)| e^{\eta\tau} d\tau + \right. \\ \left. + \sqrt{R_2(R_2 - R_1)} \int_0^\infty |\alpha'''(\tau)| e^{\eta\tau} d\tau \right].$$

We proceed to elaboration the estimates of the functions $v_j(r,t)$, $f_j(t)$, when $\alpha(\tau)$, $\alpha'(\tau)$, $\alpha''(\tau)$ and $\alpha'''(\tau)$ satisfy conditions (23)–(25), (41), (42). In this case everywhere we replace $a_1(R_1,t)$, $a_{1t}(R_1,t)$ by $a_2(R_1,t)$, $a_{2t}(R_1,t)$ according to the first equation (11). We begin with the function $v_2(r,t)$, for which inequality (28) is proved. The quantity $E(t)$ entering the right-hand side of this inequality has estimate (34), where $H_1(t)$ is given by (36) than from (37) we obtain

$$\int_{R_1}^{R_2} r Q_2^2(r,t) dr \leqslant \frac{8æ^2}{\mu_2^2} \left[a_{2t}^2(R_1,t) \int_{R_1}^{R_2} r P_4^2(r) dr + \nu_1^2 a_2^2(R_1,t) \int_{R_1}^{R_2} r \left(P_{4rr} + \frac{1}{r} P_{4r} \right)^2 dr \right] \equiv \\ \equiv d_3 a_2^2(R_1,t) + d_4 a_{2t}^2(R_1,t). \tag{53}$$

So the inequality is valid

$$H_1(t) \leqslant \frac{æ}{\sqrt{\rho_1}} |a_2(R_1,t)| + \sqrt{\frac{\rho_2}{2}} \left(\sqrt{d_3} |a_2(R_1,t)| + \sqrt{d_4} |a_{2t}(R_1,t)| \right) = \\ = \left(\frac{æ}{\sqrt{\rho_1}} + \sqrt{\frac{\rho_2 d_3}{2}} \right) |a_2(R_1,t)| + \sqrt{\frac{\rho_2 d_4}{2}} |a_{2t}(R_1,t)|$$

and estimate (34) takes the form

$$E(t) \leqslant \left[\sqrt{E(0)} + \left(\frac{æ}{\sqrt{\rho_1}} + \sqrt{\frac{\rho_2 d_3}{2}} \right) \int_0^t |a_2(R_1,t)| e^{\delta\tau} d\tau + \right. \\ \left. + \sqrt{\frac{\rho_2 d_4}{2}} \int_0^t |a_{2t}(R_1,t)| e^{\delta\tau} d\tau \right]^2 e^{-2\delta t}. \tag{54}$$

According to estimates (26), (43) the integrals in (54) have the order $e^{(\delta-\eta)t}$ and $e^{(\delta-\eta/2)t}$ for large t, therefore we obtain

$$E(t) \leqslant \gamma(t), \quad \text{where} \quad \gamma(t) \equiv d_5 \begin{cases} e^{-2\delta t}, & \delta < \eta/2, \\ t e^{-2\delta t}, & \delta = \eta/2, \\ e^{-\eta t}, & \delta > \eta/2, \end{cases} \tag{55}$$

with positive constant d_5.

Defined by equality (38) with using (53) the function $H_2(t)$ is evaluated as follows:

$$H_2(t) \leqslant \mu_2 \int_{R_1}^{R_2} r(\bar{v}_{2r}^0)^2 dr + \left(\frac{\rho_2 d_3}{2} + \frac{æ^2}{\rho_1}\right) \int_0^t a_2^2(R_1,\tau)d\tau +$$
$$+ \frac{\rho_2 d_4}{2} \int_0^t a_{2\tau}^2(R_1,\tau)d\tau \leqslant D_2 = \text{const} > 0$$

by virtue of inequalities (26), (43).

So from (13), (54), (55) we find estimate

$$|v_2(r,t)| \leqslant \frac{2æ}{\mu_2} \max_{r\in[R_1,R_2]} |P_4(r)| \left[\alpha_1(t)e^{-\eta t} + 2\left(\frac{A_1(\infty)D^2}{R_1^2 k_2 \rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}\right] + \\ + \sqrt{2}\left(\frac{2d_5}{R_1^2 \nu_2} D_2 \gamma(t)\right)^{1/4} \tag{56}$$

and $v_2(r,t)$ approaches to zero uniformly over $r \in [R_1, R_2]$ with increasing time t.

Below we need the values $f_j(0)$. From (7) we obtain the connection between them

$$\rho_1 f_1(0) = \rho_2 f_2(0) - \frac{2æ}{R_1} a_1^0(R_1).$$

The other relation follows from the second equality (3) and equation (5) (we recall that $v_j(r,0)=0$):

$$f_1(0) = -\frac{R_2^2 - R_1^2}{R_1^2} f_2(0).$$

Now we find

$$f_1(0) = \frac{2æ(R_2^2 - R_1^2)a_1^0(R_1)}{R_1^2 + \rho(R_2^2 - R_1^2)}, \quad f_2(0) = \frac{R_1^2}{R_2^2 - R_1^2} f_1(0). \tag{57}$$

Moreover the relations are valid

$$v_{1t}(r,0) = f_1(0), \quad \bar{v}_{2t}(r,0) = f_2(0) + \frac{2æ\chi_1}{\mu_2}\left(a_{1rrr}^0 + \frac{1}{r} a_{1rr}^0\right) P_4(r). \tag{58}$$

The second initial condition follows from the equations

$$|a_1(R_1,t)| = |a_2(R_1,t)| \leqslant |\alpha(t)| + 2\left(\frac{1}{R_1^2 k_2 \rho_2 c_{\rho_2}} A(t)A_1(t)\right)^{1/4}, \tag{59}$$

and (37) and replacement

$$v_2(r,t) = \bar{v}_2(r,t) - \frac{2æ a_1(R_1,t)}{\mu_2} P_4(r). \tag{60}$$

We consider the following inequality that was obtained in [1]

$$|v_{2t}(r,t)| \leqslant \frac{2æ}{\mu_2} |a_{1t}(R_1,t)| \max_{r\in[R_1,R_2]} |P_4(r)| + \sqrt{\frac{2}{R_1}}\left(\frac{2}{\rho_2\mu_2} H_3(t)E_1(t)\right)^{1/4}. \tag{61}$$

The function $E_1(t)$ on the right-hand side of inequality (61) has the form

$$E_1(t) = \frac{\rho_1}{2}\int_0^{R_1} r v_{1t}^2 dr + \frac{\rho_2}{2}\int_{R_1}^{R_2} r\bar{v}_{2t}^2 dr,$$

$$E_1(0) = \frac{\rho_1 R_1^2}{4} f_1^2(0) + \frac{\rho_2}{2} \int_{R_1}^{R_2} r\bar{v}_{2t}^2(r,0)dr,$$

where $f_1(0)$ and $\bar{v}_{2t}(r,0)$ are defined by the first (57) and the second (58) equality respectively. There is the estimate form (54) for $E_1(t)$.

$$E_1(t) \leqslant \left[\sqrt{E_1(0)} + \left(\frac{æ}{\sqrt{\rho_1}} + \sqrt{\frac{\rho_2 d_3}{2}}\right) \int_0^t |a_{2t}(R_1,\tau)|e^{\delta\tau}d\tau + \right.$$
$$\left. + \sqrt{\frac{\rho_2 d_4}{2}} \int_0^t |a_{2tt}(R_1,\tau)|e^{\delta\tau}d\tau \right]^2 e^{-2\delta t}. \tag{62}$$

Taking into account the obtained estimates (43), (51) from (53) we find using the constant d_6 the inequality

$$E_1(t) \leqslant d_6\gamma(t) \tag{63}$$

and the function $\gamma(t)$ from inequality (55).

For the function $H_3(t)$, from the right-hand side of inequality (61) we have the expression

$$H_3(t) = \mu_1 \int_0^{R_1} r(v_{1tr}^0)^2 dr + \mu_2 \int_{R_1}^{R_2} r(\bar{v}_{2tr}^0)^2 dr +$$
$$+ \frac{\rho_2}{2} \int_0^t \int_{R_1}^{R_2} rQ_3^2(r,\tau)drd\tau + \frac{æ^2}{\rho_1} \int_0^t a_2^2(R_1,\tau)d\tau, \tag{64}$$

where in our case

$$Q_3(r,t) = \frac{2æ}{\mu_2}\left[-\nu_2 a_{2t}(R_1,t)\left(P_{4rr} + \frac{1}{r}P_{4r}\right) + a_{2tt}(R_1,t)P_4(r)\right],$$
$$v_{1tr}^0(r) = 0, \quad \bar{v}_{2tr}^0 = \frac{2æ}{\mu_2} a_{2t}(R_1,0)P_{4r},$$
$$a_{2t}(R_1,0) = \chi_2\left[a_{2rr}^0(R_1) + \frac{1}{R_1}a_{2r}^0(R_1)\right].$$

It is clear that

$$\int_{R_1}^{R_2} rQ_3^2(r,t)dr \leqslant d_3 a_{2t}^2(R_1,t) + d_4 a_{2tt}^2(R_1,t)$$

with constant d_3, d_4 from (52). By virtue of the convergence of the integrals

$$\int_0^\infty (a_2^{(k)}(\tau))^2 d\tau, \quad k = 0,1,2$$

we obtain the inequality $H_3(t) \leqslant H_3(\infty)$ and estimate (61) takes the form for all $r \in [R_1, R_2]$

$$|v_{2t}(r,t)| \leqslant \frac{2æ}{\mu_2} \max_{r\in[R_1,R_2]} |P_4(r)| \left[\alpha_2(t)e^{-\eta t} + 2\left(\frac{A_3(\infty)D_1^2}{R_1^2 k_2 \rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}\right] +$$
$$+ \sqrt{2}\left(\frac{2d_6}{R_1^2\nu_2} H_3(\infty)\gamma(t)\right)^{1/4}. \tag{65}$$

The function $f_1(t)$ is the pressure gradient in the first fluid along the axis z. The function $g(t)$ on the right side of the inequality (29) has form

$$g(t) = R_1^2 v_2(R_1,t) + 2\int_{R_1}^{R_2} rv_2(r,t)dr$$

and, taking into account estimate (56), we find

$$|g(t)| \leqslant R_2^2 \left\{ \frac{2æ}{\mu_2} \max_{r\in[R_1,R_2]} |P_4(r)| \left[\alpha_1(t)e^{-\eta t} + 2\left(\frac{A_1(\infty)D^2}{R_1^2 k_2 \rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}\right] + \right.$$
$$\left. + \sqrt{2}\left(\frac{2d_5}{R_1^2\nu_2} H_2(\infty)\gamma(t)\right)^{1/4} \right\} \leqslant d_7 e^{-\omega t}, \qquad (66)$$

where $\omega = \min(\delta/2, \eta/4)$ (at $\delta = \eta/2$ in (66) there is $te^{-\omega t}$ instead of $e^{-\omega t}$ according to (54)).

Now from (29) using inequalities (65), (66) we obtain the estimate

$$|f_1(t)| \leqslant 2\nu_1 \left[S_1 d_7 e^{-\omega t} + S_2 d_7 \left| \exp\left(-\frac{\zeta_1^2\nu_1}{R_1^2}t\right) - e^{-\omega t}\right| \right] + d_8 e^{-\omega t}, \qquad (67)$$

$$S_1 = \frac{1}{7} R_1^4 + \sum_{n=1}^{\infty} |h_n^2|, \quad S_2 = \nu_1 \sum_{n=1}^{\infty} \frac{1}{\nu_1 R_1^{-2}\zeta_n^2 - \omega}\left(|h_n^1| + \frac{\zeta_n^2}{R_1^2}|h_n^2|\right),$$

at that $S_1 < \infty$ and $S_2 < \infty$. The estimate $f_2(t)$ follows from (5), inequalities (26) and (67)

$$|f_2(t)| \leqslant \rho |f_1(t)| + 2æ \left[\alpha_1(t) e^{-\eta t} + 2\left(\frac{A_1(\infty)D^2}{R_1^2 k_2\rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}\right]. \qquad (68)$$

Remark 1. *From inequality* (30), *estimates* (56) *and* (67) *it follows that the function* $v_1(r,t)$ *tends exponentially to zero with increasing time.*

$$|v_1(r,t)| \leqslant R_1 \max_{t\in[0,T]} \left| \frac{2æ}{\mu_2} \max_{r\in[R_1,R_2]} |P_4(r)| \left[\alpha_1(t)e^{-\eta t} + 2\left(\frac{A_1(\infty)D^2}{R_1^2 k_2\rho_2 c_{\rho_2}}\right)^{1/4} e^{-\eta t/2}\right] + \right.$$
$$\left. + \sqrt{2}\left(\frac{2d_5}{R_1^2\nu_2} D_2 \gamma(t)\right)^{1/4} \right| + \frac{2R_1}{\nu_1} \max_{t\in[0,T]} \left| 2\nu_1 \left[S_1 d_7 e^{-\omega t} + \right.\right. \qquad (69)$$
$$\left.\left. + S_2 d_7 \left| \exp\left(-\frac{\zeta_1^2\nu_1}{R_1^2}t\right) - e^{-\omega t}\right| \right] + d_8 e^{-\omega t} \right| \sum_{n=1}^{\infty} \frac{1}{\xi_n^3 |J_1(\xi_n)|}.$$

For the function $h_1(t)$ from (12), taking into account the first relation (3) and the inequality (56) we have the estimate

$$|h_1(t)| \leqslant \frac{R_2^2 - R_1^2}{2R_1} \left\{ \frac{2æ}{\mu_2} \max_{r\in[R_1,R_2]} |P_4(r)| \left[\int_0^t \alpha_1(\tau) e^{-\eta\tau} d\tau + \right.\right.$$
$$\left.\left. + \frac{4}{\eta}\left(\frac{A_1(\infty)D^2}{R_1^2 k_2\rho_2 c_{\rho_2}}\right)^{1/4} \left(1 - e^{-\eta t/2}\right)\right] + \sqrt{2}\left(\frac{2d_5}{R_1^2\nu_2} H_2(\infty)\right)^{1/4} \int_0^t \gamma^{1/4}(\tau) d\tau \right\} \qquad (70)$$

and $h_1(t)$ is limited at $t \to \infty$.

Thus, it is proofed

Theorem 2.1. *If the function* $\alpha(\tau)$, $\alpha'(\tau)$, $\alpha''(\tau)$, $\alpha'''(\tau)$ *satisfy conditions* (23)–(25), (41), (42), (49), *then the following estimates* (26), (27), (56), (67), (68), (69) *are valid for the functions* $a_j(r,t)$, $v_j(r,t)$, $f_j(t)$ *from which it follows that these functions tend exponentially to zero with increasing time.*

Remark 2. *Remark 6. Conditions* (23)–(25), (41), (42), (49) *physically mean that the thermal effects on the solid wall surface of cylinder at* $r = R_2$ *are very small and the braking of liquids occurs at* $t \to \infty$ *due to frictional forces.*

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Об асимптотическом поведении сопряженной задачи, описывающей ползущее осесимметричное термокапиллярное движение

Виктор К. Андреев
Евгений П. Магденко
Институт вычислительного моделирования СО РАН
Красноярск, Российская Федерация
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. В работе указаны условия для закона поведения температуры на твердой стенке цилиндра, при которых решение линейной сопряженной обратной начально-краевой задачи, описывающей двухслойное осесимметрическое ползущее движение вязких теплопроводных жидкостей, с ростом времени экспоненциально стремится к нулю.

Ключевые слова: сопряженная нелинейная обратная задача, поверхность раздела, ползущее движение.

Journal of Siberian Federal University. Mathematics & Physics 2020, 13(1), 37–47

DOI: 10.17516/1997-1397-2020-13-1-37-47
УДК 519.6

On Application of Prandtl-Obukhov Formula in the Numerical Model of the Turbulent Layer Depth Dynamics

Victor M. Belolipetskii*
Svetlana N. Genova†
Institute of Computational Modelling SB RAS
Krasnoyarsk, Russian Federation

Received 15.10.2019, received in revised form 16.11.2019, accepted 10.12.2019

Abstract. A numerical simulation of the penetration of the turbulent layer in a stably stratified fluid under the action of tangential stress was performed. For the coefficient of vertical turbulent exchange, the Prandtl–Obukhov formula is used. The results of the calculations are consistent with known experimental data and calculations by other authors.

Keywords: mathematical modeling, turbulence, stratified fluid.

Citation: V.M.Belolipetskii, S.N.Genova, On Application of Prandtl-Obukhov Formula in the Numerical Model of the Turbulent Layer Depth Dynamics, J. Sib. Fed. Univ. Math. Phys., 2020, 13(1), 37–47. DOI: 10.17516/1997-1397-2020-13-1-37-47.

Introduction

In most cases, real geophysical environments are stratified. If stratification is stable, then it prevents the development of turbulence. Unstable stratification provokes the development of turbulence. The stratification is stable at $\dfrac{\partial\rho}{\partial z} > 0$ for an incompressible fluid (the vertical distribution of fluid density is determined by the function $\rho(z)$, the z-axis is directed vertically downwards), and the stratification is unstable at $\dfrac{\partial\rho}{\partial z} < 0$. A measure of sustainability of stratified fluid is the Vaisal-Brent frequency : $N^2 = \dfrac{g}{\rho}\dfrac{\partial\rho}{\partial z}$ (c^{-2}), ($g = 981$ cm/c^2 is gravity acceleration). If N^2 is positive, the medium is stable; if N^2 is negative, it is unstable.

An example of a flow where vertical turbulent exchange plays a decisive role is the flow occurs when a turbulent liquid layer deepens in a stably stratified reservoir at the action of wind. Many works are devoted to its study (see, for example, references in [1–8]). The classical $e-\varepsilon$ –model of turbulence and its modifications are used to describe the process of the mixed layer deepening in the stratified fluid.

In this paper, the Prandtl-Obukhov formula is used to determine the coefficients of vertical turbulent exchange [9, 10].

*belolip@icm.krasn.ru
†sv@icm.krasn.ru

1. Problem statement

1.1. Statement of the problem without considering Coriolis force

The flow in a linearly stratified medium under constant shear stress is considered. Stratification is due to changes in salinity.

In the study of the process of the turbulent layer deepening simplifications are made, as a result the averaged horizontal homogeneous motion is described by a system of differential equations [2–4]:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial z}\left[\nu\frac{\partial U}{\partial z} - \langle u'w'\rangle\right], \\ \frac{\partial S}{\partial t} &= \frac{\partial}{\partial z}\left[\chi\frac{\partial S}{\partial z} - \langle S'w'\rangle\right]. \end{aligned} \tag{1}$$

Here U is a horizontal component of the averaged velocity, S is the averaged salinity, strokes mark pulsation components: $\langle u'w'\rangle$ is Reynolds shear stress, $\langle S'w'\rangle$ is the vertical vector component of flows; ν, χ_ρ are molecular viscosity and diffusion coefficients; t is time, z is vertical coordinate (directed down), t is time. In the case of a fluid linearly stratified at the initial instant of time, the dependence of the average fluid density ρ on salinity is given by the relation $\rho(S) = \rho^* + \beta(S - S^*)$. Here ρ^* is the initial value of the density on the water surface, S^* is the initial value of the salinity of the water on the surface, $\beta = const$.

The system (1) is not closed. For its closure, semi-empirical models of turbulence are used [2–4]. In this paper, it is proposed to parameterize the ratios of vertical turbulent exchange to use the Prandtl-Obukhov formula derived from stationary equations of balance of turbulence energy and its dissipation rate [10].

According to Bussinesk hypothesis the values $\langle u'w'\rangle$, $\langle S'w'\rangle$ are presented in the form of:

$$-\langle u'w'\rangle = K_{uz}\frac{\partial U}{\partial z}, \qquad -\langle S'w'\rangle = K_{Sz}\frac{\partial S}{\partial z},$$

K_{uz} is the coefficient of turbulent viscosity, K_{Sz} is the turbulent diffusion coefficient.

The Prandtl-Obukhov formula takes into account the shear mixing mechanism and stratification [9, 10]:

$$K_z = \begin{cases} (0.05\ h_1)^2\sqrt{B} + k_{min}, & B > 0, \\ k_{min}, & B \leqslant 0, \end{cases} \tag{2}$$

$$\frac{g}{\rho^*}\frac{\partial \rho}{\partial z} = \beta\ \frac{g}{\rho^*}\frac{\partial S}{\partial z}, \qquad B = \left(\frac{\partial U}{\partial z}\right)^2 - \frac{g}{\rho^*}\frac{\partial \rho}{\partial z},$$

where h_1 is the depth of the quasi-homogeneous (mixed) layer, determined by the first calculation point from the surface where the condition is satisfied

$$(0.05\ z_k)^2\sqrt{B_{|z=z_k}} < k_{min},$$

k_{min} is the minimum value of turbulent viscosity. The deepening of the turbulent layer of liquid in a reservoir by the wind influence was determined as follows:

$$h^{n+1} = h^n \ \text{ if } \ h_1 < h^n; \qquad h^{n+1} = h_1 \ \text{ if } \ h_1 > h^n,$$

where $h_1^n = h_1(t_n)$ is the quasihomogeneous layer depth in the Prandtl-Obukhov formula , $h^n = h(t_n)$ is the depth of the turbulent layer.

It is assumed that the coefficients of vertical turbulent exchange are proportional to K_z:

$$K_{uz} = \alpha_u K_z, \quad K_{Sz} = \alpha_S K_z, \quad \alpha_u = const, \quad \alpha_S = const.$$

We obtained a closed system of equations for calculating $U(t,z)$, $S(t,z)$, $h(t)$, $\rho(t)$:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \frac{\partial}{\partial z}\left[(\nu + K_{uz})\frac{\partial U}{\partial z}\right], \\ \frac{\partial S}{\partial t} &= \frac{\partial}{\partial z}\left[(\chi_\rho + K_{Sz}\frac{\partial S}{\partial z}\right]. \end{aligned} \tag{3}$$

Boundary conditions for the system (3) are: on the surface ($z = 0$)

$$(\nu + K_{uz})\frac{\partial U}{\partial z} = -\frac{\tau_w}{\rho^*}, \qquad \frac{\partial S}{\partial z} = 0, \tag{4}$$

τ_w is shear stress caused to wind load;
at the bottom ($z = H$)

$$U = 0, \qquad S = S_H = S^* + \frac{\partial S^0}{\partial z}H. \tag{5}$$

Initial conditions are:

$$U(z) = 0, \qquad S(z) = S^* + \frac{\partial S^0}{\partial z}z. \tag{6}$$

The initial salinity distribution corresponds to a linear density distribution, $\left(\dfrac{\partial S^0}{\partial z}\right) = \dfrac{1}{\beta}\dfrac{\partial \rho^0}{\partial z}$. The given relations contain empirical coefficients K_{min}, α_u, α_S determined by numerical experiments.

1.2. Statement of the problem taking into account Coriolis force

Drift currents are formed in the upper layer of the reservoir under the influence of wind. The solution of the problem of steady drift current for a deep sea of uniform density was constructed by Ekman [11]:

$$U^e = U_0 \, \exp(-\alpha z) \, \cos(\frac{\pi}{4} - \alpha z), \quad V^e = V_0 \, \exp(-\alpha z) \, \sin(\frac{\pi}{4} - \alpha z),$$

Here U^e, V^e are horizontal components of water flow velocity vector, $f = 2\,\Omega \sin(\varphi)$ is the Coriolis parameter, Ω is angular velocity of the Earth rotation, φ is latitude, $\alpha = \sqrt{\dfrac{f}{2K_z}}$, $V_0 = \dfrac{\tau_y}{\sqrt{2}\rho_0 K_z \alpha}$, wind is directed along the coordinate y ($\tau_x = 0$, $\tau_y \neq 0$,). The speed of the wind current decreases exponentially with depth. Below the horizon of $z = D$ the flow velocity is small, $D = \pi\sqrt{2K_z/f}$ is the friction depth. The main part of the kinetic energy of the drift flow is concentrated in the friction layer from 0 to D. The influence of the parameter f can be neglected for $H < D$ (H is a reservoir depth). Similarly, in the problem of deepening a turbulent layer for sufficiently large depths ($H > D$), the influence of the Coriolis forces is manifested.

The averaged horizontal homogeneous motion is described by a system of differential equations:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial z}\left[(\nu + K_{uz})\frac{\partial U}{\partial z}\right] + fV,$$
$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial z}\left[(\nu + K_{uz})\frac{\partial V}{\partial z}\right] - fU, \qquad (7)$$
$$\frac{\partial S}{\partial t} = \frac{\partial}{\partial z}\left[(\chi_S + K_{Sz}\frac{\partial S}{\partial z}\right].$$

Here U, V are horizontal components of the averaged velocity vector. The system (7) is closed by the Prandtl-Obukhov formula:

$$K_z = \begin{cases}(0.05\ h_1)^2\sqrt{B_1} + k_{min}, & B_1 > 0,\\ k_{min}, & B_1 \leqslant 0,\end{cases} \qquad (2a)$$

$$B_1 = \left(\frac{\partial U}{\partial z}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 - \frac{g}{\rho^*}\frac{\partial \rho}{\partial z}.$$

Boundary conditions for the system(7) are: on the surface ($z = 0$)

$$(\nu + K_{uz})\frac{\partial U}{\partial z} = -\frac{\tau_{wx}}{\rho^*}, \qquad (\nu + K_{uz})\frac{\partial V}{\partial z} = -\frac{\tau_{wy}}{\rho^*}, \qquad \frac{\partial S}{\partial z} = 0, \qquad (8)$$

τ_{wx}, τ_{wy} are the components of wind friction stress;
at the bottom ($z = H$)

$$U = 0, \qquad V = 0, \qquad S = S_H = S^* + \frac{\partial S^0}{\partial z}H. \qquad (9)$$

Initial conditions are:

$$U(z) = 0, \qquad V(z) = 0, \qquad S(z) = S^* + \frac{\partial S^0}{\partial z}z. \qquad (10)$$

Two mathematical models are constructed to describe the processes of vertical turbulent exchange in a stably stratified reservoir:
– Model 1 does not consider the Coriolis force (2)–(6);
– Model 2 takes into account the Coriolis force (7)–(10).

2. Numerical modeling of turbulent mixing in the upper layer of a linearly stratified fluid. Results of numerical experiments

2.1. Numerical algorithm

The numerical solution of initial-boundary value problems (2)–(6), (7)–(10) are based on an explicit scheme of the first-order accuracy.

We will show an example of the problem for velocity $U(t, z)$. For internal nodes ($j = 2, 3, \ldots, jj - 1$):

$$U_j^{n+1} = U_j^n + \Delta t\frac{K_{j+1/2}(U_{j+1}^n - U_j^n) - K_{j-1/2}(U_j^n - U_{j-1}^n)}{(\Delta z)^2},$$

on the water surface ($j = 1$), taking into account the boundary condition, we have:

$$U_1^{n+1} = U_1^n + 2\Delta t \frac{K_{3/2}(U_2^n - U_1^n) + \Delta z\ \tau_w/\rho^*}{(\Delta z)^2},$$

at the bottom $U_{jj} = 0$. Here $U_j^{n+1} = U(t^{n+1}, z_j),\ t^{n+1} = t^n + \Delta t,\ \Delta z = H/(jj-1),\ K = \nu + K_{uz}$, $K_{j+1/2} = 0.5(K_j + K_{j+1})$. For the model equation $K = K_0 = const$ stability condition [12] is

$$\Delta t \leqslant (\Delta z)^2/(2K_0).$$

Parameters of variants for numerical experiments shows in Tab. 1.

Table 1. Parameters of variants

Nomber of variant	$\dfrac{\partial \rho^0}{\partial z}$, [g/ cm^4]	τ_w, [g/(cm c^2)]	H, [cm]	u^*, [cm/c]	N_0, [c^{-1}]
1	$1.92 \cdot 10^{-3}$	0.995	30	0.9975	1.3721
2	$3.84 \cdot 10^{-3}$	2.13	30	1.459	1.94
3	$1.0 \cdot 10^{-7}$	1	4000	1	$1.0 \cdot 10^{-2}$
4	$1.0 \cdot 10^{-6}$	1	4000	1	$3.13 \cdot 10^{-2}$
5	$1.0 \cdot 10^{-8}$	1	4000	1	$3.13 \cdot 10^{-3}$
6	$1.0 \cdot 10^{-7}$	1	1500	1	$1.0 \cdot 10^{-2}$
7	$1.0 \cdot 10^{-7}$	1	1000	1	$1.0 \cdot 10^{-2}$
8	$1.0 \cdot 10^{-7}$	2	1500	1.414	$1.0 \cdot 10^{-2}$

A variant of the flow obtained by transferring the results of laboratory experiments [7] to sea conditions with a depth of H=40 m is considered in [8]. An approximation of experimental dependence is proposed

$$\widehat{h} = (15 \cdot \widehat{t}\,)^{1/3}, \tag{11}$$

where $\widehat{h} = N_0 h/u^*$ is dimensionless depth of the mixed layer, $\widehat{t} = N_0 t$ is dimensionless time, $\widehat{H} = N_0 H/u^*$ is dimensionless reservoir depth , $u^* = \sqrt{\dfrac{\tau_w}{\rho^*}}$ is friction speed, $N_0 = \sqrt{\dfrac{g}{\rho_0}\dfrac{\partial \rho_0}{\partial z}}$. At the same time, according to the authors [8], the flow parameters took values for variant 3 from Tab. 1.

2.2. Results of numerical experiments

Values of empirical coefficients are determined by numerical experiments for variants $1, 2$: $\alpha_u = 0.638 - 0.0885 \cdot \tau_w,\ \alpha_S = 0.45$ for $N_0 \sim 1$ and $\alpha_S = 1.67$ for $N_0 \ll 1$.

The first series of numerical experiments refers to variant 2. The calculations were performed on uniform grids with the number of nodes from 120 to 250, time steps from 0.01 to 0.03 s. Fig. 1 illustrates the vertical distributions of the main flow parameters U/U_{max}, K_{uz}, $\rho(z)$ at the time of 240 s. The calculation results of $\rho(z)$ according to model 1 are in good agreement with the calculations using second-order turbulence models [2]. The calculation results of U/U_{max}, K_{uz} according to model 1 are in qualitative agreement with the calculations according to the second-order turbulence models from [2].

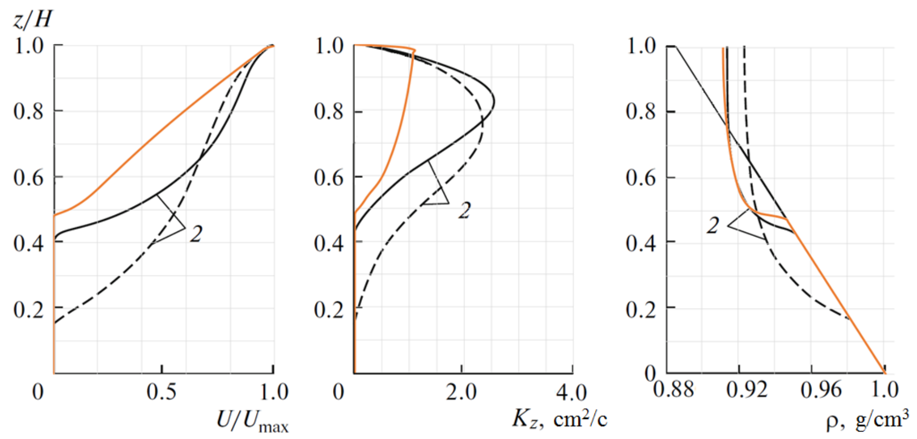

Fig. 1. Vertical distributions of the main flow characteristics for variant 2 at the time 240 s: $e-\varepsilon$-model (dashed line), improved model from work [2] (solid line), models 1,2 (yellow lines)

The process of deepening the upper mixed layer is shown in Fig. 2, where the dynamics of the dimensionless depth $\widehat{h} = N_0 h/u^*$ as a function of the dimensionless time $\widehat{t} = N_0 t$ for variant 2 presents. The proposed method gives a less intense expansion of the turbulent layer at $N_0 t < 360$ in comparison with the experiment, and at $N_0 t > 360$ the model 1 calculations approach the experiment. The calculations of variant 2 for model 2 (taking into account the Coriolis force) almost coincided with the results obtained for model 1. A more intensive expansion of the turbulent layer in comparison with the experiment was obtained by the classical $e-\varepsilon$-model. The calculations for the advanced model [2] are in good agreement with the experiment.

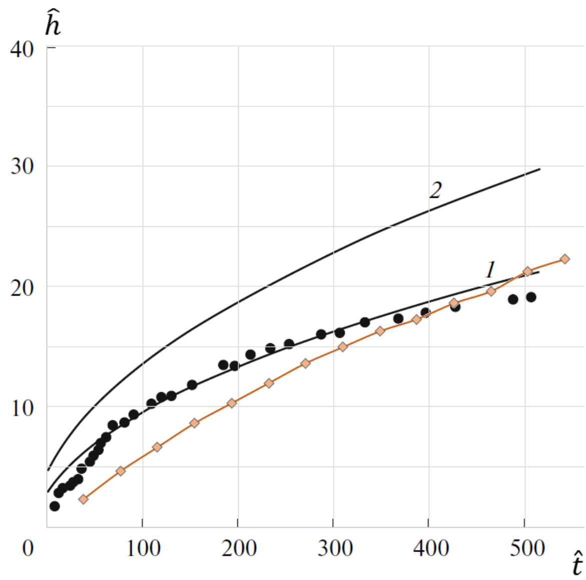

Fig. 2. Dynamics of the mixed layer depth for variant 2: advanced model (curve 1), $e-\varepsilon$-model (curve 2) from [2], model 1 (dotted line), experimental data [7] (dots)

The second series of numerical calculations relates to variant 3. The calculations were performed on uniform grids with the number of nodes from 120 to 250, time steps from 0.1 to 1.0 s. Fig. 3 shows the calculations results of the depth of the mixed layer up to the time $\widehat{t} = 1100$, obtained by the improved model [2] (curve 1), by model 1 (dotted line), experimental dependence (11) (dashed line). The calculations results by model 1 at $\widehat{t} < 600$ are underestimated compared to (11), at $\widehat{t} > 600$ they approach to the experimental dependence.

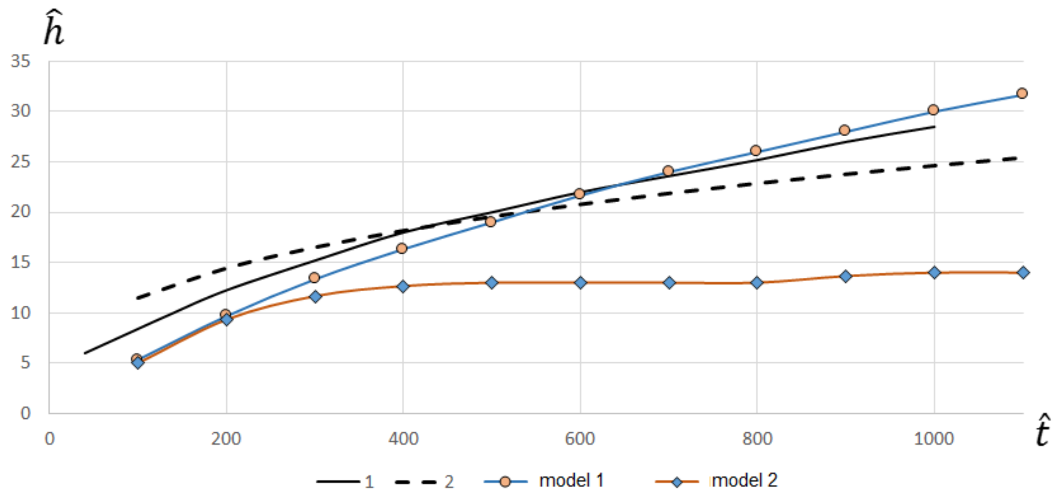

Fig. 3. Dynamics of the mixed layer depth for variant 3 at large times: improved model (curve 1), experimental data approximation (11) (dashed line) [2], model 1 (blue line), model 2 (orange line)

The Coriolis force has a significant effect on the deepening turbulent layer in a deep body of water ($H = 40$ m). The dynamics of the deepening turbulent layer by model 2 in Fig. 3 is shown by the orange line.

Numerical experiments were performed for variants 4–8. The results of numerical experiments on calculating the dynamics of a mixed turbulent layer deepening in a stably stratified reservoir using the constructed mathematical models are presented in the Figs. 4–8. The main parameters affecting the dynamics of the turbulent layer deepening in a stratified fluid are wind stress τ_w, reservoir depth H, vertical density gradient $\dfrac{\partial \rho}{\partial z}$, the Coriolis force f. Two modes are implemented for different combinations of these parameters. I — vertical mixing reaches the bottom, the results of calculations on models 1 and 2 are almost the same (Fig. 7, 8), therefore, we can restrict ourselves to model 1. II — the results of calculations for models 1 and 2 differ significantly: according to model 1, mixing reaches the bottom; according to model 2, the deepening of the bottom does not reach. In this case, when the Coriolis force is taken into account, the reservoir does not mix to the bottom and a quasistationary regime is realized $h < H$ (Fig. 4, 5). In variant 6 the solution of the problem according to model 1 and to model 2 one differs little (Fig. 6).

Thus, using simple models 1 and 2, it is possible to determine the effect of the Coriolis force on the process of deepening the turbulent layer in a stratified reservoir and to specify options when it is possible to be limited to model 1.

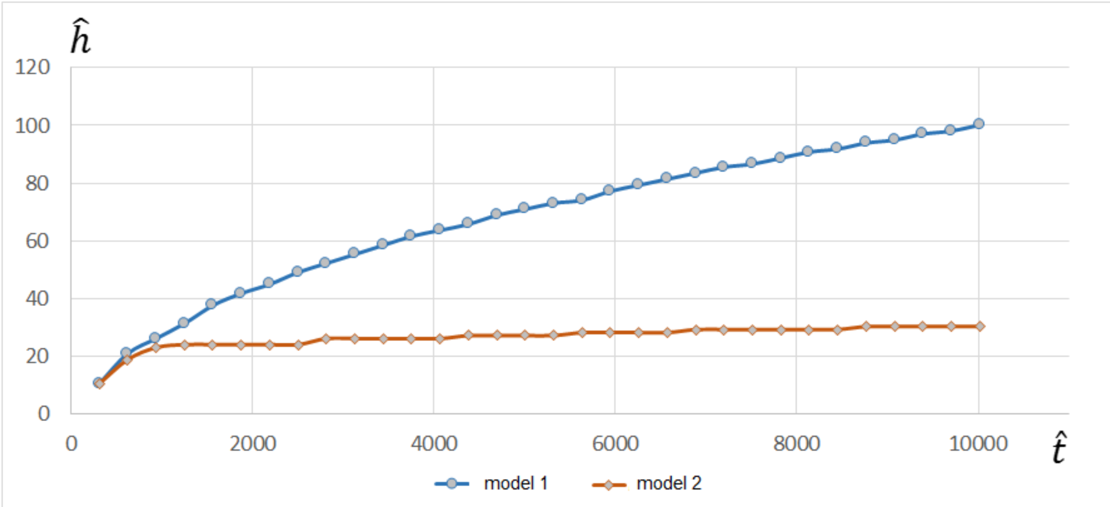

Fig. 4. Dynamics of the mixed layer depth for variant 4: model 1 (blue line), model 2 (orange line)

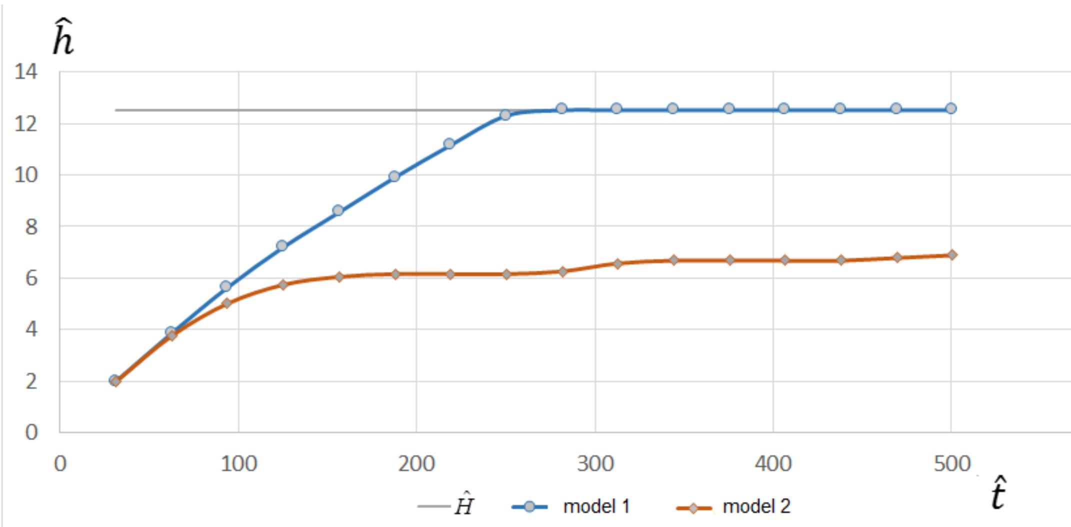

Fig. 5. Dynamics of the mixed layer depth for variant 5: model 1 (blue line), model 2 (orange line)

Conclusion

Numerical algorithms for describing the processes of vertical turbulent exchange in a stably stratified reservoir under constant shear stress are considered. These algorithms are based on the application of the Prandtl-Obukhov formula for the coefficients of vertical turbulent exchange. The Prandtl-Obukhov formula takes into account the shear mixing mechanism and stable stratification. The results of calculations of the vertical distributions of flow velocities, water density,

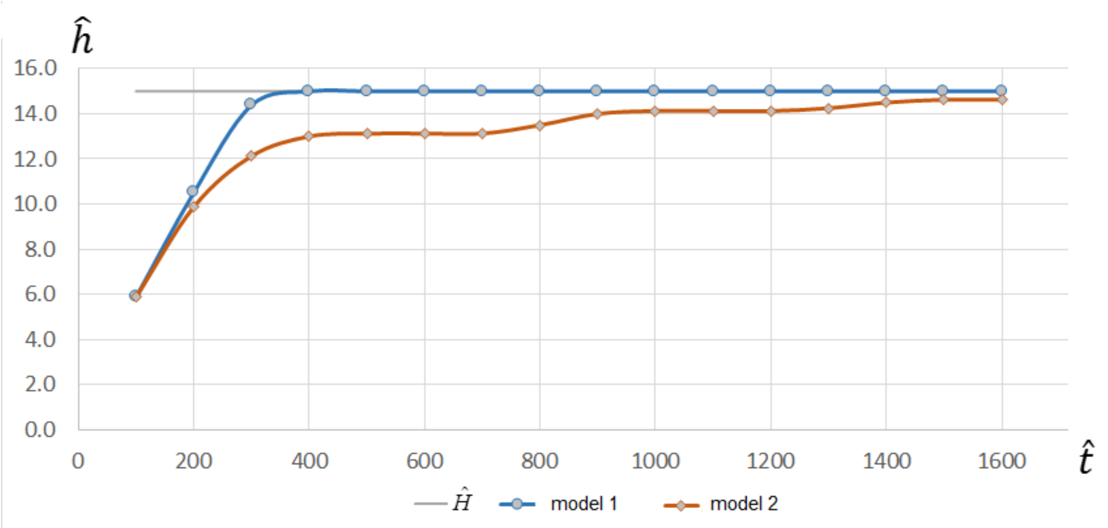

Fig. 6. Dynamics of the mixed layer depth for variant 6: model 1 (blue line), model 2 (orange line)

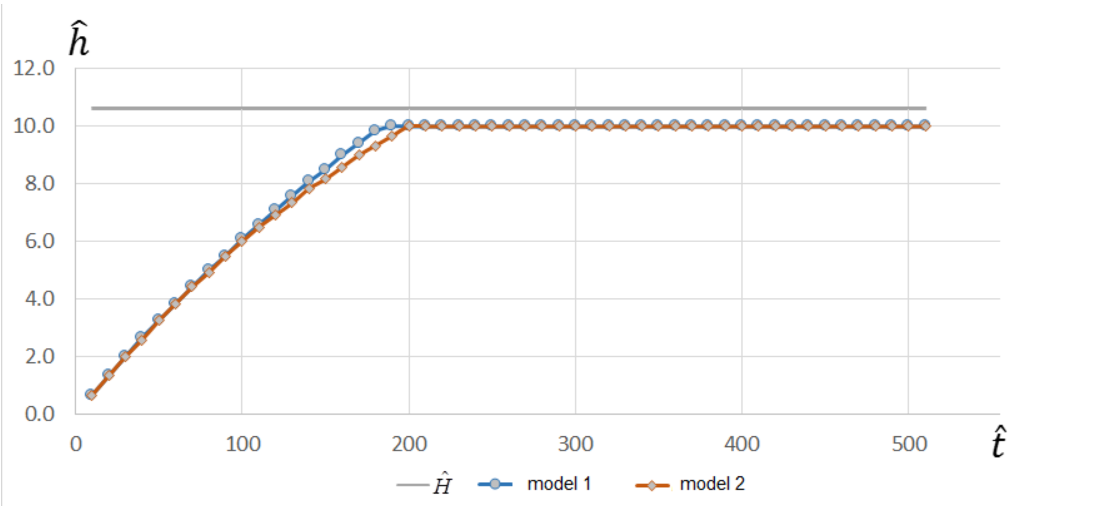

Fig. 7. Dynamics of the mixed layer depth for variant 7: model 1 (blue line), model 2 (orange line)

vertical turbulent exchange coefficients, and the dynamics of the deepening of the mixed layer according to the proposed models are consistent with experimental data and with calculations based on the $e - \varepsilon$ model and its modifications.

Using the constructed models of the dynamics of the turbulent layer deepening in a stably stratified fluid, it is possible to determine problems where the Coriolis force can be ignored.

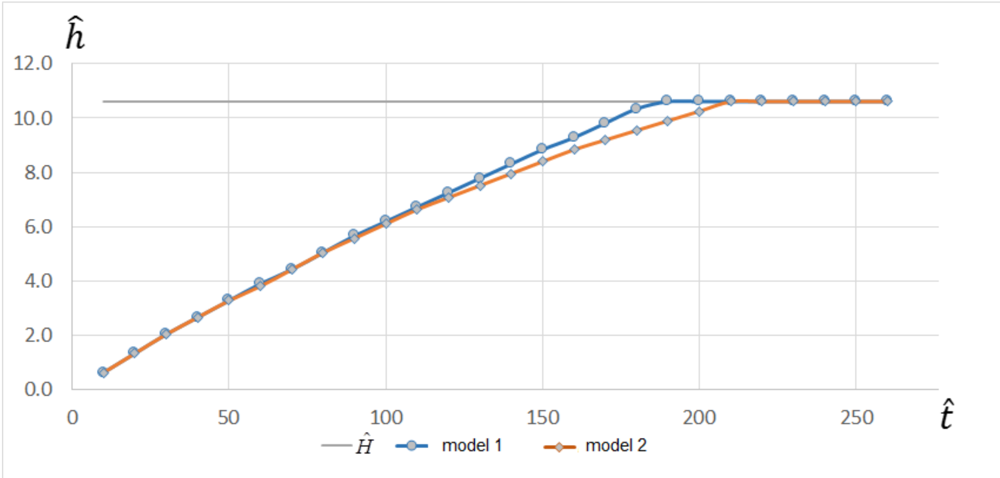

Fig. 8. Dynamics of the mixed layer depth for variant 8: model 1 (blue line), model 2 (orange line)

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О применении формулы Прандтля-Обухова в численной модели динамики заглубления турбулентного слоя

Виктор М. Белолипецкий
Светлана Н. Генова
Институт вычислительного моделирования СО РАН
Красноярск, Российская Федерация

Аннотация. Выполнено численное моделирование заглубления турбулентного слоя в устойчиво стратифицированной жидкости под действием касательного напряжения. Для коэффициента вертикального турбулентного обмена используется формула Прандтля–Обухова. Результаты расчетов согласуются с известными экспериментальными данными и расчетами других авторов.

Ключевые слова: математическое моделирование, турбулентность, стратифицированная жидкость.

DOI: 10.17516/1997-1397-2020-13-1-48-57
УДК 517.956.6; 517.44

Fundamental Solutions for a Class of Multidimensional Elliptic Equations with Several Singular Coefficients

Tuhtasin G. Ergashev*
Institute of Mathematics, Academy of Sciences of Uzbekistan
Tashkent, Uzbekistan

Received 13.09.2019, received in revised form 14.10.2019, accepted 04.12.2019

Abstract. The main result of the present paper is the construction of fundamental solutions for a class of multidimensional elliptic equations with several singular coefficients. These fundamental solutions are directly connected with multiple hypergeometric functions and the decomposition formula is required for their investigation which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables. In this paper, such a formula is proved instead of a previously existing recurrence formula.The order of singularity and other properties of the fundamental solutions that are necessary for solving boundary value problems for degenerate second-order elliptic equations are determined.

Keywords: multidimensional elliptic equation with several singular coefficients; fundamental solutions; decomposition formula.

Citation: T.G.Ergashev, Fundamental Solutions for a Class of Multidimensional Elliptic Equations with Several Singular Coefficients, J. Sib. Fed. Univ. Math. Phys., 2020, 13(1), 48–57.
DOI: 10.17516/1997-1397-2020-13-1-48-57.

Introduction

It is known that fundamental solutions have an essential role in studying partial differential equations. Formulation and solving of many local and non-local boundary value problems are based on these solutions. Moreover, fundamental solutions appear as potentials, for instance, as simple-layer and double-layer potentials in the theory of potentials.

The explicit form of fundamental solutions gives a possibility to study the considered equation in detail. For example, in the works of Barros-Neto and Gelfand [1–3] fundamental solutions for Tricomi operator, relative to an arbitrary point in the plane were explicitly calculated. In this direction we would like to note the works [4, 5], where three-dimensional fundamental solutions for elliptic equations were found. In the works [6–8] , fundamental solutions for a class of multidimensional degenerate elliptic equations with spectral parameter were constructed. The found solutions can be applied to solving some boundary value problems [9–15]. We also mention papers [16, 17] which are devoted to the study of partial differential equations with the singular coefficients and their solutions.

Let us consider the generalized Helmholtz equation with a several singular coefficients

$$L^m_{(\alpha)}(u) := \sum_{i=1}^{m} u_{x_i x_i} + \sum_{j=1}^{n} \frac{2\alpha_j}{x_j} u_{x_j} = 0 \tag{1}$$

*ergashev.tukhtasin@gmail.com https://orcid.org/0000-0003-3542-8309

in the domain $R_m^{n+} := \{(x_1, \ldots, x_m) : x_1 > 0, \ldots, x_n > 0\}$, where m is a dimension of the Euchlidean space, n is a number of the singular coefficients of equation (1); $m \geqslant 2$, $0 < n \leqslant m$; α_j are real constants and $0 < 2\alpha_j < 1$, $j = 1, \ldots, n$; $(\alpha) = (\alpha_1, \ldots, \alpha_n)$.

Various modifications of the equation (1) in the two- and three-dimensional cases were considered in many papers [4, 18–27].

Fundamental solutions for elliptic equations with singular coefficients are directly connected with hypergeometric functions. Therefore, basic properties such as decomposition formulas, integral representations, formulas of analytical continuation, formulas of differentiation for hypergeometric functions are necessary for studying fundamental solutions.

Since the aforementioned properties of hypergeometric functions of Gauss, Appell, Kummer were known [28], results on investigations of elliptic equations with one or two singular coefficients were successful. In the paper [4] when finding and studying the fundamental solutions of equation (1) for $m = 3$, an important role was played the decomposition formula of Hasanov and Srivastava [29, 30], however, the recurrence of this formula did not allow further advancement in the direction of increasing the number of singular coefficients.

In the present paper we construct all fundamental solutions for equation (1) in an explicit form and we prove a new formula for the expansion of several Lauricella hypergeometric functions by simple Gauss, with which it is possible to reveal that the found hypergeometric functions have a singularity of order $1/r^{m-2}$ at $r \to 0$. In the present paper, we assume that $m > 2$ and $0 < n \leqslant m$.

The plan of this paper is as follows. In Section 1 we briefly give some preliminary information, which will be used later. We transform the recurrence decomposition formula of Hasanov and Srivastava [29] to the form convenient for further research. Also some constructive formulas for the operator L are given. In Section 2 we describe the method of finding fundamental solutions for the considered equation and in Section 3 we show what order of singularity the found solutions will have.

1. Preliminaries

Below we give definition of Pochhammer symbol and some formulas for Gauss hypergeometric functions of one and two variables, Lauricella hypergeometric functions of three and more variables, which will be used in the next section.

A symbol $(\kappa)_\nu$ denotes the general Pochhammer symbol or the shified factorial, since $(1)_l = l!$ $(l \in N \cup \{0\};\ N := \{1, 2, 3, \ldots\})$, which is defined (for $\kappa, \nu \in C$), in terms of the familiar Gamma function, by

$$(\kappa)_\nu := \frac{\Gamma(\kappa+\nu)}{\Gamma(\kappa)} = \begin{cases} 1 & (\nu = 0;\ \kappa \in C\backslash\{0\}), \\ \kappa(\kappa+1)\ldots(\kappa+l-1) & (\nu = l \in N;\ \kappa \in C), \end{cases}$$

it being understood conventionally that $(0)_0 := 1$ and assumed tacitly that the Γ-quotient exists.

A function

$$F\left[\begin{array}{c} a, b; \\ c; \end{array} x\right] = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} x^k, \quad |x| < 1$$

is known as the Gauss hypergeometric function and an equality

$$F\left[\begin{array}{c} a, b; \\ c; \end{array} 1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad c \neq 0, -1, -2, \ldots, \mathrm{Re}(c-a-b) > 0 \tag{2}$$

holds [31, Ch.II,2.1(14)]. Moreover, the following autotransformer formula [31, Ch.II,2.1(22)]

$$F\left[\begin{array}{c} a,b; \\ c; \end{array} x\right] = (1-x)^{-b} F\left[\begin{array}{c} c-a,b; \\ c; \end{array} \frac{x}{x-1}\right] \tag{3}$$

is valid.

The hypergeometric function of n variables has a form [28, Ch.VII] (see also [32, Ch.1,1.4(1)])

$$F_A^{(n)}\left[\begin{array}{c} a,b_1,\ldots,b_n; \\ c_1,\ldots,c_n; \end{array} x_1,\ldots,x_n\right] = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{m_1!\ldots m_n!(c_1)_{m_1}\ldots(c_n)_{m_n}} x_1^{m_1}\ldots x_n^{m_n}, \tag{4}$$

where $|x_1|+\cdots+|x_n|<1$, $n\in \mathrm{N}$.

For a given multivariable function, it is useful to fund a decomposition formula which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables.

In the case of two variables for the function

$$F_2\left[\begin{array}{c} a,b_1,b_2; \\ c_1,c_2; \end{array} x,y\right] = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b_1)_i(b_2)_j}{i!j!(c_1)_i(c_2)_j} x^i y^j$$

was known expansion formula [33]

$$F_2\left[\begin{array}{c} a,b_1,b_2; \\ c_1,c_2; \end{array} x,y\right] = \sum_{k=0}^{\infty} \frac{(a)_k(b_1)_k(b_2)_k}{k!(c_1)_k(c_2)_k} x^k y^k F\left[\begin{array}{c} a+k,b_1+k; \\ c_1+k; \end{array} x\right] F\left[\begin{array}{c} a+k,b_2+k; \\ c_2+k; \end{array} y\right]. \tag{5}$$

Following the works [33, 34] Hasanov and Srivastava [29] found following decomposition formula for the Lauricella function of three variables

$$\begin{aligned} F_A^{(3)}\left[\begin{array}{c} a,b_1,b_2,b_3; \\ c_1,c_2,c_3; \end{array} x,y,z\right] &= \sum_{i,j,k=0}^{\infty} \frac{(a)_{i+j+k}(b_1)_{j+k}(b_2)_{i+k}(b_3)_{i+j}}{i!j!k!(c_1)_{j+k}(c_2)_{i+k}(c_3)_{i+j}} \times \\ &\times x^{j+k}y^{i+k}z^{i+j} F\left[\begin{array}{c} a+j+k,b_1+j+k; \\ c_1+j+k; \end{array} x\right] \times \\ &\times F\left[\begin{array}{c} a+i+j+k,b_2+i+k; \\ c_2+i+k; \end{array} y\right] F\left[\begin{array}{c} a+i+j+k,b_3+i+j; \\ c_3+i+j; \end{array} z\right] \end{aligned} \tag{6}$$

and they proved that for all $n\in N\backslash\{1\}$ is true the recurrence formula [29]

$$\begin{aligned} F_A^{(n)}\left[\begin{array}{c} a,b_1,\ldots,b_n; \\ c_1,\ldots,c_n; \end{array} x_1,\ldots,x_n\right] &= \sum_{m_2,\ldots,m_n=0}^{\infty} \frac{(a)_{m_2+\cdots+m_n}(b_1)_{m_2+\cdots+m_n}(b_2)_{m_2}\ldots(b_n)_{m_n}}{m_2!\ldots m_n!(c_1)_{m_2+\cdots+m_n}(c_2)_{m_2}\ldots(c_n)_{m_n}} \times \\ &\times x_1^{m_2+\cdots+m_n} x_2^{m_2}\ldots x_n^{m_n} F\left[\begin{array}{c} a+m_2+\cdots+m_n,b_1+m_2+\cdots+m_n; \\ c_1+m_2+\cdots+m_n; \end{array} x_1\right] \times \\ &\times F_A^{(n-1)}\left[\begin{array}{c} a+m_2+\cdots+m_n,b_2+m_2,\ldots,b_n+m_n; \\ c_2+m_2,\ldots,c_n+m_n; \end{array} x_2,\ldots,x_n\right]. \end{aligned} \tag{7}$$

Further study of the properties of the Lauricella function (4) showed that the formula (7) can be reduced to a more convenient form.

Lemma 1. *The following formula holds true at $n \in N$*

$$F_A^{(n)}\left[\begin{array}{l} a, b_1, \ldots, b_n; \\ c_1, \ldots, c_n; \end{array} x_1, \ldots, x_n\right] = \sum_{\substack{m_{i,j}=0 \\ (2\leqslant i\leqslant j\leqslant n)}}^{\infty} \frac{(a)_{N_2(n,n)}}{\underset{(2\leqslant i\leqslant j\leqslant n)}{m_{2,2}!m_{2,3}!\ldots m_{i,j}!\ldots m_{n,n}!}} \times$$
$$\times \prod_{k=1}^{n} \frac{(b_k)_{M_2(k,n)}}{(c_k)_{M_2(k,n)}} x_k^{M_2(k,n)} F\left[\begin{array}{l} a + N_2(k,n), b_k + M_2(k,n); \\ c_k + M_2(k,n); \end{array} x_k\right], \tag{8}$$

where

$$M_l(k,n) = \sum_{i=l}^{k} m_{i,k} + \sum_{i=k+1}^{n} m_{k+1,i}, \quad N_l(k,n) = \sum_{i=l}^{k+1}\sum_{j=i}^{n} m_{i,j}, \quad l \in N.$$

Proof. We carry out the proof by the method mathematical induction.

The equality (8) in the case $n = 1$ is obvious.

Let $n = 2$. Since $M_2(1,2) = M_2(2,2) = N_2(1,2) = N_2(2,2) = m_{2,2}$, we obtain the formula (5).

For the sake of interest, we will check the formula (8) in yet another value of n.

Let $n = 3$. In this case

$$M_2(1,3) = m_{2,2} + m_{2,3}, \quad M_2(2,3) = m_{2,2} + m_{3,3}, \quad M_2(3,3) = m_{2,3} + m_{3,3},$$
$$N_2(1,3) = m_{2,2} + m_{2,3}, \quad N_2(2,3) = N_2(3,3) = m_{2,2} + m_{2,3} + m_{3,3}.$$

For brevity, making the substitutions $m_{2,2} := i,\ m_{2,3} := j,\ m_{3,3} := k$, we obtain the formula (6).

So the formula (8) works for $n = 1$, $n = 2$ and $n = 3$.

Now we assume that for $n = s$ equality (8) holds; that is, that

$$F_A^{(s)}\left[\begin{array}{l} a, b_1, \ldots, b_s; \\ c_1, \ldots, c_s; \end{array} x_1, \ldots, x_s\right] = \sum_{\substack{m_{i,j}=0 \\ (2\leqslant i\leqslant j\leqslant s)}}^{\infty} \frac{(a)_{N_2(s,s)}}{\underset{(2\leqslant i\leqslant j\leqslant s)}{m_{2,2}!m_{2,3}!\ldots m_{i,j}!\ldots m_{s,s}!}} \times$$
$$\times \prod_{k=1}^{s} \frac{(b_k)_{M_2(k,s)}}{(c_k)_{M_2(k,s)}} x_k^{M_2(k,s)} F\left[\begin{array}{l} a + N_2(k,s), b_k + M_2(k,s); \\ c_k + M_2(k,s); \end{array} x_k\right]. \tag{9}$$

Let $n = s + 1$. We prove that following formula

$$F_A^{(s+1)}\left[\begin{array}{l} a, b_1, \ldots, b_{s+1}; \\ c_1, \ldots, c_{s+1}; \end{array} x_1, \ldots, x_{s+1}\right] = \sum_{\substack{m_{i,j}=0 \\ (2\leqslant i\leqslant j\leqslant s+1)}}^{\infty} \frac{(a)_{N_2(s+1,s+1)}}{\underset{(2\leqslant i\leqslant j\leqslant s+1)}{m_{2,2}!m_{2,3}!\ldots m_{i,j}!\ldots m_{s+1,s+1}!}} \times$$
$$\times \prod_{k=1}^{s+1} \frac{(b_k)_{M_2(k,s+1)}}{(c_k)_{M_2(k,s+1)}} x_k^{M_2(k,s+1)} F\left[\begin{array}{l} a + N_2(k,s+1), b_k + M_2(k,s+1); \\ c_k + M_2(k,s+1); \end{array} x_k\right] \tag{10}$$

is valid.

We write the Hasanov-Srivastava's formula (7) in the form

$$F_A^{(s+1)}\left[\begin{array}{l} a, b_1, \ldots, b_{s+1}; \\ c_1, \ldots, c_{s+1}; \end{array} x_1, \ldots, x_{s+1}\right] =$$
$$= \sum_{m_{2,2},\ldots,m_{2,s+1}=0}^{\infty} \frac{(a)_{N_2(1,s+1)}(b_1)_{M_2(1,s+1)}(b_2)_{m_{2,2}}\ldots(b_{s+1})_{m_{2,s+1}}}{m_{2,2}!\ldots m_{2,s+1}!(c_1)_{M_2(1,s+1)}(c_2)_{m_{2,2}}\ldots(c_{s+1})_{m_{2,s+1}}} \times$$
$$\times\ x_1^{M_2(1,s+1)} x_2^{m_{2,2}} \ldots x_{s+1}^{m_{2,s+1}} F\left[\begin{array}{l} a + N_2(1,s+1), b_1 + M_2(1,s+1); \\ c_1 + M_2(1,s+1); \end{array} x_1\right] \times \tag{11}$$
$$\times\ F_A^{(s)}\left[\begin{array}{l} a + N_2(1,s+1), b_2 + m_{2,2}, \ldots, b_{s+1} + m_{2,s+1}; \\ c_2 + m_{2,2}, \ldots, c_{s+1} + m_{2,s+1}; \end{array} x_2, \ldots, x_{s+1}\right].$$

By virtue of the formula (9) we have

$$F_A^{(s)}\left[\begin{array}{c} a+N_2(1,s+1), b_2+m_{2,2},\dots,b_{s+1}+m_{2,s+1}; \\ c_2+m_{2,2},\dots,c_{s+1}+m_{2,s+1}; \end{array} x_2,\dots,x_{s+1}\right]=$$

$$=\sum_{\substack{m_{i,j}=0\\(3\leqslant i\leqslant j\leqslant s+1)}}^{\infty}\frac{(a+N_2(1,s+1))_{N_3(s+1,s+1)}}{\underset{(3\leqslant i\leqslant j\leqslant s+1)}{m_{3,3}!m_{3,4}!\dots m_{i,j}!\dots m_{s+1,s+1}!}}\prod_{k=2}^{s+1}\frac{(b_k+m_{2,k})_{M_3(k,s+1)}}{(c_k+m_{2,k})_{M_3(k,s+1)}}x_k^{M_3(k,s+1)}\times \tag{12}$$

$$\times\ F\left[\begin{array}{c} a+N_2(1,s+1)+N_3(k,s+1), b_k+m_{2,k}+M_3(k,s+1); \\ c_k+m_{2,k}+M_3(k,s+1); \end{array} x_k\right].$$

Substituting from (12) into (11) we obtain

$$F_A^{(s+1)}[a,b_1,\dots,b_{s+1};c_1,\dots,c_{s+1};x_1,\dots,x_{s+1}]=$$

$$=\sum_{\substack{m_{i,j}=0\\(2\leqslant i\leqslant j\leqslant s+1)}}^{\infty}\frac{(a)_{N_2(1,s+1)+N_3(s+1,s+1)}}{\underset{(2\leqslant i\leqslant j\leqslant s+1)}{m_{2,2}!m_{2,3}!\dots m_{i,j}!\dots m_{s+1,s+1}!}}\prod_{k=1}^{s+1}\frac{(b_k)_{m_{2,k}+M_3(k,s+1)}}{(c_k)_{m_{2,k}+M_3(k,s+1)}}x_k^{m_{2,k}+M_3(k,s+1)}\times$$

$$\times\ F\left[\begin{array}{c} a+N_2(1,s+1)+N_3(k,s+1), b_k+m_{2,k}+M_3(k,s+1); \\ c_k+m_{2,k}+M_3(k,s+1); \end{array} x_k\right].$$

Further, by virtue of the following obvious equalities

$$N_2(1,s+1)+N_3(k,s+1)=N_2(k,s+1),\ 1\leqslant k\leqslant s+1,\ s\in N,$$

$$m_{2,k}+M_3(k,s+1)=M_2(k,s+1),\ 1\leqslant k\leqslant s+1,\ s\in N,$$

we finally find the equality (10). The lemma is proved. □

2. Fundamental solutions

Consider equation (1) in R_m^{n+}. Let $x:=(x_1,\dots,x_m)$ be any point and $\xi:=(\xi_1,\dots,\xi_m)$ be any fixed point of R_m^{n+}. We search for a solution of (1) as follows:

$$u(x,\xi)=P(r)w(\sigma), \tag{13}$$

where

$$\sigma=(\sigma_1,\dots,\sigma_n),\quad \tilde{\alpha}_0=\alpha_1+\dots+\alpha_n-1+\frac{m}{2},$$

$$P(r)=\left(r^2\right)^{-\tilde{\alpha}_0},\quad r^2=\sum_{i=1}^{m}(x_i-\xi_i)^2,$$

$$r_k^2=(x_k+\xi_k)^2+\sum_{i=1,i\neq k}^{m}(x_i-\xi_i)^2,\quad \sigma_k=\frac{r^2-r_k^2}{r^2},\quad k=1,2,\dots,n.$$

We calculate all necessary derivatives and substitute them into equation (1) :

$$\sum_{k=1}^{n}A_k\omega_{\sigma_k\sigma_k}+\sum_{k=1}^{n}\sum_{l=k+1}^{n}B_{k,l}\omega_{\sigma_k\sigma_l}+\sum_{k=1}^{n}C_k\omega_{\sigma_k}+D\omega=0, \tag{14}$$

where

$$A_k = P\sum_{i=1}^{m}\left(\frac{\partial\sigma_k}{\partial x_i}\right)^2, \quad B_{k,l} = 2P\sum_{i=1}^{m}\frac{\partial\sigma_k}{\partial x_i}\frac{\partial\sigma_l}{\partial x_i}, \quad k\neq l,\ k=1,\ldots,n,$$

$$C_k = P\sum_{i=1}^{m}\frac{\partial^2\sigma_k}{\partial x_i^2} + 2\sum_{i=1}^{m}\frac{\partial P}{\partial x_i}\frac{\partial\sigma_k}{\partial x_i} + 2P\sum_{j=1}^{n}\frac{\alpha_j}{x_j}\frac{\partial\sigma_k}{\partial x_j},$$

$$D = \sum_{i=1}^{m}\frac{\partial^2 P}{\partial x_i^2} + 2P\sum_{j=1}^{n}\frac{\alpha_j}{x_j}\frac{\partial P}{\partial x_j}.$$

After several evaluations we find

$$A_k = -\frac{4P(r)}{r^2}\frac{x_k}{\xi_k}\sigma_k(1-\sigma_k), \tag{15}$$

$$B_{k,l} = \frac{4P(r)}{r^2}\left(\frac{\xi_k}{x_k}+\frac{\xi_l}{x_l}\right)\sigma_k\sigma_l, \quad k\neq l, \quad l=1,\ldots,n, \tag{16}$$

$$C_k = -\frac{4P(r)}{r^2}\left\{-\sigma_k\sum_{j=1}^{n}\frac{\xi_j}{x_j}\alpha_j + \frac{\xi_k}{x_k}[2\alpha_k - \tilde{\alpha}_0\sigma_k]\right\}, \tag{17}$$

$$D = \frac{4\tilde{\alpha}_0 P(r)}{r^2}\sum_{j=1}^{n}\frac{\xi_j}{x_j}\alpha_j. \tag{18}$$

Substituting equalities (15)–(18) into (14) we obtain the following system of hypergeometric equations of Lauricella [28], which has 2^n linearly-independent solutions. Considering those solutions, from (13) we obtain 2^n fundamental solutions of equation (1):

$$1\{\, F_A^{(n)}\left[\begin{matrix} a, b_1, \ldots, b_n; \\ c_1, \ldots, c_n; \end{matrix}\ \sigma\right], \tag{19}$$

$$C_n^1\left\{\begin{array}{c} (x_1\xi_1)^{1-c_1}F_A^{(n)}\left[\begin{matrix} a+1-c_1, b_1+1-c_1, b_2, \ldots, b_n; \\ 2-c_1, c_2, \ldots, c_n; \end{matrix}\ \sigma\right], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (x_n\xi_n)^{1-c_n}F_A^{(n)}\left[\begin{matrix} a+1-c_n, b_1, \ldots, b_{n-1}, b_n+1-c_n; \\ c_1, \ldots, c_{n-1}, 2-c_n; \end{matrix}\ \sigma\right], \end{array}\right. \tag{20}$$

$$C_n^2\left\{\begin{array}{c} (x_1\xi_1)^{1-c_1}(x_2\xi_2)^{1-c_2}F_A^{(n)}\left[\begin{matrix} a+2-c_1-c_2, b_1+1-c_1, b_2+1-c_2, b_3, ..., b_n; \\ 2-c_1, 2-c_2, c_3, \ldots, c_n; \end{matrix}\ \sigma\right], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (x_1\xi_1)^{1-c_1}(x_n\xi_n)^{1-c_n}F_A^{(n)}\left[\begin{matrix} a+2-c_1-c_n, b_1+1-c_1, b_2, \ldots, b_{n-1}, b_n+1-c_n; \\ 2-c_1, c_2, \ldots, c_{n-1}, 2-c_n; \end{matrix}\ \sigma\right], \\ (x_2\xi_2)^{1-c_2}(x_3\xi_3)^{1-c_3}F_A^{(n)}\left[\begin{matrix} a+2-c_2-c_3, b_1, b_2+1-c_2, b_3+1-c_3, b_4, \ldots, b_n; \\ c_1, 2-c_2, 2-c_3, c_4, \ldots, c_n; \end{matrix}\ \sigma\right], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{(x_{n-1}\xi_{n-1})^{1-c_{n-1}}}{(x_n\xi_n)^{c_n-1}}F_A^{(n)}\left[\begin{matrix} a+2-c_{n-1}-c_n, b_1, \ldots, b_{n-2}, b_{n-1}+1-c_{n-1}, b_n+1-c_n; \\ c_1, \ldots, c_{n-2}, 2-c_{n-1}, 2-c_n; \end{matrix}\ \sigma\right], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right. \tag{21}$$

$$1\left\{(x_1\xi_1)^{1-c_1}\cdot\dots\cdot(x_n\xi_n)^{1-c_n}F_A^{(n)}\left[\begin{matrix}a+n-c_1-\dots-c_n, b_1+1-c_1,\dots,b_n+1-c_n;\\ 2-c_1,\dots,2-c_n;\end{matrix}\ \sigma\right],\right.$$

where

$$a=\tilde{\alpha}_0,\ b_i=\alpha_i,\ c_i=2\alpha_i,\ 1\leqslant i\leqslant n;\quad C_n^k=\frac{n!}{k!(n-k)!},\ 0\leqslant k\leqslant n.$$

It is easy to see that in (19) there is one function, in (20) there are $C_n^1=n$ functions, in (21) there are $C_n^2=n(n-1)/2$ functions and so on, and therefore

$$1+C_n^1+C_n^2+\dots+C_n^{n-1}+1=2^n.$$

Taking into account the symmetry property of the Lauricella function $F_A^{(n)}$ with respect to the parameters $b_1,\dots,b_n,c_1,\dots,c_n$, we can reduced the quantity of the fundamental solutions that are necessary in the study of boundary value problems: from each of the systems (19), (20), (21) and so on we take only one fundamental solution. Consequently, all $n+1$ (non-symmetric) fundamental solutions of equation (1) can be written in the form which is a convenient for further investigation:

$$q_0(x,\xi)=\gamma_0 r^{-2\tilde{\alpha}_0}F_A^{(n)}\left[\begin{matrix}\tilde{\alpha}_0,\alpha_1,\dots,\alpha_n;\\ 2\alpha_1,\dots,2\alpha_n;\end{matrix}\ \sigma\right], \tag{22}$$

$$q_k(x,\xi)=\gamma_k\prod_{i=1}^{k}(x_i\xi_i)^{1-2\alpha_i}\cdot r^{-2\tilde{\alpha}_k}F_A^{(n)}\left[\begin{matrix}\tilde{\alpha}_k,1-\alpha_1,\dots,1-\alpha_k,\alpha_{k+1},\dots,\alpha_n;\\ 2-2\alpha_1,\dots,2-2\alpha_k,2\alpha_{k+1},\dots,2\alpha_n;\end{matrix}\ \sigma\right],\ k=\overline{1,n}, \tag{23}$$

where

$$\tilde{\alpha}_k=\frac{m}{2}+k-1-\alpha_1-\dots-\alpha_k+\alpha_{k+1}+\dots+\alpha_n,\quad k=\overline{1,n},$$

$$\gamma_k=2^{2\tilde{\alpha}_k-m}\frac{\Gamma(\tilde{\alpha}_k)}{\pi^{m/2}}\prod_{i=k+1}^{n}\frac{\Gamma(\alpha_i)}{\Gamma(2\alpha_i)}\prod_{j=1}^{k}\frac{\Gamma(1-\alpha_j)}{\Gamma(2-2\alpha_j)},\quad k=\overline{0,n}.$$

3. Singularity properties of fundamental solutions

Let us show that the fundamental solutions (22) and (23) have a singularity at $r=0$.

We choose a solution $q_0(x,\xi)$ and we use the expansion for the hypergeometric function of Lauricella (8). As a result, a solution defined by (22) can be written as follows

$$q_0(x,\xi)=\gamma_0 r^{-2\tilde{\alpha}_0}\sum_{\substack{m_{i,j}=0\\(2\leqslant i\leqslant j\leqslant n)}}^{\infty}\frac{(\tilde{\alpha}_0)_{N_2(n,n)}}{\underset{(2\leqslant i\leqslant j\leqslant n)}{m_{2,2}!m_{2,3}!\dots m_{i,j}!\dots m_{n,n}!}}\times$$
$$\times\prod_{k=1}^{n}\frac{(\alpha_k)_{M_2(k,n)}}{(2\alpha_k)_{M_2(k,n)}}\left(1-\frac{r_k^2}{r^2}\right)^{M_2(k,n)}F\left[\begin{matrix}\tilde{\alpha}_0+N_2(k,n),\alpha_k+M_2(k,n);\\ 2\alpha_k+M_2(k,n);\end{matrix}\ 1-\frac{r_k^2}{r^2}\right]. \tag{24}$$

By virtue of formula (3) we rewrite (24) as

$$q_0(x,\xi)=\frac{\gamma_0}{r^{m-2}}\prod_{k=1}^{n}r_k^{-2\alpha_k}\cdot f_0\left(r^2,r_1^2,\dots,r_n^2\right),$$

where

$$f_0\left(r^2,r_1^2,\dots,r_n^2\right)=\sum_{\substack{m_{i,j}=0\\(2\leqslant i\leqslant j\leqslant n)}}^{\infty}\frac{(\tilde{\alpha}_0)_{N_2(n,n)}}{\underset{(2\leqslant i\leqslant j\leqslant n)}{m_{2,2}!m_{2,3}!\cdots m_{i,j}!\cdots m_{n,n}!}}\times$$

$$\times \prod_{k=1}^{n} \frac{(\alpha_k)_{M_2(k,n)}}{(2\alpha_k)_{M_2(k,n)}} \left(\frac{r^2}{r_k^2} - 1\right)^{M_2(k,n)} F \left[\begin{matrix} 2\alpha_k - \tilde{\alpha}_0 + M_2(k,n) - N_2(k,n), \alpha_k + M_2(k,n); \\ 2\alpha_k + M_2(k,n); \end{matrix} 1 - \frac{r^2}{r_k^2} \right].$$

Below we show that $f_0\left(r^2, r_1^2, \ldots, r_n^2\right)$ will be constant at $r \to 0$.

For this aim we use an equality (2) and following inequality

$$N_2(k,n) - M_2(k,n) := \sum_{i=2}^{k} \left(\sum_{j=i}^{n} m_{i,j} - m_{i,k} \right) \geqslant 0, \quad 1 \leqslant k \leqslant n \leqslant m.$$

Then we get

$$\lim_{r\to 0} f_0\left(r^2, r_1^2, \ldots, r_n^2\right) = \frac{1}{\Gamma^n(\tilde{\alpha}_0)} \prod_{k=1}^{n} \frac{\Gamma\left(2\alpha_k\right)\Gamma\left(\tilde{\alpha}_0 - \alpha_k\right)}{\Gamma\left(\alpha_k\right)}. \tag{25}$$

Expressions (24) and (25) give us the possibility to conclude that the solution $q_0(x,\xi)$ reduces to infinity of the order r^{2-m} at $r \to 0$. Similarly it is possible to be convinced that solutions $q_k(x;\xi),\ k = 1, 2, \ldots, n$ also reduce to infinity of the order r^{2-m} when $r \to 0$.

It can be directly checked that constructed functions (22) and (23) possess following properties

$$\left(x_j^{2\alpha_j} \frac{\partial q_0\left(x,\xi\right)}{\partial x_j} \right)\Bigg|_{x_j=0} = 0, \quad q_n\left(x,\xi\right)|_{x_j=0} = 0, \quad 1 \leqslant j \leqslant n,$$

$$q_k\left(x,\xi\right)|_{x_j=0} = 0, \quad 1 \leqslant j \leqslant k, \quad \left(x_j^{2\alpha_j} \frac{\partial q_k\left(x,\xi\right)}{\partial x_j} \right)\Bigg|_{x_j=0} = 0, \quad k+1 \leqslant j \leqslant n, \quad 1 \leqslant k \leqslant n-1.$$

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Фундаментальные решения многомерного эллиптического уравнения с несколькими сингулярными коэффициентами

Тухтасин Г. Эргашев
Институт математики Академии наук Узбекистана
Ташкент, Узбекистан

Аннотация. Основным результатом настоящей работы является построение фундаментальных решений для одного класса эллиптических уравнений с несколькими сингулярными коэффициентами. Поскольку эти решения напрямую связаны с гипергеометрическими функциями многих переменных Лауричелла, то для изучения свойств найденных фундаментальных решений требуется найти формулу разложения, которая выражала бы многомерную гипергеометрическую функцию в виде суммы произведений нескольких более простых гипергеометрических функций с меньшим числом переменных. В этой работе такая формула доказана вместо ранее существовавшей рекуррентной формулы и определен порядок особенности фундаментальных решений.

Ключевые слова: многомерное эллиптическое уравнение с несколькими сингулярными коэффициентами, фундаментальные решения, формула разложения.

DOI: 10.17516/1997-1397-20-13-1-58-70
УДК 517.5

Berry Phase for Time-Dependent Coupled Harmonic Oscillators in the Noncommutative Phase Space via Path Integral Techniques

Leila Khiari*
LRPPS, University Kasdi Merbah
Ouargla, Algeria

Tahar Boudjedaa†
University of Jijel
Jijel, Algeria

Abdenacer Makhlouf‡
University of Haute Alsace
Mulhouse, France

Mohammed Tayeb Meftah§
LRPPS, University Kasdi Merbah
Ouargla, Algeria

Received 08.05.2019, received in revised form 15.10.2019, accepted 25.12.2019

Abstract. The purpose of this paper is the description of Berry's phase, in the Euclidean Path Integral formalism, for 2D quadratic system: two time dependent coupled harmonic oscillators. This treatment is achieved by using the adiabatic approximation in the commutative and noncommutative phase space.

Keywords: Berry's phase, noncommutative phase space, coupled oscillators.

Citation: L.Khiari, T.Boudjedaa, A.Makhlouf, M.Tayeb Meftah, Berry Phase for Time-Dependent Coupled Harmonic Oscillators in the Noncommutative Phase Space via Path Integral Techniques, J. Sib. Fed. Univ. Math. Phys., 2020, 13(1), 58–70.
DOI: 10.17516/1997-1397-20-13-1-58-70.

1. Introduction and preliminaries

The classical geometry is based on the duality between the geometry and the commutative algebra. In commutative algebra, the product of two algebraic quantities is independent from the order. In Quantum Mechanics, following Heisenberg's viewpoint, the geometry of the states space describing a microscopic system, an atom for example, has a new property such as the momentum and the position are non-commuting operators [1–7]:

$$[x_i, x_j] = i\theta_{ij}, \qquad [p_i, p_j] = i\sigma_{ij}, \qquad [x_i, p_j] = i\delta_{ij}. \tag{1}$$

The purpose of noncommutative geometry is to generalize the duality of space geometry [8–10] and algebra to the more general situation where the algebra is not commutative. This leads to

*leila.khiari2010@gmail.com
†boudjedaa@gmail.com
‡abdenacer.makhlouf@uha.fr
§mewalid@yahoo.com

change two fundamental concepts of mathematics, those of space and symmetry and adjusts all the mathematical tools in these new paradigms.

The prime interest of the theory comes entirely from new and unexpected phenomena that have no counterpart in the case of commutative geometry. The commutative Riemannian geometry which provides a framework of general relativity was generalized by Einstein to "quantum" version.

The passage of the Riemann geometry to the noncommutative geometry [11] is the transition from the measurement of distances to the use of operators algebra. This gives a notion of a spectral nature of geometric space which is more flexible.

The noncommutative geometry treats both the noninteger dimension space [12,13], an infinite dimension space, especially "quantum" space and finally the space-time itself. If we take into account, not only the electromagnetic strength (which led to Poincaré, Einstein and Minkowski model of spacetime), but also the weak and strong forces, the use of the noncommutative spacetime properties becomes necessary.

Furthermore, Feynman paths integrals method encounters substantial difficulties when used in a noncommutative space because it is basically meaningless to talk about path in a noncommutative spacetime. Therefore the formulation of path integrals must done not in the space of noncommutative coordinates itself but in the space of noncommutative phase space (mixed space). This is required by the spirit of the Feynman path integrals construction.

Indeed, we consider a space-time (2 + 1)-dimensions, which can be easily generalized to higher dimensions. So, to conciliate our work in the canonical formalism, we used a full basis of commuting operators. We shall take as space, the configuration space (x_1, p_2) built on commuting operators such that we avoid the noncommutativity.

In this paper we are mainly concerned with two coupled harmonic oscillators with arbitrary time dependent frequencies and masses leading to use some time-dependent transformations. The originality of this work, is the description of the system in the noncommutative mixed phase space by using the path integral techniques to extract the "Berry's phase". We recall that Berry's phase has attracted the attention of many physicists, it was first discovered in 1956, and rediscovered in 1984 by Berry who has published a paper [14] which has until now deeply influenced the physical community. Therein he considers cyclic evolutions of systems under special conditions, namely adiabatic ones. He finds that an additional phase factor occurs in contrast to the well known dynamical phase factor. This phenomenon can be described by "global change without local change". Berry points out the geometrical character of this phase which is not negligible because of its non integrable character [15]. This was not the first time such a phase factor appears, for instance, considerations of the Born-Oppenheimer approximation done by Mead and Truhlar in 1979 revealed also this additional phase factor but it had been neglected [16]. Berry showed that this was not correct because the phase is a gauge invariant and therefore can not be gauged away.

A brief outline of the present paper is as follows: in the next section, we give the construction of the path integral in the noncommutative phase space. In Section 3, we present two applications, the first one is the time dependent coupled harmonic oscillators in commutative phase space, the second application deals with the time-dependent coupled harmonic oscillators in noncommutative phase space. In each case, Berry's phase (geometric phase) was derived as well as the dynamic phase. A conclusion is provided in the last section.

2. Path integral in noncommutative phase space

In this section we must be concerned with Feynman's path integral formalism, which is described by a Hamiltonian $H(x, p)$ made up the cartesian coordinates x_i, and their canonically conjugate momenta p_j. Nevertheless, unlike the usual case, "coordinates and momenta" are assumed to obey the noncommutative rules.

$$[x_i, x_j] = i\theta_{ij}, \qquad [p_i, p_j] = i\boldsymbol{\Sigma}_{ij}, \qquad [x_i, p_j] = i\delta_{ij}, \tag{2}$$

where $\boldsymbol{\Theta}$ and $\boldsymbol{\Sigma}$ are two-antisymmetric matrix such as $\boldsymbol{\Theta}_{12} = \theta$ and $\boldsymbol{\Sigma}_{12} = \sigma$. To these commutation relations correspond the deformed Poisson brackets in classical phase space defined as

$$\{x_i, x_j\} = \theta_{ij}, \qquad \{p_i, p_j\} = \boldsymbol{\Sigma}_{ij}, \qquad \{x_i, p_j\} = \delta_{ij}, \tag{3}$$

where $\boldsymbol{\Theta}$ and $\boldsymbol{\Sigma}$ are noncommutative parameters.

Acting on the Heisenberg algebra (2), it is easy to found the Path integral in noncommutative phase space. But in this work, we propose to build the path integral while maintaining the spirit of Feynman's construction. On this basis, we choose the mixed phase space i.e. $Q^T = (x_1, p_2)$ and $P^T = (x_2, p_1)$.

A path integral formalism in noncommutative mixed coordinates is

$$K_{\theta\sigma}(Q^i, Q^f, T) = \int DQDP \exp\left[i \int \left[PJ_{\theta\sigma}^{-1}\dot{Q} - H\left(Q, P\right)\right] dt\right], \tag{4}$$

where $Q^T = (x_1, p_2)$, $P^T = (x_2, p_1)$ and $(J_{\theta,\sigma})_{ij} = \begin{pmatrix} \theta & 1 \\ -1 & -\sigma \end{pmatrix}$ is the symplectic form. For simplicity, the propagator (4), using linear canonical transformation which known as Bopp-shift in form matrix, may be written as

$$K_{\theta\sigma}(Q^i, Q^f, T) = \int DQDP \exp\left[i \int \left[PJ_{0,0}^{-1}\dot{Q} - H_{\theta\sigma}\left(Q, \bar{A}P\right)\right] dt\right], \tag{5}$$

where

$$J_{\mathbf{0},\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 1 & \theta \\ \sigma & 1 \end{pmatrix}.$$

As θ and σ are very small parameters, we set $\bar{A} = 1 + \beta$, with $\beta = \begin{pmatrix} 0 & \theta \\ \sigma & 0 \end{pmatrix}$, then

$$H_{\theta\sigma}\left(Q, \bar{A}P\right) = H(Q, P) + H_\beta(Q, P), \tag{6}$$

with $H_\beta(Q, P)$ is now a small perturbation added to the Hamiltonian such that we can use Taylor's expansion of the Hamiltonian

$$H_\beta(Q, P) \simeq (\beta P)^T \frac{\partial H(Q, P)}{\partial P} + \sum_{ij} \frac{1}{2!} \left(\beta P\right)_i^T \left(\beta P\right)_j \frac{\partial^2 H(Q, P)}{\partial P_i \partial P_j}. \tag{7}$$

The Feynman's formalism for a general potential in noncommutative phase space is given by

$$K_{\theta\sigma}(Q^i, Q^f, T) = \int DQDP \exp\left[i \int dt \left(PJ^{-1}\dot{Q} - H(Q, P) - H_\beta(Q, P)\right)\right]. \tag{8}$$

The main goal of this paper is to find Berry phase of the 2-dimensional coupled harmonic oscillators in two cases the first one in commutative phase space, and the second case in noncommutative phase space, under the Euclidean path integral formalism. The original premise for Berry's phase is the adiabatic theorem of quantum mechanics [17], which deals with a system coupled to a slowly changing environment: the Hamiltonian system $H\left(t\right)$ varies adiabatically.

To extract the Berry's phase from the propagator (8) , we follow the method used by Kashiwa to obtain the Berry's phase for one dimension harmonic oscillator , in which it summarizes as follows [18].

1. Consider the Euclidean kernel for the given Hamiltonian: ($t \to -it$) while keeping the external variables unchanged.
2. Examine the large T limit of the kernel.
3. Find the imaginary part of $O(T)$ from the exponent of the kernel.

So, In the Euclidean space ($t \to -it$) and for adiabatic approximation, we set $s = \dfrac{t}{T}$ with T very large, the(8) is given by:

$$K_{\theta\sigma}(Q^i, Q^f, T) = \int DQDP \exp\left[T \int ds \left(\frac{i}{T} PJ^{-1}\dot{Q} - H(Q,P) - H_\beta(Q,P)\right)\right], \tag{9}$$

this latter is the Feynman's formalism in non-commutative phase space. where, we put $\theta, \sigma \to 0$, will return (9) to the commutative phase space (the usual phase space) ,i.e, the (9) is given

$$K(Q^i, Q^f, T) = \int DQDP \exp\left[T \int ds \left(\frac{i}{T} PJ^{-1}\dot{Q} - H(Q,P)\right)\right]. \tag{10}$$

2.1. Time-dependent coupled harmonic oscillators in commutative phase space

Consider a pair of coupled general time-dependent oscillators with same frequencies and masses whose Hamiltonian in commutative phase space takes the form [21].

The quantum mechanical evolution of the system can be described by the Feynman propagator, in the mixed phase space $Q^T = (x_1, p_2)$ and $P^T = (x_2, p_1)$ (formulation of Feynman's path integral), which is defined formally by

$$K(Q^f, Q^i; t) = \int DQDP =$$
$$= \exp\left[\int \left((PJ_{0,0}^{-1}\dot{Q} - \frac{1}{2} PM(t)P - \frac{1}{2} QW(t)Q - P\lambda(t)Q\right) dt\right]. \tag{11}$$

The matrices $M(t)$, $W(t)$ and $\lambda(t)$ are time dependent functions given respectively by

$$\begin{pmatrix} \mu_1(t) & 0 \\ 0 & \mu_2(t) \end{pmatrix}, \begin{pmatrix} \omega_1^2(t) & 0 \\ 0 & \omega_2^2(t) \end{pmatrix}, \begin{pmatrix} 0 & \lambda_2(t) \\ \lambda_1(t) & 0 \end{pmatrix}, \tag{12}$$

where

$$\mu_1(t) = m(t)\,\omega^2(t), \qquad \mu_2(t) = \frac{1}{m(t)}, \tag{13}$$

$$\omega_1(t) = \omega(t)\sqrt{m(t)}, \qquad \omega_2(t) = \frac{1}{\sqrt{m(t)}}. \tag{14}$$

Now, we follow the steps [18] that we have mentioned previously, we take $t \to -it$, and in order to specify the adiabatic parameter $\dfrac{1}{T}$, we introduce a scaled times $s = t/T$ in (11), and after using change the variable $P \to JP$ on a level of Lagrange in (11), we get

$$K(Q^i, Q^f, T) = \int DQDP =$$
$$= \exp\left[T \int \left(\frac{i}{T} P\dot{Q} - \frac{1}{2} P\left(J_{0,0} M J_{0,0}^{-1}\right) P - \frac{1}{2} QW(s)Q - P\left(J\lambda(s)\right)Q\right) ds\right]. \tag{15}$$

If we want to transform a Hamiltonton into a simpler and more convenient one, this is possible by using time-dependent canonical conversion, as the latter is very useful and effective in researching the properties of dynamical systems described by a time-dependent Hamiltonian. To simplify the Action given in (15), Let us introduce the canonical transformations which define the new phase space $(Q, P) \to (X, \Pi)$ [22–24] given by,

$$\begin{aligned}
\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{\mu_2(s)}} & 0 \\ 0 & \frac{1}{\sqrt{\mu_1(s)}} \end{pmatrix} \left[\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} - \begin{pmatrix} \beta_1(s) & 0 \\ 0 & \beta_2(s) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right], \\
\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} &= \begin{pmatrix} \sqrt{\mu_2(s)} & 0 \\ 0 & \sqrt{\mu_1(s)} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},
\end{aligned} \tag{16}$$

where the functions $\beta(s)$ can be conveniently chosen to make separation of variables straightforward possible. As a result of the transformation, and after using the gaussian integration over Π, one may write (15) as

$$\begin{aligned}
K(Q_j^f, Q_j^i; T) = \int DX \exp \frac{i\left[X(\beta(s))X\right]_0^T}{2} = \\
= \exp\left\{ T \int \left(-\frac{1}{2T^2} \dot{X}\dot{X} - \frac{1}{2}\left(\Omega_1^c\right)^2(s) X_1^2 - \frac{1}{2}\left(\Omega_2^c\right)^2(s) X_2^2 \right) ds \right\},
\end{aligned} \tag{17}$$

where

$$\left(\Omega^c_{(1,2)}(s)\right)^2 = \mu_{(2,1)}\omega^2_{(1,2)} - \beta^2_{(1,2)} + \frac{i}{T}\mu_{(2,1)}\frac{d}{ds}\left(\frac{\beta_{(1,2)}}{\mu_{(2,1)}}\right) \tag{18}$$

with

$$\beta_1(s) = \lambda_1(s) - \frac{i}{2T}\frac{\dot{\mu}_2(s)}{\mu_2(s)}, \tag{19}$$

$$\beta_2(s) = -\lambda_2(s) - \frac{i}{2T}\frac{\dot{\mu}_2(s)}{\mu_2(s)}. \tag{20}$$

Therefore, the propagator $K(Q_j^f, Q_j^i; T)$ in (17) is now reduced to the sum of the propagators for two uncoupled general time-dependent oscillators of frequencies $\Omega^c_{(1,2)}(s)$, and same masses $m_{(1,2)}(s) = 1$.

We know that the WKB-approximation, $\hbar$-expansion with $\hbar \to 0$ is almost equivalent to the adiabatic approximation, $\frac{1}{T}$-expansion with $T \to \infty$, It is known as well as that the easiest way to do the WKB approximation is the path integral, this is confirmed by Kashiwa in his article [18], for this reason we see that the most appropriate way to perform an addiabatic approximation is the path integral.

It is clear that we can see that the adiabatic approximation in (15), so it is very easy to treat this latter directly by the semi-classical methods, in which case we resort to the Van Vleck-Pauli formula [18].

Hence, we have the Van Vleck-Pauli formula

$$K(Q^f, Q^i; T) = \sqrt{\det\left(\frac{i}{2\pi}\frac{\partial^2 S}{\partial X^f \partial X^i}\right)} \exp\left(-S\left(X^f, X^i; T\right)\right), \tag{21}$$

where, $S\left(X^f, X^i; T\right)$ is the classical action defined by

$$S\left(X^f, X^i; T\right) = \frac{i\left[X(\beta(s))X\right]_0^T}{2} + \int_0^T L\left(X^f, X^i; T\right) dt, \tag{22}$$

with

$$L = \frac{-1}{2T^2}\left(\frac{dX_j}{ds}\right)\left(\frac{dX_j}{ds}\right) - \frac{1}{2}\left(\Omega_j^c(s)\right)^2 (X_j)^2, \quad j = 1, 2 \tag{23}$$

Now, that we have done all the steps described in APPENDIX , we get the propagator that accompanies this latter (23),

$$K(Q^f, Q^i; T) = \prod_{j=1}^{2} K(Q_j^f, Q_j^i; T), \tag{24}$$

with

$$\begin{aligned} K(Q_j^f, Q_j^i; T) &= \left\{ \prod_{j=1}^{2} \frac{\left(\mathbf{w}_j^c(T)\mathbf{w}_j^c(0)\right)^{\frac{1}{2}}}{2\sinh\Theta_j(T)} \right\} \exp\left(-\frac{\sqrt{\mathbf{w}_j^c(T)\mathbf{w}_j^c(0)}}{2\sinh\Theta_j(T)} = \right. \\ &= \left\{ \left(\sqrt{\frac{\mathbf{w}_j^c(T)}{\mathbf{w}_j^c(0)}} \left(X_j^f\right)^2 + \sqrt{\frac{\mathbf{w}_j^c(0)}{\mathbf{w}_j^c(T)}} \left(X_j^i\right)^2 \right) \cosh\Theta_j(s) - 2X_j^f X_j^i \right\} = \\ &\left. = -\exp\frac{i\left[X\beta(s)X\right]_0^T}{2} \right) \end{aligned} \tag{25}$$

and where X_1 and X_2 are given by (16).

As it is known, informations on the ground state can be derived by setting $T \to \infty$ in (25). In fact, when we take this limit we obtain:

$$\begin{aligned} K(Q^f, Q^i; T) \underset{T\to\infty}{\sim} &\left(\prod_{j=1}^{2} \left(\mathbf{w}_j^c(T)\mathbf{w}_j^c(0)\right)^{\frac{1}{4}} \right) e^{-\frac{1}{2}\Sigma_{j=1}^{2}\Theta_j(T)} \times \\ &\times \prod_{j=1}^{2} \exp\left\{ \left(\left(\mathbf{w}_j^c(T) + i\beta(T)\right)\left(X_j^f\right)^2 + \left(\mathbf{w}_j^c(0) - i\beta(0)\right)\left(X_j^i\right)^2 \right) \right\}. \end{aligned} \tag{26}$$

In this formula, the imaginary part of $\Theta_1(T)$ and $\Theta_2(T)$ given by (63), corresponds to the Berry phase,

$$\gamma_{(1,2)}(T) = \frac{1}{4}\int dt \left(\frac{\mu_{(2,1)}(t)}{\mathbf{w}_{(1,2)}(t)} \frac{d}{dt} \left(\frac{\lambda_{(1,2)}(t)}{\mu_{(2,1)}(t)} \right) \right) \tag{27}$$

or

$$\gamma_1(T) = \frac{1}{4}\int_0^T dt \left(\frac{1}{m(t)\sqrt{\omega^2(t) - \lambda_1^2(t)}} \frac{d}{dt}\left(m(t)\lambda_1(t)\right) \right), \tag{28}$$

$$\gamma_2(T) = \frac{1}{4}\int_0^T dt \left(\frac{m(t)\,\omega^2(t)}{\sqrt{\omega^2(t) - \lambda_2^2(t)}} \frac{d}{dt}\left(\frac{\lambda_2(t)}{m(t)\,\omega^2(t)} \right) \right). \tag{29}$$

whereas the real parts of $\Theta_1(T)$ and $\Theta_2(T)$, correspond to the dynamical phase.

2.2. Time-dependent coupled harmonic oscillator in noncommutative phase space

To specify a particular system in the context of non-commutative quantum mechanics it is necessary to define the Hamiltonian $H_{\theta\sigma} = H_{nc}$. The latter must be chosen so that it is reduced to standard Hamiltonian. We consider a system of two coupled harmonic oscillators where the hamiltonian $H(t)$ is an explicit function of time, via the frequency $\omega(t)$ and the mass $m(t)$ which are functions of time. In the case of noncommutative phase space, this system is described by the following hamiltonian:

$$\begin{aligned} H_{\theta\sigma}\left(x_1,x_2;p_1,p_2\right) &\simeq \left(\frac{1}{2m\left(t\right)}+\frac{m\left(t\right)\theta^2\omega^2\left(t\right)}{2}\right)p_1^2+\frac{p_2^2}{2m(t)}+\frac{1}{2}m(t)\omega^2(t)x_1^2+\\ &+\left(\frac{m\left(t\right)\omega^2\left(t\right)}{2}+\frac{\sigma^2}{2m\left(t\right)}\right)x_2^2+\lambda_1\left(t\right)\sigma x_1x_2+\lambda_2\theta p_1p_2+\\ &+\left(\frac{\sigma}{m\left(t\right)}+\theta m\left(t\right)\omega^2\left(t\right)\right)p_1x_2+\lambda_1\left(t\right)p_1x_1+\lambda_2\left(t\right)p_2x_2, \end{aligned} \tag{30}$$

where θ and σ are the deformed parameters defined above in Section 2.

We rewrite (30) using mixed coordinates $Q=(Q_1,Q_2)=(x_1,p_2)$ and $P=(P_1,P_2)=(x_2,p_1)$. Therefore the compact form of the above Hamiltonian is:

$$H_{\theta\sigma}(Q,P)=\frac{1}{2}PM(t)P+\frac{1}{2}QW(t)Q+P\lambda(t)Q, \tag{31}$$

where

$$M(t)=\begin{pmatrix}\mu_1(t) & b\left(t\right)\\ b\left(t\right) & \mu_2(t)\end{pmatrix},\quad W\left(t\right)=\begin{pmatrix}\omega_1^2\left(t\right) & 0\\ 0 & \omega_2^2\left(t\right)\end{pmatrix},\quad \lambda\left(t\right)=\begin{pmatrix}\lambda_1\left(t\right)\sigma & \lambda_2\left(t\right)\\ \lambda_1\left(t\right) & \lambda_2\left(t\right)\theta\end{pmatrix} \tag{32}$$

with

$$\mu_1(t) = \left(m\left(t\right)\omega^2\left(t\right)+\frac{\sigma^2}{m\left(t\right)}\right),\quad \mu_2(t)=\left(\frac{1}{m\left(t\right)}+m\left(t\right)\theta^2\omega^2\left(t\right)\right), \tag{33}$$

$$b\left(t\right)=\left(\frac{\sigma}{m\left(t\right)}+\theta m\left(t\right)\omega^2\left(t\right)\right), \tag{34}$$

$$\omega_1\left(t\right) = \sqrt{m\left(t\right)}\omega\left(t\right),\qquad \omega_2\left(t\right)=\frac{1}{\sqrt{m\left(t\right)}}. \tag{35}$$

We suggested setting $\theta=-\sigma$ and $m^2\left(t\right)\omega^2\left(t\right)=1$ to facilitate calculations, avoid repetition and reduce steps.

After this simplification, the matrix becomes as follows:

$$M(t)=\begin{pmatrix}\mu_1(t)=\dfrac{1}{m\left(t\right)}\left(1+\sigma^2\right) & 0\\ 0 & \mu_2(t)=\dfrac{1}{m\left(t\right)}\left(1+\sigma^2\right)\end{pmatrix}. \tag{36}$$

The quantum mechanical evolution of the hamiltonian (30) can be described by the propagator, in the non-commutative phase space formulation of Feynman's path integral, which is defined by:

$$\begin{aligned} &K_{\theta\sigma}(Q^f,Q^i;T)=\int DQDP=\\ &=\exp\left[T\int\left(\left(\frac{i}{T}P\dot{Q}-\frac{1}{2}\left(P\left(J^{-1}M(s)J\right)P+QW(s)Q+2P\left(J^{-1}\lambda(s)\right)Q\right)\right)ds\right]. \end{aligned} \tag{37}$$

In this subsection, we are mainly interested to find the Berry's phase in the Euclidean path integral formalism in non-commutative phase space.

And to find the Berry phase in non-commutative phase space, we will follow the same steps we took in the case of the commutative phase space of two coupled harmonic oscillators, then we compare between the both applications in order to highlight the impact of the deformation parameters (θ, σ) on the hamiltonian.

We can see the path integral (37) is not trivial but can be important. In the case of making this equation (37) to a more easily form, it is useful to use the time-dependent canonical transformation. This transformation leads to an effective diagonal Hamiltonian in terms of non-commutative coordinates.

In order to remove the matrix $\left(J^{-1}M(s)J\right)$ and $\left(J^{-1}\lambda(s)\right)$, we can use the time-dependent canonical transformation $(P,Q) \to (\Pi, X)$ similar to that provided in [23] and [24]:

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\mu}} & 0 \\ 0 & \frac{1}{\sqrt{\mu}} \end{pmatrix} \left(\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} - \begin{pmatrix} \beta_{11}(s) & \beta_{12}(s) \\ \beta_{21}(s) & \beta_{22}(s) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \right), \tag{38}$$

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\mu} & 0 \\ 0 & \sqrt{\mu} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}. \tag{39}$$

Hence, the new propagator for the system becomes

$$K_{\theta\sigma}(Q^f, Q^i; T) = \int DXD\Pi \exp\left[\frac{i\left(X\left(\beta(s) + \beta^t(s)\right)X\right)}{2}\right]_0^T \times$$
$$\times \exp(T) \int ds \left(\frac{i}{T}\Pi\dot{X} - \frac{1}{2}\Pi\Pi - \frac{1}{2}X\Omega^{nc}(s)X\right), \tag{40}$$

where

$$(\Omega_1^{nc})^2 = \mu\omega_1^2(t) + \lambda_1^2 - \sigma^2\left(\lambda_1\lambda_2 - \frac{1}{2}\lambda_2^2\right) - \frac{i}{T}\mu\frac{d}{ds}\left(\frac{\beta_{11}}{\mu}\right), \tag{41}$$

$$(\Omega_2^{nc})^2 = \mu\omega_2^2(t) + \lambda_2^2 - \sigma^2\left(\lambda_1\lambda_2 - \frac{1}{2}\lambda_1^2\right) - \frac{i}{T}\mu\frac{d}{ds}\left(\frac{\beta_{22}}{\mu}\right), \tag{42}$$

$$\Omega_3^{nc} = -\frac{i}{T}\mu\frac{d}{ds}\left(\frac{\beta_{12}}{\mu}\right) - \frac{i}{T}\mu\frac{d}{ds}\left(\frac{\beta_{21}}{\mu}\right) \tag{43}$$

and the matrices elements of $\beta(s)$ are

$$\beta_{(11)}(s) = \lambda_{(1,2)} - \frac{i}{2T}\frac{\dot{\mu}}{\mu}, \tag{44}$$

$$\beta_{(22)}(s) = -\lambda_{(1,2)} - \frac{i}{2T}\frac{\dot{\mu}}{\mu}, \tag{45}$$

$$\beta_{(12,21)}(s) = -\sigma\lambda_{(2,1)} \tag{46}$$

in this case we took $\dot{\sigma} = 0$, and in put $\lambda_1 = -\lambda_2 = \lambda$ the relations (41), (42) and (43) becomes:

$$\left(\Omega_{(1,2)}^{nc}\right)^2(s) = \left(\mathbf{w}_{(1,2)}^{nc}\right)^2(s) - \frac{1}{T}\tilde{\omega}_{(1,2)}^{nc}(s), \tag{47}$$

where

$$\left(\mathbf{w}_{(1,2)}^{nc}\right)^2(s) = \left(\mathbf{w}_{(1,2)}^{c}\right)^2(s) - \frac{3\sigma^2}{2}\lambda^2(s) \tag{48}$$

and

$$\tilde{\omega}_{(1,2)}^{nc}(s) = -i\mu\frac{d}{ds}\left(\frac{\lambda(s)}{\mu}\right), \tag{49}$$

where Ω^c and Ω^{nc} are commutative and noncommutative frecuency.

The Π-integration in (40) is easily performed to give the new propagator:

$$K_{\theta\sigma}(Q^f, Q^i; T) = \sqrt{2\pi} \int DX \exp\left[\frac{i\left(X\left(\beta\left(s\right) + \beta^t\left(s\right)\right)X\right)}{2}\right]_0^T = $$
$$= \exp\left[T \int ds \Big(\sum_{j=1,2} \left(\frac{-1}{2T^2}\left(\dot{X}^2 - \Omega_j^2(s) X_j^2\right)\right)\Big)\right] \tag{50}$$

The final expression of the propagator, for the system of two coupled harmonic oscillators in non-commutative phase space governed by the Hamiltonian (30), is given by

$$K(Q_j^f, Q_j^i; T) = \left\{\prod_{j=1}^{2} \frac{\left(\mathbf{w}_j^{nc}(T)\mathbf{w}_j^{nc}(0)\right)^{\frac{1}{2}}}{2\sinh\Theta_j(T)}\right\} \exp\left(-\frac{\sqrt{\mathbf{w}_j^{nc}(T)\mathbf{w}_j^{nc}(0)}}{2\sinh\Theta_j(T)} \times \right.$$
$$\times \left\{\left(\sqrt{\frac{\mathbf{w}_j^{nc}(T)}{\mathbf{w}_j^{nc}(0)}}\left(X_j^f\right)^2 + \sqrt{\frac{\mathbf{w}_j^{nc}(0)}{\mathbf{w}_j^{nc}(T)}}\left(X_j^i\right)^2\right)\cosh\Theta_j(s) - 2X_j^f X_j^i\right\} -$$
$$\left. - \exp\left[\frac{i\left(X\left(\beta\left(s\right) + \beta^t\left(s\right)\right)X\right)}{2}\right]_0^T\right), \tag{51}$$

with

$$\Theta_{(1,2)}(s) = T\int_0^s d\tau\left(\mathbf{w}_{(1,2)}^{nc}(\tau) - \frac{i}{2T}\frac{\mu(\tau)}{\mathbf{w}_{(1,2)}^{nc}(s)}\frac{d}{d\tau}\left(\frac{\lambda(\tau)}{\mu(\tau)}\right)\right). \tag{52}$$

When we put $T \to \infty$ we have a real and imaginary part. This last corresponds to the Berry's phase which are as follows

$$\gamma_{\theta\sigma}^{(1,2)}(s) = \frac{T}{4}\int d\tau\left(\frac{\mu(\tau)\frac{d}{d\tau}\left(\frac{\lambda(\tau)}{\mu(\tau)}\right)}{\sqrt{\left(\mathbf{w}_{(1,2)}^{c}\right)^2(s) - \frac{3\sigma^2}{2}\lambda^2(s)}}\right). \tag{53}$$

Finally, we have found the Berry phase in the non-commutative state where depends this latter to the deformed parameters θ and σ, after we used a method of adiabatical approximation. In the case of $\theta = \sigma = 0$, we can obtain exactly Berry's phase in the commutative phase space case.

Conclusion

In this paper we applied the path integral construction [1] in the noncommutative phase space, in which the structure of the phase space is deformed by introducing two deformation parameters θ and σ. We present an alternative treatment (via path integral formalism) for the problem of the coupled harmonic oscillators in two dimensions with time-dependent mass and frequency. We study two cases: the first one in commutative phase space and the second in noncommutative phase space. The treatment is based on the use of time-dependent canonical transformation and auxiliary time-dependent transformation by path integral techniques. To each canonical transformation correspond a new mass and a new frequency.

We know that the Berry phase is limited to the adiabatic approximation. We have calculated Berry phase in each case following the semi-classical solution via path integral. The result are two functions $\gamma^{(1)}(t)$ and $\gamma^{(2)}(t)$ in terms of the system parameters are $m(t)$, $\omega(t)$, $\lambda_1(t)$ and

$\lambda_2(t)$ in the commutative case, but, in the case of the non-commutative phase space, the result are two functions also $\gamma_{\theta\sigma}^{(1)}(t)$ and $\gamma_{\theta\sigma}^{(2)}(t)$ in the terms of the system parameters $m(t)$, $\omega(t)$, $\lambda_1(t)$ and $\lambda_2(t)$, in addition to the deformations parameters θ and σ, It is easy to see that we find the result of the Berry phase in commutative case in the limit $\theta, \sigma \to 0$.

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Appendix

Rewrite equation (18) as follows

$$\left(\Omega_j^c\left(s\right)\right)^2=\left(\mathbf{w}_j^c\left(s\right)\right)^2-\frac{1}{T}\tilde{w}_j\left(s\right),\quad j=1,2, \tag{54}$$

where

$$\mathbf{w}_{(1,2)}^c\left(s\right)=\left(\mu_{(2,1)}\omega_{(1,2)}^2-\lambda_{(1,2)}^2\left(s\right)\right)^{\frac{1}{2}} \tag{55}$$

and

$$\tilde{w}_{(1,2)}\left(s\right)=-i\mu_{(2,1)}\frac{d}{ds}\left(\frac{\beta_{(1,2)}}{\mu_{(2,1)}}\right). \tag{56}$$

From the Lagrangian (23) we extract the motion equations

$$\left(\frac{d^2X_{(1,2)}}{ds^2}\right)-T^2\Omega_{(1,2)}^2(s)X_{(1,2)}=0. \tag{57}$$

The boundary conditions are

$$X_{(j)}(0)=\frac{Q_{(1,2)}^i}{\sqrt{\mu_{(2,1)}\left(0\right)}},\quad X_{(j)}(T)=\frac{Q_{(1,2)}^f}{\sqrt{\mu_{(2,1)}\left(T\right)}},\qquad j=1,2. \tag{58}$$

Now the only task needed here is to find a classical solution $X_{(j)}^c$ of Equation (57), we consider for this the two Ansatz

$$X_{(1,2)}=e^{T\int_0^s d\sigma\rho^{(1,2)}(\sigma)}a_{(1,2)};\qquad \rho^{(1,2)}(s)=\sum_{n=0}^{\infty}\rho_n^{(1,2)}(0)\left(\frac{1}{T}\right)^n, \tag{59}$$

where $a_{(1,2)}$ is a given constant. Substituting equation (59) into (57), and using (54), lead to

$$T^2\left(\rho_0^{(1,2)}\right)^2+T\dot{\rho}_0^{(1,2)}+\left(\rho_1^{(1,2)}\right)^2-T^2\left(\mathbf{w}_{(1,2)}^c\left(s\right)\right)^2-T\tilde{w}_{(1,2)}\left(s\right)=0. \tag{60}$$

By identification of coefficients with respect to T, and by restricting to $O(T^2)$ and $O(T)$, we obtain

$$\begin{cases} \rho_0^{(1,2)} = \pm \mathbf{w}^c_{(1,2)}(s) & O(T^2) \\ \rho_1^{(1,2)} = \dfrac{-\dot{\rho}_0^{(1,2)} + \tilde{w}_{(1,2)}(s)}{2\rho_0^{(1,2)}} & O(T). \end{cases} \tag{61}$$

Finally, by taking into account the boundary condition (58), we get

$$X_{(1,2)}(s) = \frac{1}{\sqrt{\mathbf{w}^c_{(1,2)}(s)} \sinh \Theta_{(1,2)}(T)} \left\{ \sqrt{\frac{\mathbf{w}^c_{(1,2)}(T)}{\mu_{(1,2)}(T)}} Q^f_{(1,2)} \sinh \Theta_{12}(s) + \right.$$
$$\left. + \sqrt{\frac{\mathbf{w}^c_{(1,2)}(0)}{\mu_{(1,2)}(0)}} Q^i_{(1,2)} \sinh \bar{\Theta}_{(1,2)}(s) \right\} \times \left\{ 1 + O\left(\frac{1}{T}\right) \right\}, \tag{62}$$

where

$$\Theta_{(1,2)}(s) = T \int_0^s d\tau \left(\mathbf{w}^c_{(1,2)}(\tau) - \frac{i}{2T} \frac{\mu_{(2,1)}(\tau)}{\mathbf{w}^c_{(1,2)}(\tau)} \frac{d}{d\tau} \left(\frac{\lambda_{(1,2)}(\tau)}{\mu_{(2,1)}(\tau)} \right) \right) \tag{63}$$

and $\bar{\Theta}_j(s) = \Theta_j(T) - \Theta_j(s)$. The action $S\left(X^f, X^i; T\right)$ could be computed using the last solution. Indeed, integration by parts in the kinetic term of the action and the use of the motion equation (57) give

$$S\left(X^f, X^i; T\right) = \frac{i\left[X\beta\left(s\right)X\right]_0^T}{2} + \int_0^T \left(\frac{-1}{2T^2} \left(\frac{dX_j}{ds}\right) \left(\frac{dX_j}{ds}\right) - \frac{1}{2}\Omega_j^2(s) X_j^2 \right) ds =$$
$$= \frac{i\left[X\beta\left(s\right)X\right]_0^T}{2} + \frac{1}{2T}\left[X_{(1,2)} \frac{dX_{(1,2)}}{ds}\right]_0^T \simeq S_1 + S_2. \tag{64}$$

Straightforward calculation provides

$$S_2 = \frac{\sqrt{\mathbf{w}^c_{(1,2)}(T)\mathbf{w}^c_{(1,2)}(0)}}{2\sinh \Theta_{(1,2)}(T)} \times$$
$$\times \left\{ \left(\sqrt{\frac{\mathbf{w}^c_{(1,2)}(T)}{\mathbf{w}^c_{(1,2)}(0)}} \left(X^f_{(1,2)}\right)^2 + \sqrt{\frac{\mathbf{w}^c_{(1,2)}(0)}{\mathbf{w}^c_{(1,2)}(T)}} \left(X^i_{(1,2)}\right)^2 \right) \coth \Theta_{(1,2)}(T) - 2X^f_{(1,2)} X^i_{(1,2)} \right\}, \tag{65}$$

yielding the determinant to

$$\sqrt{\det\left(\frac{i}{2\pi} \frac{\partial^2 S}{\partial X^f_{(1,2)} \partial X^i_{(1,2)}}\right)} = \frac{\sqrt{\mathbf{w}^c_{(1,2)}(T)\mathbf{w}^c_{(1,2)}(0)}}{2\sinh \Theta_{(1,2)}(T)} . \sqrt{\frac{\mathbf{w}^c_{(1,2)}(T)\mathbf{w}^c_{(1,2)}(0)}{\mu_{(1,2)}(0)\mu_{(1,2)}(T)}} \tag{66}$$

Фаза Берри для нестационарных связанных гармонических осцилляторов в некоммутативном фазовом пространстве с помощью методов интеграла по траектории

Лейла Хиари
Университет Касди Мербах
Уаргла, Алжир

Тахар Буджедаа
Университет Джиджель
Джиджель, Алжир

Абденасер Махлуф
Университет Высшего Эльзаса
Мюлуз, Франция

Мохамед Таеб Мефта
Университет Касди Мербах
Уаргла, Алжир

Аннотация. Целью данной работы является описание фазы Берри в формализме евклидова интеграла по путям для двумерной квадратичной системы: двух связанных во времени гармонических осцилляторов. Эта обработка достигается с помощью адиабатического приближения в коммутативном и некоммутативном фазовом пространстве.

Ключевые слова: фаза Берри, некоммутативное фазовое пространство, связанные осцилляторы.

DOI: 10.17516/1997-1397-2020-13-1-71-78
УДК 517.9

Physical Basis of Quasi-optimal Seismoacoustic Pulse Generating for Geophysical Prospecting in Shallow Water and Transit Zones. Part 2. The Layout of Aqueous Seismic Source and the Results of Experiments

Danil S. Kudinov[*]
Oleg A. Maykov[†]
Siberian Federal University
Krasnoyarsk, Russian Federation
Pavel V. Balandin[‡]
LLC Evenkiyageofizika
Tyumen, Russian Federation

Received 09.09.2019, received in revised form 03.11.2019, accepted 20.12.2019

Abstract. The article discusses theoretical aspects of seismic wave excitation of in the aquatic environment, addresses the problems of instrumental implementation of a fundamentally new source of seismic vibrations that can work: in the water area, in tidal and coastal zones. The scientific substantiation of the developed seismic source (SS) design is given.

The results of the seismic influence simulation of hydrodynamic resistance on the media, as well as the formation of the “added mass” are given. The results were obtained using the developed mathematical model of the motion of the radiating surface. Based on the experimental work, a comparative analysis of the energy efficiency of the developed seismic source model and the serial sample of the VEM-50 "Yenisei" water seismic source was made. Experimental results were obtained at the geophysical well of the test and training area.

Keywords: seismic source, seismoacoustics, pseudorandom sequence, scrambling sequence, shallow water, transit zone.

Citation: D.S.Kudinov, O.A.Maykov, P.V.Balandin, Physical Basis of Quasi-optimal Seismoacoustic Pulse Generating for Geophysical Prospecting in Shallow Water and Transit Zones. Part 2. The Layout of Aqueous Seismic Source and the Results of Experiments, J. Sib. Fed. Univ. Math. Phys., 2020, 13(1), 71–78.
DOI: 10.17516/1997-1397-2020-13-1-71-78.

1. Rationale of aqueous seismic source structure

To conduct investigational studies of effective excitation of seismic vibrations in an aqueous medium. The experimental model of electromagnetic seismic source (EMSS) was created. There is the short-stroke electromagnetic drive inside the outrigger float is loaded on the reaction mass

[*]kudinovdanil@yandex.ru
[†]maykov_oleg@bk.ru
[‡]balandin_p_v@mail.ru

of the outrigger float M and forces on outrigger buttom, which is radiator seismic vibrations. The 3D model of EMSS is shown in Fig. 1.

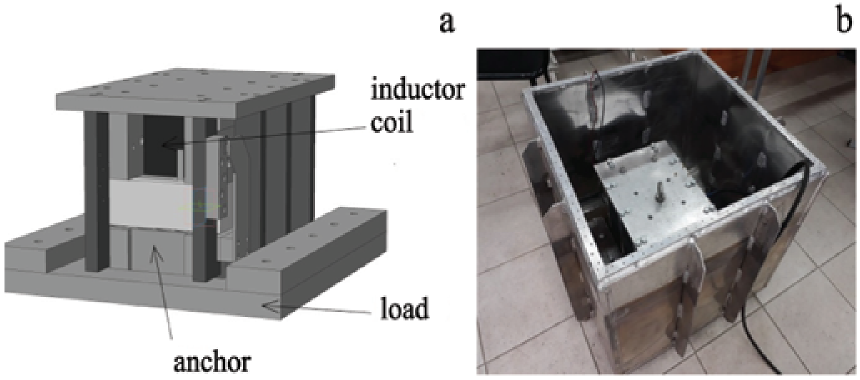

Fig. 1. The EMSS model for water areas: a — 3D model; b — physical configuration of experimental model and outrigger float

The initial data are set based on the required characteristics of the EMSS. Let us conduct an estimated calculation of the design parameters of the seismic source:

— electromagnetic traction force, F_{EM}=25000 N;

— the magnetic gap between the anchor and inductor, δ=4 mm;

— current impulse time, τ=3 ms;

— power-supply voltage of energizing coil, U=1200 V.

The magnetic core is made of electrical steel, which provides maximum value of induction B=2 T, then the magnetic force in the core H_{st}=150 A/m. The steel grade selection is based on need for ensure maximum value of induction of a magnetic field B when the magnetizing force is the least [3]. The required cross section of the magnet core is determined by the electromagnet traction force F_{EM}=25 kN:

$$S_{MC} = \frac{F_{EM} \cdot \mu}{B^2} = 78.5 \cdot 10^{-4} m^2. \tag{1}$$

We use a square magnetic core for the inductor with dimensions of 9×9=0.81 m^2. The field line length of the magnet core l=0.6 m. The required magnetization current is determined by the formula [1]:

$$I = I_{MC} + I_\delta = H_{st} \cdot l + B\frac{2\delta}{\mu_0} = 12890A, \tag{2}$$

where I_{MC} is magnetization current of magnetic core; I_δ is magnetization current of the magnetic gap.

The voltage of the magnetizing coil is determined by the formula [1]:

$$U_L = B \cdot n \cdot S_{MC}/\tau_{CP}, \tag{3}$$

where n is number of turns in the magnetizing coil; τ_{CP} is magnetization current pulse duration.

The limit numbers of turns n for the area of magnetic core in case of S_{MC}=0.0081 m^2 and τ_{CP}=1 ms is determined by setting a maximum allowable voltage U_L=1200 V. It is determined from the formula (3).

$$n = \frac{U_L \cdot \tau_{CP}}{B \cdot S_{MC}} = 74.$$

The inductance of the inductor magnetizing coil L is determined by the formula:

$$L = \frac{1,26 \cdot 10^{-6} \cdot n^2 \cdot S_{MC}}{\delta} = 6,5\, mH. \tag{4}$$

The active resistance r of the inductor magneting coil, wen an average length of the turn l=0.5 m, and in case of a copper ribbon of coil has a specific resistance ρ_W=0.018 $(\Omega \cdot m)/mm^2$ and cross-sectional area $S_W = 40 \times 0.5 = 20$ mm^2 is:

$$r = \frac{l_c \cdot n \cdot \rho_W}{S_W} = 0.032\, \Omega.$$

The actual induction in the magnetic gap will be by 20% less due to dispersion [1], therefore, the gap induction will be B= 1.5 T. Then an actual electromagnetic traction force F_{EM} will be:

$$F_{EM} = \frac{B^2 \cdot S_{MC}}{2\mu_0} = 16580\, N. \tag{5}$$

2. The experimental technique and the results of comparison tests

The experiments were carried out at a 100 m deep well with a known structure of a geological cross-section in the water basin of a test-and-training geophysical range (Minusinsk, Krasnoyarsk Territory). The MSK SGD-SLM "Gnome" seismic station was used with the SGD-SLM/G3 measuring acoustic to record the seismic signal level in the well. The acoustic probe consistently lowered to a depth of 50 m and 100 m. The VEM-50 "Yenisei" water seismic source with a peak force of 500 kN was used as a comparative sample. A series of 10 impacts was made for both seismic sources and for each position of the acoustic probe in the well.

An experimental verification of the calculated characteristics of the seismic source was carried out by recording the parameters of the force and pressure in water using by accelerometer and hydrophone. The accelerometer was located on the impact plate. The hydrophone was placed in water to a depth h=1 m under the emitter plate (Fig. 2).

The measured acceleration magnitude of the emitter is a=450 m/s^2. Then a total mass of the emitter (anchor + float) is 38 kg and the dimension of the dispersion plate is 60×60 cm, a scalar electromagnetic traction force F_{EM} applied to the emitter is:

$$F_{EM} = m \cdot a = 17100\, N.$$

Moreover, a pressure applied by the base plate at the interface is:

$$P_{EM} = \frac{F_{EM}}{S} = 0.475\, kg/cm^2.$$

The estimated values are in good agreement with a theoretical values of the force obtained according to the formula (5). A pressure value satisfies a limiting condition $P_{EM} < P_{max} = 1$ kg/cm^2, causing parasitic cavitation processes in the emitter zone.

Fig. 2. EMSS model lowered in a test water tank

The impulse action is bipolar, as shown in the Fig. 3. On this basis it can be concluded that the choice of EMSS design was correct. The peak voltage of the positive half-wave on the hydrophone (the radiating plate moves upward), with a shock force of F_{EM}=1 kN, is U_g=700 mV. When a hydrophone sensitivity is γ=39 $\mu V/Pa$, the pressure P_g at a depth of 1 m will be:

$$P_g = \frac{U_g}{\gamma} \approx 18\,kPa = 0.018\,kg/cm^2.$$

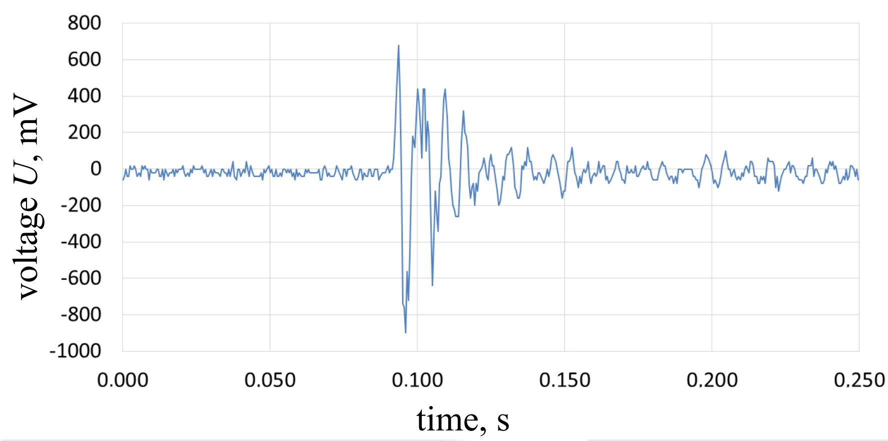

Fig. 3. The signal level recorded hydrophone

In Fig. 3 the residual oscillation is a multiplere reflection of the signal from the bottom and walls of the test water tank.

3. Investigation of the seismic activity of EMSS during bipolar excitation of the medium

The level of signals recorded in the well at a depth of h=100 m from the VEM-50 and EMSS seismic sources located in the test water tank are chown in Figs. 4 a, b. The results of the comparative tests are presented in the Tab. 1. There are levels of the recorded signals from the samples at different depths.

Table 1. The voltage at the geophone output when testing VEM and EMSS

Number of observations	VEM amplitude signal, mV		EMSS amplitude signal, mV	
	U_{max1}	U_{max2}	U_{max1}	U_{max2}
1	0.0939	0.1085	0.0091	0.0140
2	0.0948	0.1068	0.0083	0.0143
3	0.0956	0.1060	0.0090	0.0143
4	0.0946	0.1079	0.0099	0.0147
5	0.0951	0.1093	0.0097	0.0150
6	0.0939	0.1095	0.0083	0.0143
7	0.0951	0.1092	0.0095	0.0123
8	0.0920	0.1113	0.0097	0.0153
9	0.0946	0.1119	0.0087	0.0130
10	0.0937	0.1095	0.0095	0.0143

An analysis of the signals received at a depth of 50 meters and 100 meters from the EMSS showed that the period of a bipolar pulse at both recording points has an identical pulse duration τ=16 ms. Thus, the pulse duration does not change with increasing signal propagation depth. This indicates the effectiveness of using a bipolar seismic source. This effect may be due to a fact that in the VEM-50 source a "negative" half-wave is not emitted, and an impact energy is spent on a movement of the reactive mass. A negative half-wave was recorded by a seismic probe in the well and an accelerometer, which is fixed to the emitting surface. It is expected that it is formed in the medium due to the resonance properties of the source. Pulse duration is constant in EMSS, because the oscillation period is formed by a bipolar excitation signal. Theoretically, the advantages of bipolar excitation were investigated using an electric model of the geological medium. This medium is presented as a resonant oscillatory system [4].

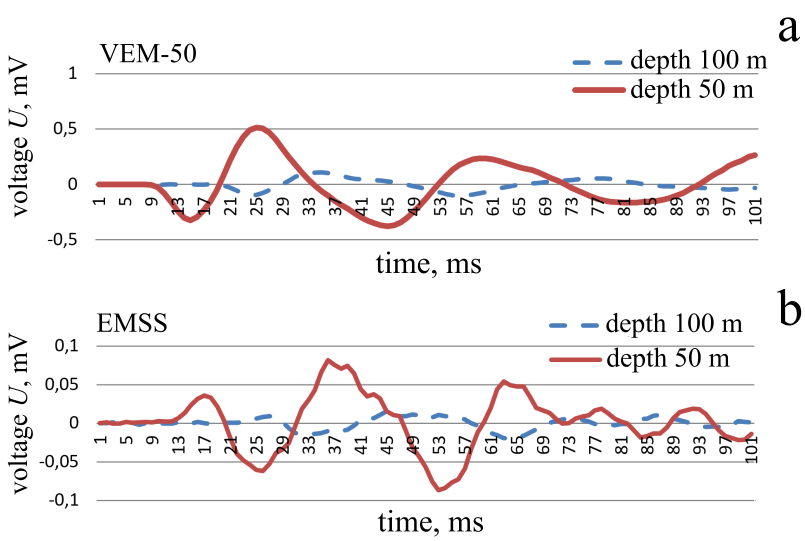

Fig. 4. The seismic signal recorded by the well acoustic sonde: a — from the VEM-50 seismic source; b — from the experimental model of the EMSS

The amplitude values of signals at depths of 100 and 50 m are presented in the Tab. 1 in order to evaluate energy potential VEM-50 seismic source and EMSS as well their seismic efficiency. We have defined the concept of seismic efficiency of a seismic sources as the ratio of an acoustic pressure amplitude value at a specified space point to the action force of the emitter on a medium:

$$\xi = \frac{P_0}{F_{EM}}, \tag{6}$$

where P_0 is acoustical pressure at a specified point in the geological environment. The pressure P_0 can be calculated by forming [5]:

$$P_0 = \rho \cdot c \cdot \vartheta_0, \tag{7}$$

where
c is compressional velocity of elastic P-waves;
ρ is density;
ϑ_0 is vibrational particle velocity.

The estimated ratio for seismic efficiency can be obtained by calculating ϑ_0 through the measured signal value at the well seismic probe:

$$\xi = \frac{\rho \cdot c \cdot U_{av}}{F_{EM} \cdot \gamma_S}, \tag{8}$$

where γ_S is acoustic sonde sensitivity $\left[\frac{V \cdot c}{m}\right]$; U_{av} is the average value of the signal level at the seismic station input circuit [V]. We determine the efficiency of EMSS in comparison with VEM-50 bu the formula:

$$\frac{\xi_{EMSS}}{\xi_{VEM}} = \frac{F_{EM}^{VEM} \cdot U_{av}^{EMSS}}{F_{EM}^{EMSS} \cdot U_{av}^{VEM}} \approx 3.5. \tag{9}$$

Thus, the seismic efficiency of the EMSS is 3.5 times higher than the seismic efficiency of the VEM-50. Therefore, expected signal level for the VEM-50 will be provided when the force is:

$$F_{EM}^{EMSS} = \frac{F_{EM}^{VEM}}{3.5} = 142\, kN,$$

During synchronous operation of N EMSS emitters will be:

$$N \geqslant \frac{142\, kN}{17.1\, kN} = 9.$$

Thus, a system configuration of the antenna array of synchronously operating emitters 9 (3×3) mounted on a floating platform will be equvalent of VEM-50 . At the same time, the mass of the VEM-50 seismic source is approximately equal to $M_{VEM} \approx 5700$ kg, and the mass of one EMSS module complete with a reference float does not exceed m=180 kg. The experiment showed that the variant of a seismic source with an electromagnetic drive and bipolar excitation has higher energy efficiency and allows you to generate a signal with an identical amplitude to the VEM-50 source when the force acts on the medium 3 times less. In addition, EMSS has a higher probe pulse frequency.

According to the results of a series of 10 impacts, their amplitude identity is $\pm 5\%$, which is sufficient reason to create a seismic source with good repeatability of the characteristics of seismic probe pulses. This parameter will ensure coherent integration mode in order to increase the signal-to-noise ratio without increasing the peak power of the seismic source. This will make it possible to reduce the negative acoustic effect on the environment when working in water areas and transition zone. Along with passive hydrocarbon search methods, similar studies have been actively conducted around the world in the last decade [6–9].

Conclusion

Analysis of the problem in the field of water seismic exploration works has revealed the advantage of EM sources over air-guns sources and explosions in terms of environmental friendliness and has made it possible to determine the disadvantages of pulsed EM seismic sources. Theoretical justification of effectiveness of using EMSS with bipolar excitation in comparison with seismic sources with an electromagnetic drive of the VEM-50 "Yenisei" series (SS VEM) is presented in article.

Investigational studies and comparative field tests of experimental models of EMSS and VEM-50 seismic source represent essential scientific results. These results have confirmed that the choice of the proposed theoretical and mathematical models of sources functioning in the aquatic environment, their design, as well as the effectiveness of using bipolar excitation to ensure environmentally friendly and high-resolution seismic exploration works was made correctly works was made correctly. In particular, the development of a mathematical model of the source in an aquatic environment is issentional result in terms of particle. It describes the dependence of the movement of the base plate-emitter in a liquid under the action of an external driving force. The obtained results will be the basis for further research in the field of seismic signals based on a large database of $B \geqslant 1$ (orthonormal or noise-like signals).

The further studies include the completion of EMSS model of array configuration and comparative field tests at a geophysical training range and field of known geological section structure and comparative field test are carried out in comparison with pseudonoise signal seismic exploration and classical pulsed technology. Also research in the are of the propagation of unipolar, bipolar and M-sequence long pulses will be conducted.

Additionally, we plan to clarify the concept of added mass of water (medium), using a system of sensors based on accelerometers and hydrophones. The research of the ratios between the parameters of the added mass and the hydrodynamic indicators of water will allow us to choose the optimal size of the EMSS emitter.

The reported study was funded by RFBR and the government of Krasnoyarsk region according to the research project no. 18-45-242003. "Research and development of methods for distance increasing of the geological sounding by seismoacoustic pseudonoise orthonormal sequences signals". The authors also thank the Siberian Federal University and "Evenkiyageofizika" OJSC for their help in organizing field studies at the geophysical training and testing ground.

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Физические основы формирования энергетически квазиоптимального импульсного сейсмоакустического воздействия для геофизических исследований в условиях мелководья и транзитных зон. Часть 2. Конструкция водного источника и результаты экспериментов

Данил С. Кудинов
Олег А. Майков
Сибирский федеральный университет
Красноярск, Российская Федерация
Павел В. Баландин
ООО "Эвенкиягеофизика"
Тюмень, Российская Федерация

Аннотация. В статье дается научное обоснование разработанной конструкции сейсмоисточника (СИ). Оценивается эффективность двухполярного возбуждения водной среды СИ в сравнении с известными водными импульсными источниками серии «Енисей». Приводятся результаты моделирования влияния гидродинамического сопротивления среды СА воздействию, а также формирования «присоединенной массы». Результаты получены на основе разработанной математической модели движения излучающей поверхности. На основе проведенных опытных работ сделан сравнительный анализ энергетической эффективности работы макета СИ и серийного образца водного СИ ВЭМ-50 модельного ряда «Енисей». Экспериментальные результаты получены на геофизической скважине учебно-испытательного полигона Сибирского федерального университета.

Ключевые слова: сейсмоисточник, сейсмоакустика, псевдослучайная последовательность, мелководье, транзитная зона.

DOI: 10.17516/1997-1397-2020-13-1-79-86.
УДК 532.517.4

Similarity in the Far Swirling Momentumless Turbulent Wake

Alexey V. Shmidt*
Institute of computational modelling SB RAS
Krasnoyarsk, Russian Federation

Received 14.02.2019, received in revised form 15.10.2019, accepted 20.12.2019

Abstract. A self–similar solution to one model of the far momentumless swirling turbulent wake is proposed in the paper.

Keywords: far swirling turbulent wake, self–similar solution, shooting method.

Citation: A.V.Shmidt, Similarity in the Far Swirling Momentumless Turbulent Wake, J. Sib. Fed. Univ. Math. Phys., 2020, 13(1), 79–86.
DOI: 10.17516/1997-1397-2020-13-1-79-86.

Introduction

Turbulent swirling wakes are usually generated during flow past around a body. Swirls are inserted into the flow by propulsors and they can be formed in various technological devices. An overview of papers devoted to experimental and numerical investigations of the swirling turbulent wakes is presented in [1, 2]. The similarity laws of the swirling turbulent flow decay are investigated [3]. Asymptotic and numerical analysis of the swirling turbulent wakes was performed [4–6]. The classical $k-\varepsilon$ model of turbulence was used in these studies. It was shown that even if the tangential component of the mean velocity is small it significantly affects the flow pattern in the turbulent wake and this influence can be traced at sufficiently large distances behind a body.

The streamwise component of the excess momentum J and angular momentum M are important integral characteristics of the swirling turbulent wake. The case $J = 0$, $M = 0$ corresponds to the swirling turbulent wake behind a self–propelled body. This configuration can be implemented in a wake behind the self–propelled body of revolution (the thrust of a body propulsor compensates the hydrodynamic drag force) with compensation of the swirl introduced by a propulsor.

Numerical modelling of the swirling momentumless turbulent wake ($J = 0$) with nonzero angular momentum was carried out on the basis of the second–order semi–empirical models of turbulence [7–9]. Furthermore, a comparison with experimental data [7] obtained in a wind tunnel in the wake past an ellipsoid of revolution was performed. The drag was compensated by momentum of a swirling jet exhausted from its trailing part and the swirl introduced by the jet was balanced out by the rotation of the body part in the opposite direction.

Experimental results on the swirling turbulent wake for $M \neq 0$ were presented [10–14]. It should be noted that there is a discrepancy in the results obtained by different authors.

*schmidt@icm.krasn.ru

Experimental data on the swirling turbulent wakes with various total excess momentum and angular momentum were presented [11].

The swirling momentumless turbulent wake with nonzero angular momentum was numerically simulated [15–20]. Simulations were based on simplified $e - \varepsilon$ model of turbulence [15, 16] and on the hierarchy of improved semi–empirical second–order models of turbulence [18–20]. Good agreement with experimental data was obtained [11]. Numerical analysis of decay of a swirling turbulent wake corresponding to the case $J = 0$ and $M \neq 0$ was carried out [20]. It was shown that at distances of about 1000 diameters behind the body the flow becomes substantially self–similar. Simplified mathematical models of a far momentumless swirling turbulent wake were constructed and their applicability in the case of large distances from the body was proved.

Self–similar solutions of certain semi–empirical models of free turbulent shear flows were constructed on the basis of group–theoretic analysis and modified shooting method [21–24]. The obtained results are in agreement with the experimental data. In addition to that, a comparison of obtained self–similar solutions of the three–dimensional far momentumless turbulent wake model in a passive stratified medium with results of direct numerical solutions of the complete model was conducted [22]. Moreover, it was found that solutions obtained by the shooting method play the role of an attracting set for solutions obtained by direct numerical calculations of the complete model.

The purpose of this study is to construct self–similar solutions of the simplified model of the far swirling momentumless turbulent wake ($J = 0$, $M \neq 0$) [20] on the basis of the previously developed approach [21–24].

1. Problem statement

In order to demonstrate the flow pattern, a scheme of the experimental set-up (Fig. 1) adapted from [11] is presented. In Fig. 1, the wake develops along x axis, r is the radius, U_0 is the

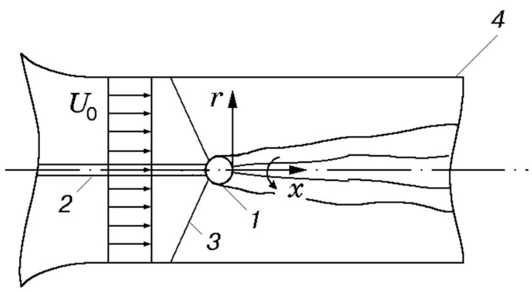

Fig. 1. Scheme of the experimental setup (1 is the sphere, 2 is the tube which delivers the air to form a swirling jet that flows from the rear of the sphere, 3 are tension members, 4 is the throat of the wind tunnel)

undisturbed flow velocity. A special nozzle for a tangential air flow exhausting was built into the trailing edge of the sphere to provide swirling stream behind the sphere.

The following semi–empirical model of turbulence [20] is used to describe the flow in a far momentumless swirling turbulent wake:

$$U_0 \frac{\partial U_1}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(C_u r \frac{e^2}{\varepsilon}\frac{\partial U_1}{\partial r}\right) + \frac{\partial}{\partial x}\int_r^\infty \frac{W^2}{r'}dr', \tag{1}$$

$$U_0 \frac{\partial W}{\partial x} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(C_w r^3 \frac{e^2}{\varepsilon} \frac{\partial (W/r)}{\partial r} \right), \tag{2}$$

$$U_0 \frac{\partial e}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left(C_e r \frac{e^2}{\varepsilon} \frac{\partial e}{\partial r} \right) + C_u r^2 \frac{e^2}{\varepsilon} \left(\frac{\partial (W/r)}{\partial r} \right)^2 - \varepsilon, \tag{3}$$

$$U_0 \frac{\partial \varepsilon}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left(C_\varepsilon r \frac{e^2}{\varepsilon} \frac{\partial \varepsilon}{\partial r} \right) + C_{\varepsilon 1} C_u r^2 e \left(\frac{\partial (W/r)}{\partial r} \right)^2 - C_{\varepsilon 2} \frac{\varepsilon^2}{e}. \tag{4}$$

Here $U_1 = U - U_0$ is the deficit of the mean longitudinal velocity component, W is the mean tangential velocity component, k is the kinetic energy of turbulence, and ε is the kinetic energy dissipation. It is assumed that the fluid is incompressible and the flow is steady. Moreover, in what follows the undisturbed flow velocity U_0 is taken to be unity. The empirical constants are as follows [20]:

$$C_u = C_w = 0.25, \quad C_e = 0.147, C_\varepsilon = 0.113, \quad C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92.$$

Model (1)–(4) is a simplification of more complicated mathematical model that includes a system of averaged equations of motion, continuity, transport of normal Reynolds stresses and turbulence energy dissipation rate in a rotationally–symmetrical flow in the approximation of a thin shear layer [9, 18, 19, 25]. Moreover, turbulent tangential stresses are determined from nonequilibrium algebraic relations [9, 19, 26]. The simplification introduced in [20] is based on the fact that absolute axial value of the longitudinal velocity component decreases much faster than the maximum absolute value of the tangential velocity component. Therefore, at large distances from the body one can neglect the contribution of this quantity to the term that describes production of turbulence energy . Simplification is based on the far wake approximation and on the replacement of equations for the transfer of normal stresses by a single equation for the turbulence energy balance. In addition, the ratio of the turbulence energy production term to the kinetic energy dissipation is set equal to zero in expressions for turbulent viscosity coefficients (this ratio does not exceed 0.1 in the far wake).

Conservation of total excess momentum and angular momentum follow from equations (1)–(4) and initial and boundary conditions for the considered flow:

$$J = 2\pi\rho \int_0^\infty \left(U_0 U_1 - \int_r^\infty \frac{W^2}{r'} dr' \right) r dr = 0, \tag{5}$$

$$M = 2\pi\rho \int_0^\infty r^2 U_0 W dr = M_0 \neq 0, \tag{6}$$

here ρ is the fluid density.

It was shown that at large distances behind the body a flow becomes close to self–similar [20]. Therefore, it is natural to seek the self–similar reductions of model (1)–(4).

2. Self–similar reduction

A group analysis is used to construct self-similar solutions [27]. The Lie algebra basis of equations (1)–(4) consists of the following infinitesimal generators:

$$X_1 = \frac{\partial}{\partial x}, \ X_2 = \frac{\partial}{\partial U_1}, \ X_3 = x \frac{\partial}{\partial x} - 2U_1 \frac{\partial}{\partial U_1} - W \frac{\partial}{\partial W} - 2e \frac{\partial}{\partial e} - 3\varepsilon \frac{\partial}{\partial \varepsilon},$$

$$X_4 = r\frac{\partial}{\partial r} + 2U_1\frac{\partial}{\partial U_1} + W\frac{\partial}{\partial W} + 2e\frac{\partial}{\partial e} + 2\varepsilon\frac{\partial}{\partial \varepsilon}.$$

Using the linear combination of operators X_3 and X_4 it is not difficult to obtain the following representation for a solution of the initial model (1)–(4):

$$U_1 = x^{2\alpha-2}U_2(t),\ W = x^{\alpha-1}W_1(t),\ e = x^{2\alpha-2}K(t),\ \varepsilon = x^{2\alpha-3}E(t),\ t = r/x^{\alpha}, \qquad (7)$$

here t is a self–similar variable, α is an arbitrary constant appearing in the linear combination of operators X_3 and X_4.

Using the law of conservation (6) and representation (7) for W, it is not difficult to show that α should be equal to 0.25. Let us remark that decay laws (7) of required functions are in an agreement with the results of numerical calculations of the initial model [20]. They are presented in Fig. 2. In this figure, D is the diameter of a body; $L_{1/2} \sim x^{\alpha}$ is the characteristic scale of the wake width; $|U_{10}|$ is the absolute axial value of the defect of longitudinal velocity component; $|W_m|$ is the maximum absolute value of the tangential velocity component; e_0 is the axial value of kinetic energy of turbulent disturbances; ε_0 is the axial value of kinetic energy dissipation. Markers correspond to experimental data. It can be noted that the decay law of an axial value of the defect of longitudinal velocity component changes at about 1000 diameters behind the body. This is apparently due to the fact that swirl term in equation (1) is negligible at large distances from the body (see, [11]).

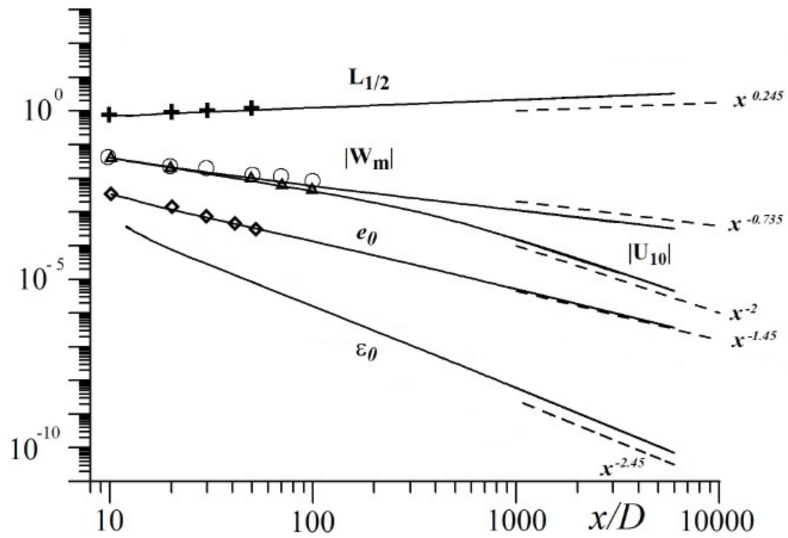

Fig. 2. Variation of dimensionless scale turbulence characteristics in the swirling momentumless wake versus distance from the body

In this case the Lie algebra basis of equations (1)–(4) consists of the following infinitesimal generators:

$$X_1 = \frac{\partial}{\partial x},\quad X_2 = \frac{\partial}{\partial U_1},\quad X_3 = r\frac{\partial}{\partial W},\quad X_4 = x\frac{\partial}{\partial x} - W\frac{\partial}{\partial W} - 2e\frac{\partial}{\partial e} - 3\varepsilon\frac{\partial}{\partial \varepsilon},$$

$$X_5 = U_1\frac{\partial}{\partial U_1},\quad X_6 = r\frac{\partial}{\partial r} + W\frac{\partial}{\partial W} + 2e\frac{\partial}{\partial e} + 2\varepsilon\frac{\partial}{\partial \varepsilon}.$$

Corresponding self–similar representation of solution has the form

$$U_1 = x^{\beta}U_2(t), \quad W = x^{\alpha-1}W_1(t), \quad e = x^{2\alpha-2}K(t), \quad \varepsilon = x^{2\alpha-3}E(t), \quad t = r/x^{\alpha}. \tag{8}$$

Here β is an arbitrary constant. Let us remark that in the self–propulsion mode ($J = 0$, $M = 0$) the mean tangential velocity component decreases much faster (see, [9]).

Using representation (7), we obtain a reduction of initial mathematical model (1)–(4) to the following system of ordinary differential equations:

$$C_u\frac{K^2U_2''}{E} + \left(C_u\frac{K}{E}\left(2K' - \frac{KE'}{E} + \frac{K}{t}\right) + \alpha t\right)U_2' - 2(\alpha-1)\left(U_2 + \int_t^{\infty}\frac{W_2^2}{t'}dt'\right) - \alpha W_1^2 = 0, \tag{9}$$

$$C_u\frac{K^2W_1''}{E} + \left(C_u\frac{K}{E}\left(2K' - \frac{KE'}{E} + \frac{K}{t}\right) + \alpha t\right)W_1' + \left(C_u\frac{K}{E}\left(2K' - \frac{KE'}{E} + \frac{K}{t}\right) - \right.$$
$$\left. -\alpha + 1\right)W_1 = 0, \tag{10}$$

$$C_e\frac{K^2K''}{E} + 2C_e\frac{KK'^2}{E} + \left(C_e\frac{K^2}{E}\left(\frac{E'}{E} - \frac{1}{t}\right) + \alpha t\right)K' + C_u\frac{K^2}{E}\left(W_1' - \frac{W_1}{t}\right)^2 +$$
$$+ 2(\alpha-1)K - E = 0, \tag{11}$$

$$C_\varepsilon\frac{K^2E''}{E} - C_\varepsilon\frac{K^2E'^2}{E^2} + \left(C_\varepsilon\frac{K}{E}\left(2K' + \frac{K}{t}\right) + \alpha t\right)E' - C_uC_{\varepsilon 1}K\left(W_1' - \frac{W_1}{t}\right)^2 - C_{\varepsilon 2}\frac{E^2}{K} -$$
$$- (2\alpha - 3)E = 0. \tag{12}$$

For representation (8) the first equation of the reduced system has the following form:

$$C_u\frac{K^2U_2''}{E} + \left(C_u\frac{K}{E}\left(2K' - \frac{KE'}{E} + \frac{K}{t}\right) + \alpha t\right)U_2' - \beta U_2 = 0. \tag{13}$$

Reduced system (10)–(13) is solved numerically.

3. Calculation results

Solutions of reduced system (10)–(13) have to satisfy the following conditions:

$$U_2'(0) = W_1(0) = K'(0) = E'(0) = 0,$$
$$U_2(a) = W_1(a) = K(a) = E(a) = 0.$$

The first group of conditions takes into account that flow is symmetric with respect to the Ox axis. The second group of conditions follows from the requirement that flow is undisturbed outside the turbulent wake domain (all required functions have to take zero values in this domain). The value of a related with the turbulent wake semi-width can be either set equal to unity in calculations because equations of reduced system (10)–(13) are invariant with respect to the transformation of extension or taken from the experimental data. It should also be noted that coefficients of system (10)–(13) have singularities in boundary conditions.

For numerical solution of the boundary value problem the modified shooting method was used, together with the asymptotic expansion of the solution in the vicinity of the singular point $t = a$

$$U_2 = c_1|t-a|^{\alpha_1} + o\left(|t-a|^{\alpha_1}\right), \quad W_1 = c_2|t-a|^{\alpha_2} + o\left(|t-a|^{\alpha_2}\right), \quad K = c_3|t-a|^{\alpha_3} + o\left(|t-a|^{\alpha_3}\right),$$

$$E = c_4|t-a|^{\alpha_4} + o\left(|t-a|^{\alpha_4}\right), \quad \alpha_3 = \frac{C_\varepsilon}{2C_\varepsilon - C_e}, \quad \alpha_1 = \alpha_2 = \frac{C_e}{C_u}\alpha_3, \quad \alpha_4 = 2\alpha_3 - 1, \quad c_3 = \frac{C_e \alpha_3 c_2^2}{a\alpha}.$$

As the result of calculations the following values were obtained:

$$\beta = -2.09, \quad U_2(0) = 0.35378, \quad W_1'(0) = -0.916, \quad K(0) = 1.14357, \quad E(0) = 1.32668.$$

In Fig. 3 self-similar profiles of solutions obtained by the shooting method are compared with numerical results obtained on the basis of the full model of equations (1)–(4) [20] (1–3 are numerical results [20], 4 are self-similar solutions obtained by the shooting method).

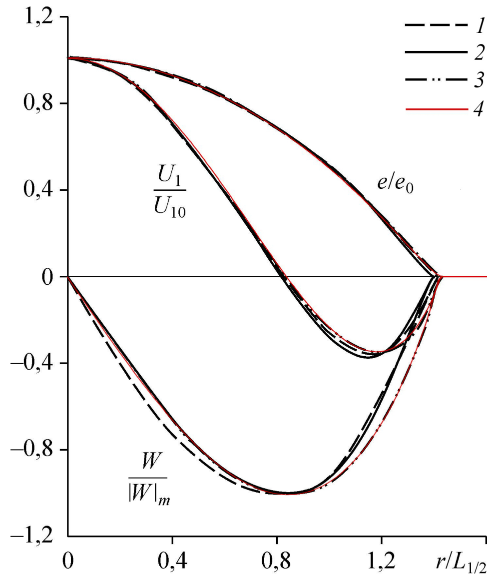

Fig. 3. Self-similar normalized profiles of the deficit of the mean longitudinal velocity component, the mean tangential velocity component, and the kinetic energy of turbulence

Self-similar distributions presented in Fig. 3 are very close to numerical results [20]. Therefore, this indicates the applicability of simplified mathematical models [20] to simulations of the far field of swirling momentumless turbulent wake.

Self-similar profiles of turbulence energy and tangential velocity components are finite bell-shaped functions. At the same time the self–similar profile of the defect of the velocity longitudinal component has regions of negative and positive values. This is in agreement with conservation laws (5) and (6).

The study was supported by the Russian Foundation for Basic Research (grant no. 17-01-00332)

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Автомодельность дальнего закрученного безымпульсного турбулентного следа

Алексей В. Шмидт
Институт вычислительного моделирования СО РАН
Красноярск, Российская Федерация

Аннотация. В работе проведено построение автомодельного решения полуэмпирической модели дальнего безымпульсного закрученного турбулентного следа.

Ключевые слова: дальний закрученный турбулентный след, автомодельное решение, метод стрельбы.

DOI: 10.17516/1997-1397-2020-13-1-87-96
УДК 517.55

Hypergeometric Series and the Mellin-Barnes Integrals for Zeros of a System of Laurent Polynomials

Vladimir R. Kulikov*
Siberian Federal University
Krasnoyarsk, Russian Federation

Received 13.09.2019, received in revised form 06.11.2019, accepted 20.12.2019

Abstract. In the article we present a criterion for convergence of the Mellin-Barnes integral for zeros of a system of Laurent polynomials. Also we give a hypergeometric series for these zeros.

Keywords: Mellin-Barnes integrals, hypergeometric series, Laurent polynomials.

Citation: V.R.Kulikov, Hypergeometric Series and the Mellin-Barnes Integrals for Zeros of a System of Laurent Polynomials, J. Sib. Fed. Univ. Math. Phys., 2020, 13(1), 87–96.
DOI: 10.17516/1997-1397-2020-13-1-87-96.

Introduction

In 1921 Hj. Mellin wrote down an integral representing a solution $y(x)$ of a reduced algebraic equation of the form

$$y^n + x_1 y^{n-1} + \ldots + x_{n-1} y - 1 = 0.$$

This integral has a non-empty domain of convergence, it is defined by conditions on arguments $\theta_j = \arg x_j$. A complete description of the convergence domain has been obtained relatively recently in the paper by I. A. Antipova [3].

In the present paper we study the same problem in several variables. Consider a system of algebraic equations of the form

$$\mathbf{y}^{\omega^{(j)}} + \sum_{\lambda \in \Lambda^{(j)}} x_\lambda^{(j)} \mathbf{y}^\lambda - 1 = 0,\ j = 1, \ldots, n, \tag{1}$$

where $\Lambda^{(j)} \subset \mathbb{Z}^n$, and $\omega^{(j)}$ is a column vector, the matrix made of columns $\omega^{(j)}$ we denote by Ω. Let us also introduce the notation $\Lambda := \bigsqcup_{j=1}^{n} \Lambda^{(j)}$ for a disjunctive union of sets $\Lambda^{(j)}$, the cardinality of Λ we denote by N. By $\overline{\Lambda^{(j)}}$ we shall denote the set $\Lambda^{(j)} \cup \{\omega^{(j)}\}$, analogously $\overline{\Lambda} = \bigsqcup_{j=1}^{n} \overline{\Lambda^{(j)}}$.

The set of coefficients of the system (1) runs over the vector space $\mathbb{C}^\lambda \cong \mathbb{C}^N_{\mathbf{x}}$, where coordinates of points $\mathbf{x} = (x_\lambda)$ are indexed by the elements $\lambda \in \Lambda$. A group of coordinates corresponding to indices $\lambda \in \Lambda^{(i)}$ we, as a rule, write as $x_\lambda^{(i)}$, having identified $\mathbb{C}^\Lambda$ with $\mathbb{C}^{\Lambda^{(1)}} \times \ldots \times \mathbb{C}^{\Lambda^{(n)}}$; sometimes for elements of $\mathbb{C}^{\Lambda^{(i)}}$ we use the notation x_λ, $\lambda \in \Lambda^{(i)}$. Denote also by X the diagonal matrix with x_λ on the diagonal ($X = \operatorname{diag}[\mathbf{x}]$).

*v.r.kulikov@mail.ru https://orcid.org/0000-0002-2291-0449

The set Λ we will also treat as a matrix

$$\Lambda = \left(\Lambda^{(1)}, \ldots, \Lambda^{(n)}\right) = \left(\lambda^1, \ldots, \lambda^N\right),$$

whose columns are the vectors $\lambda^k = \left(\lambda_1^k, \ldots, \lambda_n^k\right)$ of exponents of monomials of the system (1). Here we mean that a block $\Lambda^{(i)}$ of the matrix Λ corresponds to the ith equation of the system (1); enumeration of columns λ^k in each block $\Lambda^{(i)}$ is arbitrary but fixed.

Denote by χ the characteristic matrix of the set Λ.

In this notation the system (1) can be written in a matrix form:

$$\mathbf{y}^{\Omega} + \mathbf{y}^{\Lambda} X \chi^T - I = 0. \tag{2}$$

We are interested in a branch of a solution $\mathbf{y}(\mathbf{x}) = (y_1(\mathbf{x}), \ldots, y_n(\mathbf{x}))$ of the system (1) with the condition $\mathbf{y}(\mathbf{0}) = (1, \ldots, 1)$, which we call the principal solution. Following [2, 4], to a monomial $\mathbf{y}^{\mu} = y_1^{\mu_1} \ldots y_n^{\mu_n}$ of the principal solution $\mathbf{y} = \mathbf{y}(\mathbf{x})$ of the system we put into the correspondence the Mellin-Barnes integral:

$$\mathbf{y}^{\mu}(\mathbf{x}) \to \frac{1}{(2\pi i)^N} \int\limits_{\gamma + i\mathbb{R}^N} \frac{\Gamma(\mathbf{u})\Gamma(\Omega^{-1}\mu - \Omega^{-1}\Lambda\mathbf{u})}{\Gamma(\Omega^{-1}\mu - \Omega^{-1}\Lambda\mathbf{u} + \chi\mathbf{u} + I)} Q(\mathbf{u}) \mathbf{x}^{-\mathbf{u}} du, \tag{3}$$

where the vector γ is from the polyhedron

$$\{\mathbf{u} \in \mathbb{R}^N_{>0} : \langle \varphi_j, \mathbf{u} \rangle < \mu_j, j = 1, \ldots, n\},$$

and $Q(\mathbf{u})$ is a polynomial given by the determinant

$$Q(\mathbf{u}) = \det\left(\mathrm{diag}[\Omega^{-1} \cdot (\mu - \Lambda \cdot \mathbf{u})] + \Omega^{-1} \cdot \Lambda \cdot \mathrm{diag}[\mathbf{u}]\chi^T\right). \tag{4}$$

The integral (3) is obtained by a formal computation of the Mellin transform of $\mathbf{y}^{\mu}(\mathbf{x})$ using linearization.

Consider the following matrices made of exponents of monomials of the system (1):

$$\begin{pmatrix} \lambda_1^{(1)} & \cdots & \lambda_1^{(n)} \\ \vdots & \ddots & \vdots \\ \lambda_n^{(1)} & \cdots & \lambda_n^{(n)} \end{pmatrix}, \tag{5}$$

where each column vector $\lambda^{(j)} = \left(\lambda_1^{(j)} \ldots \lambda_n^{(j)}\right)^T$ runs over the corresponding set $\overline{\Lambda^{(j)}}$.

Theorem 1. *The integral* (3) *corresponding to a system of algebraic equations* (1) *has a non-empty domain of convergence and represents the monomial function of the solution if and only if the determinants of all matrices of the form* (5) *are non-zero and have the same sign.*

Note that in [9] and [10] analogous results have been obtained for systems with a diagonal matrix ω and non-negative exponents of monomials $\lambda \in \Lambda^{(j)}$. Applications of these results for study of discriminants of systems are given in [11].

Convergence of the Mellin-Barnes integral

In this section we prove the convergence of the Mellin-Barnes integral (3) under hypothesis of Theorem 1.

Recall that a multiple Mellin-Barnes integral has the form

$$\Phi(\mathbf{z}) = \frac{1}{(2\pi i)^m} \int\limits_{\gamma + i\mathbb{R}^m} \frac{\Gamma(A\cdot \mathbf{s} + \mathbf{c})}{\Gamma(B\cdot \mathbf{s} + \mathbf{d})} \mathbf{z}^{-\mathbf{s}} ds, \tag{6}$$

where $A \in \mathbb{R}^{m\times p}$, $B \in \mathbb{R}^{m\times q}$, $\mathbf{c} \in \mathbb{C}^p$, $\mathbf{d} \in \mathbb{C}^q$, $\mathbf{z}, \mathbf{s} \in \mathbb{C}^m$, and the vector γ is chosen such that the integration set $\gamma + i\mathbb{R}^m$ does not contain poles of Γ functions of the numerator.

We shall assume that the variable z varies in a Riemannian covering of the complex algebraic torus $(\mathbb{C}\setminus\{0\})^m$, consequently, the factors in the integral kernel are defined as

$$z_j^{-s_j} = e^{-s_j \log z_j}, \ \arg z_j \in \mathbb{R}.$$

Denote $\theta = \mathrm{Arg} z = (\arg z_1, \ldots, \arg z_m)$ and introduce the function

$$g(\mathbf{v}) = \sum_{j=1}^{p} |\langle A_j, \mathbf{v}\rangle| - \sum_{k=1}^{q} |\langle B_k, \mathbf{v}\rangle|,$$

where A_j and B_k are rows of matrices A and B, respectively.

The next theorem gives a description of the convergence domain of a multiple Mellin-Barnes integral.

Theorem 2 (Nilsson, Passare, Tsikh)**.** *For an integration set $\gamma + i\mathbb{R}^m$ that does not contain singularities of the integrand the convergence domain of the Mellin-Barnes integral* (6) *has the form* $\mathrm{Arg}^{-1}(U)$, *where*

$$U = \bigcap_{\|v\|=1} \left\{\theta \in \mathbb{R}^m : |\langle \mathbf{v}, \theta\rangle| < \frac{\pi}{2} g(\mathbf{v})\right\}. \tag{7}$$

In the case when the set U is not empty it coincides with the interior Θ° of the polyhedron

$$\Theta = \left\{\theta \in \mathbb{R}^m : |\langle \mathbf{v}_\nu, \theta\rangle| \leqslant \frac{\pi}{2} g(\mathbf{v}_\nu), \nu = 1, \ldots, d\right\}, \tag{8}$$

where $\pm\mathbf{v}_1, \ldots, \pm\mathbf{v}_d$ is the set of unit vectors generating the fan K defined by a decomposition of $\mathbb{R}^m$ by hyperplanes $\langle A_j, \mathbf{v}\rangle = 0$, $j = 1, \ldots, p$ and $\langle B_k, \mathbf{v}\rangle = 0$, $k = 1, \ldots, q$.

Thus, the convergence domain of the integral (6) is not empty if the function $g(v)$ is positive on the compact set (sphere) $\|\mathbf{v}\| = 1$. Since $g(\mathbf{v})$ is homogeneous, this is equivalent to its positivity for $\mathbf{v} \neq 0$.

As has been established in earlier papers, the convergence domain of a Mellin-Barnes integral does not depend on the presence of a polynomial factor $Q(u)$.

For the integral (3) the function $g(v)$ is

$$g(v) = \|\mathbf{v}\| + \|\Omega^{-1}\Lambda\mathbf{v}\| - \|(\chi - \Omega^{-1}\Lambda)\mathbf{v}\|,$$

where $\|\mathbf{v}\| = |v_1| + \ldots + |v_N|$.

The matrix $\Phi = \Omega^{-1}\Lambda$ inherits its block structure from the matrix Λ compatible with the characteristic matrix χ; the blocks of this matrix are denoted by $\Phi^{(j)}$. In [9] it has been shown that in this case the function $g(\mathbf{v})$ vanishes only for $v = 0$ if and only if all diagonal minors of matrices $\varphi = \left(\varphi^{(1)}, \dots, \varphi^{(n)}\right)$ are positive, here $\varphi^{(j)}$ is an arbitrary column vector of the matrix $\Phi^{(j)}$.

Consider a diagonal minor of order p of this matrix, it may be obtained as the determinant of a product of two rectangular matrices:

$$\varphi^{j_1,\dots,j_p}_{j_1,\dots,j_p} = \det\left((\Omega^{-1})_{j_1,\dots,j_p} \cdot \lambda^{j_1,\dots,j_p}\right),$$

where $\lambda = \Omega\varphi$.

By the Cauchy-Binet formula, the determinant of a product of two such matrices is a sum of products of minors of these matrices:

$$\varphi^{j_1,\dots,j_p}_{j_1,\dots,j_p} = \sum_{1\leqslant k_1<\dots<k_p\leqslant n} (\Omega^{-1})^{k_1\dots,k_p}_{j_1,\dots,j_p} \cdot \lambda^{j_1,\dots,j_p}_{k_1\dots,k_p}.$$

By definition,

$$\Omega^{-1} = \frac{\mathrm{adj}\Omega^T}{|\Omega|}.$$

Therefore, by the Jacobi identity the minor $(\Omega^{-1})^{k_1\dots,k_p}_{j_1,\dots,j_p}$ can be computed as:

$$(\Omega^{-1})^{k_1\dots,k_p}_{j_1,\dots,j_p} = \frac{(-1)^\sigma}{|\Omega|}\Omega^{j_{p+1},\dots,j_n}_{k_{p+1}\dots,k_n},$$

where σ is the order of the permutation

$$\begin{pmatrix} j_1 & \dots & j_n \\ k_1 & \dots & k_n \end{pmatrix}.$$

Substituting these expressions into the formula for the minor $\varphi^{j_1,\dots,j_p}_{j_1,\dots,j_p}$ and using the Laplace expansion along several columns we get

$$\varphi^{j_1,\dots,j_p}_{j_1,\dots,j_p} = \frac{|A|}{|\Omega|} > 0,$$

where A is the matrix whose columns with numbers j_s are equal to $\lambda^{(j_s)}$, and all the remaining columns are the corresponding columns of Ω, i.e. A is a matrix of the form (5).

Since the choice of a matrix φ and an order of a minor p is arbitrary, it follows that all determinants of matrices of the form (5) has the same sign. Thus, we have proved that under the hypothesis of Theorem 1 the Mellin-Barnes integral corresponding to a solution of the system (1) converges.

Solution of the system as a Taylor series

Consider the system (2):

$$\mathbf{y}^\Omega + \mathbf{y}^\Lambda X\chi^T - I = 0.$$

The following statement holds

Theorem 3. *The monomial $\widehat{\mathbf{y}}^\mu(\mathbf{x})$ of the principal solution of the system (2) is given by the Taylor series*

$$\widehat{\mathbf{y}}^l(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^N_{\geqslant}} c_\mathbf{k}\mathbf{x}^\mathbf{k} \tag{9}$$

with the coefficients

$$c_k = \frac{(-1)^{|\mathbf{k}|}}{\mathbf{k}!}\frac{\Gamma\big(\Omega^{-1}\cdot\mu+\Omega^{-1}\cdot\Lambda\cdot\mathbf{k}\big)}{\Gamma(\Omega^{-1}\cdot\mu+\Omega^{-1}\cdot\Lambda\cdot\mathbf{k}-\chi\cdot\mathbf{k}+I)}\cdot Q(\mathbf{k}), \tag{10}$$

where $Q(\mathbf{k}) = \det\big(\mathrm{diag}\big[\Omega^{-1}\cdot(\mu+\Lambda\cdot\mathbf{k})\big]-\Omega^{-1}\cdot\Lambda\cdot\mathrm{diag}[\mathbf{k}]\cdot\chi^T\big)$.

Proof. First, let us linearize the system (2). Consider it as a system in $\mathbb{C}^N_{\mathbf{x}}\times\mathbb{C}^n_{\mathbf{y}}$ and make in $\mathbb{C}^{N+n}$ the following change of variables:

$$y = W^{-\Omega^{-1}};\ \ x = \xi\odot W^{\Omega^{-1}\Lambda-\chi}.$$

In the new variable the system becomes

$$W = \xi\chi^T + I.$$

Represent the inverse $\xi(\mathbf{x})$ as an implicit function given by the system of equations

$$F(\xi,\mathbf{x}) = \xi\odot W^{\Omega^{-1}\Lambda-\chi}-\mathbf{x}.$$

Zeroes of these functions define the change of linearization. Therefore, the monomial function of the solution can be found by A. P. Yuzhakov's logarithmic residue formula. According to this formula

$$y^\mu(\mathbf{x}) = \frac{1}{(2\pi i)^N}\int\limits_{\Gamma_\varepsilon}\frac{y^\mu(\xi)\Delta(\xi)d\xi}{F^I(\xi,\mathbf{x})}, \tag{11}$$

where $\Gamma_\varepsilon = \{\xi\in\mathbb{C}^N : |\xi_\lambda| = \varepsilon, \lambda\in\Lambda\}$, $\Delta(\xi)$ is the Jacobian of the mapping $F(\xi,x)$ with respect to variables ξ (notice that the Jacobian does not contain variables x), $F^I(\xi,x)$ denotes the product $F_1(\xi,\mathbf{x})\cdot\ldots\cdot F_N(\xi,\mathbf{x})$. The radius ε is chosen in such a way that the corresponding polydisc lies outside zeroes of the Jacobian $\Delta(\xi)$.

Lemma 1. *The Jacobian of* $F(\xi,\mathbf{x})$ *with respect to* ξ *is*

$$\Delta(\xi) = W^{(\Omega^{-1}\Lambda-\chi)I-I}\det\big(E+\Omega^{-1}\Lambda\cdot\Xi\cdot\chi^T\big), \tag{12}$$

here and further on $\Xi = \mathrm{diag}[\xi]$.

Proof. The component of the mapping $F(\xi,\mathbf{x})$ with the index $\lambda^{(j)}\in\Lambda^{(j)}$, $j = 1,\ldots,n$ has the form

$$F_{\lambda^{(j)}} = \xi_{\lambda^{(j)}}\prod_{k=1}^n W_k^{(\Omega^{-1}\Lambda-\chi)_k^{\lambda^{(j)}}}$$

here $(\Omega^{-1}\Lambda-\chi)_k^{\lambda^{(j)}}$ denotes the k-th component of the column with the index $\lambda^{(j)}$ of the matrix $\Omega^{-1}\Lambda-\chi$.

Non-diagonal elements of the Jacobian are

$$\frac{\partial F_{\lambda^{(j)}}}{\partial\xi_{\eta^{(i)}}} = \xi_{\lambda^{(j)}}(\Omega^{-1}\Lambda-\chi)_i^{\lambda^{(j)}}\prod_{k=1}^n W_k^{(\Omega^{-1}\Lambda-\chi)_k^{\lambda^{(j)}}-\delta_k^i},\ \ \lambda^{(j)}\in\Lambda^{(j)},\ \ \eta^{(i)}\in\Lambda^{(i)},$$

and if $i = j$ then $\lambda^{(j)}\neq\eta^{(i)}$.

Its diagonal elements has the form

$$\frac{\partial F_{\lambda^{(j)}}}{\partial \xi_{\lambda^{(j)}}} = \prod_{k=1}^{n} W_k^{(\Omega^{-1}\Lambda-\chi)_k^{\lambda^{(j)}}} \left(1 + \xi_{\lambda^{(j)}} (\Omega^{-1}\Lambda - \chi)_j^{\lambda^{(j)}} W_j^{-1}\right), \quad \lambda^{(j)} \in \Lambda^{(j)},$$

where δ_k^j is the Kronecker delta.

From each row of the Jacobian we factor out

$$W^{(\Omega^{-1}\Lambda-2\chi)^{\lambda^{(j)}}} = \prod_{k=1}^{n} W_k^{(\Omega^{-1}\Lambda-\chi)_k^{\lambda^{(j)}} - \delta_k^j}.$$

Then, before the Jacobian we have the factor $W^{(\Omega^{-1}\Lambda-2\chi)I}$, and the elements of the Jacobian become

$$\frac{\partial F_{\lambda^{(j)}}}{\partial \xi_{\eta^{(i)}}} = \xi_{\lambda^{(j)}} (\Omega^{-1}\Lambda - \chi)_i^{\lambda^{(j)}}, \quad \lambda^{(j)} \in \Lambda^{(j)}, \quad \eta^{(i)} \in \Lambda^{(i)},$$

outside the diagonal, and

$$\frac{\partial F_{\lambda^{(j)}}}{\partial \xi_{\lambda^{(j)}}} = W_j + \xi_{\lambda^{(j)}} (\Omega^{-1}\Lambda - \chi)_j^{\lambda^{(j)}}, \quad \lambda^{(j)} \in \Lambda^{(j)},$$

on the diagonal.

In each i-th block-column of the obtained determinant subtract one column of this block from all other columns of the block, the chosen columns we shall call *marked*, and their indices are denoted by ${}'\eta^{(i)}$, while ${}'\Lambda^{(i)} := \Lambda(i) \setminus \{{}'\eta^{(i)}\}$ denote indices of not marked columns of the ith block.

The elements of the Jacobian then take the form:

$$\frac{\partial F_{\lambda^{(j)}}}{\partial \xi_{\eta^{(i)}}} = 0; \; \lambda^{(j)} \in \Lambda^{(j)}, \;\; \eta^{(i)} \in {}'\Lambda^{(i)}, \; i \neq j,$$

for those in non-diagonal blocks in not marked columns.

$$\frac{\partial F_{{}'\eta^{(j)}}}{\partial \xi_{\eta^{(j)}}} = -W_j, \quad \eta^{(j)} \in {}'\Lambda^{(j)},$$

for elements in diagonal blocks in marked columns.

$$\frac{\partial F_{\lambda^{(j)}}}{\partial \xi_{\lambda^{(j)}}} = W_j, \quad \lambda^{(j)} \in {}'\Lambda^{(j)},$$

for diagonal elements in not marked columns.

$$\frac{\partial F_{{}'\eta^{(j)}}}{\partial \xi_{{}'\eta^{(j)}}} = W_j + \xi_{{}'\eta^{(j)}} (\Omega^{-1}\Lambda - \chi)_j^{{}'\eta^{(j)}},$$

for diagonal elements in marked columns.

$$\frac{\partial F_{\lambda}}{\partial \xi_{{}'\eta^{(i)}}} = \xi_{\lambda} (\Omega^{-1}\Lambda - \chi)_i^{\lambda}, \quad \lambda \in \Lambda \setminus \{\eta^{(i)}\},$$

for all other elements in marked columns, and

$$\frac{\partial F_{\lambda^{(j)}}}{\partial \xi_{\eta^{(j)}}} = 0; \;\; \lambda^{(j)}, \eta^{(j)} \in {}'\Lambda^{(j)}, \;\; \lambda^{(j)} \neq \eta^{(j)},$$

for all remaining elements.

Now in each jth block-row add to the row with the index $'\eta^{(j)}$ all other rows of this block to get

$$\frac{\partial F_{'\eta^{(j)}}}{\partial \xi_{'\eta^{(i)}}} = W_j \delta_i^j + \sum_{\lambda^{(j)} \in \Lambda^{(j)}} \xi_{\lambda^{(j)}} (\Omega^{-1}\Lambda - \chi)_i^{\lambda^{(j)}},$$

$$\frac{\partial F_{'\eta^{(j)}}}{\partial \xi_{\eta^{(j)}}} = 0, \quad \eta^{(j)} \in {}'\Lambda^{(j)}.$$

Since non-zero elements are now only in marked columns and on the principal diagonal, the Jacobian can be reduced to a determinant of order n:

$$\frac{\partial F}{\partial \xi} = W^{(\Omega^{-1}\Lambda-\chi)I-I} \det\left(E + \Omega^{-1}\Lambda \cdot \Xi \cdot \chi^T\right).$$

This proves the lemma. □

Substitute now the expression for the Jacobian as well as the expression for $\mathbf{y}^{\mu}(\xi)$ into (11)

$$\mathbf{y}^{\mu}(\mathbf{x}) = \frac{1}{(2\pi i)^N} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\Omega^{-1}\mu} \cdot W^{(\Omega^{-1}\Lambda-\chi)I-I} \det\left(E + \Omega^{-1}\Lambda \cdot \Xi \cdot \chi^T\right)}{(\xi \odot W^{(\Omega^{-1}\Lambda-\chi)} - \mathbf{x})^I} d\xi \tag{13}$$

and reduce the fraction in the integrand

$$\mathbf{y}^{\mu}(\mathbf{x}) = \frac{1}{(2\pi i)^N} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\Omega^{-1}\mu-I} \det\left(E + \Omega^{-1}\Lambda \cdot \Xi \cdot \chi^T\right)}{\xi^I (I - \mathbf{x} \odot \xi^{-E} \odot W^{\chi-\Omega^{-1}\Lambda})^I} d\xi. \tag{14}$$

There exists δ such that for any $\xi \in \Gamma_\varepsilon$ and $\|\mathbf{x}\| < \delta$ we have the inequality $\mathbf{x} \odot \xi^{-E} \odot W^{\chi-\Omega^{-1}\Lambda} < I$, therefore we can represent the integrand as a geometric series

$$\begin{aligned} \mathbf{y}^{\mu}(\mathbf{x}) = & \frac{1}{(2\pi i)^N} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\Omega^{-1}\mu-I} \det\left(E + \Omega^{-1}\Lambda \cdot \Xi \cdot \chi^T\right)}{\xi^I} \times \\ & \times \left(\sum_{\mathbf{k} \in \mathbb{Z}^N_{\geqslant}} \mathbf{x}^{\mathbf{k}} \cdot \xi^{-\mathbf{k}} \cdot W^{(\chi-\Omega^{-1}\Lambda)\mathbf{k}} \right) d\xi. \end{aligned} \tag{15}$$

Now change the order of integration in (15):

$$\mathbf{y}^{\mu}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^N_{\geqslant}} c_{\mathbf{k}} \mathbf{x}^{\mathbf{k}},$$

here the coefficients $c_{\mathbf{k}}$ are given by

$$c_{\mathbf{k}} = \frac{1}{(2\pi i)^N} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\Omega^{-1}(\mu+\Lambda\mathbf{k})+\chi\mathbf{k}-I}}{\xi^{\mathbf{k}+I}} \det\left(E + \Omega^{-1}\Lambda \cdot \Xi \cdot \chi^T\right) d\xi.$$

The coefficients of the obtained series can be computed by the Cauchy formula

$$c_{\mathbf{k}} = \frac{1}{\mathbf{k}!} \frac{\partial^{\mathbf{k}}}{\partial \xi^{\mathbf{k}}} \left(W^{-\Omega^{-1}(\mu+\Lambda\mathbf{k})+\chi\mathbf{k}-I} \det\left(E + \Omega^{-1}\Lambda \cdot \Xi \cdot \chi^T\right) \right) \Big|_{\xi=0}.$$

The computation of the derivatives gives

$$c_{\mathbf{k}} = \frac{(-1)^{\mathbf{k}}}{\mathbf{k}!} \frac{\Gamma(\Omega^{-1}(\mu + \Lambda \cdot \mathbf{k}))}{\Gamma(\Omega^{-1}(\mu + \Lambda \cdot \mathbf{k}) - \chi \cdot k + I)} Q(k),$$

where $Q(\mathbf{k}) = \det(\mathrm{diag}[\Omega^{-1}(\mu + \Lambda\mathbf{k})] - \Omega^{-1}\Lambda \cdot \mathrm{diag}[\mathbf{k}] \cdot \chi^T)$. □

Computation of the Mellin-Barnes integral

In this section we show that a convergent integral corresponding to a solution of a system of algebraic equations can be computed as a hypergeometric series, which coincides with a hypergeometric series representing solution of this system.

Consider a system of two trinomials

$$y_1^4 + x_1 y_1^2 y_2^{-1} - 1 = 0,$$

$$y_2^4 + x_2 y_1^{-1} y_2^2 - 1 = 0\,.$$

It is easy to check that this system satisfies the condition for the convergence of the Mellin-Barnes integral, which for the monomial of the solution $y^\mu(x)$ has the form

$$\frac{1}{(2\pi i)^2}\int\limits_{\gamma+i\mathbb{R}^2}\frac{\Gamma(u_1)\Gamma(u_2)\Gamma(\frac{1}{4}(\mu_1-2u_1+u_2))\Gamma(\frac{1}{4}(\mu_2+u_1-2u_2))}{\Gamma\left(\frac{1}{4}(\mu_1+2u_1+u_2)+1\right)\Gamma\left(\frac{1}{4}(\mu_2+u_1+2u_2)+1\right)}Q(u)x^{-u}du, \tag{16}$$

where

$$Q(u)=\frac{1}{16}\left(\mu_1\mu_2+\mu_1u_1+\mu_2u_2\right).$$

To compute the integral we use the principle of separating cycles of A. K. Tsikh. This principle applies for computation of integrals

$$\frac{1}{(2\pi i)^s}\int\limits_{\Delta_g}\frac{h(\mathbf{z})d\mathbf{z}}{f_1(\mathbf{z})\dots f_s(\mathbf{z})} \tag{17}$$

of the Grothendieck type where poles of the meromorphic integrand are associated to a proper holomorphic mapping $\mathbf{f}=(f_1,\dots,f_s):\mathbb{C}^s\to\mathbb{C}^s$, and the integration set Δ_g is the distinguished boundary of the polyhedron $\Pi_\mathbf{g}$ associated to another proper holomorphic mapping $\mathbf{g}=(g_1,\dots,g_s):\mathbb{C}^s\to\mathbb{C}^s$. When the mappings $\mathbf{f}$ and $\mathbf{g}$ coincide, the integral (17) is equal to a sum of Grothendieck residues of the integrand over all zeroes of $\mathbf{f}$ in Π_g. Indeed, in this case the distinguished boundary of Δ_g is homologous to a sum of local cycles separating local divisors $D_j = f_j = 0$, $j=1,\dots,s$, i.e. those cycles that are involved into definition of a local residue of Grothendieck. In the problem of representation of the integral (17) by a sum of local residues a principal role is played by the following notion.

Definition 1. *A polyhedron $\Pi_\mathbf{g}$ is called compatible with a family of hypersurfaces (divisors) $\{D_j\}$, if the jth facet of the polyhedron $\Pi_\mathbf{g}$ does not intersect D_j for all j, $j=1,\dots,s$.*

Theorem 4 (The principle of separating cycles)**.** *If a polyhedron $\Pi_\mathbf{g}$ is bounded in compatible with the family of polar divisors $\{D_j\}$, then the integral* (17) *is equal to a sum of Grothendieck residues in $\Pi_\mathbf{g}$.*

In case of an unbounded polyhedron, an additional condition of rapid decrease of the integrand in $\Pi_\mathbf{g}$ is required, similar to that in the classical Jordan lemma where instead of $\Pi_\mathbf{g}$ we have a half-plane. Such a condition is given in [8] and [7].

In the integral (16) the vertical subspace $\gamma+i\mathbb{R}^2$ can be seen as a distinguished boundary of some polyhedron. Note that in the case $N>1$ the number of such polyhedra is infinite. Our task is to divide polar hypersurfaces in (16) into 2 divisors and attach to $\gamma+i\mathbb{R}^2$ a polyhedron compatible with the obtained family.

As a polyhedron we take

$$\Pi = \{\mathrm{Re} u_1 < \gamma_1, \mathrm{Re} u_2 < \gamma_2\}.$$

The family of divisors we organize as follows: polar sets of Gamma functions $\Gamma(u_1)$ and $\Gamma\big(\frac{1}{4}(\mu_2 + u_1 - 2u_2)\big)$ (red and violet) we put into one divisor, while polar sets of $\Gamma(u_2)$ and $\Gamma\big(\frac{1}{4}(\mu_1 - 2u_1 + u_2)\big)$ (blue and green) into another. By gray lines we depict zeroes of the integrand (singularities of Gamma functions in the denominator and zeroes of $Q(u)$) (Fig. 1).

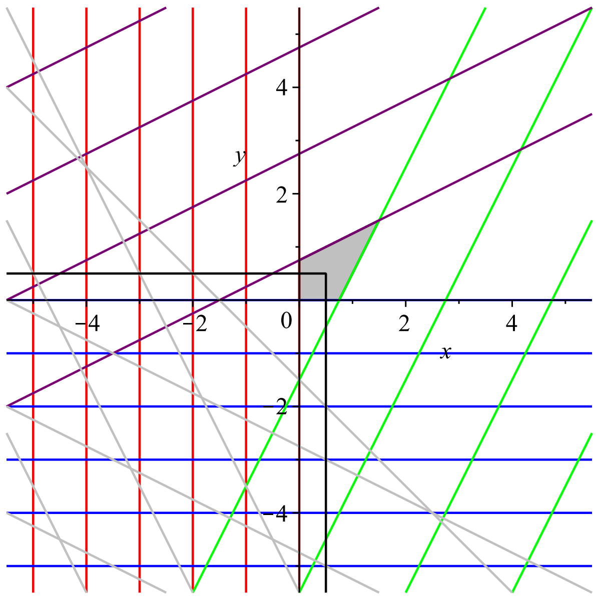

Fig. 1. Families of polar divisors of the integrand

It is easy to see that at the intersection points of oblique polar sets with vertical and horizontal ones inside the polyhedron the residue is 0. Thus, in this case the integral (16) is a sum of residues over all points of the lattice $\mathbb{Z}^2_{\leqslant 0}$ where the integrand has a pole of the first order.

A pole at a point of $\mathbb{Z}^2_{\leqslant 0}$ gives an expression $c_{\mathbf{k}}\mathbf{x}^{\mathbf{k}}$, where $c_{\mathbf{k}}$ is defined in Theorem 3 by the formula (10). Summing up over all points of $\mathbb{Z}_{\leqslant 0}$ we obtain the series from Theorem 3.

Thus, we have shown that the integral (3) under hypothesis of Theorem 1 has a non-empty convergence domain and represents a monomial of a solution of the system of algebraic equations (1). Moreover, computations show that the integral (16) coincides with the solution in the form of a hypergeometric series.

The research is carried out with the financial support of the RFBR, project no. 18-31-00193.

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Гипергеометрические ряды и интегралы Меллина-Барнса для нулей системы полиномов Лорана

Владимир Р. Куликов
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. В работе приведен критерий сходимости интеграла Меллина-Барнса, представляющего нули системы полиномов Лорана. Представлена формула в виде кратного ряда гипергеометрического типа.

Ключевые слова: интегралы Меллина-Барнса, гипергеометрические ряды, полиномы Лорана.

DOI: 10.17516/1997-1397-2020-13-1-97-103
УДК 538.9

Application of DMA 242 C for Quasi-Static Measurements of Piezoelectric Properties of Solids

Pavel P. Turchin*
Siberian Federal University
Krasnoyarsk, Russian Federation
Kirensky Institute of Physics
Federal Research Center KSC SB RAS
Krasnoyarsk, Russian Federation
Vladimir I. Turchin
Sergey V. Yurkevich
Pavel O. Sukhodaev
Irina S. Raikova
Siberian Federal University
Krasnoyarsk, Russian Federation

Received 09.09.2019, received in revised form 16.10.2019, accepted 06.12.2019

Abstract. An experimental device for quasi-static measurements of piezoelectric moduli d_{ijk}, based on the possibilities of precision variations in mechanical stresses with the device DMA 242 C in the frequency range 0-100 Hz has been developed. A special sample holder and a charge amplifier are used in the measuring scheme. The measurements of piezoelectric moduli values of trigonal piezoelectric single crystalls $La_3Ga_5SiO_{14}$ (P321) and $YAl_3(BO_3)_4$ (R32), as well as hexagonal ZnO ($P6_3mc$) have been carried out.

Keywords: piezoelectric modules, quasistatic method, multiferroics.
Citation: P.P.Turchin, V.I.Turchin, S.V.Yurkevich, P.O.Sukhodaev, I.S.Raikova, Application of DMA 242 C for Quasi-Static Measurements of Piezoelectric Properties of Solids, J. Sib. Fed. Univ. Math. Phys., 2020, 13(1), 97–103.
DOI: 10.17516/1997-1397-2020-13-1-97-103.

Introduction

A wide range of piezoelectric materials applications determines the development of physical methods for the experimental determination of their piezoelectric properties [1–4]. The resonance [1, 2, 5], ultrasonic [2, 5] and quasi-static [6, 7] methods for determining piezoelectric constants, along with measurements of direct [8] and inverse [9] piezoelectric effects in crystals are known. Depending on thermodynamic boundary conditions and point symmetry of a material, each of these methods has a set of ratios and cuts for the separate determination of piezoelectric constants. In particular, for ultrasound measurements of piezoelectric moduli e_{ijk} of a number of single crystals, such an analysis is carried out in [10, 11]. The complex of experimental tools and calculated ratios determine the accuracy and future prospects of a selected method.

The existing experimental methods are subjected to constant modifications and improvements due to the continuous development of piezoelectric applications [12], as well as appearance of new materials, for instance, multiferroics [13], and the necessity to study their magnetoelectric and other properties.

*pturchin@sfu-kras.ru

The possibilities for applying precision device DMA 242 C to create periodic mechanical stresses when determining piezoelectric moduli d_{ijk} by the quasi-static method are studied in this paper. The measured values of the piezoelectric constants of trigonal and hexagonal crystals are compared with the scientific literature data. The values of piezoelectric moduli of yttrium aluminum borate obtained by the authors with the application of the developed experimental method are given.

1. Experimental quasi-static method

DMA 242 C device [14] provides a precise change in the applied dynamic load with a frequency of 0–100 Hz to the sample in the range of 0–8 N with simultaneous possibility to control static load in the same range. The direction of load application coincides with the vertical axis of the device pusher.

The block diagram of the experimental method with the use of DMA 242 C for quasi-static measurements of piezoelectric moduli is given in Fig. 1. The test sample is placed in the measuring chamber of the device between two dielectric holders. The lower one is installed on the sample holder DMA 242 C 7, and the upper one is free, and the load from the device pusher is applied to it. The uniform distribution of mechanical stresses in the sample is achieved by applying lubricant between the sample and the holder. Electrodes for measuring the charge are applied on the horizontal facets of the sample for measuring the longitudinal piezoelectric effect and on the vertical ones for measuring transverse piezoelectric moduli. The electric surface charges of the piezoelectric sample under load are converted by the charge amplifier 4 to the voltage recorded by the oscilloscope 3. The value of the measured piezoelectric modulus is determined by the equation of state, and in this experimental diagram is determined according to the ratio

$$d_{i\lambda} = \frac{q}{F} = \frac{US_l}{K_a F S_e}, \tag{1}$$

where $d_{i\lambda}$ is the measured piezoelectric modulus, q is the charge on the electrodes, F is the dynamic force amplitude, U is the voltage amplitude at the output of the charge amplifier, S_l is the load application area, S_e is the surface area of the electrodes, K_a is conversion factor of the charge amplifier.

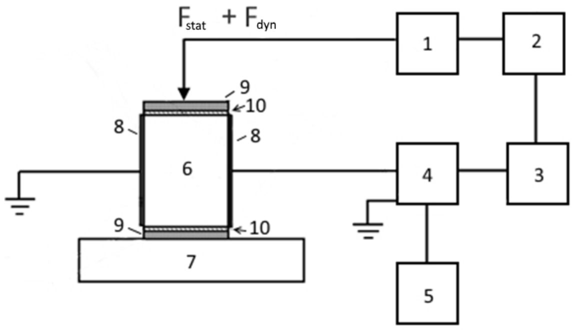

Fig. 1. Block diagram of the quasi-static method for measuring piezoelectric moduli. A variant of measurement of the transverse to the direction of the pressure of electric polarization is presented. 1 — DMA 242 C, 2 — personal computer, 3 — oscilloscope DPO 7104, 4 — charge amplifier LE-41, 5 — power supply, 6 — sample, 7 — sample holder, 8 — electrodes, 9 — dielectric holders, 10 — lubricant layer

The accuracy of the device performance and the piezoelectric moduli measurement were controlled by measuring piezoelectric constants under conditions of different dynamic loads and different measuring frequencies. Fig. 2 demonstrates linear dependence of the measured charge on the magnitude of the dynamic load, and Fig, 3 shows the constancy of the determined voltage for different measurement frequencies.

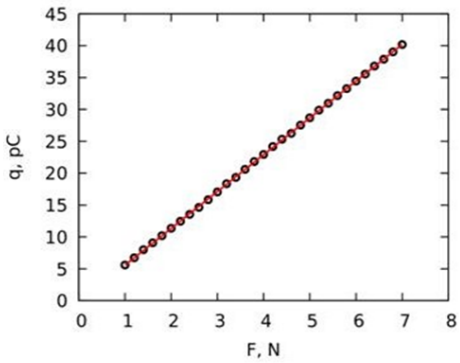

Fig. 2. Dependence of the charge values on the sample facets on the dynamic load magnitude

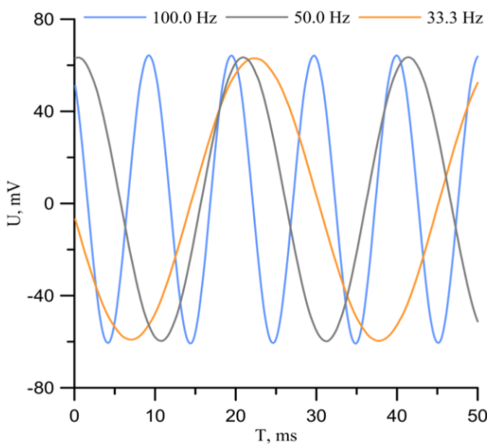

Fig. 3. Values of voltage U on the sample when measuring at different frequencies

2. Calculated ratios for determining piezoelectric moduli

The implemented quasi-static method has been applied to determine the piezoelectric moduli d_{ijk} of the well-studied single crystals of langasite $La_3Ga_5SiO_{14}$ [12] and zincite ZnO [15], as well as yttrium alumoborate $YAl_3(BO_3)_4$ [16], which belongs to the rare-earth oxiborates $RMe_3(BO_3)_4$ (where R=Y, La-Lu; Me=Fe, Al, Cr, Ga, Sc), but does not have a magnetic subsystem. Due to the latter circumstance, yttrium alumoborate is a kind of standard of electroelastic interaction for the whole family of oxiborates. However, there is no data on the piezoelectric properties of this single crystal in scientific literature.

Moduli d_{11} and d_{14} were determined for trigonal (point symmetry group 32) langasite and yttrium alumoborate, and for hexagonal (6mm) zinc oxide — d_{33}, d_{31}. The geometry of the samples required for the separate determination of these components is given in Fig. 4.

In quasi-static measurements, the longitudinal piezo modulus d_{11} was determined by applying mechanical compressive stress along axis X_1 and registering a charge in the same direction (Fig. 4 (a, b)). To determine piezoelectric moduli d_{33} and d_{31}, the mechanical compressive

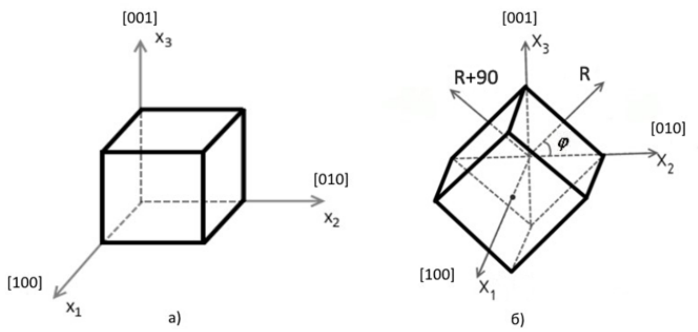

Fig. 4. Sample orientation for quasi-static measurements

stress was applied along X_3, and the charge was recorded along axes X_3 and X_1, respectively (Fig. 4 (a)). The piezo modulus d_{14} was found by determining the charge in the direction of axis X_1 when mechanical stresses were applied to facets R and R + 90 (Fig. 4 (b)). In this case, from the rule of the orthogonal tensor transformation [17, 18]

$$d_{i'j'k'} = a_{i'l}a_{j'm}a_{k'n}d_{lmn}, \tag{2}$$

for crystals of the point symmetry group 32 for the direction R we find

$$d_{12}^{R} = d_{14}sin\varphi cos\varphi - d_{11}cos^2\varphi, \tag{3}$$

for the direction R+90

$$d_{12}^{R+90} = -d_{14}sin\varphi cos\varphi - d_{11}sin^2\varphi. \tag{4}$$

The angle φ for langasit was 45 degrees, and for yttrium alumoborate 48.05 degrees. The sign of the piezoelectric modulus was found by determining the direction of the electric induction vector as related to the axes of the crystal-physical coordinate system, the selection rules of which are described in [18].

3. Experimental results

The linear dimensions of the samples were about 5–6 mm, the accuracy of the facets' orientation was not worse than ±3', the flatness of the opposite facets was 3 microns. The positive direction of the axis X_1 of the crystal-physical coordinate system for the crystals of symmetry 32 was chosen from acoustic measurements [12, 18], under condition that $C_{14}<0$. The sign of the piezoelectric constants was controlled by the direct measurement of the piezoelectric effect. Mechanical compressive stress was considered negative. All measurements were carried out at room temperature.

The obtained values of piezoelectric moduli are summarized in the Tab. 1 and compared with the data of other authors for langasite and zinc oxide. According to the Table, the experimental values of the determined piezoelectric moduli for zinc oxide and langasite correlate with the values of the piezoelectric moduli obtained by other authors. This fact suggests the correct operation of the device for quasi-static measurements of the piezoelectric moduli with the application of DMA 242 C and confirms the accuracy of determining piezoelectric moduli of yttrium alumoborate.

Conclusion

The carried out research has demonstrated the possibility to apply DMA 242 C to control dynamic loads on the crystal in the process of quasi-static measurements of single crystal piezo-

Table 1. Piezoelectric moduli

Modulus	Experimental value $d_{i\lambda}$, 10^{-12} C/N	Literature values $d_{i\lambda}$, 10^{-12} C/N	References
ZnO			
d_{31}	–5.2±0.2	–5.12	[19]
		–5.0	[20]
		–5.2	[21]
		–3.7	[22]
d_{33}	11.7±0.2	12.3	[19]
		12.4	[20]
		10.6	[21]
		8.0	[22]
$La_3Ga_5SiO_{14}$			
d_{11}	–6.1±0.2	–6.16	[23]
		6.25	[24]
		6.20	[25]
		6.1	[26]
d_{14}	–5.4±0.3	5.36	[23]
		–3.65	[24]
		5.4	[26]
$YAl_3(BO_3)_4$			
d_{11}	–6.0±0.3	–	–
d_{14}	–7.2±0.4	–	–

electric moduli. This application is based on the modification of the measuring cell of the device with the use of additional dielectric sample holders, as well as on the use of the standard charge amplifier in the measuring diagram. The accuracy and repeatability of the results are based on the creation of uniform mechanical stresses in the sample. The obtained values of the piezoelectric langasite and zincite moduli correspond to scientific literature data. The device developed on the basis of DMA 242 C was applied to determine the piezoelectric moduli of yttrium aluminum borate.

The reported study was funded by Russian Foundation for Basic Research project no. 18-42-240016, Government of Krasnoyarsk Territory, Krasnoyarsk Regional Fund of Science, to the research project: "Electromechanical Properties and Anisotropy of Acoustic Wave Propagation in Yttrium Aluminoborates Single Crystals".

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Применение DMA 242 C для квазистатических измерений пьезоэлектрических свойств твердых тел

Павел П. Турчин
Сибирский федеральный университет
Красноярск, Russian Federation
Институт физики им. Л. В. Киренского СО РАН
Красноярск, Российская Федерация
Владимир И. Турчин
Сергей В. Юркевич
Павел О. Суходаев
Ирина С. Райкова
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. Разработана экспериментальная установка для квазистатических измерений пьезомодулей d_{ijk}на основе возможностей прецизионных изменений механических напряжений прибором DMA 242 C в диапазоне частот 0-100 Гц. В измерительной схеме применен специальный держатель образцов и усилитель заряда. Выполнены измерения значений пьезомодулей тригональных пьезоэлектриков $La_3Ga_5SiO_{14}$ (P321) and $YAl_3(BO_3)_4$ (R32). Результаты исследований коррелируют с данными других авторов.

Ключевые слова: пьезоэлектрические модули, квазистатический метод, мультиферроики.

Journal of Siberian Federal University. Mathematics & Physics 2020, 13(1), 104–113

DOI: 10.17516/1997-1397-2020-13-1-104-113
УДК 512.554

Minimal Proper Quasifields with Additional Conditions

Olga V. Kravtsova*
Siberian Federal University
Krasnoyarsk, Russian Federation

Received 10.10.2019, received in revised form 22.11.2019, accepted 26.12.2019

Abstract. We investigate the finite semifields which are distributive quasifields, and finite near-fields which are associative quasifields. A quasifield Q is said to be a minimal proper quasifield if any of its sub-quasifield $H \neq Q$ is a subfield. It turns out that there exists a minimal proper near-field such that its multiplicative group is a Miller–Moreno group. We obtain an algorithm for constructing a minimal proper near-field with the number of maximal subfields greater than fixed natural number. Thus, we find the answer to the question: Does there exist an integer N such that the number of maximal subfields in arbitrary finite near-field is less than N? We prove that any semifield of order p^4 (p be prime) is a minimal proper semifield.

Keywords: quasifield, semifield, near-field, subfield.

Citation: O.V.Kravtsova, Minimal polynomials in finite semifields, J. Sib. Fed. Univ. Math. Phys., 2020, 13(1), 104–113.
DOI: 10.17516/1997-1397-2020-13-1-104-113.

1. Introduction and preliminaries

Closely related problems of classification and construction of projective translation planes and their coordinatizing quasifields have been studied from the beginning of the 20th century (Dickson [1], Veblen and Maclagan-Wedderburn [2]; see also [3, 4]). Recall that a set L with a binary operation $\circ$ is called a *loop* if L contains a neutral element and equations $a \circ x = b$ and $x \circ a = b$ are uniquely solvable for any $a, b \in L$ [5, 6]. So, a group is an associative loop. A set Q with binary operations of addition $+$ and multiplication $\cdot$ is called a *right quasifield* [3] if the following conditions are satisfied

1) $(Q, +)$ is an abelian group with zero 0,
2) $Q^* = (Q \setminus \{0\}, \cdot)$ is a loop with an identity e,
3) $x0 = 0$ for any $x \in Q$,
4) Q satisfies the right distributivity $(x + y)z = xz + yz$ for any $x, y, z \in Q$,
5) if $a, b, c \in Q$ and $a \neq b$ then the equation $xa = xb + c$ has an unique solution in Q.

A left quasifield is defined in the same way by replacing the right distributivity with the left distributivity. Any associative right quasifield is called a right *near-field*. Any distributive quasifield is called a *semifield*.

As for finite quasifields the following problems are studied (see also [7]).

(A) *Enumerate maximal subfields and their possible orders.*

*ol71@bk.ru https://orcid.org/0000-0002-6005-2393

(B) *Find finite quasifields Q with not-one-generated loop Q^*.*

The **hypothesis** is as follows: *a loop Q^* of any finite semifield Q is generated by one element.*

(C) *Define a spectra of a loop Q^* if Q is a finite quasifield or a semifield.*

(D) *Find the automorphism group $\operatorname{Aut} Q$.*

The problems were studied earlier for certain semifields and quasifields of small orders [7–9]. See also Theorem 3.4 in Section 3. In Section 2 of the paper the question **(A)** is studied on maximal subfields for the finite near-fields.

Clearly that a field is a trivial example of a quasifield. Any finite quasifield which is not a field is said to be a *proper* quasifield. A quasifield Q is called a *minimal proper quasifield* if any of its sub-quasifield $H \neq Q$ is a subfield. For instance, any of non-trivial quasifields of order p^2 (p is a prime number) is evidently a minimal proper quasifield. Therefore, by the well-known Zassenhaus theorem, studies of question **(A)** are reduced to Dickson near-fields.

According to Dancs [10, 11] and Felgner [12], the maximal subfield of Dickson near-field containing the center is unique. Certain near-fields have only two or three maximal subfields [13]. However, earlier V. M. Levchuk noted that the answer to the following question is unknown:

Does there exist an integer N such that the number of maximal subfields in arbitrary finite near-field is less than N?

The Dancs description of sub-near-fields in a Dickson near-field is used (see also [13]). Developing Dancs and Felgner approach, the method of construction of some minimal proper near-fields is proposed (Theorem 2.1). Main theorem 2.2 in Section 2 provides the negative answer to the question above even in the class of minimal proper near-fields.

In the case of a finite semifield (Section 3), it is proved that any semifield W of order p^4 is a minimal proper semifield, and any of its sub-semifields $H \neq W$ is a subfield of order p or p^2 (Theorem 3.3). A semifield of order $p^3 > 8$ is also a minimal proper semifield. According to Knuth's theorem [14], such semifield contains only the prime subfield.

2. Subfields in finite near-fields

First examples of finite near-fields were constructed by Dickson in 1906. All finite near-fields were described by Zassenhaus [15] in 1936. His construction of *Dickson near-field* is based on the special expansion of a Galois field $GF(q)$, $q = p^l$ for a prime p. The additive group of a Galois field $GF(q^n)$ is used and it is characterized by the *Dickson pair* (q, n), where

1) any prime divisor of n divides $q - 1$;

2) if $q \equiv 3 \pmod 4$ then $n \not\equiv 0 \pmod 4$.

By Zassenhaus theorem [15], *all finite near-fields are Dickson near-fields, except seven near-fields of order p^2 where $p = 5, 7, 11$ (two near-fields), 23, 29 and* 59 (see also [6]).

Clearly that the prime subfield $P = \{ke \mid k \in \mathbb{Z}\}$ of any finite near-field Q is in the *kernel*

$$K(Q) = \{x \in Q \mid x(y + z) = xy + xz,\ (y + z)x = yx + zx\ \forall y, z \in Q\}.$$

However, the center $Z(Q)$ is not necessary a subfield. In fact, it was shown [13, Th. 1] that *for any finite near-field the center coincides with the kernel except Zassenhaus near-fields Q of orders 5^2, 7^2, 11^2 and 29^2 with $|Z(Q^*)| = 2$, 2, 2 and 14, respectively.*

According to [16], the prime subfield is a unique maximal subfield in a near-field of order p^r for any prime number r. So, in this case the near-field is a minimal proper near-field, and question **(A)** is reduced to the case of Dickson near-fields, where $r = ln$ is not prime number.

The class of all Dickson near-fields of order q^n with the center $GF(q)$, $q = p^l$ is denoted by $DF(q, n)$. The well-known correspondence between the subfields in a Galois field $GF(p^m)$ and the divisors of m may be generalized to Dickson near-fields and their sub-near-fields (see [10,11]). The following lemma describes this generalized correspondence.

Lemma 1. *For any sub-near-field H of a Dickson near-field $Q \in DF(p^l, n)$ there are $h \,|\, (ln)$ and $0 < j \leqslant n$ such that $|H| = p^h$, $H \in DF(p^z, h/z)$, $z = \mathrm{GCD}(jl, h)$ and*

$$j \equiv \frac{p^{ln} - 1}{p^h - 1} \,(\bmod\, n). \tag{1}$$

Inversely, if $h \,|\, (ln)$ then Q contains the unique sub-near-field H of order p^h.

Felgner [12] proved that any Dickson near-field Q has the unique maximal subfield $M(Q)$ containing the center $Z(Q)$. By [13], if $|M(Q)| = q^\lambda$, then for the canonical decomposition of n and λ we have:

$$n = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r}, \qquad \lambda = p_1^{[n_1/2]} p_2^{[n_2/2]} \dots p_r^{[n_r/2]}.$$

Example 1. Let Q be any near-field of order 2^{180} from the class $DF(2^4, 45)$. Lemma 1 is used to construct the lattice of sub-near-fields of Q (see Fig. 1). The commutative sub-near-fields, i.e. subfields, are shown in colour. The near-field Q contains three maximal subfileds, their orders are 2^{45}, 2^{30} and $2^{12} = |M(Q)|$. Maximal sub-near-fields of orders 2^{90}, 2^{36}, 2^{60} are not subfields.

Further, examples of minimal proper Dickson near-fields Q will be given. Next we consider the following Lemma.

Lemma 2. *Let H be a sub-near-field of order p^h in a Dickson near-field $Q \in DF(p^l, n)$ and $H \in DF(p^z, h/z)$. Then $(h/z)|n$.*

Proof. It is enough to consider the case where $k = (ln)/h$ is a prime number. Let k divides n. Then $n = kn'$, $h = ln'$,

$$z = \mathrm{GCD}\,(jl, h) = \mathrm{GCD}\,(jl, ln') = l \cdot \mathrm{GCD}\,(j, n') = ln'',$$

where n'' divides n. So, we have

$$\frac{h}{z} = \frac{ln'}{ln''} = \frac{n'}{n''} | n.$$

Let k divides l. Then $l = kl'$, $h = l'n$,

$$z = \mathrm{GCD}\,(jl, h) = \mathrm{GCD}\,(jkl', l'n) = l' \cdot \mathrm{GCD}\,(jk, n) = l'n',$$

where n' divides n. So,

$$\frac{h}{z} = \frac{l'n}{l'n'} = \frac{n}{n'}$$

divides n. □

Let us denote the set of all prime divisors of $m \in \mathbb{N}$ by $\pi(m)$. Firstly, we consider the case of the minimal expansion degree $n = 2$.

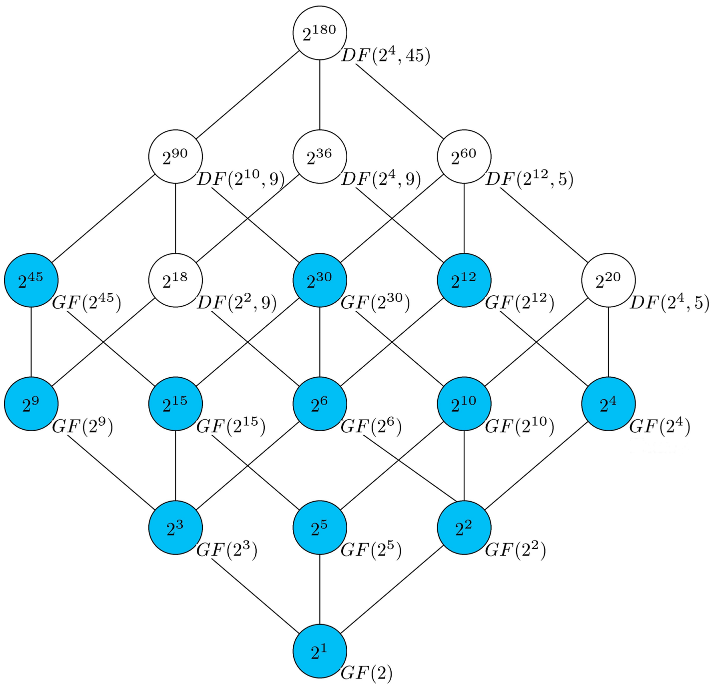

Fig. 1. Sub-near-fields lattice in the Dickson near-field of order 2^{180} with the center $GF(2^4)$

Lemma 3. *The center $Z(Q) \simeq GF(p^l)$ is the unique maximal subfield in any finite Dickson near-field $Q \in DF(p^l, 2)$.*

Proof. It is evident that $p > 2$ and $Z(Q)$ is a maximal subfield of Q. Let H be another maximal subfield of Q. Then $H \not\subset Z(Q)$ and $|H| = p^{2l'}$, where l' divides l. Let us consider the sub-near-fields sequence

$$Q = H_0 \supset H_1 \supset \cdots \supset H_{k-1} \supset H_k = H,$$

where $|H_i| = p^{h_i}$ and h_{i-1}/h_i are prime numbers. The maximality of subfield H leads to $H_k \in DF(p^{2l'}, 1)$ and $H_{k-1} \in DF(p^{l''}, 2)$, where $l'|l''$, $l''|l$ and $l''/l' = m$ is a prime number. Let us determine parameter j (1) for the sub-near-field H_k in H_{k-1} and obtain

$$j = \frac{p^{2l''} - 1}{p^{2l'} - 1} = \frac{p^{2ml'} - 1}{p^{2l'} - 1} = p^{2l'(m-1)} + p^{2l'(m-2)} + \cdots + p^{2l'} + 1 \equiv m (\bmod 2),$$

that is $j = 1$ if $m > 2$ and $j = 2$ if $m = 2$.

If $m > 2$ then $z = \mathrm{GCD}(jl'', h) = \mathrm{GCD}(l'm, 2l') = l'$ so $H_k \in DF(p^{l'}, 2)$ and H_k is not a subfield. This is contradictory to the supposition.

If $m = 2$ then $z = \mathrm{GCD}(jl'', h) = \mathrm{GCD}(2l'', 2l') = 2l'$. We have $H_k \in DF(p^{2l'}, 1)$ and $H_{k-1} \in DF(p^{2l'}, 2)$, where $2l'$ divides l, so H_k is in the center $Z(Q)$ and it is not a maximal subfield. This is contradictory to the supposition. □

If the expansion degree n is greater than two then one can choose the prime number p such that a Dickson near-field $Q \in DF(q, n)$ is a minimal proper near-field.

Theorem 2.1. *There exist infinitely many minimal proper near-fields $Q \in DF(q, n)$ for any fixed prime number $n > 2$.*

Proof. Let $n > 2$ be a prime number. Let us consider the field $GF(n)$ and choose its primitive element p_0, $p_0^{n-1} \equiv 1(\bmod n)$ and $p_0^m \not\equiv 1(\bmod n)$ for any $0 < m < n-1$. The arithmetical progression $\{p_0 + nt\}_{t=0}^{\infty}$ contains infinitely many prime numbers. Let $p = p_0 + nt$ be one of them. Then (p^{n-1}, n) is a Dickson pair. Indeed,

$$p^{n-1} = (p_0 + nt)^{n-1} \equiv p_0^{n-1} \equiv 1(\bmod n),$$

that is, n divides $q - 1 = p^{n-1} - 1$. Now let Q be any near-field from the class $DF(p^{n-1}, n)$. Let us consider all its maximal sub-near-fields. Number n is a prime number. It is clear that the center $Z(Q) \simeq GF(p^{n-1})$ is a maximal sub-near-field in Q. Suppose that $H \neq Z(Q)$ is another maximal sub-near-field of Q. Then $|H| = p^h$, where $h = nl'$ and $k = (n-1)/l'$ is a prime number. Let us determine parameters j and z (1) for H and obtain

$$j \equiv \frac{p^{(n-1)n} - 1}{p^{l'n} - 1}(\bmod n),$$

$$p^{(n-1)n} - 1 \equiv 0(\bmod n), \qquad p^n \equiv p(\bmod n),$$
$$p^{l'n} - 1 = (p^n)^{l'} - 1 \equiv p^{l'} - 1(\bmod n) \not\equiv 0(\bmod n),$$

so $j = n$. Further, $z = \mathrm{GCD}(jl, h) = \mathrm{GCD}(n(n-1), nl') = nl' = h$ and $H \in DF(p^h, 1)$, that is, H is a subfield of Q. So, all maximal sub-near-fields of Q are subfields, and their number is equal to $|\pi(n-1)| + 1$. □

The following theorem proposes a method to construct the minimal proper near-field where the number of maximal subfields is greater than any fixed integer.

Theorem 2.2. *For any $s \in \mathbb{N}$ there exists a minimal proper Dickson near-field that has more than s maximal subfields.*

Proof. Let s be any integer. Let us consider the product of s different prime numbers $N = r_1 \cdot r_2 \cdot \dots \cdot r_s$. Then the arithmetical progression $\{1 + Nt\}_{t=1}^{\infty}$ contains infinitely many prime numbers. Let $n = 1 + Nt_0$ be one of them. According to Theorem 2.1, one can choose the prime number p such that the class $DF(p^{n-1}, n)$ contains a minimal proper near-field Q. The number of maximal subfields in Q is equal to $1 + |\pi(n-1)| \geqslant 1 + s$. □

Example 2. Using these results, one can give an example of a minimal proper near-field with five maximal subfields. Let $n = 2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211$. It is a prime number. The Galois field $GF(211)$ contains the primitive element 3: $3^{210} \equiv 1(\bmod 211)$ and $3^m \not\equiv 1(\bmod 211)$ for any $0 < m < 210$. Then the near-field $Q \in DF(3^{210}, 211)$ contains five subfields H_i of orders 3^{h_i}, $i = 1, \dots, 5$, where

$$h_1 = \frac{210 \cdot 211}{2}, \quad h_2 = \frac{210 \cdot 211}{3}, \quad h_3 = \frac{210 \cdot 211}{5}, \quad h_4 = \frac{210 \cdot 211}{7}, \quad h_5 = \frac{210 \cdot 211}{211}.$$

Indeed, the calculation of j and z (1) shows that $j = n$ and $z = h_i$ so $h_i/z = 1$ and $H_i \simeq GF(3^{h_i})$. Numbers n/h_i are all prime numbers so these subfields are maximal sub-near-fields in Q.

A minimal proper near-field with exactly one maximal subfield is also determined.

Lemma 4. *A Dickson near-field Q is a minimal proper near-field that has unique maximal subfield iff Q is from one of classes $DF(p, r)$, $DF(p, r^2)$, $DF(p^r, r)$, where p and r are possible prime number.*

Proof. According to [16], in a near-field $Q \in DF(p,r)$ the center $\mathbb{Z}_p$ is a unique maximal subfield, and Q has no another sub-near-fields by Lemma 1. For a near-field $Q \in DF(p,r^2)$ or $Q \in DF(p^r,r)$ of order p^{r^2} the unique maximal sub-near-field is the subfield $M(Q)$ because $\lambda = r$. Inversely, let $Q \in DF(p^l,n)$ satisfies the condition. It is clear that ln is a degree of one prime number, $ln = r^t$. The maximal subfield $M(Q)$ which contains the center $GF(p^l)$ has the order p^λ, where $\lambda = r^{[t/2]}$. If $H \neq Q$ is a maximal sub-near-field in Q of order $p^{r^{t-1}}$ then $H = M(Q)$. So, $t = 1$ or $t = 2$. The case $DF(p^r,1)$ is evidently corresponds to the field $GF(p^r)$. □

Clearly, that the multiplicative group Q^* of the minimal proper near-field $Q \in DF(2^2,3)$ is a *Miller–Moreno group* [17]. On the other hand, the multiplicative group Q^* of the near-field Q from Example 2 is not a Miller–Moreno group.

3. Subfields in finite semifields

Let $\langle W,+,\circ\rangle$ be a semifield of order p^n (p is a prime number). The universal method to determine a finite semifield (see, for example, [3,18]) is to introduce n-dimensional linear space over the field $\mathbb{Z}_p$ with a multiplication law

$$x \circ y = x \cdot \theta(y) \qquad (x, y \in W).$$

Here, θ is an injective linear mapping from W to $GL_n(p) \cup \{0\}$ with the property $\theta(e) = E$ (the identity matrix) for some vector $e \in W$ (neutral under the multiplication $\circ$). Then, the set $R = \{\theta(y) \mid y \in W\}$ is called a *spread set* of a semifield W. The notation $W = W(n,p,\theta)$ is used. Elements of the prime subfiled $P \simeq \mathbb{Z}_p$ correspond to the scalar matrices $kE = \theta(k \circ e) \in R$. Note that $k \circ a$ ($k \in \mathbb{N}$, $a \in W$) is the sum of k items equal to a. According to the definition of a semifield, the following result is evident (see also [19]).

Lemma 5. *Let W be a semifield of order p^n, and $R \subset GL_n(p) \cup \{0\}$ is its spread set. Then, for any non-scalar matrix $A \in R$ the characteristic polynomial $\chi_A(x) \in \mathbb{Z}_p[x]$ has no linear divisors $x - \lambda$.*

Proof. Indeed, let $A = \theta(a)$, $a \in W$, and $x - \lambda$ divides $\chi_A(x)$. If $b \in W$ is a correspondent eigenvector then

$$b\theta(a) = \lambda b, \quad b\theta(a - \lambda \circ e) = 0, \quad b \circ (a - \lambda \circ e) = 0, \quad b \neq 0.$$

So, we have $a = \lambda \circ e$ because a semifield has no zero divisors. □

In what follows the results on minimal polynomials in finite semifields which were proved in [20] are used. For any polynomial $f(x) \in \mathbb{Z}_p$,

$$f(x) = c_m x^m + c_{m-1}x^{m-1} + \cdots + c_2x^2 + c_1x + c_0 \quad (c_i \in \mathbb{Z}_p,\ i = 0,1,\ldots,m),$$

and any element $a \in W$ the *right-* and *left-ordered value* of the polynomial are defined:

$$f(a)) = c_m \circ a^{m)} + c_{m-1} \circ a^{m-1)} + \cdots + c_2 \circ a^2 + c_1 \circ a + c_0 \circ e,$$
$$f((a) = c_m \circ a^{(m} + c_{m-1} \circ a^{(m-1} + \cdots + c_2 \circ a^2 + c_1 \circ a + c_0 \circ e.$$

Here, $a^{s)}$ and $a^{(s}$ are the *right-* and *left-ordered degrees* of an element a, respectively. They are determined inductively by the rule

$$a^{s)} := a^{s-1)} \circ a, \quad a^{(s} := a \circ a^{(s-1}, \quad a^{1)} := a = a^{(1}.$$

Evidently, in the case of degree $\leqslant 2$, the right- and the left-ordered values $f(a))$ and $f((a)$ are equal.

The *right-ordered minimal polynomial* of an element $a \in W(n,p,\theta)$ is said to be a monic polynomial $\mu_a^r(x) \in \mathbb{F}_p[x]$ of the minimal degree such that $\mu_a^r(a)) = 0$. The *left-ordered minimal polynomial* $\mu_a^l(x)$ is defined in a similar way. According to [20], we have

Lemma 6. *If $a \in W(n,p,\theta)$ and $A = \theta(a)$ then the right-ordered minimal polynomial of an element a is a factor of the minimal polynomial of the matrix A.*

Now consider semifields of small orders p^3 and p^4 and their subfields. It is well-known [14] that a semifield of order p^2 or 8 is a field. So, it is clear that any semifield of order $p^3 > 8$ is a minimal proper semifield. Let us specify the possible orders of subfields in such a semifield.

Lemma 7. *Let W be a semifield of order p^n with the multiplicative identity e. If a non-zero element $a \in W$ has the right-ordered minimal polynomial $\mu_a^r(x) \in \mathbb{Z}_p[x]$ then $deg(\mu_a^r) = 1$ iff a belongs to the prime subfield P and $deg(\mu_a^r) = 2$ iff $K = \{\alpha_1 \circ e + \alpha_2 \circ a \mid \alpha_1, \alpha_2 \in \mathbb{Z}_p\}$ is a subfield in W of order p^2.*

Proof. The first proposition is evident. Let $deg(\mu_a^r) = 2$. Then the system of vectors e, a is linear independent over $\mathbb{Z}_p$, $a^2 \in K$, so $|K| = p^2$. Moreover, K is closed with respect to multiplication and multiplication in K is associative. Inversely, if K is a subfield of order p^2 then $a \notin P$ and $a^2 \in K$. □

Corollary 1. *Let W be a semifield of order p^n. The subset of elements with the minimal polynomial of degree 1 or 2 (together with 0) is the union of all subfields of order p^2 in W.*

Let us note that for a sub-semifield (or a subfield) U of order p^m in a semifield W of order p^n the condition $m|n$ need not be satisfied. This fact can be explained by the absence of multiplicative associativity: in general a semifield W is not a linear space over U. Moreover, a finite semifield may contain more than one sub-semifields (subfields) of the same order.

For example, there exists the semifield of order 32 containing the subfield of order 4, and also semifields of order 81 with three disjoint subfields of order 9 (see [7, 9, 21]).

The evident examples of subfields in the finite semifields are the *left, middle and right nuclei* [3]

$$\begin{aligned} N_l &= \{x \in W \mid x \circ (y \circ z) = (x \circ y) \circ z \ \forall y, z \in W\}, \\ N_m &= \{x \in W \mid y \circ (x \circ z) = (y \circ x) \circ z \ \forall y, z \in W\}, \\ N_r &= \{x \in W \mid y \circ (z \circ x) = (y \circ z) \circ x \ \forall y, z \in W\}, \end{aligned}$$

the nucleus $N = N_l \cap N_m \cap N_r$ and the center $Z = \{z \in N \mid z \circ x = x \circ z \ \forall x \in W\}$. Let us consider now another example of a semifield of order p^4 with a subfield of order p^2 (see [18]).

Lemma 8. *Let W be a semifield of order p^4 and φ be an involutory automorphism of W. Then the stabilizer $U = \{x \in W \mid \varphi(x) = x\}$ is a subfield of order p^2.*

It is natural to assume that for a semifield of order p^n any sub-semifield is of order p^m, where $m \leqslant n/2$. Let us show that it is true, at least, for the semifields of order p^3 and p^4.

Theorem 3.3. *For a semifield W of order p^n, where $n = 3$ or $n = 4$, any proper sub-semifield is a subfield of order p^m, $m \leqslant n/2$.*

Proof. Let W be a semifield of order p^3, U be its subfield of order p^2, and the element $a \in U$ does not belong to the prime subfield P. Then, its minimal polynomial $\mu_a(x) \in \mathbb{Z}_p[x]$ is of degree two and it divides the minimal polinomial $\mu_A(x)$ of the correspondent matrix $A = \theta(a) \in GL_3(p)$ from the spread set. Then, the characteristic polynomial $\chi_A(x)$ of matrix A has a linear factor that is impossible.

Let W be a semifield of order p^4, U be its sub-semifield of order p^3. It was proved above that it does not contain the subfields of order p^2. So, any of its elements $a \in U$, not from the prime subfield P, has the right-ordered minimal polynomial $\mu_a^r(x) \in \mathbb{Z}_p[x]$ of degree 3. Then, the characteristic polynomial $\chi_A(x)$ of the correspondent matrix $A = \theta(a) \in GL_4(p)$ from the spread set has a linear factor. $\square$

One can generalize the obtained result using the notion of the *right-cyclic* semifield. An element a of a semifield W of order p^n is called *right-cyclic* over $\mathbb{Z}_p$, if elements

$$e, a, a^2, a^{3)}, \ldots, a^{n-1)}$$

form a base of W as a n-dimensional linear space over $\mathbb{Z}_p$. So, the semifield W is called *right-cyclic* over $\mathbb{Z}_p$. A left-cyclic element and a left-cycllic semifield are defined in a similar way. Let us note that all known up to now finite semifields are right- and left-cyclic even non-primitive semifields of order 32 and 64 (see, for example, [19,22,23] and [7].

Corollary 2. *A semifield W of order p^n contains no right-cyclic over $\mathbb{Z}_p$ sub-semifields of order p^{n-1}.*

Proof. It is enough to consider the right-ordered minimal polynomial of a right-cyclic element a of a sub-semifield of order p^{n-1}. The characteristic polynomial of correspondent matrix $A = \theta(a)$ from a spread set has a linear factor. $\square$

Let us now illustrate these results by the examples of semifields of order 5^4 and 13^4 with additional condition to autotopisms. Remind that the triple of automorphisms $\langle \alpha, \beta, \gamma \rangle$ of the additive group $(W, +)$ is called an *autotopism* of a semifield W if for all $x, y \in W$ the equality $x^\alpha \circ y^\beta = (x \circ y)^\gamma$ is satisfied. It is simple to prove (see [18]) that fixed α and γ defines the automorphism β.

Let W be a semifield of order p^4 (p is a prime number, $p \equiv 1 (\operatorname{mod} 4)$) determined as a 4-dimensional linear space over $\mathbb{Z}_p$. Now consider its mappings

$$\begin{aligned} &\alpha_1 : (x_1, x_2, x_3, x_4) \to (-ix_1, -ix_2, ix_3, ix_4), \\ &\alpha_2 : (x_1, x_2, x_3, x_4) \to (-x_3, -x_4, x_1, x_2), \qquad x_j \in \mathbb{Z}_p,\ j = 1, 2, 3, 4, \end{aligned} \tag{2}$$

where $i \in \mathbb{Z}_p$, $i^2 = -1$. Let $\sigma_1 = \langle \alpha_1, \beta_1, \alpha_1 \rangle$, $\sigma_2 = \langle \alpha_2, \beta_2, \alpha_2 \rangle$ be the autotopism of W, where α_1 and α_2 are defined by (2), and $H = \langle \sigma_1, \sigma_2 \rangle$ be the autotopism subgroup. Then, H is isomorphic to the quaternion group Q_8. It can be verified by direct calculation. Let us denote the numbers of non-isomorphic and non-isotopic semifields of order p^4 admitting H by $n(p)$ and $n'(p)$, respectively. Questions **(A)**–**(D)** from the introduction can be solved with the use of computer constructions.

Theorem 3.4. *Let W be a semifield of order p^4, where $p = 5$ or $p = 13$, which admit an autotopism subgroup $H = \langle \sigma_1, \sigma_2 \rangle \simeq Q_8$. Then W is not commutative. It is left- and right-primitive, and it has the center of order p and the left nucleus N_l of order p^2, and*

$$n(5) = 9, \quad n'(5) = 3, \qquad n(13) = 99, \quad n'(13) = 33.$$

The number of maximal subfields of order p^2 in W equals 1, 2 or $p+2$. The automorphism group $Aut\, W$ is the cyclic group $\mathbb{Z}_2$ or $\mathbb{Z}_{p+1}$.

Note that any such semifield is a minimal proper semifield but it may contain more than one subfiled of order p^2. It is anomalous property in comparison with the properties of finite fields and finite near-fields. The theorem does not concern the question **(C)** on the spectra of elements of multiplicative loop because of its complicated statement. But it is another anomalous property of finite semifields that the spectra contain the integers which does not divide the order of W^*.

This work was funded by Russian Foundation for Basic Research (project no. 19-01-00566 A).

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Минимальные собственные квазиполя с дополнительными условиями

Ольга В. Кравцова
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. Мы рассматривем конечные полуполя, то есть дистрибутивные квазиполя, и конечные почти-поля, то есть ассоциативные квазиполя. Квазиполе Q называем минимальным собственным квазиполем, если всякое его подквазиполе $H \neq Q$ является подполем. Оказывается, существует минимальное собственное почти-поле, мультипликативная группа которого есть группа Миллера–Морено. Найден алгоритм построения минимального собственного почти-поля, в котором количество максимальных подполей больше любого заданного числа. Таким образом, получен ответ на вопрос: существует ли такое натуральное число N, что количество максимальных подполей в произвольном почти-поле меньше N? Доказано, что всякое полуполе порядка p^4 (p – простое) есть минимальное собственное полуполе.

Ключевые слова: квазиполе, полуполе, почти-поле, подполе.

DOI: 10.17516/1997-1397-2020-13-1-114-130
УДК 517.55+512.761

Analytic Continuation for Solutions to the System of Trinomial Algebraic Equations

Irina A. Antipova[*]
Ekaterina A. Kleshkova[†]
Vladimir R. Kulikov[‡]
Siberian Federal University
Krasnoyarsk, Russian Federation

Received 05.10.2019, received in revised form 09.11.2019, accepted 24.12.2019

Abstract. In the paper, we deal with the problem of getting analytic continuations for the monomial function associated with a solution to the reduced trinomial algebraic system. In particular, we develop the idea of applying the Mellin-Barnes integral representation of the monomial function for solving the extension problem and demonstrate how to achieve the same result following the fact that the solution to the universal trinomial system is polyhomogeneous. As a main result, we construct Puiseux expansions (centred at the origin) representing analytic continuations of the monomial function.

Keywords: algebraic equation, analytic continuation, Puiseux series, discriminant locus, Mellin–Barnes integral.

Citation: I.A.Antipova, E.A. Kleshkova, V.R.Kulikov, Analytic Continuation for Solutions to the System of Trinomial Algebraic Equations, J. Sib. Fed. Univ. Math. Phys., 2020, 13(1), 114–130.
DOI: 10.17516/1997-1397-2020-13-1-114-130.

Introduction

We consider a system of n trinomial algebraic equations of the form

$$\sum_{\alpha\in A^{(i)}} a_\alpha^{(i)} y^\alpha = 0,\ i = 1,\dots,n, \tag{1}$$

with the unknown $y = (y_1,\dots,y_n) \in (\mathbb{C}\setminus 0)^n$ and variable coefficients $a_\alpha^{(i)}$, where $A^{(i)} \subset \mathbb{Z}^n$ are fixed three-element subsets and $y^\alpha = y_1^{\alpha_1}\cdot\ldots\cdot y_n^{\alpha_n}$ is a monomial. Without loss of generality we assume, that all sets $A^{(i)}$ contain the zero element $\bar{0}$ (this may be achieved by dividing the ith equation in (1) by a monomial with the exponent in $A^{(i)}$, see the system (2) below). We call (1) the *universal trinomial system* since any trinomial algebraic system is a result of the substitution of polynomials in new variables for coefficients $a_\alpha^{(i)}$.

When $n = 1$, the system (1) is a scalar trinomial equation. It has a special place in the centuries-old history of algebraic equations. As early as 1786, Bring proved that every quintic polynomial could be reduced to the trinomial form $y^5 + ay + b$ using the Tschirnhaus transformation. At the turn of the XIX–XX centuries, the dependence of norms of roots on coefficients

[*]iantipova@sfu-kras.ru https://orcid.org/0000-0003-1382-0799
[†]ekleshkova@gmail.com https://orcid.org/0000-0002-7443-2979
[‡]v.r.kulikov@mail.ru https://orcid.org/0000-0002-2291-0449

of the trinomial equation with fixed support was actively studied. Although algebraic characterisation of the mentioned dependence was given by Bohl already in 1908, the geometric view on the problem has been formed much later. In the recent study by Theobald and de Wolff [15], a geometrical and topological characterisation for the space of univariate trinomials was provided by reinterpreting the problem in terms of the amoeba theory.

Of particular interest is the *reduced system* of n trinomial equations

$$y^{\omega^{(i)}} + x_i y^{\sigma^{(i)}} - 1 = 0, \ i = 1, \dots, n, \tag{2}$$

with the unknown $y = (y_1, \dots, y_n)$, equation supports $A^{(i)} := \{\omega^{(i)}, \sigma^{(i)}, \overline{0}\} \subset \mathbb{Z}^n_{\geqslant}$ and variable complex coefficients $x = (x_1, \dots, x_n)$. It is assumed that a matrix ω composed of column vectors $\omega^{(1)}, \dots, \omega^{(n)}$ is nondegenerate.

Let $y(x) = (y_1(x), \dots, y_n(x))$ be a multivalued algebraic vector-function of solutions to the system (2). We call a branch of $y(x)$ defined by conditions $y_i(0) = 1, \ i = 1, \dots, n$ the *principal solution* to the system (2). Having determined the principal solution $y(x)$, we consider the following monomial function

$$y^d(x) := y_1^{d_1}(x) \cdot \ldots \cdot y_n^{d_n}(x), \ d = (d_1, \dots, d_n) \in \mathbb{Z}^n_+. \tag{3}$$

Our goal is to obtain Puiseux expansions (centred at the origin) representing analytic continuations of the Taylor series for the monomial $y^d(x)$ of the principal solution to the system (2). Puiseux type parameterizations of an algebraic variety via the amoeba of the discriminant locus of the variety canonical projection were studied in [6]. The existence of such parameterizations for plane curves was proved by Puiseux [12]: this fact is known as the Newton-Puiseux theorem which states that one can find local parameterizations of the form $x = t^k$, $y = \varphi(t)$, where φ is a convergent power series. We aim at investigating Puiseux expansions for analytic continuations of (3) which may fail to "recognize" some pieces of the discriminant set. It means that the convergence domain G of a series projects onto the domain $\mathrm{Log}(G)$ containing a certain collection of connected components of the discriminant amoeba complement. An example in Section 1 illustrates how the series converging in the preimage $\mathrm{Log}^{-1}(E_0)$ of the component E_0 of the amoeba complement admits an analytic continuation to the domain G for which $\mathrm{Log}(G)$ covers components E_1, E_2 and an amoeba tentacle separating them, see Fig. 1. This analytic continuation is given by another series expansion.

When $n = 1$, analytic continuations for the Taylor series of the principal solution to the universal algebraic equation (not necessarily a trinomial) were found in [3], where the Mellin-Barnes integral representation for the solution was used as a tool of the analytic continuation. This integral, with indicating the convergence region of it, was wholly studied in [2]. While a power series converges in a polycircular domain, a Mellin-Barnes integral converges in a sectorial domain which is defined only by conditions for arguments $\arg x_i$ of variables x_i. Remark that the intersection of these domains is always nonempty. Consequently, a series expansion of the solution to the equation admits an analytic continuation into the sectorial domain by means of the integral. Of course, we may follow this approach to getting analytic extensions for the monomial (3) in a case when the corresponding Mellin-Barnes integral represents it. Herein we can obtain analytic continuations of the Taylor series in the form of Puiseux series via the multidimensional residues technique.

However, we can get the same series following the fact that the solution $y(a)$ to the system (1) is polyhomogeneous. This means that via some monomial transformation of coefficients the system (1) can be reduced to the form (2) or to another system which, similarly, has only one variable coefficient in each equation. We perceive any reduced system of equations as the general (homogeneous) system (1) written in suitable coordinates. The transition from one reduced system to another enables us to obtain series continuations for monomials of coordinates of solutions to these systems.

The paper is organized as follows. In Section 1 we review the technique of the calculation of multidimensional Mellin-Barnes integrals which is based on the separating cycle principle formulated in [16] (see also [17]). We present an example which illustrates what computational issues can arise in this way of getting analytic extensions. In Section 2 we discuss the procedure of the dehomogenization (reduction) of the system (1) and obtain the Taylor series expansions for the monomials of the principal solutions to all reduced systems associated with the system (1). Theorem 1 gives these series as a result of the application of the logarithmic residue formula [5] and the linearization procedure for each reduced system. The idea of using the logarithmic residue formula for getting the Taylor expansions was developed in [8], where the special instance of the reduced polynomial system with the diagonal matrix ω was considered. In Section 3 we use Taylor expansions derived in Theorem 1 and appropriate monomial transformations to obtain the desired Puiseux series which are supposed to be the analytic continuations of the Taylor series for the monomial $y^d(x)$ of the principal solution to (2) (Theorem 2). Finally, we discuss the example from Section 1 again in terms of the result of Theorem 2.

1. Mellin-Barnes integral as a tool of analytic continuation

Traditionally, the Mellin-Barnes integrals are regarded as the inverse Mellin transform for special meromorphic functions, which are rations of products of a finite number of superpositions of gamma functions with affine functions. Their role in the theory of algebraic equations was revealed first by Mellin in [9], where he wrote down without any proof the integral representation for the solution to the universal algebraic equation later investigated in [2]. In our study we consider such integrals in the extended sense, having in mind the presence of a polynomial factor in the integrand besides gamma-functions.

The Mellin integral transform for monomials of a solution to the reduced polynomial system was studied in [1] and [14]. Following [14], we associate the Mellin-Barnes integral

$$\frac{1}{(2\pi i)^n}\int\limits_{\gamma+i\mathbb{R}^n}\prod_{j=1}^{n}\frac{\Gamma(z_j)\Gamma\Big(\frac{d_j}{\omega_j}-\frac{1}{\omega_j}\langle\sigma_j,z\rangle\Big)}{\Gamma\Big(\frac{d_j}{\omega_j}-\frac{1}{\omega_j}\langle\sigma_j,z\rangle+z_j+1\Big)}Q(z)x^{-z}\,dz \tag{4}$$

with the monomial $y^d(x)$. In (4) x^{-z} denotes the product $x_1^{-z_1}\cdot\ldots\cdot x_n^{-z_n}$, σ_j is the jth row of the matrix σ composed of column vectors $\sigma^{(1)},\ldots,\sigma^{(n)}$, γ belongs to the domain

$$U=\{u\in\mathbb{R}^n_+:\langle\sigma_j,u\rangle<d_j,\ j=1,\ldots,n\},$$

and $Q(z)$ is a polynomial represented by the determinant

$$Q(z)=\frac{1}{\det\omega}\det\Big|\Big|\delta_i^j(d_j-\langle\sigma_j,z\rangle)+\sigma_j^{(i)}z_i\Big|\Big|_{i,j=1}^n,$$

where δ_i^j is the Kronecker symbol. Here it is assumed that ω is a diagonal matrix with elements $\omega_1,\ldots,\omega_n$ on the diagonal.

Remark that the integral (4) can have the empty convergence domain. It follows from [7] that its convergence domain is nonempty if and only if all the diagonal minors of the matrix σ

are positive. In this case, the integral (4) represents the monomial $y^d(x)$ of the principal solution to the system (2), and it can be used as a tool of the constructive analytic continuation of power series.

Let us show how to calculate the integral (4). A method of the calculation is based on the separating cycle principle formulated in [16] and developed in [17]. This principle deals with the calculation of the Grothendieck-type integrals

$$\frac{1}{(2\pi i)^n} \int\limits_{\Delta_g} \frac{h(z)dz}{f_1(z)\dots f_n(z)}, \tag{5}$$

where the integration set Δ_g is the skeleton of the polyhedron Π_g associated with the holomorphic proper mapping $g : (g_1, \dots, g_n) : \mathbb{C}^n \to \mathbb{C}^n$, and the integrand has poles on divisors $D_j = \{z : f_j(z) = 0\}$, $j = 1, \dots, n$. The polyhedron Π_g is the preimage $g^{-1}(G)$ of the domain $G = G_1 \times \dots \times G_n$, where each G_j is a domain on the complex plane with the piecewise smooth boundary. We associate a facet $\sigma_j = \{z : g_j(z) \in \partial G_j,\ g_k(z) \in G_k,\ k \neq j\}$ of the polyhedron Π_g with $j \in \{1, \dots, n\}$.

Definition. *A polyhedron Π_g is said to be compatible with the set of divisors $\{D_j\}$, if for each $j = 1, \dots, n$ the corresponding facet σ_j of the polyhedron Π_g does not intersect the divisor D_j .*

Assume further that the intersection $Z = D_1 \cap \dots \cap D_n$ is discrete. The local residue with respect to the family of divisors $\{D_j\}$ at each point $a \in Z$ (the Grothendieck residue) is defined by the integral (see [16])

$$\mathrm{res}_{f,a}\Omega = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_a(f)} \Omega,$$

where Ω is the integrand in (5), and $\Gamma_a(f)$ is a cycle given in the neighborhood U_a of the point a as follows

$$\Gamma_a(f) = \{z \in U_a : |f_1(z)| = \varepsilon_1, \dots, |f_n(z)| = \varepsilon_n\}, \quad \varepsilon_j << 1.$$

If a is a simple zero of the mapping f, i.e. the Jacobian $J_f = \partial f / \partial z$ is nonzero at the point a, then the local residue is calculated by the formula

$$\mathrm{res}_{f,a}\Omega = \frac{h(a)}{J_f(a)}. \tag{6}$$

Theorem 1 (principle of separating cycles)**.** *If the polyhedron Π_g is bounded and compatible with the family of polar divisors $\{D_j\}$, then the integral (5) equals to the sum of Grothendieck residues in the domain Π_g.*

One can reduce the integral (4) to the canonical form (5) in the following way. We interpret the vertical integration subspace $\gamma + i\mathbb{R}^n$ as the skeleton of some polyhedron. For instance, in the case $n = 1$, it can be the skeleton of only two polyhedra: the right and left halfplanes with the separating line $\gamma + i\mathbb{R}$. For $n > 1$ this subspace may serve as the skeleton of an infinite number of polyhedra. Our objective is to divide all the set of $2n$ families of polar hyperplanes of the integral (4)

$$\begin{aligned} &L_j : z_j = -\nu, \\ &L_{n+j} : \frac{d_j}{\omega_j} - \frac{1}{\omega_j}\langle \sigma_j, z\rangle = -\nu, \quad j = 1, \dots, n, \quad \nu \in \mathbb{Z}_{\geqslant} \end{aligned}$$

into n divisors and construct a polyhedron compatible with this family of divisors. We consider polyhedra of the type

$$\Pi_g = \{z \in \mathbb{C}^n : \mathrm{Re} g_j(z) < r_j,\ j = 1, \dots, n\},$$

where $g_j(z)$ are linear functions with real coefficients. It is clear that $\Pi_g = \pi + i\mathbb{R}^n$ where π is a simplicial n-dimensional cone in the real subspace $\mathbb{R}^n \subset \mathbb{C}^n$. Remark that in the case of an unbounded polyhedron, besides the compatibility condition of the polyhedron and polar divisors, one should require a sufficiently rapid decrease of the integrand Ω in the polyhedron Π_g. For the integral (4) the nonconfluence property provides the decrease of the integrand, see [10] and [17]. We recall that the nonconfluence property for the hypergeometric Mellin-Barnes integral means that sums of coefficients of the variable z_j over all gamma-factors in the numerator and the denominator are equal.

Now, applying the technique discussed above, we construct analytic continuations for the solution to the following system of equations

$$\begin{cases} y_1^4 + x_1 y_1^2 y_2 - 1 = 0, \\ y_2^4 + x_2 y_1 y_2^2 - 1 = 0. \end{cases} \tag{7}$$

For the description of the convergence domains of power series and Mellin-Barnes integrals we introduce the following mappings from $(\mathbb{C} \setminus 0)^n$ into $\mathbb{R}^n$:

$$\mathrm{Log} : (x_1, \dots, x_n) \longrightarrow (\log|x_1|, \dots, \log|x_n|)\,,$$

$$\mathrm{Arg} : (x_1, \dots, x_n) \longrightarrow (\arg x_1, \dots, \arg x_n)\,.$$

The monomial $y_1(x) \cdot y_2(x)$ of the principle solution to the system (7) admits the Taylor series representation

$$\sum_{|k| \geqslant 0} \frac{(-1)^{|k|}}{k!} \frac{\Gamma\left(\frac{1}{4} + \frac{1}{2}k_1 + \frac{1}{4}k_2\right)\Gamma\left(\frac{1}{4} + \frac{1}{4}k_1 + \frac{1}{2}k_2\right)}{\Gamma\left(\frac{5}{4} - \frac{1}{2}k_1 + \frac{1}{4}k_2\right)\Gamma\left(\frac{5}{4} + \frac{1}{4}k_1 - \frac{1}{2}k_2\right)} \frac{1}{16}(1 + k_1 + k_2) x_1^{k_1} x_2^{k_2}, \tag{8}$$

which converges in some neighborhood of the origin, see Theorem 1 below. In turn, the Mellin-Barnes integral of the form

$$\frac{1}{(2\pi i)^2} \int\limits_{\gamma + i\mathbb{R}^2} \frac{\Gamma(z_1)\Gamma(z_2)\Gamma\left(\frac{1}{4} - \frac{1}{2}z_1 - \frac{1}{4}z_2\right)\Gamma\left(\frac{1}{4} - \frac{1}{4}z_1 - \frac{1}{2}z_2\right)}{\Gamma\left(\frac{5}{4} + \frac{1}{2}z_1 - \frac{1}{4}z_2\right)\Gamma\left(\frac{5}{4} - \frac{1}{4}z_1 + \frac{1}{2}z_2\right)} \frac{(1 - z_1 - z_2)}{16} x^{-z} dz, \tag{9}$$

where γ is a point in the open quadrangle

$$U = \left\{u \in \mathbb{R}^2_+ : 2u_1 + u_2 < 1,\ u_1 + 2u_2 < 1\right\},$$

represents the monomial $y_1(x) \cdot y_2(x)$ in a sectorial domain $\mathrm{Arg}^{-1}(\Theta)$ determined by

$$\Theta = \left\{(\theta_1, \theta_2) \in \mathbb{R}^2 :\ |\theta_1| < \frac{\pi}{2},\ |\theta_2| < \frac{\pi}{2}, |2\theta_2 - \theta_1| < \frac{3\pi}{4}, |\theta_2 - 2\theta_1| < \frac{3\pi}{4}\right\}, \tag{10}$$

here $\theta_1 = \arg x_1, \theta_2 = \arg x_2$. Fig. 2 shows the domain Θ which is the interior of the convex octagon. The general description of convergence domains of multiple Mellin-Barnes integrals gives Theorem 4.4.25 in the book [13]. Thus, the integral (9) gives the analytic continuation of the series (8) into the sectorial domain $\mathrm{Arg}^{-1}(\Theta)$.

We next calculate the integral (9) using the principle of separating cycles. It admits a representation as a sum of local residues of the integrand

$$\Omega = \frac{\Gamma(z_1)\Gamma(z_2)\Gamma\left(\frac{1}{4}-\frac{1}{2}z_1-\frac{1}{4}z_2\right)\Gamma\left(\frac{1}{4}-\frac{1}{4}z_1-\frac{1}{2}z_2\right)}{\Gamma\left(\frac{5}{4}+\frac{1}{2}z_1-\frac{1}{4}z_2\right)\Gamma\left(\frac{5}{4}-\frac{1}{4}z_1+\frac{1}{2}z_2\right)}\frac{(1-z_1-z_2)}{16}x_1^{-z_1}x_2^{-z_2}dz_1\,dz_2 \tag{11}$$

in some polyhedron, which contains the vertical imagine integration subspace $\gamma + i\mathbb{R}^2$ as the skeleton. Furthermore, the polyhedron and polar divisors of Ω should satisfy the compatibility conditions.

The form Ω has four families of polar complex lines:

$$\begin{aligned} &L_1 : z_1 = -\nu,\\ &L_2 : z_2 = -\nu,\\ &L_3 : \frac{1}{4}-\frac{1}{4}(2z_1+z_2) = -\nu,\\ &L_4 : \frac{1}{4}-\frac{1}{4}(z_1+2z_2) = -\nu,\ \ \nu\in\mathbb{Z}_{\geq}. \end{aligned} \tag{12}$$

Figs. 3 and 4 show the intersection of the real subspace with families (12), and also with

$$\begin{aligned} &L_5 : \frac{5}{4}+\frac{1}{2}z_1-\frac{1}{4}z_2 = -\nu,\\ &L_6 : \frac{5}{4}-\frac{1}{4}z_1+\frac{1}{2}z_2 = -\nu, \end{aligned}$$

which are polar sets of gamma-functions in the denominator of the form (11). The quadrangle U, to which the point γ belongs, is coloured in grey.

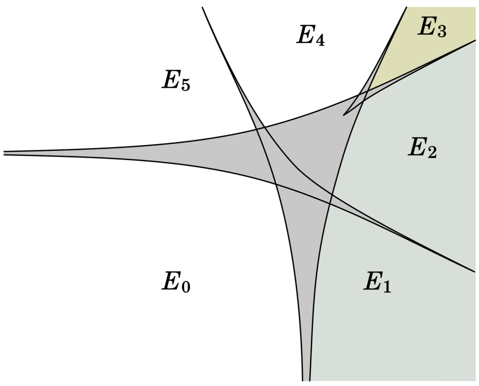

Fig. 1. The discriminant amoeba of the system (7) and its complement components E_ν

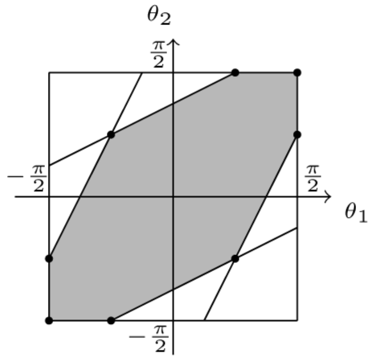

Fig. 2. The domain Θ

First, given the set of all polar lines of the integrand Ω, we form two divisors $D_1 = \{L_2, L_3\}$ and $D_2 = \{L_1, L_4\}$. We next construct a polyhedron $\Pi_1 = \pi_1 + i\mathbb{R}^2$ compatible with this set of divisors, with the skeleton $\gamma + i\mathbb{R}^2$. Fig. 3 shows a two-dimensional cone (sector) $\pi_1 \subset \mathbb{R}^2$ generated by rays which are parallel to the real sections of L_3 and L_4. It forms the polyhedron Π_1. Second, we consider divisors $D_1^{'} = \{L_3, L_4\}$ and $D_2^{'} = \{L_2\}$. A cone π_2 generated by rays which are parallel to the real sections of L_3 and L_2 forms a polyhedron $\Pi_2 = \pi_2 + i\mathbb{R}^2$, compatible with the set of divisors $D_1^{'}$, $D_2^{'}$, see Fig. 4.

We can see in Fig. 3 that families L_5, L_6 as well as L_1, L_2, L_3, L_4 come into the polyhedron Π_1, so in the cone π_1 there are points at which two, three and even four lines intersect. However, the form Ω has nonzero residues only at points $z(k) = (z_1(k), z_2(k))$ with coordinates

$$\begin{aligned} z_1(k) &= \frac{1}{3} + \frac{8}{3}k_1 - \frac{4}{3}k_2, \\ z_2(k) &= \frac{1}{3} - \frac{4}{3}k_1 + \frac{8}{3}k_2,\ k = (k_1, k_2) \in \mathbb{Z}^2_{\geqslant}. \end{aligned} \tag{13}$$

The intersection points (13) of lines L_3, L_4 are indicated in Fig. 3 by a black colour. Hence, the sum of local residues at points $z(k)$ yields the Puiseux series

$$P_1(x) = \frac{1}{x_1^{1/3} x_2^{1/3}} \sum_{k \in \mathbb{Z}^2_{\geqslant}} c_k x_1^{-8/3k_1 + 4/3k_2} x_2^{4/3k_1 - 8/3k_2} \tag{14}$$

with coefficients

$$c_k = \frac{(-1)^{|k|}}{k!} \frac{\Gamma\left(\frac{1}{3} + \frac{8}{3}k_1 - \frac{4}{3}k_2\right)\Gamma\left(\frac{1}{3} - \frac{4}{3}k_1 + \frac{8}{3}k_2\right)}{\Gamma\left(\frac{4}{3} + \frac{5}{3}k_1 - \frac{4}{3}k_2\right)\Gamma\left(\frac{4}{3} - \frac{4}{3}k_1 + \frac{5}{3}k_2\right)} \frac{1}{9}(1 - 4k_1 - 4k_2). \tag{15}$$

Four families of lines L_2, L_3, L_4 and L_5 come into the polyhedron Π_2, see Fig. 2. However, the form Ω has nonzero residues only at points $z(k) = (z_1(k), z_2(k))$ with coordinates

$$\begin{aligned} z_1(k) &= \frac{1}{2} + 2k_1 + \frac{1}{2}k_2, \\ z_2(k) &= -k_2,\ k = (k_1, k_2) \in \mathbb{Z}^2_{\geqslant}. \end{aligned} \tag{16}$$

Points (16) are black in Fig. 4, where lines L_2, L_3 intersect. The sum of residues at $z(k)$ yields the Puiseux series

$$P_2(x) = \frac{1}{x_1^{1/2}} \sum_{k \in \mathbb{Z}^2_{\geqslant}} c_k x_1^{-2k_1 - 1/2k_2} x_2^{k_2} \tag{17}$$

with coefficients

$$c_k = \frac{(-1)^{|k|}}{k!} \frac{\Gamma\left(\frac{1}{2} + 2k_1 + \frac{1}{2}k_2\right)\Gamma\left(\frac{1}{8} - \frac{1}{2}k_1 + \frac{3}{8}k_2\right)}{\Gamma\left(\frac{3}{2} + k_1 + \frac{1}{2}k_2\right)\Gamma\left(\frac{7}{8} - \frac{1}{2}k_1 - \frac{5}{8}k_2\right)} \frac{1}{16}(1 - 4k_1 + k_2). \tag{18}$$

We remark that arguments of Γ–functions in coefficients of the series (8) and also in (15) and (18) can be real nonpositive numbers, which are poles for the function Γ. So, by a ration of two Γ–functions we mean a meromorphic function with removable singularities at those points. For instance, we mean

$$\frac{\Gamma(-1)}{\Gamma(0)} = \frac{\Gamma(-1)}{-\Gamma(-1)} = -1.$$

So, series (14) and (17) are analytic extensions of the series (8).

We now characterize domains of convergence of Puiseux series obtained above in the logarithmic scale. According to the two-sided Abel lemma for hypergeometric series [10], there exists a relationship between the structure of the convergence domain of this series and its support. Since series (14) and (17) represent branches of the multivalued algebraic function $y_1(x) \cdot y_2(x)$ with singularities on the discriminant set of the system (7), projections of convergence domains of such series on the space of variables $\log|x_1|$, $\log|x_2|$ are unions of several components of the discriminant amoeba complement, see Fig. 1. We recall that the amoeba of the algebraic set

$V \subset \mathbb{C}^n$ is defined to be the image of V under the mapping Log. In this way, the series (14) converges in the domain $G_1 = \mathrm{Log}^{-1}(E_3)$, where E_3 is an amoeba complement component. The projection $\mathrm{Log}(G_2)$ of the convergence domain G_2 of the series (17) covers two components E_1, E_2 and an amoeba tentacle separating them, see Fig. 1.

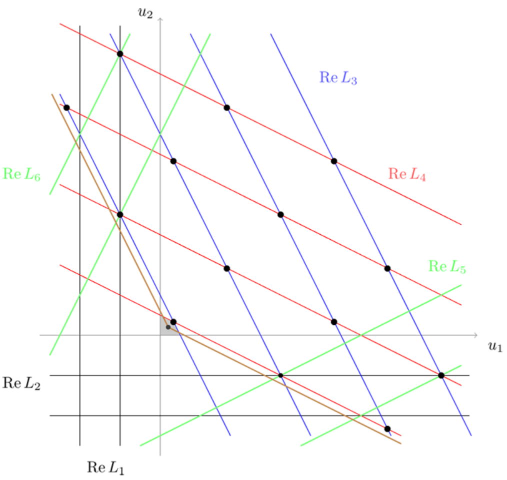

Fig. 3. The real section of polar divisors. The cone π_1

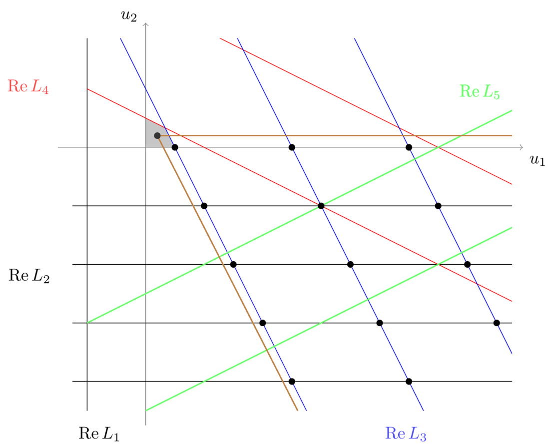

Fig. 4. The real section of polar divisors. The cone π_2

2. Taylor series for monomials of solutions to reduced systems

We consider the system of n trinomials (1) with unknowns $y_1, \ldots, y_n$, variable coefficients $a = (\ldots, a_\alpha^{(i)}, \ldots)$ and the set of supports $A^{(1)}, \ldots, A^{(n)}$, the same as the system (2) has.

Let us denote by A the disjunctive union of sets $A^{(i)}$. It consists of $3n$ elements, and we interpret it as the $(n \times 3n)$ – matrix

$$A = \left(A^{(1)}, \ldots, A^{(n)}\right) = (\alpha^1, \ldots, \alpha^{3n}),$$

with columns α^k which are exponents of monomials of the system (1). We order elements $\alpha \in A$, and, correspondingly, coefficients $a_\alpha^{(i)}$, $\alpha \in A$ of the system (1). The set of coefficients $a = (a_\alpha)$ is a vector space $\mathbb{C}^A \simeq \mathbb{C}^{3n}$.

The system (1) can be reduced by an appropriate change of coefficients in such a way that only one variable coefficient remains in each equation, and the other ones will be constant as in the system (2). Herein, supports $A^{(1)}, \ldots, A^{(n)}$ remain the same, and the solution to the system (1) can be restored by the solution to any reduced system. On the whole, the reduction procedure (dehomogenization) of the system is based on the polyhomogeniety property of the solution $y(a) = (y_1(a), \ldots, y_n(a))$, which can be expressed as follows:

$$y\left(\ldots \lambda_0^{(i)} \lambda^\alpha a_\alpha^{(i)} \ldots\right) = \left(\lambda_1^{-1} y_1\left(..a_\alpha^{(i)}..\right), \ldots, \lambda_n^{-1} y_n\left(..a_\alpha^{(i)}..\right)\right), \tag{19}$$

where $\lambda_0 = \left(\lambda_0^{(1)}, \ldots, \lambda_0^{(n)}\right)$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C} \setminus 0)^n$, see [4].

In each set $A^{(i)}$ we fix a pair of elements $\mu^{(i)},\ \nu^{(i)}$ and form the $n\times n$-matrix

$$\varkappa := \left(\mu^{(1)}-\nu^{(1)},\dots,\mu^{(n)}-\nu^{(n)}\right) \tag{20}$$

with columns $\mu^{(i)}-\nu^{(i)}$. The matrix $\varkappa$ is assumed to be nondegenerate. Each fixed set of n pairs $\mu^{(i)}$, $\nu^{(i)}$ corresponds to the reduced system of trinomials

$$r^{(i)}_{\beta^{(i)}}y^{\beta^{(i)}}+y^{\mu^{(i)}}-y^{\nu^{(i)}}=0,\ \ i=1,\dots,n, \tag{21}$$

with new unknown $y=(y_1,\dots,y_n)$, variable coefficients $r=\left(r^{(i)}_{\beta^{(i)}}\right)\in\mathbb{C}^n$ and $\beta^{(i)}\in A^{(i)}$. In each set $A^{(i)}$, we can choose an unordered pair $\mu^{(i)},\nu^{(i)}$ in three ways. Hence, we consider at most 3^n ways of the reduction of the system (1) to the form (21). If $\mu^{(i)}=\omega^{(i)}$, $\nu^{(i)}=\overline{0}$ and $\beta^{(i)}=\sigma^{(i)}$ for all $i\in\{1,\dots,n\}$, then we get the system (2).

Consider a branch of the solution to the system (21) under condition $y_i(0)=1$ and call it the principle solution. For the vector $d=(d_1,\dots,d_n)\in\mathbb{R}^n_+$ we introduce the monomial function $y^d(r):=y_1^{d_1}(r)\cdot\ldots\cdot y_n^{d_n}(r)$ of coordinates of the principal solution to the system (21). Concerning the system (21), we use the following notations: β is the matrix formed of columns $\beta^{(i)}$ and $\overline{\beta}$ is the matrix with columns $\left(\beta^{(i)}-\nu^{(i)}\right)$. Moreover, the symbol $\Gamma(b)$ we will use for the short writing of the product $\prod\limits_{k=1}^{n}\Gamma\left(b_k\right)$, where $b=(b_1,\dots,b_n)$ is a vector. The diagonal matrix with components of the vector b on the main diagonal we denote by $\mathrm{diag}[b]$ and the I denotes the vector with unit coordinates.

Theorem 1. *The monomial $y^d(r)$ of the principle solution to the system (21) admits the Taylor series representation with coefficients*

$$c_k=\frac{(-1)^{|k|}}{k!}\frac{\Gamma\left(\varkappa^{-1}d+\varkappa^{-1}\overline{\beta}k\right)}{\Gamma\left(\varkappa^{-1}d+\varkappa^{-1}\overline{\beta}k-k+I\right)}Q(k),\ \ k\in\mathbb{Z}^n_{\geqslant}, \tag{22}$$

where $Q(k)$ is the determinant of the matrix $\left(\mathrm{diag}\left[\varkappa^{-1}d+\varkappa^{-1}\overline{\beta}k\right]-\varkappa^{-1}\overline{\beta}\ \mathrm{diag}\left[k\right]\right)$, $k!:=k_1!\cdot\ldots\cdot k_n!$ and $\mid k\mid:=k_1+\ldots+k_n$.

Proof. Following [4], we carry out the linearization of the system (21). For that we regard (21) as a system of equations in the space $\mathbb{C}^n_r\times\mathbb{C}^n_y$ with coordinates $r=(r^{(i)}_{\alpha^{(i)}})$, $y=(y_1,\dots,y_n)$, and introduce in $\mathbb{C}^n\times\mathbb{C}^n$ the change of variables $(\xi,W)\to(r,y)$ by setting

$$y=W^{-\varkappa^{-1}},\ \ r=\xi\odot W^{\varkappa^{-1}\bar{\beta}-E}, \tag{23}$$

where $\xi=(\xi_1,\dots,\xi_n)$, $W=(W_1,\dots,W_n)$, $\odot$ denotes the Hadamard (coordinate-wise) product and E is the unit matrix. As a result of this change of variables, the system (21) can be written in the vector form as follows

$$W=\xi+I. \tag{24}$$

Equations of the system (24) are linear, so the change of variables (23) is called the linearization. Coordinates of the solution to the system (21) in new variables $\xi=(\xi_1,\dots,\xi_n)$, $W=(W_1,\dots,W_n)$ take the form

$$y_j(r(\xi))=(W_1,\dots,W_n)^{-\left(\varkappa^{-1}\right)^{(j)}},$$

where $W_i=1+\xi_i$, $\left(\varkappa^{-1}\right)^{(j)}$ is the jth column of the inverse matrix $\varkappa^{-1}$ for the matrix $\varkappa$.

We represent the inversion $\xi(r)$ of the linearization (23) as an implicit mapping given by the following set of equations

$$F(\xi, r) = (F_1(\xi, r), \ldots, F_n(\xi, r)) = \xi \odot W^{\varkappa^{-1}\bar{\beta}-E} - r = 0. \tag{25}$$

Calculate the vector $y(\xi)$ at the value of the mapping $\xi(r)$. To this end, following the idea implemented in [8] for a system of polynomials with a diagonal matrix ω, we apply the logarithmic residue formula, see [5, Th. 20.1, 20.2]. It yields the following integral

$$y^d(r) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\varepsilon} \frac{y^d(\xi)\Delta(\xi)d\xi}{F(\xi, r)},$$

where $\Gamma_\varepsilon = \{\xi \in \mathbb{C}^n : |\xi_j| = \varepsilon, j = 1, \ldots, n\}$, $\Delta(\xi)$ is the Jacobian of the mapping (25) with respect to ξ and $F(\xi, r)$ denotes the product $F_1(\xi, r) \cdot \ldots \cdot F_n(\xi, r)$. The radius ε we choose in such a way that the corresponding polycylinder lies outside the zero set of the Jacobian $\Delta(\xi)$.

Lemma 1. *The Jacobian of the mapping $F(\xi, r)$ with respect to ξ is*

$$\Delta(\xi) = W^{(\varkappa^{-1}\bar{\beta})I-2I} \det\left(E + \mathrm{diag}[\xi]\varkappa^{-1}\bar{\beta}\right).$$

Proof. The jth component of the mapping $F(\xi, r)$ has the following form:

$$F_j = F_j(\xi, r) = \xi_j \prod_{k=1}^{n} W_k^{(\varkappa^{-1}\bar{\beta}-E)_k^{(j)}} - r_j.$$

The calculation of the derivative of F_j with respect to ξ_j looks as follows:

$$\frac{\partial F_j}{\partial \xi_j} = \prod_{k=1}^{n} W_k^{(\varkappa^{-1}\bar{\beta}-E)_k^{(j)}} + \xi_j(\varkappa^{-1}\bar{\beta} - E)_j^{(j)} \prod_{k=1}^{n} W_k^{(\varkappa^{-1}\bar{\beta}-E)_k^{(j)}-\delta_k^j} =$$
$$=(1 + \xi_j(\varkappa^{-1}\bar{\beta})_j^{(j)}) \prod_{k=1}^{n} W_k^{(\varkappa^{-1}\bar{\beta})_k^{(j)}-2\delta_k^j},$$

and the derivative with respect to ξ_i, when $i \neq j$, is equal to

$$\frac{\partial F_j}{\partial \xi_i} = \xi_j(\varkappa^{-1}\bar{\beta} - E)_i^{(j)} \prod_{k=1}^{n} W_k^{(\varkappa^{-1}\bar{\beta})_k^{(j)}-\delta_k^j-\delta_k^i},$$

where δ_k^j, δ_k^i denote the Kronecker symbols.

Extracting common factors in the rows and columns of the obtained determinant, we get the assertion of the lemma. □

Remark that at the origin the Jacobi matrix for the mapping $F(\xi, r)$ is the unit matrix. Hence, the Jacobian $\Delta(\xi)$ does not vanish in the neighborhood of the origin and conditions of Theorems 20.1, 20.2 from [5] hold.

The monomial $y^d(r)$ after the change of variables takes the following form:

$$y^d(\xi) = W^{-\varkappa^{-1}d}.$$

Consequently, application of the logarithmic residue formula yields the integral representation:

$$y^d(r) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\varkappa^{-1}d+(\varkappa^{-1}\bar{\beta})I-2I}}{F(\xi, r)} \det\left(E + \mathrm{diag}[\xi]\varkappa^{-1}\bar{\beta}\right) d\xi. \tag{26}$$

Expand the kernel of the integral (26) into a multiple geometric series. To this end, we use the coordinate notations:

$$y^d(r) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\varkappa^{-1}d+(\varkappa^{-1}\bar\beta)I-2I}}{\prod\limits_{j=1}^{n}\left(\xi_j \prod\limits_{k=1}^{n} W_k^{(\varkappa^{-1}\bar\beta-E)_k^{(j)}} - r_j\right)} \det\left(E+\operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right) d\xi =$$

$$= \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\varkappa^{-1}d+(\varkappa^{-1}\bar\beta)I-2I}}{W^{(\varkappa^{-1}\bar\beta)I-I}\prod\limits_{j=1}^{n}\xi_j\left(1-\frac{r_j}{\xi_j\prod\limits_{k=1}^{n}W_k^{(\varkappa^{-1}\bar\beta-E)_k^{(j)}}}\right)} \det\left(E+\operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right) d\xi =$$

$$= \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\varkappa^{-1}d-I}}{\prod\limits_{j=1}^{n}\xi_j\left(1-\frac{r_j}{\xi_j\prod\limits_{k=1}^{n}W_k^{(\varkappa^{-1}\bar\beta-E)_k^{(j)}}}\right)} \det\left(E+\operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right) d\xi.$$

Since there exists such a number δ that for all $\xi\in\Gamma_\varepsilon$ and $\|r\|<\delta$ the inequality

$$\frac{r_j}{\xi_j\prod\limits_{k=1}^{n}W_k^{(\varkappa^{-1}\bar\beta-E)_k^{(j)}}} < 1$$

is valid, the integral (26) admits the following representation:

$$y^d(r) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_\varepsilon} \frac{W^{-\varkappa^{-1}d-I}\det\left(E+\operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right)}{\prod\limits_{j=1}^{n}\xi_j} \left(\sum_{k\in\mathbb{Z}^n_\geqslant}\prod_{j=1}^{n}\left(\frac{r_j}{\xi_j W^{(\varkappa^{-1}\bar\beta-E)^{(j)}}}\right)^{k_j}\right) d\xi.$$

Changing the order of summation and integration in the last integral, we get the series

$$y^d(r) = \sum_{k\in\mathbb{Z}^n_\geqslant}\left(\frac{1}{(2\pi i)^n}\int\limits_{\Gamma_\varepsilon}\frac{W^{-\varkappa^{-1}(d+\bar\beta k)+k-I}}{\xi^{k+I}}\det\left(E+\operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right)d\xi\right) r^k.$$

The coefficient c_k of the series is determined by the expression in parentheses. It can be calculated by the Cauchy integral formula. As a result, we get:

$$c_k = \frac{1}{k!}\frac{\partial^k}{\partial\xi^k}\left(W^{-\varkappa^{-1}(d+\bar\beta k)+k-I}\det\left(E+\operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right)\right)\Big|_{\xi=0}.$$

We bring the factor $W^{-\varkappa^{-1}(d+\bar\beta k)+k-I}$ into the determinant in such a way that each row of it still to depend on one variable ξ_j. We obtain

$$c_k = \frac{1}{k!}\frac{\partial^k}{\partial\xi^k}\det\left(\operatorname{diag}\left[W^{\operatorname{diag}\left[-\varkappa^{-1}(d+\bar\beta k)+k-I\right]}\right]\times\left(E+\operatorname{diag}[\xi]\varkappa^{-1}\bar\beta\right)\right)\Big|_{\xi=0}.$$

We next use the multilinearity property of the determinant and the fact that each row depends only on one variable ξ_j. As a result, we have

$$c_k = \frac{1}{k!}\det\left\|\frac{\partial^{k_j}}{\partial\xi_j^{k_j}}W_j^{\left(-\varkappa^{-1}(d+\bar\beta k)\right)_j+k_j-1}\left(\delta_i^j+\xi_j(\varkappa^{-1}\bar\beta)_j^{(i)}\right)\Big|_{\xi_j=0}\right\|_{i,j=1}^{n}.$$

Finally, we perform calculations in the above determinant:

$$\frac{\partial^{k_j}}{\partial \xi_j^{k_j}} W_j^{\left(-\varkappa^{-1}(d+\bar{\beta}k)\right)_j + k_j - 1} \left(\delta_i^j + \xi_j (\varkappa^{-1}\bar{\beta})_j^{(i)} \right) \Bigg|_{\xi_j=0} =$$

$$=(-1)^{k_j} \left(\left(\varkappa^{-1}(d+\bar{\beta}k)\right)_j \delta_i^j - k_j (\varkappa^{-1}\bar{\beta})_j^{(i)} \right) \prod_{m=1}^{k_j-1} \left(\left(\varkappa^{-1}(d+\bar{\beta}k)\right)_j - k_j + m \right) =$$

$$=(-1)^{k_j} \frac{\Gamma\left(\left(\varkappa^{-1}(d+\bar{\beta}k)\right)_j\right)}{\Gamma\left(\left(\varkappa^{-1}(d+\bar{\beta}k)\right)_j - k_j + 1\right)} \left(\left(\varkappa^{-1}(d+\bar{\beta}k)\right)_j \delta_i^j - k_j (\varkappa^{-1}\bar{\beta})_j^{(i)} \right).$$

Taking out the common factor in each row of the determinant and taking into account the factor $\frac{1}{k!}$, we get the view of the coefficient c_k declared in formula (22). □

Coefficients of the Taylor series for the monomial $y^d(x)$ of the principal solution to the system (2) one can find by formula (22) setting $\varkappa = \omega$, $\overline{\beta} = \sigma$. Thus, the series is as follows:

$$y^d(x) = \sum_{k \in \mathbb{Z}_{\geqslant}^n} \frac{(-1)^{|k|}}{k!} \frac{\Gamma(\omega^{-1}d + \omega^{-1}\sigma k)}{\Gamma(\omega^{-1}d + \omega^{-1}\sigma k - k + I)} P(k) x^k, \tag{27}$$

where $P(k) = \det\left(\mathrm{diag}\left[\omega^{-1}d + \omega^{-1}\sigma k\right] - \omega^{-1}\sigma \ \mathrm{diag}\,[k]\right)$.

3. Puiseux series

We fix n couples $\mu^{(i)}$, $\nu^{(i)} \in A^{(i)}$ of exponents of the system (2) and compose the matrix

$$\varkappa = \left(\varkappa_j^{(i)}\right) = \left(\mu_j^{(i)} - \nu_j^{(i)}\right),$$

assuming that it is nondegenerate. In accordance with the choice of the set of pairs $\mu^{(i)}$, $\nu^{(i)}$, let us devide the set $\{1, \dots, n\}$ on three disjoint subsets:

$$\begin{aligned} J &= \{j : \nu^{(j)} = \bar{0},\ \mu^{(j)} = \omega^{(j)}\}, \\ L &= \{l : \nu^{(l)} = \bar{0},\ \mu^{(l)} = \sigma^{(l)}\}, \\ T &= \{t : \nu^{(t)} = \sigma^{(t)},\ \mu^{(t)} = \omega^{(t)}\}. \end{aligned} \tag{28}$$

We introduce two matrices

$$\Phi := \varkappa^{-1} \cdot \sigma, \ \ \Psi := \varkappa^{-1} \cdot \omega,$$

with rows $\varphi_1, \dots, \varphi_n$ and $\psi_1, \dots, \psi_n$ respectively. Moreover, we consider truncated rows

$$\varphi_l^J,\ \psi_l^L,\ \psi_l^T,\ l \in L,$$

$$\varphi_t^J,\ \psi_t^L,\ \psi_t^T,\ t \in T,$$

which consist of entries of rows φ_l, ψ_l, $l \in L$ and φ_t, ψ_t, $t \in T$ indexed by elements of sets J, L and T. Respectively, we introduce truncated vectors k^J, k^L, k^T for the vector $k = (k_1, \dots, k_n)$. The scalar product of vectors we denote as follows $\langle \cdot, \cdot \rangle$.

Theorem 2. *For any collection of n couples $\mu^{(i)}$, $\nu^{(i)} \in A^{(i)}$ with the nondegeneracy condition of the corresponding matrix $\varkappa$ there exists an analytic continuation of the Taylor series for the monomial $y^d(x)$ of the principal solution to the system (2) in the form of the Puiseux series*

$$\sum_{k\in\mathbb{Z}^n_{\geqslant}} \tilde{c}_k x^{m(k)},$$

which has the support consisting of points $m(k) = (m_1(k), \dots, m_n(k))$ with coordinates

$$\begin{aligned}
&m_j(k) = k_j,\ j \in J,\\
&m_l(k) = -\langle \varphi_l^J, k^J\rangle - \langle \psi_l^L, k^L\rangle + \langle \psi_l^T, k^T\rangle - \langle d, \varkappa_l^{-1}\rangle,\ l \in L,\\
&m_t(k) = \langle \varphi_t^J, k^J\rangle + \langle \psi_t^L, k^L\rangle - \langle \psi_t^T, k^T\rangle + \langle d, \varkappa_t^{-1}\rangle,\ t \in T,
\end{aligned}$$

and coefficients $\tilde{c}_k$ expressed in terms of coefficients (22) as follows

$$\tilde{c}_k = e^{i\pi \sum\limits_{t\in T}(k_t + m_t(k))} c_k.$$

Proof. We start the proof with finding the monomial change of variables $r = r(a)$ reducing the system (1) to the form (21). To this end, we get the Smith normal form S_q for the matrix $\varkappa$, multiplying it on the left and right by unimodular matrices C and F as follows:

$$C\varkappa F = S_q, \tag{29}$$

here the S_q is a diagonal matrix with integers $q_1, \dots, q_n$ on the diagonal, and $q_j \mid q_{j+1}$, $1 \leqslant j \leqslant n-1$, see [11]. It follows from (29) that the inverse matrix $\varkappa^{-1}$ admits the representation

$$\varkappa^{-1} = F S_q^{-1} C. \tag{30}$$

As it was mentioned above, the solution $y(a)$ of the system (1) is polyhomogeneous. We find the polyhomogeneity parameters $\lambda_0^{(i)}$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ such that

$$\begin{gathered}
\lambda_0^{(i)} \lambda^{\mu^{(i)}} a^{(i)}_{\mu^{(i)}} = 1,\\
\lambda_0^{(i)} \lambda^{\nu^{(i)}} a^{(i)}_{\nu^{(i)}} = -1,
\end{gathered} \tag{31}$$

for $i = 1, \dots, n$. For that, we solve the following system of equations:

$$\lambda^{\varkappa^{(i)}} = g_i,\ i = 1, \dots, n, \tag{32}$$

where

$$g_i = -\frac{a^{(i)}_{\nu^{(i)}}}{a^{(i)}_{\mu^{(i)}}}.$$

Using the relation (30), we can write the solution of the system (32) in the matrix form as follows

$$\lambda = g^{\varkappa^{-1}} = g^{FS_q^{-1}C} = \left(\left(g^{f^{(1)}}\right)^{\frac{1}{q_1}}, \dots, \left(g^{f^{(n)}}\right)^{\frac{1}{q_n}}\right)^C,$$

where the vector g has coordinates g_i, and $f^{(1)}, \dots, f^{(n)}$ are columns of the matrix F. By choosing for each i all q_i values of the radical $\left(g^{f^{(i)}}\right)^{\frac{1}{q_i}}$, we yield all branchers of the matrix radical $g^{\varkappa^{-1}}$. There are $|\det \varkappa| = q_1 \cdot \ldots \cdot q_n$ of them.

For each $i \in \{1,\dots,n\}$ we find the parameter $\lambda_0^{(i)}$, using one of relations (31). If $\nu^{(i)} = \overline{0}$, then $\lambda_0^{(i)} = -\dfrac{1}{a_0^{(i)}}$. For $\mu^{(i)} = \omega^{(i)}$ we get

$$\lambda_0^{(i)} = \frac{1}{a^{(i)}_{\omega^{(i)}}} \cdot \left(\left(g^{f^{(1)}}\right)^{\frac{1}{q_1}}, \dots, \left(g^{f^{(n)}}\right)^{\frac{1}{q_n}} \right)^{-C\omega^{(i)}}.$$

If $i \in J$, then the coefficient $r^{(i)}_{\sigma^{(i)}}$ of the system (21) can be expressed in terms of coefficients a of the system (1) in two ways:

$$\begin{aligned} r^{(i)}_{\sigma^{(i)}} &= -\frac{a^{(i)}_{\sigma^{(i)}}}{a^{(i)}_0} \cdot \left(g^{f^{(1)}}\right)^{\frac{\langle c_1, \sigma^{(i)}\rangle}{q_1}} \cdot \ldots \cdot \left(g^{f^{(n)}}\right)^{\frac{\langle c_n, \sigma^{(i)}\rangle}{q_n}}, \\ r^{(i)}_{\sigma^{(i)}} &= \frac{a^{(i)}_{\sigma^{(i)}}}{a^{(i)}_{\omega^{(i)}}} \cdot \left(g^{f^{(1)}}\right)^{\frac{\langle c_1, \sigma^{(i)}-\omega^{(i)}\rangle}{q_1}} \cdot \ldots \cdot \left(g^{f^{(n)}}\right)^{\frac{\langle c_n, \sigma^{(i)}-\omega^{(i)}\rangle}{q_n}}. \end{aligned} \tag{33}$$

If $i \in L$, then the coefficient $r^{(i)}_{\omega^{(i)}}$ of the system (21) can be expressed in terms of coefficients a of the system (1) as follows

$$r^{(i)}_{\omega^{(i)}} = -\frac{a^{(i)}_{\omega^{(i)}}}{a^{(i)}_0} \cdot \left(g^{f^{(1)}}\right)^{\frac{\langle c_1, \omega^{(i)}\rangle}{q_1}} \cdot \ldots \cdot \left(g^{f^{(n)}}\right)^{\frac{\langle c_n, \omega^{(i)}\rangle}{q_n}}. \tag{34}$$

For $i \in T$ the relation is as follows

$$r^{(i)}_{\overline{0}} = \frac{a^{(i)}_{\overline{0}}}{a^{(i)}_{\omega^{(i)}}} \cdot \left(g^{f^{(1)}}\right)^{-\frac{\langle c_1, \omega^{(i)}\rangle}{q_1}} \cdot \ldots \cdot \left(g^{f^{(n)}}\right)^{-\frac{\langle c_n, \omega^{(i)}\rangle}{q_n}}. \tag{35}$$

In formulae (33)–(35) vectors $c_1,\dots,c_n$ are rows of the matrix C.

In particular, if for all $i \in \{1,\dots,n\}$ we choose $\mu^{(i)} = \omega^{(i)}$, $\nu^{(i)} = \overline{0}$, then $L = \varnothing$, $T = \varnothing$ and $\varkappa = \omega$. The matrix ω is nondegenerate by assumption and the system (21) coincides with the system (2). In this case, we get the change of variables $x = x(a)$. It can be written in two ways:

$$\begin{aligned} x_i &= -\frac{a^{(i)}_{\sigma^{(i)}}}{a^{(i)}_{\overline{0}}} \cdot \left(h^{v^{(1)}}\right)^{\frac{\langle u_1, \sigma^{(i)}\rangle}{p_1}} \cdot \ldots \cdot \left(h^{v^{(n)}}\right)^{\frac{\langle u_n, \sigma^{(i)}\rangle}{p_n}}, \\ x_i &= \frac{a^{(i)}_{\sigma^{(i)}}}{a^{(i)}_{\omega^{(i)}}} \cdot \left(h^{v^{(1)}}\right)^{\frac{\langle u_1, \sigma^{(i)}-\omega^{(i)}\rangle}{p_1}} \cdot \ldots \cdot \left(h^{v^{(n)}}\right)^{\frac{\langle u_n, \sigma^{(i)}-\omega^{(i)}\rangle}{p_n}}. \end{aligned} \tag{36}$$

In formulae (36) the vector h has coordinates $h_i = -\dfrac{a^{(i)}_{\overline{0}}}{a^{(i)}_{\omega^{(i)}}}$, vectors $u_1,\dots,u_n$ are rows of the unimodular matrix U, in turn, vectors $v^{(1)},\dots,v^{(n)}$ are columns of the unimodular matrix V such that $\omega = US_pV$, where $S_p = \operatorname{diag}[p_1,\dots,p_n]$, $p_j \mid p_{j+1}$, $1 \leqslant j \leqslant n-1$.

Remark that $g_i = h_i$ for $i \in J$. Furthermore, if $i \in L$ then $g_i = -\dfrac{a^{(i)}_{\overline{0}}}{a^{(i)}_{\sigma^{(i)}}}$, and for $i \in T$ we have $g_i = -\dfrac{a^{(i)}_{\sigma^{(i)}}}{a^{(i)}_{\omega^{(i)}}}$. Getting these ratios from (36), we substitute the expressions for g_i into (33)–(35).

As a result, we get coordinates of the monomial transformation $r = r(x)$ for the transition from the system (21) to the system (2):

$$\begin{aligned} r^{(j)}_{\sigma(j)} &= x_j \prod_{l\in L} x_l^{-\varphi_l^{(j)}} \cdot \prod_{t\in T} (-x_t)^{\varphi_t^{(j)}},\ j \in J,\\ r^{(j)}_{\omega(j)} &= \prod_{l\in L} x_l^{-\psi_l^{(j)}} \cdot \prod_{t\in T} (-x_t)^{\psi_t^{(j)}},\ j \in L,\\ r^{(j)}_{\overline{0}} &= -\prod_{l\in L} x_l^{\psi_l^{(j)}} \cdot \prod_{t\in T} (-x_t)^{-\psi_t^{(j)}},\ j \in T. \end{aligned} \tag{37}$$

According to the polyhomogeneity property (19), the division of the jth coordinate of the solution to the system (1) on $\lambda_j \neq 0$ is compensated by the multiplication of the coefficient $a^{(i)}_\alpha$ on λ^α. So taking into account (32) we obtain the relationship between monomials $y^d(x)$ and $y^d(r)$ of the following form:

$$y^d(x) = \prod_{j=1}^{n} \frac{g_j^{\langle d,\, \varkappa_j^{-1}\rangle}}{h_j^{\langle d,\, \omega_j^{-1}\rangle}} y^d(r), \tag{38}$$

where $\varkappa_j^{-1}$, ω_j^{-1} are jth rows of matrices $\varkappa^{-1}$ and ω^{-1} correspondingly. Using relations (36), and the fact that $g_j = h_j$ for $j \in J$, we write (38) as follows:

$$y^d(x) = \prod_{l\in L} x_l^{-\langle d,\, \varkappa_l^{-1}\rangle} \prod_{t\in T} \left(e^{i\pi} x_t\right)^{\langle d,\, \varkappa_t^{-1}\rangle} y^d(r). \tag{39}$$

Hence, making the substitution (37) in the expansion (22) and taking into account the relation (39), we conclude, that the support S of the required Puiseux series consists of points $m(k) = (m_1(k), \ldots, m_n(k))$ with coordinates

$$\begin{aligned} m_j(k) &= k_j,\ j \in J,\\ m_l(k) &= -\langle \varphi_l^J, k^J\rangle - \langle \psi_l^L, k^L\rangle + \langle \psi_l^T, k^T\rangle - \langle d, \varkappa_l^{-1}\rangle,\ l \in L,\\ m_t(k) &= \langle \varphi_t^J, k^J\rangle + \langle \psi_t^L, k^L\rangle - \langle \psi_t^T, k^T\rangle + \langle d, \varkappa_t^{-1}\rangle,\ t \in T. \end{aligned}$$

The coefficient $\tilde{c}_k$ of the Puiseux series is expressed in terms of the coefficient (22) by the following formula

$$\tilde{c}_k = e^{i\pi \sum\limits_{t\in T} (k_t + m_t(k))} c_k.$$

□

As mentioned in Section 1, by the two-sided Abel lemma for hypergeometric series [10] the cone of the support S of the series defines the logarithmic image $\mathrm{Log}(G)$ of the convergence domain G of the series. This means that the geometry of the domain G is closely related to the structure of the amoeba $\mathcal{A}$ of the discriminant hypersurface ∇ of the system (2). The amoeba $\mathcal{A}$ can be obtained from the amoeba $\mathcal{A}'$ of the discriminant set of the system (21) via an affine transform associated with the change of variables $r = r(x)$. Consequently, the recession cone of the set $\mathrm{Log}(G)$ for the Puiseux series of the monomial $y^d(x)$ is the image of the negative orthant $-\mathbb{R}^n_+$ under an affine transform.

In conclusion, we return to the example from Section 1 to make the following remark. By Theorem 2 we associate the Puiseux series (14) with couples of exponents:

$$(2,1), (0,0) \in A^{(1)},\quad (1,2), (0,0) \in A^{(2)},$$

and, accordingly, the series (17) with the set

$$(2,1),(0,0)\in A^{(1)},\quad (0,4),(0,0)\in A^{(2)}.$$

The first author was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS"(no. 18-1-7-60-1). The second author was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics "BASIS"(no. 18-1-7-60-2). The third author was supported by the grant of the Ministry of Education and Science of the Russian Federation (no. 1.2604.2017/PCh).

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DOI: 10.1007/BF02677509

Аналитические продолжения решений систем триномиальных алгебраических уравнений

Ирина А. Антипова
Екатерина А. Клешкова
Владимир Р. Куликов
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. Статья посвящена исследованию аналитических продолжений мономиальной функции координат решения приведенной триномиальной алгебраической системы. В частности, показано, как техника интегральных представлений Меллина-Барнса и свойство полиоднородности решения универсальной триномиальной системы применяются для разрешения задачи аналитического продолжения. Таким образом, получены разложения Пюизо (с центром в нуле), представляющие аналитические продолжения ряда Тейлора указанной мономиальной функции.

Ключевые слова: алгебраическое уравнение, аналитическое продолжение, ряд Пюизо, дискриминант, интеграл Меллина-Барнса.