ISSN 1997-1397 (Print)
ISSN 2313-6022 (Online)

Журнал Сибирского федерального университета

Математика и физика

Journal of Siberian Federal University

Mathematics & Physics

2020 13 (4)

ISSN 1997-1997-1397 (Print)

ISSN 2313-6022 (Online)

2020 13 (4)

ЖУРНАЛ СИБИРСКОГО ФЕДЕРАЛЬНОГО УНИВЕРСИТЕТА
Математика и Физика

JOURNAL OF SIBERIAN FEDERAL UNIVERSITY
Mathematics & Physics

Издание индексируется Scopus (Elsevier), Emerging Sources Citation Index (WoS, Clarivate Analytics), Российским индексом научного цитирования (НЭБ), представлено в международных и российских информационных базах: Ulrich's periodicals directiory, ProQuest, EBSCO (США), Google Scholar, MathNet.ru, КиберЛенинке.

Включено в список Высшей аттестационной комиссии «Рецензируемые научные издания, входящие в международные реферативные базы данных и системы цитирования».

Все статьи представлены в открытом доступе http://journal.sfu-kras.ru/en/series/mathematics_physics.

Журнал Сибирского федерального университета.
Математика и физика
Journal of Siberian Federal University. Mathematics & Physics

Учредитель: Федеральное государственное автономное образовательное учреждение высшего образования "Сибирский федеральный университет"(СФУ)
Главный редактор: А.М. Кытманов. Редакторы: В.Е. Зализняк, А.В. Щуплев.
Компьютерная верстка: Г.В. Хрусталева
№ 4. 26.08.2020. Индекс: 42327. Тираж: 1000 экз. Свободная цена
Адрес редакции и издательства: 660041 г. Красноярск, пр. Свободный, 79, оф. 32-03.
Отпечатано в типографии Издательства БИК СФУ
660041 г. Красноярск, пр. Свободный, 82а.
Свидетельство о регистрации СМИ ПИ № ФС 77-28724 от 27.06.2007 г., выданное Федеральной службой по надзору в сфере массовых коммуникаций, связи и охраны культурного наследия
http://journal.sfu-kras.ru
Подписано в печать 15.08.20. Формат 84×108/16. Усл.печ. л. 11,9.
Уч.-изд. л. 11,6. Бумага тип. Печать офсетная.
Тираж 1000 экз. Заказ 11411
Возрастная маркировка в соответствии с Федеральным законом № 436-ФЗ:16+

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DOI: 10.17516/1997-1397-2020-13-4-387-397
УДК 512.6

Centers and Limit Cycles of Generalized Kukles Polynomial Differential Systems: Phase Portraits and Limit Cycles

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Received 06.04.2020, received in revised form 25.05.2020, accepted 16.06.2020

Abstract. In this work, we give the seven global phase portraits in the Poincaré disc of the Kukles differential system given by

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= x + ax^8 + bx^4y^4 + cy^8,\end{aligned}$$

where $x, y \in \mathbb{R}$ and $a, b, c \in \mathbb{R}$ with $a^2 + b^2 + c^2 \neq 0$.

Moreover, we perturb these system inside all classes of polynomials of eight degrees, then we use the averaging theory up sixth order to study the number of limit cycles which can bifurcate from the origin of coordinates of the Kukles differential system.

Keywords: limit cycle, generalized Kukles differential system, averaging method, phase portrait.

Citation: A. Belfar, R. Benterki, Centers and Limit Cycles of Generalized Kukles Polynomial Differential Systems: Phase Portraits and Limit Cycles, J. Sib. Fed. Univ. Math. Phys., 2020, 13(4), 387–397. DOI: 10.17516/1997-1397-2020-13-4-387-397.

1. Introduction and statement of the main results

We consider the so-called *Kukles homogeneous differential system.* Giné [5]

$$\dot{x} = -y, \quad \dot{y} = x + Q_n(x, y), \tag{1}$$

which has a center at the origin, where $Q_n(x, y)$ denotes a homogeneous real polynomial of degree n.

In 1999 Volokitin and Ivanov [12] conjectured that systems (1) have a center at the origin definitely if they are symmetric with respect to one of the coordinate axes. For $n = 2$ and $n = 3$, the authors of the conjecture knew that it holds. Giné [5] in 2002 proved the conjecture for $n = 4$ and $n = 5$. Giné et al. [6, 7] proved the conjecture for all n under an additional assumption, that the authors believe that it is redundant.

The phase portraits for quadratic systems with center written in the form (1), are known, see Vulpe [13]. The phase portraits of cubic differential systems symmetry with respect to a straight line are also known and in particular those of system (1) with $n = 3$, see Buzzi et al. [3], see also Malkin [11]; Vulpe Sibirskii [13] and Żołądek [14, 15]. The phase portraits of systems (1) with $n = 4$ follows from Benterki and Llibre [1]. Llibre and Silva [9, 10] classified the phase portraits of the systems (1) for $n = 5, 6$. The phase portraits of systems (1) with $n = 7$ follows from Benterki and Llibre [2].

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In our work, we classify the global phase portraits of the polynomial differential system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + ax^8 + bx^4y^4 + cy^8. \end{aligned} \tag{2}$$

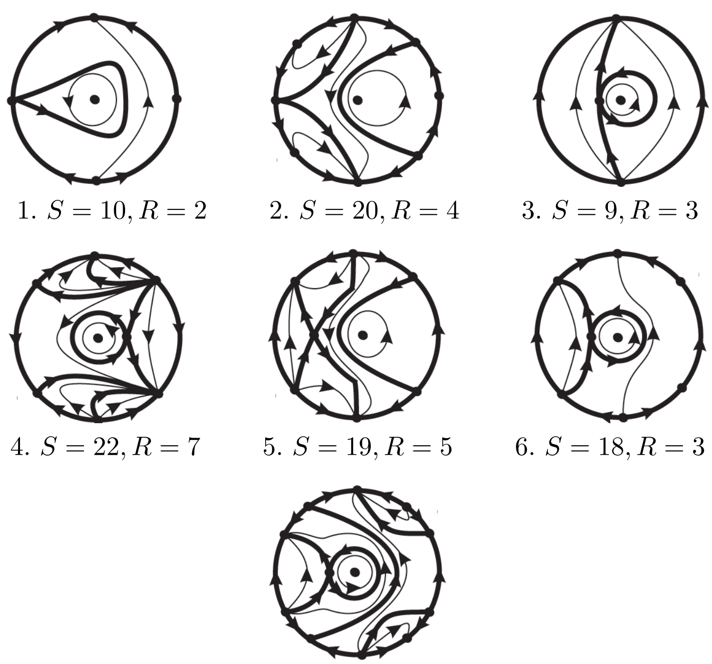
The first main objective of this work is to study the phase portraits in the Poincaré disc of the differential system (2).

The second objectif, is to give the number of limit cycles which can bifurcate from the origin of coordinates of system (2) when we perturb them inside all classes of polynomial of eight degree, and we do this by using the averaging theory up sixth order.

In Section 2 we give more information about the global phase portraits of the polynomial differential system (2).

Our first main result is given in the following Theorem.

Theorem 1. *The set of all global phase portraits in the Poincaré disc of the differential system (2) with $a^2 + b^2 + c^2 \neq 0$, are topologically equivalent to the phase portraits given in Fig. 1.*



1. $S = 10, R = 2$ 2. $S = 20, R = 4$ 3. $S = 9, R = 3$

4. $S = 22, R = 7$ 5. $S = 19, R = 5$ 6. $S = 18, R = 3$

7. $S = 29, R = 6$

Fig. 1. Global phase portraits of differential system (2)

Theorem 1 is proved in Section 3.

When we perturbed the polynomial differential system (2) with polynomials of degree eight, we get

$$\begin{aligned} \dot{x} &= -y + \sum_{s=1}^{6} \varepsilon^s \sum_{0\leqslant i+j\leqslant 8} \alpha_{ij}^{(s)} x^i y^j, \\ \dot{y} &= x + ax^8 + bx^4y^4 + cy^8 + \sum_{s=1}^{6} \varepsilon^s \sum_{0\leqslant i+j\leqslant 8} \beta_{ij}^{(s)} x^i y^j, \end{aligned} \tag{3}$$

where $i, j \in \mathbb{N}$. For more information about the averaging theory of higher order see Section 5.

Our second main result is given in the following theorem.

Theorem 2. *The number of limit cycles of the differential system* (3) *with* $\varepsilon \neq 0$ *is*

(a) 0 *if we use the averaging theory of order* 1 *or* 2*,*

(b) 1 *if we use the averaging theory of order* 3 *or* 4*,*

(c) 2 *if we use the averaging theory of order* 5 *or* 6*.*

We give the Proof of Theorem 2 in Section 5.

2. Preliminaries

In this section, we give some basic results which are necessary to study the behavior of the trajectories of a planar differential systems near infinity. Let $X(x, y) = (P(x, y), Q(x, y))$ represent a vector field to each system which we are going to study its phase portraits, then for doing this we use the so called a Poincaré compactification. We consider the Poincaré sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, and we define the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ (with $T_{(0,0,1)}\mathbb{S}^2$ the tangent space of $\mathbb{S}^2$ at the point $(0, 0, 1)$), such that for each point $q \in T_{(0,0,1)}\mathbb{S}^2$, $T_{(0,0,1)}\mathbb{S}^2(q)$ associaltes the two intersection points of the straight line which connects the point q and $(0, 0)$). The equator $\mathbb{S}^1 = \{(x, y, z) \in \mathbb{S}^2 : z = 0\}$ represent the infinity points of $\mathbb{R}^2$. In summary we get a vector field $\mathcal{X}'$ defined in $\mathbb{S}^2 \setminus \mathbb{S}^1$, which is formed by to symmetric copies of $\mathcal{X}$, and we prolong it to a vector field $p(X)$ on $\mathbb{S}^2$. By studiying the dynamics of $p(X)$ near $\mathbb{S}^1$ we get the dynamics of $\mathcal{X}$ at infinity. We need to do the calculations on the Poincaré sphere near the local charts $U_i = \{Y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{Y \in \mathbb{S}^2 : y_i < 0\}$ for $i = 1, 2, 3$; with the associated diffeomorphisms $F_i : U_i \longrightarrow \mathbb{R}^2$ and $G_i : V_i \longrightarrow \mathbb{R}^2$ for $i = 1, 2, 3$. After a rescaling in the independent variable in the local chart (U_1, F_1) the expression for $p(X)$ is

$$\dot{u} = v^n \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \qquad \dot{v} = -v^{n+1} P\left(\frac{1}{v}, \frac{u}{v}\right);$$

in the local chart (U_2, F_2) the expression for $p(X)$ is

$$\dot{u} = v^n \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \qquad \dot{v} = -v^{n+1} Q\left(\frac{u}{v}, \frac{1}{v}\right);$$

and for the local chart (U_3, F_3) the expression for $p(X)$ is

$$\dot{u} = P(u, v), \qquad \dot{v} = Q(u, v).$$

3. Study of phase portraits

In what follows we shall study the phase portraits of the polynomial differential system (2) with $(a, b, c) \neq (0, 0, 0)$.

Remark 3. *System* (2) *is invariant under the change* $(t, x, y) \to (-t, -x, y)$*. Hence, the phase portrait of system* (2) *is symmetric with respect to the* x*-axis.*

Remark 4. *System* (2) *is also invariant under the change*

$$(x, y, t, a, b, c) \to (-x, y, -t, -a, -b, -c)$$

then we only need to study the phase portrait of system (2) *when (*$a = 0$, $b \geqslant 0$ *and* $c \geqslant 0$*), (*$a = 0$, $b > 0$ *and* $c < 0$*),(* $a > 0$, $b \geqslant 0$ *and* $c \geqslant 0$*), (*$a < 0$, $b > 0$ *and* $c = 0$*), (*$a > 0$, $b \geqslant 0$ *and* $c < 0$*), (*$a > 0$, $b^2 - 4ac = 0$*, and* $b < 0$*) and (*$a > 0$, $b^2 - 4ac > 0$, $b < 0$ *and* $c > 0$*).*

3.1. Configurations of singular points

To study the phase portrait of system (2) we identify all the finite singular points and their local phase portrait. We go through the same steps to study the local phase portrait for the infinite ones.

3.1.1. Finite singular points

We identify the finite singular points of the generalized kukles polynomial differential system (2) in the following Proposition.

Proposition 5. *The differential system* (2) *has*

(i) Two finite singular points, a center at $(0,0)$ *and a hyperbolic saddle at* $(-\sqrt[7]{1/a},0)$*, if* $a \neq 0$*;*

(ii) one singular point at $(0,0)$ *wich is a center, if* $a = 0$.

Proof. Clearly when $a \neq 0$ the system has two equilibria the origin, with eigenvalues $\pm i$, then we take into acount the symmetry of system (2) with respect to x-axis, we conclude that the origin is a center. The second equilibria is $(-\sqrt[7]{1/a},0)$ with eigenvalues $\pm\sqrt{7}$. So it is a hyperbolic saddle. □

3.1.2. Infinite singular points

By using the preliminaries given in Section 2 we study the infinite singular points and their nature in the Poincaré disc.

Proposition 6. *In the chart* U_1 *system* (2) *has*

(a) The origin as a linearly zero infinite singular point, and its local phase portrait consists of four hyperbolic sectors, if $a = 0$, $b \geqslant 0$ *and* $c \geqslant 0$*;*

(b) three infinite singular points, the origin mentioned in the previous case and two saddle-nodes at $\left(\pm\sqrt[4]{-b/c},0\right)$*, if* $a = 0$, $b > 0$ *and* $c < 0$*;*

(c) no singularity, if $a > 0$, $b \geqslant 0$ *and* $c \geqslant 0$*;*

(d) two infinite semi-hyperbolic saddle-nodes, $\left(\pm\sqrt[4]{-a/b},0\right)$*, if* $c = 0$, $b > 0$ *and* $a < 0$ *;*

(e) two infinite semi-hyperbolic saddle-nodes at $\left(\pm\sqrt[4]{\dfrac{-b-\sqrt{b^2-4ac}}{2c}},0\right)$*, if* $a > 0$, $b \geqslant 0$ *and* $c < 0$*;*

(f) two infinite linearly zero singular points $(\pm\sqrt[4]{-2a/b},0)$*, such that their local phase portraits consist of two hyperbolic and two parabolic sectors, if* $a > 0$, $c > 0$, $b^2 = 4ac$ *and* $b < 0$*;*

(g) four infinite semi-hyperbolic saddle-nodes,

$$\left(\pm\sqrt[4]{\frac{-b+\sqrt{b^2-4ac}}{2c}},0\right),\ \textit{and}\ \left(\pm\sqrt{\frac{b+\sqrt{b^2-4ac}}{-2c}},\ 0\right),$$

if $a > 0$, $b^2 - 4ac > 0$, $b < 0$ *and* $c > 0$.

The origin of the chart U_2 *is*

(h) a hyperbolic node, which is stable if $c > 0$ *and unstable if* $c < 0$*;*

(i) a linearly zero singular point, such that its local phase portrait consists of four parabolic sectors, if $c = 0$.

Proof. The differential system (2) in the chart U_1 is given by

$$\begin{aligned}\dot{u} &= a + bu^4 + cu^8 + v^7 + u^2v^7,\\ \dot{v} &= uv^8.\end{aligned} \tag{4}$$

If $b > 0$, $a = 0$ and $c \geqslant 0$ system (4) is written as follows

$$\begin{aligned}\dot{u} &= bu^4 + cu^8 + v^7 + u^2v^7,\\ \dot{v} &= uv^8.\end{aligned} \tag{5}$$

System (2) has one linealrly zero singular point at the origin. Then to study its local phase portrait we have to do blow-up's. We take the directional blow-up $(u, v) \to (u, w)$ with $w = v/u$ and by doing the rescaling of the time $u^3dt = ds$ we have

$$\begin{aligned}\dot{u} &= bu + cu^5 + u^4w^7 + u^6w^7,\\ \dot{w} &= -bw - cu^4w - u^3w^8,\end{aligned} \tag{6}$$

this system has one hyperbolic saddle at $(0, 0)$, with eigenvalues $\pm b$. Returning through the change of variables to system (4), we conclude that the local phase portrait at the origin trained by four hyperbolic sectors.

If $a = 0$, $b = 0$ and $c > 0$, and after taking a rescaling of the time $u^6dt = ds$ we get the following system

$$\begin{aligned}\dot{u} &= cu^2 + uw^7 + u^3w^7,\\ \dot{w} &= -cuw - w^8.\end{aligned} \tag{7}$$

System (7) has a linearly zero singular point at the origin. Doing blow- up's by performing the directional $(u, w) \to (u, z)$ with $z = w/u$ and by doing rescaling of the time $udt = ds$ we get

$$\begin{aligned}\dot{u} &= cu + u^7z^7 + u^9z^7,\\ \dot{z} &= -2cz - 2u^6z^8 - u^8z^8.\end{aligned} \tag{8}$$

System (8) has one hyperbolic saddle at the origin with eigenvalues c and $-2c$. Returning through the change of variables to system (7), we conclude that the local phase portrait at the origin formed by four hyperbolic sectors.

If $b > 0$, $a = 0$ and $c = 0$ we have the following system

$$\begin{aligned}\dot{u} &= bu^4 + u^7w^7 + u^9w^7,\\ \dot{w} &= -bu^3w - u^6w^8.\end{aligned} \tag{9}$$

Doing a change of variable $u^3dt = ds$, we get the following system

$$\begin{aligned}\dot{u} &= bu + u^4w^7 + u^6w^7,\\ \dot{w} &= -bw - u^3w^8.\end{aligned} \tag{10}$$

System (10) has one hyperbolic saddle at the origin with eigenvalues b and $-b$. Returning through the change of variables, we know that the local phase portrait at the origin of system (4), when $a = 0$, $b > 0$ and $c = 0$, consists of four hyperbolic sectors. Then the statement (a) holds.

If $b > 0$, $a = 0$ and $c < 0$ system (6) has in addition to the origin (the same case in satement (a)) two infinite semi-hyperbolic singular points, namely $(\pm\sqrt[4]{-b/c}, 0)$, with eigenvalues

$\pm 4b\sqrt[4]{-b^3/c^3}$ and 0. Applying Theorem 2.19 of [4] we know that these points are saddle-nodes. Then statement (b) holds.

If $a > 0$, $b \geqslant 0$ and $c \geqslant 0$ system (4) has no singular point.

If $c = 0$, $b > 0$ and $a < 0$ system (4) becomes

$$\begin{aligned} \dot{u} &= a + bu^4 + v^7 + u^2v^7, \\ \dot{v} &= uv^8, \end{aligned} \tag{11}$$

this system has two semi-hyperbolic singular points, $(\pm\sqrt[4]{-a/b}, 0)$ with eigenvalues $\lambda_1 = \pm 4b(-a^3/b^3)^{(1/4)}$ and $\lambda_2 = 0$. We perform the translation $u = z \pm (-a/b)^{(1/4)}$ to system (11). Applying Theorem 2.19 of [4] we know that the points are saddle-nodes. Then (d) is proved.

If $a > 0$, $c > 0$, $b^2 = 4ac$, and $b < 0$ we get the following system

$$\begin{aligned} \dot{u} &= (2a + bu^4)^2/(4a) + (1 + u^2)v^7, \\ \dot{v} &= uv^8. \end{aligned} \tag{12}$$

This system has two singular points $(\pm\sqrt[4]{(-2a)/b}, 0)$ which are linearly zero. We study at first the point $(\sqrt[4]{(-2a)/b}, 0)$ after performing the translation $u = z + \sqrt[4]{(-2a)/b}$. Doing blow-up's by taking the directional $(z, v) \to (z, w)$ with $w = v/z$, and eleminating the common factor z between $\dot{z}$ and $\dot{w}$, we get the following differential system

$$\begin{aligned} \dot{z} = &- 8bz\sqrt{\frac{-2a}{b}} - 24(\frac{-2a}{b})^{\frac{1}{4}}bz^2 - 34bz^3 + \frac{1}{a}14(-2a/b)^{\frac{3}{4}}b^2z^4 \\ &+ \frac{7}{a}\sqrt{\frac{-2a}{b}}b^2z^5 + \frac{2^{\frac{5}{4}}}{a}(\frac{-a}{b})^{\frac{1}{4}}b^2z^6 + w^7z^6 + \sqrt{\frac{-2a}{b}}w^7z^6 \\ &+ w^7z^8 + \frac{b^2z^7}{4a} + 2^{\frac{5}{4}}(\frac{-a}{b})^{\frac{1}{4}}w^7z^7, \\ \dot{w} = &\,8\sqrt{\frac{-2a}{b}}bw + 24(\frac{-2a}{b})^{\frac{1}{4}}bwz + 34bwz^2 - \frac{14}{a}(\frac{-2a}{b})^{\frac{3}{4}}b^2wz^3 \\ &- \frac{7}{a}\sqrt{\frac{-2a}{b}}b^2wz^4 - \frac{2}{a}(-2a/b)^{\frac{1}{4}}b^2wz^5 - w^8z^5 - \sqrt{\frac{-2a}{b}}w^8z^5 \\ &- \frac{b^2wz^6}{4a} - (\frac{-2a}{b})^{\frac{1}{4}}w^8z^6. \end{aligned} \tag{13}$$

For $z = 0$, system (13) has one hyperbolic saddle at the origin with eigenvalues $-8\sqrt{2}b\sqrt{(-a/b)}$ and $8\sqrt{2}b\sqrt{(-a/b)}$. Going back through the change of variables to system (12), we conclude that the local phase portrait at the singular point $(\sqrt[4]{(-2a)/b}, 0)$ formed by two hyperbolic and two parabolic sectors. We get the same local phase portrait for the singular point $(-\sqrt[4]{(-2a)/b}, 0)$ as the singular point $(\sqrt[4]{(-2a)/b}, 0)$.

If $a > 0$, $b^2 - 4ac > 0$, $b < 0$ and $c > 0$ system (4) has four semi-hyperbolic singular points, $\Big(\pm\sqrt[4]{(-b + \sqrt{b^2 - 4ac})/2c}, 0\Big)$, with eigenvalues $\lambda_1 = 2^{\frac{5}{4}}(b^2 - 4ac)^{\frac{1}{4}}\Big(\dfrac{-b - \sqrt{b^2 - 4ac}}{c}\Big)^{\frac{3}{4}}$ and $\lambda_2 = 0$, and the points $\Big(\pm\sqrt[4]{(-b - \sqrt{b^2 - 4ac})/2c}, 0\Big)$, with eigenvalues $\lambda_1 = -2^{\frac{5}{4}}(b^2 - 4ac)^{(1/4)}\Big(\dfrac{-b - \sqrt{b^2 - 4ac}}{c}\Big)^{\frac{3}{4}}$ and $\lambda_2 = 0$. Hence the four singular points are semi-hyperbolic. After transforming these points to the origin we apply Theorem 2.19 of [4] we know that these points are saddle-nodes.

In the chart U_2 the differential system (2) becomes

$$\begin{aligned} \dot{u} &= -cu - bu^3 - v^3 - au^5 - u^2v^3, \\ \dot{v} &= -cv - buv - au^4v - bu^2v - uv^4. \end{aligned} \tag{14}$$

If $c \neq 0$ the origin is a hyperbolic node of system (14), with eigenvalues $-c$ and $-c$, then it is stable if $c > 0$ and unstable if $c < 0$. So statement (e) holds.

If $c = 0$, system (14) becomes

$$\begin{aligned} \dot{u} &= -bu^3 - v^3 - au^5 - u^2v^3, \\ \dot{v} &= -buv - au^4v - bu^2v - uv^4. \end{aligned} \tag{15}$$

The origin is a linearly zero singular point of the differential system (15). We have to do blow-up's to know the local phase portrait at this point. We take the directional blow-up $w = v/u$, and by doing the rescaling $udt = ds$ and we get the system

$$\begin{aligned} \dot{u} &= -bu^2 - au^4 - u^2w^3 - u^4w^3, \\ \dot{w} &= -bw + uw^4. \end{aligned} \tag{16}$$

When $u = 0$; the origin is the only singular point of system (16), with eigenvalues 0 and $-b$. Then, it is a semi-hyperbolic singular point. By using Theorem 2.19 of [4] we conclude that the origin is a saddle-node. Going back through the change of variables to system (15), we know that its local phase portrait formed by four parabolic sectors. □

4. Phase portraits on the Poincaré disc

Taking into account the results on the finite and infinite singular points given in Subsections 3.1.1 and 3.1.2, respectively, we shall obtain the different phase portraits of the system (2) that we describe in what follows.

Theorem 7. *The phase portraits in the Poincaré disc of the two compactified polynomial differential systems $p(\mathcal{X})$ and $p(\mathcal{Y})$ are topologically equivalent if and only if their separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are topologically equivalent.*

Case 1. When $a = 0$, $b \geqslant 0$ and $c \geqslant 0$ system (2) has one finite singular point, a center at $(0, 0)$. And from statement (a) of Proposition 6 we know that in the chart U_1 the system has one singular point at $(0, 0)$ wich is linearly zero and its local phase portrait consists of four hyperbolic sectors. From statement (h) of Proposition 6, the origin of U_2 is a hyperbolic stable node. So, the phase portrait of system (2) is given by Fig. 1 (1), and its immediate that $S = 10$ and $R = 2$.

Case 2. When $a = 0$, $b > 0$ and $c < 0$ system (2) has one finite singular point at the origin of coodrinates, wich is a center. From statement (b) of Proposition 6 and in the local chart U_1 the system has three infinite singular points, a linearly zero singularity at the origin such that its local phase portrait consists of four hyperbolic sectors, and two semi-hyperbolic saddle-nodes. In U_2 and from statement (h) of Proposition 6, the origin is a hyperbolic unstable node. Then, the phase portrait in this case is topologically equivalent to Fig. 1 (2), and its immediate that $S = 20$ and $R = 4$.

Case 3. When $a > 0$, $b \geqslant 0$ and $c \geqslant 0$ system (2) has two finite singular points, a center at $(0, 0)$ and a hyperbolic saddle at $(-\sqrt[7]{1/a}, 0)$. From statement (c) of Proposition 6 the system has no singular points in the local chart U_1. In U_2 and from statement (h) of Proposition 6, the origin is a hyperbolic stable node if $c > 0$, and from statement (i) of the same proposition the origin is a linearly zero singular point and its local phase portrait formed by four parabolic sectors if $c = 0$. So, in this case the phase portrait is topologically equivalent to Fig. 1 (3), and its immediate that $S = 9$ and $R = 3$.

Case 4. When $a < 0$, $b > 0$ and $c = 0$, this case and the following cases 5, 6 and 7 have the same finite singularities as the case 3. From statement (d) of Proposition 6 the system has two infinite semi-hyperbolic saddle-nodes in the local chart U_1. In U_2 and from statement (i) of Proposition 6, the origin is a linearly zero singular point with local phase portrait formed by four parabolic sectors. Therefore in this case, the phase portrait of system (2) is topologically equivalent to the Fig. 1 (4), and its immediate that $S = 22$ and $R = 7$.

Case 5. When $a > 0$, $b \geqslant 0$ and $c < 0$ system (2) and from statement (e) of Proposition 6 we obtain that the system has two infinite semi-hyperbolic saddle-nodes in the local chart U_1. The origin of the chart U_2 is a hyperbolic unstable node. So, the phase portrait of system (2) is topologically equivalent to Fig. 1 (5), and its immediate that $S = 19$ and $R = 5$.

Case 6. When $a > 0$, $b^2 = 4ac$ and $b < 0$ and from statement (f) of Proposition 6 the system has two infinite linearly zero singular points in the local chart U_1 and their local phase portraits consist of two hyperbolic and two parabolic sectors. In U_2 and from statement (h) of Proposition 6, the origin is a hyperbolic unstable node. Then, we conclude that the phase portrait of system (2) is topologically equivalent to Fig. 1 (6), and its immediate that $S = 18$ and $R = 3$.

Case 7. When $a > 0$, $b^2 - 4ac > 0$, $b < 0$ and $c > 0$ and from statement (g) of Proposition 6 we obtain that the system has four infinite semi-hyperbolic saddle-nodes in the local chart U_1. In U_2 and from statement (h) of Proposition 6, the origin is a hyperbolic unstable node. Therefore in this case, the phase portrait of system (2) is topologically equivalent to the Fig. 1 (7), and its immediate that $S = 29$ and $R = 6$.

To know the number of zeros of a real polynomial, we are going to use the following Theorem.

Descartes Theorem. *Consider the real polynomial* $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \cdots + a_{i_r}x^{i_r}$ *with* $0 \leqslant i_1 < i_2 < \cdots < i_r$ *and* $a_{i_j} \neq 0$ *real constants for* $j \in \{1, 2, \cdots, r\}$. *When* $a_{i_j}a_{i_{j+1}} < 0$, *we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of signs is* m, *then* $p(x)$ *has at most* m *positive real roots. Moreover, it is always possible to choose the coefficients of* $p(x)$ *in such a way that* $p(x)$ *has exactly* $r - 1$ *positive real roots.*

5. Proof of Theorem 2

Consider system (2), we shall study which periodic solutions of the center become limit cycles when we perturb the center inside the class of polynomial differential systems of degree 8. This study will be done by applying the averaging theory, we work as follows.

Before doing the scaling $x = \varepsilon X$, $y = \varepsilon Y$, with ε is a small parameter we get a new differential system $(\dot{X}, \dot{Y})$. After we perform the polar change of coordinates $X = r\cos\theta$, $Y = r\sin\theta$, then we get a differential system $(\dot{r}, \dot{\theta})$. We take the independent variable the angle θ we get the differential equation $dr/d\theta$, and by doing a Taylor expansion up to 6-th order in ε we obtain the differential equation

$$r' = \frac{dr}{d\theta} = \sum_{i=0}^{6} \varepsilon^i F_i(\theta, r) + O(\varepsilon^7). \tag{17}$$

The functions $F_i(\theta, r)$ $i = 1, \ldots, 6$ of the differential system (17) are analytic, and since the independent variable θ appears through the sinus and cosinus of θ, they are 2π-periodic. Hence the assumptions for applying the averaging theory given in [8] are satisfied.

To know how the averaging theory for differential equation works we advice the lecture to see [8].

We give only the expression of the function $F_1(r,\theta)$. The explicit expression of $F_i(r,\theta)$ with $i=2,\dots,6$ is quite large so we omit them.

The functions $F_i(\theta,r)$ $i=1,\dots,6$ and $R(t,x,\varepsilon)$ of system (17) are analytic, and since the variable appears through sinus and cosinus of θ, they are 2π-periodic. Hence the assumptions of Theorem 10 are satisfied.

The expression of $F_1(r,\theta)$ is

$$F_1(r,\theta)=+\beta_{00}^{(2)}\sin\theta+\frac{r}{2}(\alpha_{10}^{(1)}+\beta_{01}^{(1)}+(\alpha_{10}^{(1)}-\beta_{01}^{(1)})\cos 2\theta+(\alpha_{01}^{(1)}+\beta_{10}^{(1)})\sin 2\theta).$$

Using the formulas given in section 4.1 of [8] the averaged function of first order is

$$f_1(r)=(\alpha_{10}^{(1)}+\beta_{01}^{(1)})r.$$

Clearly equation $f_1(r)=0$ has no positive zeros. Thus the averaging method of first order does not provide limit cycles.

We put $\alpha_{10}^{(1)}=-\beta_{01}^{(1)}$ we obtain $f_1(r)\equiv 0$. We apply the averaging theory of second order, we get the averaging function of second order.

$$f_2(r)=(\alpha_{10}^{(2)}+\beta_{01}^{(2)})r.$$

We see that the equation $f_2(r)=0$ has no positive zeros, it follows that there is no limit cycle by applying the averaging theory of second order.

To apply the averaging method of third order we must put $\alpha_{10}^{(2)}=-\beta_{01}^{(2)}$, and we get $f_2(r)\equiv 0$. The third averaging function is

$$\begin{aligned}f_3(r)= &-(\beta_{11}^{(1)}\beta_{00}^{(2)}-\beta_{01}^{(3)}+2\beta_{00}^{(2)}\alpha_{20}^{(1)}-2\beta_{02}^{(1)}\alpha_{00}^{(2)}-\alpha_{11}^{(1)}\alpha_{00}^{(2)}-\alpha_{10}^{(3)})r\\ &+(1/4)(3\beta_{03}^{(1)}+\beta_{21}^{(1)}+3\alpha_{30}^{(1)}+\alpha_{12}^{(1)})r^3.\end{aligned}$$

So, $f_3(r)$ can have at most one positive real root. Then we have the proof of the theorem for $k=3$.

To apply the averaging method of fourth order, we need to have $f_3(r)\equiv 0$, for that we set $\alpha_{10}^{(3)}=\beta_{11}^{(1)}\beta_{00}^{(2)}-\beta_{01}^{(3)}+2\beta_{00}^{(2)}\alpha_{20}^{(1)}-2\beta_{02}^{(1)}\alpha_{00}^{(2)}-\alpha_{11}^{(1)}\alpha_{00}^{(2)}$ and $\alpha_{12}^{(1)}=-(3\beta_{03}^{(1)}+\beta_{21}^{(1)}+3\alpha_{30}^{(1)})$.

The averaging function of fourth order is

$$f_4(r)=r(A_1+A_2r^2),$$

where

$$\begin{aligned}A_1= &\Big(\beta_{10}^{(1)}\beta_{11}^{(1)}\beta_{00}^{(2)}+2\beta_{01}^{(1)}\beta_{02}^{(1)}\beta_{00}^{(2)}-\beta_{00}^{(2)}\beta_{11}^{(2)}-\beta_{11}^{(1)}\beta_{00}^{(3)}+2\beta_{10}^{(1)}\beta_{00}^{(2)}a_{20}^{(1)}-2\beta_{00}^{(3)}\alpha_{20}^{(1)}\\ &+\beta_{01}^{(1)}\beta_{00}^{(2)}\alpha_{11}^{(1)}-\beta_{01}^{(1)}\beta_{11}^{(1)}\alpha_{00}^{(2)}+2\beta_{02}^{(2)}\alpha_{00}^{(2)}+2\beta_{02}^{(1)}\alpha_{01}^{(1)}\alpha_{00}^{(2)}-2\beta_{01}^{(1)}\alpha_{20}^{(1)}\alpha_{00}^{(2)}\\ &+\alpha_{01}^{(1)}\alpha_{11}^{(1)}\alpha_{00}^{(2)}-2\beta_{00}^{(2)}\alpha_{20}^{(2)}+\alpha_{00}^{(2)}\alpha_{11}^{(2)}+2\beta_{02}^{(1)}\alpha_{00}^{(3)}+\alpha_{11}^{(1)}\alpha_{00}^{(3)}+\alpha_{10}^{(4)}+\beta_{01}^{(4)}\Big).\end{aligned}$$

$$\begin{aligned}A_2= &\frac{-1}{4}\Big(\beta_{20}^{(1)}\beta_{11}^{(1)}+\beta_{11}^{(1)}\beta_{02}^{(1)}+\beta_{10}^{(1)}\beta_{21}^{(1)}+2\beta_{01}^{(1)}\beta_{12}^{(1)}-3\beta_{03}^{(2)}-\beta_{21}^{(2)}+\beta_{21}^{(1)}\alpha_{01}^{(1)}\\ &-\alpha_{20}^{(1)}\alpha_{11}^{(1)}-2\beta_{02}^{(1)}\alpha_{02}^{(1)}-\alpha_{11}^{(1)}\alpha_{02}^{(1)}+2\beta_{01}^{(1)}\alpha_{21}^{(1)}+3\beta_{10}^{(1)}\alpha_{30}^{(1)}+3\alpha_{01}^{(1)}\alpha_{30}^{(1)}\\ &-3\alpha_{30}^{(2)}+2\beta_{20}^{(1)}\alpha_{20}^{(1)}-\alpha_{12}^{(2)}\Big).\end{aligned}$$

According to the expression of the function f_4 we conclude that we can get at most one limit cycle.

Solving $A_1 = 0$ and $A_2 = 0$ we obtain $f_4(r) \equiv 0$, so we can apply the averaging theory of order 5, and its corresponding averaging function is $f_5(r) = r(B_1 + B_2r^2 + B_3r^4)$.

The explicit expression of B_i, with $i = 1, 2, 3$ is quite large so we omit them.

The rank of the largest square matrix of the Jacobian matrix $\mathcal{B} = (B_1, B_2, B_3)$ is 3. Then the coefficients B_1, B_2 and B_3 are linearly independent in their variables. By the Descartes Theorem (or by the roots of a quadratic polynomial in the variable r^2) it follows that we can get at most two positive real roots of $f_5(r)$. So statement (c) holds. Solving $B_1 = 0$, $B_2 = 0$ and $B_3 = 0$ we obtain $f_5(r) \equiv 0$.

Now if we apply the averaging method of sixth order we get

$$f_6(r) = \left(K_1 + K_2r^2 + K_3r^4\right) r.$$

In this case and for the same reason as the previous one we will not give the explicit expression of K_i, with $i = 1, 2, 3$ because it is quite large.

The rank of the Jacobian matrix $\mathcal{K} = (K_1, K_2, K_3)$ with respect to its variables is 3. We have three of the coefficients K_i, $i = 1, 2, 3$ which are linearly independent in their variables. Therefore by Descartes Theorem, it follows that $f_6(r) = 0$ can has 2 positive real solutions. Consequently, the differential system (2) has at least 2 limit cycles. This ends the proof of the theorem.

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Центры и предельные циклы обобщенно-дифференциальных полиномиальных систем Куклеса: фазовые портреты и предельные циклы

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Аннотация. В этой работе мы даем семь глобальных фазовых портретов в диске Пуанкаре дифференциальной системы Куклеса, заданной как

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= x + ax^8 + bx^4y^4 + cy^8,\end{aligned}$$

где $x, y \in \mathbb{R}$, $a, b, c \in \mathbb{R}$ и $a^2 + b^2 + c^2 \neq 0$.

Кроме того, мы возмущаем эту систему внутри всех классов многочленов восьмой степени, а затем используем теорию усреднения до шестой степени для изучения числа предельных циклов, которые могут раздвоиться от начала координат дифференциальной системы Куклеса.

Ключевые слова: предельный цикл, обобщенная дифференциальная система Куклеса, метод усреднения, фазовый портрет.

Journal of Siberian Federal University. Mathematics & Physics 2020, 13(4), 398–413

DOI: 10.17516/1997-1397-2020-13-4-398-413
УДК 519.644.7

On Error Estimates in S_p for Cubature Formulas Exact for Haar Polynomials

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Received 12.03.2020, received in revised form 8.05.2020, accepted 15.05.2020

Abstract. On the spaces S_p, an upper and lower estimates for the norm of the error functional cubature formulas possessing the Haar d-property are obtained for the n-dimensional case.

Keywords: Haar d-property, error estimates for cubature formulas, function spaces S_p.

Citation: K.A. Kirillov, On Error Estimates in S_p for Cubature Formulas Exact for Haar Polynomials, J. Sib. Fed. Univ. Math. Phys., 2020, 13(4), 398–413. DOI: 10.17516/1997-1397-2020-13-4-398-413.

Introduction

The problem of constructing and analyzing cubature formulas that are exact for a given set of functions was earlier considered primarily as applied to the computation of integrals exact for algebraic and trigonometric polynomials. For example, the approximate integration formulas of algebraic accuracy can be found in [1, 2]. The cubature formulas exact for trigonometric polynomials in particular were studied in [3–7].

The approximate integration formulas exact for the system of Haar functions can be found in the monograph [8]. The accuracy of approximate integration formulas for finite Haar sums was used in [8] to derive error estimates for these formulas.

A description of all minimal weighted quadrature formulas possessing the Haar d-property, i.e., formulas exact for Haar functions of groups with indices not exceeding a given number d, was given in [9]. The error estimates for quadrature formulas possessing the Haar d-property in the case of the weight function $g(x) \equiv 1$ were obtained in [10]. In particular, in the mentioned paper the upper estimate for the norm of the error functional $\|\delta_N\|_{S_p^*}$ was found for the quadrature formulas having the Haar d-property:

$$\|\delta_N\|_{S_p^*} \leqslant \left(2^d\right)^{-\frac{1}{p}},$$

and the lower estimate for the norm of the error functional $\|\delta_N\|_{S_p^*}$ was obtained for the quadrature formulas exact for constants:

$$\|\delta_N\|_{S_p^*} \geqslant 2^{-\frac{1}{p}} N^{-\frac{1}{p}}.$$

The problem of constructing cubature formulas possessing the Haar d-property, i.e., formulas exact for Haar polynomials of degree at most d, was solved in the two-dimensional case in

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[11–15] under the condition that the weight function $g(x_1, x_2) \equiv 1$. The error estimates for these cubature formulas was derived in [16]. In particular, in [16] the upper estimate for the norm of the error functional $\|\delta_N\|_{S_p^*}$ was obtained for the mentioned cubature formulas:

$$\|\delta_N\|_{S_p^*} \leqslant 2^{\frac{1}{p}} (2^d)^{-\frac{1}{p}}.$$

In the present paper the error estimates of cubature formulas with arbitrary positive coefficients at the nodes, similar to the estimates given above for the one- and two-dimensional cases, are derived in the n-dimensional case. As a result, we find the upper estimates for the error functional δ_N of the cubature formulas possessing the Haar d-property:

$$|\delta_N[f]| \leqslant 2^{\frac{n-1}{p}} \left(2^d\right)^{-\frac{1}{p}} \|f\|_{S_p}, \qquad \|\delta_N\|_{S_p^*} \leqslant 2^{\frac{n-1}{p}} \left(2^d\right)^{-\frac{1}{p}},$$

and we obtain the lower estimate for the norm of the error functional $\|\delta_N\|_{S_p^*}$ for the cubature formulas exact for any constant:

$$\|\delta_N\|_{S_p^*} \geqslant \left(2^{n+1} - n - 1\right)^{-\frac{1}{p}} N^{-\frac{1}{p}}.$$

1. Basic definitions

In this paper, we use the original definition of the functions $\chi_{m,j}(x)$ introduced by A. Haar [17].

The binary intervals of rank m are the intervals $l_{m,1} = \left[0, \dfrac{1}{2^{m-1}}\right)$, $l_{m,2^{m-1}} = \left(\dfrac{2^{m-1}-1}{2^{m-1}}, 1\right]$, $m = 2, 3, \ldots$, and $l_{m,j} = \left(\dfrac{j-1}{2^{m-1}}, \dfrac{j}{2^{m-1}}\right)$, $m = 3, 4, \ldots$, $j = 2, \ldots, 2^{m-1}-1$. By a binary interval of the 1st rank we will consider the interval $l_{1,1} = [0, 1]$. The binary segments of rank m are the closed intervals $\overline{l_{m,j}} = \left[\dfrac{j-1}{2^{m-1}}, \dfrac{j}{2^{m-1}}\right]$, $m = 1, 2, \ldots$, $j = 1, \ldots, 2^{m-1}$.

The left and right halves of $l_{m,j}$ (without its midpoint) are denoted by $l_{m,j}^-$ and $l_{m,j}^+$, respectively. Obviously, $l_{m,j}^- = l_{m+1,2j-1}$, $l_{m,j}^+ = l_{m+1,2j}$.

In [17], the Haar functions $\chi_{m,j}(x)$ are defined by:

$$\chi_{m,j}(x) = \begin{cases} 2^{\frac{m-1}{2}}, & x \in l_{m,j}^-, \\ -2^{\frac{m-1}{2}}, & x \in l_{m,j}^+, \\ 0, & x \in [0,1] \setminus \overline{l_{m,j}}, \\ \left\{\chi_{m,j}(x-0) + \chi_{m,j}(x+0)\right\}/2, & x \text{ is an interior discontinuity point}, \end{cases} \tag{1}$$

$m = 1, 2, \ldots$, $j = 1, \ldots, 2^{m-1}$.

Thus, the Haar system of functions is constructed in groups: the mth group contains 2^{m-1} functions $\{\chi_{m,j}(x)\}$, where $m = 1, 2, \ldots$, $j = 1, \ldots, 2^{m-1}$. The Haar system of functions includes the function $\chi_1(x) \equiv 1$ too, which is outside of any group.

In the one-dimensional case, the Haar polynomials of degree d are by definition the functions

$$P_d(x) = a_0 + \sum_{m=1}^{d} \sum_{j=1}^{2^{m-1}} a_m^{(j)} \chi_{m,j}(x),$$

where $d = 1, 2, \ldots,\ a_0,\ a_m^{(j)} \in \mathbb{R},\ m = 1, \ldots, d,\ j = 1, \ldots, 2^{m-1}$, and

$$\sum_{j=1}^{2^{d-1}} \left\{a_d^{(j)}\right\}^2 \neq 0.$$

By the 0-degree Haar polynomials we will consider real constants.

In the n-dimensional case, the Haar polynomials of degree d are the functions

$$P_d(x_1, \ldots, x_n) = a_0 +$$
$$+ \sum_{s=1}^{n} \sum_{1 \leqslant i_1 < \ldots < i_s \leqslant n} \sum_{m_1 + \ldots + m_s \leqslant d} \sum_{j_1=1}^{2^{m_1-1}} \cdots \sum_{j_s=1}^{2^{m_s-1}} a_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1, \ldots, i_s) \chi_{m_1,j_1}(x_{i_1}) \ldots \chi_{m_s,j_s}(x_{i_s}),$$

where $d = 1, 2, \ldots,\ \ a_0,\ a_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1, \ldots, i_s) \in \mathbb{R},\ 1 \leqslant i_1 < \ldots < i_s \leqslant n,\ \ m_1 + \ldots + m_s \leqslant d$, $s = 1, \ldots, n,\ j_k = 1, \ldots, 2^{m_k-1},\ k = 1, \ldots, s$, and

$$\sum_{s=1}^{n} \sum_{1 \leqslant i_1 < \ldots < i_s \leqslant n} \sum_{m_1 + \ldots + m_s = d} \sum_{j_1=1}^{2^{m_1-1}} \cdots \sum_{j_s=1}^{2^{m_s-1}} \left\{a_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1, \ldots, i_s)\right\}^2 \neq 0.$$

The same way as in the one-dimensional case, by 0-degree Haar polynomials we will consider real constants.

Consider the following cubature formula

$$I[f] = \int_0^1 \cdots \int_0^1 f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \approx \sum_{k=1}^{N} C_k f\left(x_1^{(k)}, \ldots, x_n^{(k)}\right) = Q_N[f], \tag{2}$$

where $\left(x_1^{(k)}, \ldots, x_n^{(k)}\right) \in [0,1]^n$ are the nodes, the coefficients C_k at the nodes are real, $k = 1, \ldots, N$.

The cubature formula (2) is said to possess the Haar d–property (or just the d-property) if it is exact for any Haar polynomial $P(x_1, \ldots, x_n)$ of degree at most d, i.e., $Q_N[P] = I[P]$. Such a formula with the least possible number of nodes is called a minimal cubature formula with the d-property.

We recall the definition of the linear normed space S_p in the n-dimensional case introduced by I. M. Sobol' [8].

Let p be a fixed number with $1 \leqslant p < +\infty$. The set of functions $f(x_1, \ldots, x_n)$ defined in the unit n-dimensional cube $[0,1]^n$ and representable as a Fourier–Haar series

$$f(x_1, \ldots, x_n) = c_0 +$$
$$+ \sum_{s=1}^{n} \sum_{1 \leqslant i_1 < \ldots < i_s \leqslant n} \sum_{m_1=1}^{\infty} \cdots \sum_{m_s=1}^{\infty} \sum_{j_1=1}^{2^{m_1-1}} \cdots \sum_{j_s=1}^{2^{m_s-1}} c_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1, \ldots, i_s) \chi_{m_1,j_1}(x_{i_1}) \ldots \chi_{m_s,j_s}(x_{i_s}) \tag{3}$$

with real coefficients $c_0,\ c_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1, \ldots, i_s)$ $(1 \leqslant i_1 < \ldots < i_s \leqslant n,\ \ m_1, \ldots, m_s = 1, 2, \ldots,$ $s = 1, \ldots, n,\ j_k = 1, \ldots, 2^{m_k-1},\ k = 1, \ldots, s)$ satisfying the conditions

$$A_p^{(i_1,\ldots,i_s)}(f) =$$
$$= \sum_{m_1=1}^{\infty} \cdots \sum_{m_s=1}^{\infty} 2^{\frac{m_1-1}{2} + \ldots + \frac{m_s-1}{2}} \left\{ \sum_{j_1=1}^{2^{m_1-1}} \cdots \sum_{j_s=1}^{2^{m_s-1}} \left|c_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1, \ldots, i_s)\right|^p \right\}^{\frac{1}{p}} \leqslant A_{i_1,\ldots,i_s}, \tag{4}$$

(where $A_{i_1,\dots,i_s}$ are real constants, $1 \leqslant i_1 < \ldots < i_s \leqslant n$, $1 \leqslant s \leqslant n$) is called the class $S_p(A_1, \dots, A_n, \dots, A_{i_1,\dots,i_s} \dots, A_{1,\dots,n})$.

It was proved in [8] that the set of functions $f(x_1, \dots, x_n)$ belonging to all the classes $S_p(A_1, \dots, A_n, \dots, A_{i_1,\dots,i_s} \dots, A_{1,\dots,n})$ (with all possible $A_1, \dots, A_n, \dots,$ $A_{i_1,\dots,i_s}, \dots, A_{1,\dots,n}$, while p being fixed) equipped with the norm

$$\|f\|_{S_p} = \sum_{s=1}^{n} \sum_{1\leqslant i_1<\ldots<i_s\leqslant n} A_p^{(i_1,\dots,i_s)}(f), \tag{5}$$

forms a linear normed space, which is denoted by S_p. All the functions $f(x_1, \dots, x_n)$ that differ by constant terms are regarded as a single function.

The coefficients c_0, $c^{(j_1,\dots,j_s)}_{m_1,\dots,m_s}(i_1, \dots, i_s)$ $(1 \leqslant i_1 < \ldots < i_s \leqslant n$, $m_1, \dots, m_s = 1, 2, \dots,$ $s = 1, \dots, n$, $j_k = 1, \dots, 2^{m_k-1}$, $k = 1, \dots, s)$ in the representation of the function $f(x_1, \dots, x_n)$ as a series (3) are called the Fourier–Haar coefficients of this function.

In [8] it was proved that the series (3) converges absolutely and uniformly.

2. Derivation of estimates for the norm of the error functional of cubature formulas in S_p

Let (2) be a cubature formula with the coefficients C_k at the nodes satisfying the inequalities $C_k > 0$, $k = 1, 2, \dots, N$. We denote the error functional of the cubature formula (2) by $\delta_N[f]$ so that

$$\delta_N[f] = I[f] - Q_N[f] = \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_n)\, dx_1 \ldots dx_n - \sum_{k=1}^{N} C_k f\left(x_1^{(k)}, \dots, x_n^{(k)}\right), \tag{6}$$

where the function $f \in S_p$, $p > 1$. It was shown in [8] that any such function is continuous at all points which coordinates are not binary rational numbers. Hence the integral $\int_0^1 \ldots \int_0^1 f(x_1, \dots, x_n)\, dx_1 \ldots dx_n$ exists not only in the Lebesgue sense, but also in the Riemann sense.

Let

$$\varSigma^{(i_1,\dots,i_s)}_{m_1,\dots,m_s}(q) = 2^{-\frac{m_1-1}{2}-\ldots-\frac{m_s-1}{2}} \left\{ \sum_{j_1=1}^{2^{m_1-1}} \cdots \sum_{j_s=1}^{2^{m_s-1}} \left| \sum_{k=1}^{N} C_k\, \chi_{m_1,j_1}\!\left(x_{i_1}^{(k)}\right) \ldots \chi_{m_s,j_s}\!\left(x_{i_s}^{(k)}\right) \right|^q \right\}^{\frac{1}{q}}, \tag{7}$$

where $q > 1$, $1 \leqslant i_1 < \ldots < i_s \leqslant n$, $m_1, \dots, m_s = 1, 2, \dots,$ $s = 1, \dots, n$.

Lemma 1. *If the cubature formula (2) is exact for any constant and $f \in S_p$, then for the absolute value of the error functional satisfies the inequality*

$$\begin{aligned} \left|\delta_N[f]\right| \leqslant \sum_{s=1}^{n} \sum_{1\leqslant i_1<\ldots<i_s\leqslant n} \sum_{m_1=1}^{\infty} \cdots \sum_{m_s=1}^{\infty} \Bigg\{ 2^{\frac{m_1-1}{2}+\ldots+\frac{m_s-1}{2}} \times \\ \times \left[\sum_{j_1=1}^{2^{m_1-1}} \cdots \sum_{j_s=1}^{2^{m_s-1}} \left| c^{(j_1,\dots,j_s)}_{m_1,\dots,m_s}(i_1, \dots, i_s) \right|^p \right]^{\frac{1}{p}} \varSigma^{(i_1,\dots,i_s)}_{m_1,\dots,m_s}(q) \Bigg\}. \end{aligned} \tag{8}$$

Proof. The series (3) is substituted into (6). Since the series (3) converges uniformly and since the cubature formula (2) is exact for any constant, we have:

$$\delta_N[f] = -\sum_{s=1}^{n} \sum_{1\leqslant i_1<\ldots<i_s\leqslant n} \sum_{m_1=1}^{\infty} \ldots \sum_{m_s=1}^{\infty} \sum_{j_1=1}^{2^{m_1-1}} \ldots \sum_{j_s=1}^{2^{m_s-1}} \Big\{ c_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1,\ldots,i_s)\times \\ \times \sum_{k=1}^{N} C_k \chi_{m_1,j_1}\left(x_{i_1}^{(k)}\right)\ldots\chi_{m_s,j_s}\left(x_{i_s}^{(k)}\right)\Big\}. \tag{9}$$

Since the series in (3) is absolutely convergent, it follows that the series in (9) also absolutely converges. Applying the triangle inequality to the expression on the right-hand side of (9), we obtain:

$$\left|\delta_N[f]\right| \leqslant \sum_{s=1}^{n} \sum_{1\leqslant i_1<\ldots<i_s\leqslant n} \sum_{m_1=1}^{\infty} \ldots \sum_{m_s=1}^{\infty} \sum_{j_1=1}^{2^{m_1-1}} \ldots \sum_{j_s=1}^{2^{m_s-1}} \Big| c_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1,\ldots,i_s)\times \\ \times \sum_{k=1}^{N} C_k \chi_{m_1,j_1}\left(x_{i_1}^{(k)}\right)\ldots\chi_{m_s,j_s}\left(x_{i_s}^{(k)}\right)\Big|. \tag{10}$$

Now we apply the Hölder inequality to the sums over $j_1,\ldots,j_s$ on the right-hand side of (10). Taking into account (7), we obtain the inequality (8). □

It was shown in [9] that there exist Haar polynomials of one variable of degree m that satisfy the equality:

$$\kappa_{m,j}(x) = \begin{cases} 2^m, & x \in l_{m+1,j}, \\ 2^{m-1}, & x \in \overline{l_{m+1,j}} \setminus l_{m+1,j}, \\ 0, & x \in [0,1] \setminus \overline{l_{m+1,j}}, \end{cases} \tag{11}$$

where $m = 1,2,\ldots$ and $j = 1,2,\ldots,2^m$. It was also proved in [9] that the functions $\kappa_{m,1}(x),\ldots,\kappa_{m,2^m}(x)$ form a basis in the linear space of Haar polynomials of degree at most m.

The definition of the Haar functions (1) and relation (11) imply the following equalities:

$$\chi_{m,j}(x_i) = 2^{-\frac{m+1}{2}}\Big[\kappa_{m,2j-1}(x_i) - \kappa_{m,2j}(x_i)\Big], \tag{12}$$

$$\kappa_{m,2j-1}(x_i) + \kappa_{m,2j}(x_i) = 2\kappa_{m-1,j}(x_i), \tag{13}$$

$i = 1,\ldots,n,\ m = 1,2,\ldots,\ j = 1,\ldots,2^{m-1}$.

Let

$$K_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(x_{i_1},\ldots,x_{i_s}) = \kappa_{m_1,j_1}(x_{i_1})\ldots\kappa_{m_s,j_s}(x_{i_s}), \tag{14}$$

$1 \leqslant i_1 < \ldots < i_s \leqslant n,\ m_1,\ldots,m_s = 1,2,\ldots,\ s = 1,\ldots,n,\ j_r = 1,\ldots,2^{m_r-1},\ r = 1,\ldots,s.$

Lemma 2. *For any ordered set $(i_1,\ldots,i_s)$, $1 \leqslant i_1 < \ldots < i_s \leqslant n$, $1 \leqslant s \leqslant n$, and for any positive integer M there exists at least one ordered set $(M_1,\ldots,M_s)$ satisfying the inequality $M_1 + \ldots + M_s \geqslant M$ such that*

$$\Sigma_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}(q) = \sup_{m_1+\ldots+m_s\geqslant M} \Sigma_{m_1,\ldots,m_s}^{(i_1,\ldots,i_s)}(q). \tag{15}$$

Proof. For a fixed positive integer M, we choose $(\widetilde{m}_1,\ldots,\widetilde{m}_s)$ in accordance with condition that the sum $m_1+\ldots+m_s$ is minimum among all ordered sets $(m_1,\ldots,m_s)$ such that $m_1+\ldots+$ $+m_s \geqslant M$ and each of the closed s-dimensional binary parallelepipeds $\overline{l_{m_1+1,j_1}} \times \ldots \times \overline{l_{m_s+1,j_s}}$ contains at most one node of the cubature formula (2).

If the coordinates of the nodes of the cubature formula (2) $x_{i_r}^{(k)} \notin \{2^{-\widetilde{m}_r}(2j_r-1):\ j_r = 1,\dots,2^{\widetilde{m}_r-1}\}$, $k=1,\dots,N$, then we set $\widehat{m}_r=\widetilde{m}_r$. Otherwise, we set $\widehat{m}_r = 1+\max\{m_r\in\mathbb{N}:$ there exists $x_r^{(K)} = 2^{-m_r}(2j_r^{(K)}-1)$, $1\leqslant j_r^{(K)}\leqslant 2^{m_r-1}$, $1\leqslant K\leqslant N\}$, $r=1,\dots,s$.

Then, for all ordered sets $(m_1,\dots,m_s)$ such that $m_1+\ldots+m_s\geqslant \widehat{m}_1+\ldots+\widehat{m}_s$ the following three conditions are satisfied:

– the inequality $m_1+\ldots+m_s\geqslant M$ holds;

– each of the closed s-dimensional binary parallelepipeds $\overline{l_{m_1+1,j_1}}\times\ldots\times\overline{l_{m_s+1,j_s}}$ contains at most one node of the cubature formula (2);

– the coordinates of every node of the cubature formula (2) differ from the points $\{2^{-m_r}(2j_r-1)\} = \operatorname{supp}\{\kappa_{m_r,2j_r-1}\}\cap\operatorname{supp}\{\kappa_{m_r,2j_r}\}$, $j_r=1,\dots,2^{m_r-1}$, $r=1,\dots,s$.

By virtue of (7), (12), we have:

$$\varSigma^{(i_1,\dots,i_s)}_{\widehat{m}_1,\dots,\widehat{m}_s}(q) = 2^{-\widehat{m}_1-\ldots-\widehat{m}_s}\left\{\sum_{j_1=1}^{2^{\widehat{m}_1-1}}\cdots\sum_{j_s=1}^{2^{\widehat{m}_s-1}}\left|\sum_{k=1}^{N}C_k\times\right.\right.$$
$$\left.\left.\times\left[\kappa_{\widehat{m}_1,2j_1-1}\left(x_{i_1}^{(k)}\right)-\kappa_{\widehat{m}_1,2j_1}\left(x_{i_1}^{(k)}\right)\right]\cdots\left[\kappa_{\widehat{m}_s,2j_s-1}\left(x_{i_s}^{(k)}\right)-\kappa_{\widehat{m}_s,2j_s}\left(x_{i_s}^{(k)}\right)\right]\right|^q\right\}^{\frac{1}{q}}, \tag{16}$$

$1\leqslant i_1<\ldots<i_s\leqslant n,\ 1\leqslant s\leqslant n.$

According to the choice of $(\widehat{m}_1,\dots,\widehat{m}_s)$, the coordinates

$$x_{i_1}^{(k)},\dots,x_{i_s}^{(k)}\quad (k=1,\dots,N) \tag{17}$$

of every node of the cubature formula (2) differ from the points $\{2^{-\widehat{m}_r}(2j_r-1)\} = \operatorname{supp}\{\kappa_{\widehat{m}_r,2j_r-1}\}\cap\operatorname{supp}\{\kappa_{\widehat{m}_r,2j_r}\}$, $j_r=1,\dots,2^{m_r-1}$, $r=1,\dots,s$, and each of the closed s-dimensional binary parallelepipeds

$$\overline{l_{\widehat{m}_1+1,j_1}}\times\ldots\times\overline{l_{\widehat{m}_s+1,j_s}} \tag{18}$$

contains at most one node of the cubature formula (2) (by this fact every binary segment $\overline{l_{\widehat{m}_r+1,j_r}} = \operatorname{supp}\{\kappa_{\widehat{m}_r,j_r}\}$ contains a projection at most one of node of the cubature formula), $j_r=1,\dots,2^{m_r}$, $r=1,\dots,s$. Then the equality (16) can be rewritten as

$$\varSigma^{(i_1,\dots,i_s)}_{\widehat{m}_1,\dots,\widehat{m}_s}(q) = 2^{-\widehat{m}_1-\ldots-\widehat{m}_s}\left\{\sum_{j_1=1}^{2^{\widehat{m}_1}}\cdots\sum_{j_s=1}^{2^{\widehat{m}_s}}\left[\sum_{k=1}^{N}C_k\kappa_{\widehat{m}_1,j_1}\left(x_{i_1}^{(k)}\right)\cdots\kappa_{\widehat{m}_s,j_s}\left(x_{i_s}^{(k)}\right)\right]^q\right\}^{\frac{1}{q}} =$$
$$= 2^{-\widehat{m}_1-\ldots-\widehat{m}_s}\left\{\sum_{k=1}^{N}\sum_{j_1=1}^{2^{\widehat{m}_1}}\cdots\sum_{j_s=1}^{2^{\widehat{m}_s}}\left[C_k\kappa_{\widehat{m}_1,j_1}\left(x_{i_1}^{(k)}\right)\cdots\kappa_{\widehat{m}_s,j_s}\left(x_{i_s}^{(k)}\right)\right]^q\right\}^{\frac{1}{q}}, \tag{19}$$

$1\leqslant i_1<\ldots<i_s\leqslant n,\ 1\leqslant s\leqslant n.$ Here we use the fact that the sum

$$\sum_{k=1}^{N}C_k\kappa_{\widehat{m}_1,j_1}\left(x_{i_1}^{(k)}\right)\cdots\kappa_{\widehat{m}_s,j_s}\left(x_{i_s}^{(k)}\right)$$

contains at most one nonzero term for any ordered set $(j_1,\dots,j_s)$.

Consider the coordinates (17) of nodes of the cubature formula (2) satisfying the equality

$$x_{i_r}^{(k)} = 2^{-\widehat{m}_r}j_r,\ 1\leqslant j_r\leqslant 2^{\widehat{m}_r},\ 1\leqslant r\leqslant s. \tag{20}$$

The following $(s+1)$ cases are possible for the quantity of such coordinates of the nodes.

1. Equality (20) does not hold for any of the coordinates (17) of the nodes (for definiteness, the numbers of such nodes are denoted by $k = 1, \ldots, N_1$).

2. Only one coordinate in (17) satisfies equality (20) (let $k = N_1 + 1, \ldots, N_2$ be the numbers of nodes whose coordinates satisfy this condition).

3. Exactly two coordinates in (17) satisfy equality (20) (to be specific, we assume that the coordinates of the nodes with numbers $k = N_2 + 1, \ldots, N_3$ obey this condition).

. . . .

$s + 1$. Equality (20) holds for all s coordinates (17) (let $k = N_s + 1, \ldots, N$ be the numbers of nodes whose coordinates satisfy this condition).

Moreover, each of the nodes with the numbers $k = N_r+1, \ldots, N_{r+1}$ belongs to exact 2^r closed s-dimensional binary parallelepipeds of the form (18), where $r = 0, 1, \ldots, s$, $N_0 = 0$, $N_{s+1} = N$.

Given the above, as well as the equality (11), the relation (19) can be rewritten as

$$\Sigma^{(i_1,\ldots,i_s)}_{\widehat{m}_1,\ldots,\widehat{m}_s}(q) = 2^{-\widehat{m}_1-\ldots-\widehat{m}_s}\left[\sum_{k=1}^{N_1}\left(2^{\widehat{m}_1+\ldots+\widehat{m}_s}C_k\right)^q + 2\sum_{k=N_1+1}^{N_2}\left(2^{\widehat{m}_1+\ldots+\widehat{m}_s-1}C_k\right)^q + \right.$$
$$\left. +4\sum_{k=N_2+1}^{N_3}\left(2^{\widehat{m}_1+\ldots+\widehat{m}_s-2}C_k\right)^q + \ldots + 2^s\sum_{k=N_s+1}^{N}\left(2^{\widehat{m}_1+\ldots+\widehat{m}_s-s}C_k\right)^q\right]^{\frac{1}{q}} = \tag{21}$$
$$= \left[\sum_{k=1}^{N_1}{C_k}^q + 2^{1-q}\sum_{k=N_1+1}^{N_2}{C_k}^q + 2^{2(1-q)}\sum_{k=N_2+1}^{N_3}{C_k}^q + \ldots + 2^{s(1-q)}\sum_{k=N_s+1}^{N}{C_k}^q\right]^{\frac{1}{q}},$$

$1 \leqslant i_1 < \ldots < i_s \leqslant n,\ 1 \leqslant s \leqslant n.$

Since this reasoning holds not only for $(\widehat{m}_1, \ldots, \widehat{m}_s)$, but also for any ordered set $(m_1, \ldots, m_s)$ such that $m_1 + \ldots + m_s \geqslant \widehat{m}_1 + \ldots + \widehat{m}_s$ it is true that the value $\Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q)$ does not depend on $m_1, \ldots, m_s$ for all $(m_1, \ldots, m_s)$ satisfying the inequality $m_1 + \ldots + m_s \geqslant \widehat{m}_1 + \ldots + \widehat{m}_s$. Therefore, $\sup\limits_{m_1+\ldots+m_s\geqslant M}$ in the equality (15) reduces to $\max\limits_{M\leqslant m_1+\ldots+m_s\leqslant\widehat{m}_1+\ldots+\widehat{m}_s}$, whence we obtain the assertion of the lemma. □

Let q be a number related to p by

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{22}$$

Let us prove the following theorem.

Theorem 1. *If the cubature formula (2) is exact for any constant, then its error functional satisfy the following relations:*

$$|\delta_N[f]| \leqslant \|f\|_{S_p} \sup_{m_1,\ldots,m_s\in\mathbb{N}} \Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q),\ \ f \in S_p, \tag{23}$$

$$\|\delta_N\|_{S_p^*} = \sup_{m_1,\ldots,m_s\in\mathbb{N}} \Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q). \tag{24}$$

If the cubature formula (2) possesses the Haar d-property, then

$$|\delta_N[f]| \leqslant \|f\|_{S_p} \sup_{m_1+\ldots+m_s>d} \Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q),\ \ f \in S_p, \tag{25}$$

$$\|\delta_N\|_{S_p^*} = \sup_{m_1+\ldots+m_s>d} \Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q). \tag{26}$$

Proof. Let the cubature formula (2) be exact for any constant. By virtue of (4), (5), the inequality (23) follows from (8). Using (23), we obtain:

$$\|\delta_N\|_{S_p^*} \leqslant \sup_{m_1,\ldots,m_s\in\mathbb{N}} \Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q).$$

In order to establish that this inequality can not be improved, we use the technique applied in [8]. For $M = s$, we fix the ordered set $(M_1,\ldots,M_s)$, the existence of which was proved in Lemma 2. We introduce the following notation:

$$\Theta^{(i_1,\ldots,i_s)}_{j_1,\ldots,j_s} = 2^{-\frac{M_1-1}{2}-\ldots-\frac{M_s-1}{2}} \sum_{k=1}^{N} C_k\,\chi_{M_1,j_1}\left(x^{(k)}_{i_1}\right)\ldots\chi_{M_s,j_s}\left(x^{(k)}_{i_s}\right).$$

Then, according to Lemma 2, we have

$$\sup_{m_1+\ldots+m_s\geqslant M} \Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q) = \Sigma^{(i_1,\ldots,i_s)}_{M_1,\ldots,M_s}(q) = \left[\sum_{j_1=1}^{2^{M_1-1}}\cdots\sum_{j_s=1}^{2^{M_s-1}}\left|\sum_{k=1}^{N}\Theta^{(i_1,\ldots,i_s)}_{j_1,\ldots,j_s}\right|^q\right]^{\frac{1}{q}}. \tag{27}$$

Consider the function

$$f^{(i_1,\ldots,i_s)}_{M_1,\ldots,M_s}(x_1,\ldots,x_n) = \sum_{j_1=1}^{2^{M_1-1}}\cdots\sum_{j_s=1}^{2^{M_s-1}} \operatorname{sign}\Theta^{(i_1,\ldots,i_s)}_{j_1,\ldots,j_s}\left|\Theta^{(i_1,\ldots,i_s)}_{j_1,\ldots,j_s}\right|^{q-1}\chi_{M_1,j_1}(x_{i_1})\ldots\chi_{M_s,j_s}(x_{i_s}),$$

$1\leqslant i_1<\ldots<i_s\leqslant n,\ 1\leqslant s\leqslant n$. For this function, the Fourier–Haar coefficients are given by

$$c_0=0,\quad c^{(j_1,\ldots,j_s)}_{m_1,\ldots,m_s}(i_1,\ldots,i_s) = \begin{cases} \operatorname{sign}\Theta^{(i_1,\ldots,i_s)}_{j_1,\ldots,j_s}\left|\Theta^{(i_1,\ldots,i_s)}_{j_1,\ldots,j_s}\right|^{q-1}, & m_1=M_1,\ldots,m_s=M_s,\\ 0 & \text{otherwise.}\end{cases}$$

Then, taking into account the relation (4) and the equality $(q-1)p = q$, which follows from (22), we have:

$$A_p^{(i_1,\ldots,i_s)}\left(f^{(i_1,\ldots,i_s)}_{M_1,\ldots,M_s}\right) = 2^{\frac{M_1-1}{2}+\ldots+\frac{M_s-1}{2}}\left[\sum_{j_1=1}^{2^{M_1-1}}\cdots\sum_{j_s=1}^{2^{M_s-1}}\left|\Theta^{(i_1,\ldots,i_s)}_{j_1,\ldots,j_s}\right|^q\right]^{\frac{1}{p}}. \tag{28}$$

At the same time, according to (9),

$$\delta_N\left[f^{(i_1,\ldots,i_s)}_{M_1,\ldots,M_s}\right] = -\sum_{j_1=1}^{2^{M_1-1}}\cdots\sum_{j_s=1}^{2^{M_s-1}}\left[\operatorname{sign}\Theta^{(i_1,\ldots,i_s)}_{j_1,\ldots,j_s}\left|\Theta^{(i_1,\ldots,i_s)}_{j_1,\ldots,j_s}\right|^{q-1}\times\right.$$
$$\left.\times\sum_{k=1}^{N}C_k\,\chi_{M_1,j_1}\left(x^{(k)}_{i_1}\right)\ldots\chi_{M_s,j_s}\left(x^{(k)}_{i_s}\right)\right] = -2^{\frac{M_1-1}{2}+\ldots+\frac{M_s-1}{2}}\sum_{j_1=1}^{2^{M_1-1}}\cdots\sum_{j_s=1}^{2^{M_s-1}}\left|\Theta^{(i_1,\ldots,i_s)}_{j_1,\ldots,j_s}\right|^q.$$

The last relation, combined with (27) and (28), shows that

$$\left|\delta_N\left[f^{(i_1,\ldots,i_s)}_{M_1,\ldots,M_s}\right]\right| = 2^{\frac{M_1-1}{2}+\ldots+\frac{M_s-1}{2}}\left[\sum_{j_1=1}^{2^{M_1-1}}\cdots\sum_{j_s=1}^{2^{M_s-1}}\left|\Theta^{(i_1,\ldots,i_s)}_{j_1,\ldots,j_s}\right|^q\right]^{\frac{1}{p}}\times$$
$$\times\left[\sum_{j_1=1}^{2^{M_1-1}}\cdots\sum_{j_s=1}^{2^{M_s-1}}\left|\Theta^{(i_1,\ldots,i_s)}_{j_1,\ldots,j_s}\right|^q\right]^{\frac{1}{q}} = A_p^{(i_1,\ldots,i_s)}\left(f^{(i_1,\ldots,i_s)}_{M_1,\ldots,M_s}\right)\Sigma^{(i_1,\ldots,i_s)}_{M_1,\ldots,M_s}(q).$$

Note that $A_p^{(k_1,\ldots,k_s)}\left(f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\right) = 0$ for all ordered sets $(k_1,\ldots,k_s) \neq (i_1,\ldots,i_s)$. Then $\|f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\|_{S_p} = A_p^{(i_1,\ldots,i_s)}\left(f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\right)$, and

$$\left|\delta_N\left[f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\right]\right| = \Sigma_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}(q)\left\|f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\right\|_{S_p},$$

which implies the equality (24).

If the cubature formula (2) possesses the Haar d-property, then by virtue of its accuracy for Haar polynomials of degree at most d, the equality (9) becomes

$$\delta_N[f] = -\sum_{s=1}^{n}\sum_{1\leqslant i_1<\ldots<i_s\leqslant n}\sum_{m_1+\ldots+m_s>d}\sum_{j_1=1}^{2^{m_1-1}}\cdots\sum_{j_s=1}^{2^{m_s-1}}\left\{c_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1,\ldots,i_s)\times\right.$$
$$\left.\times\sum_{k=1}^{N}C_k\chi_{m_1,j_1}\left(x_{i_1}^{(k)}\right)\ldots\chi_{m_s,j_s}\left(x_{i_s}^{(k)}\right)\right\}.$$

Hence, the inequality (8) can be written as

$$\left|\delta_N[f]\right| \leqslant \sum_{s=1}^{n}\sum_{1\leqslant i_1<\ldots<i_s\leqslant n}\sum_{m_1+\ldots+m_s>d}\left\{2^{\frac{m_1-1}{2}+\ldots+\frac{m_s-1}{2}}\times\right.$$
$$\left.\times\left[\sum_{j_1=1}^{2^{m_1-1}}\cdots\sum_{j_s=1}^{2^{m_s-1}}\left|c_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(i_1,\ldots,i_s)\right|^p\right]^{\frac{1}{p}}\Sigma_{m_1,\ldots,m_s}^{(i_1,\ldots,i_s)}(q)\right\}.$$

Then the inequality (23) becomes (25). Proceeding as in the proof of the equality (24), we construct the function $f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}(x_1,\ldots,x_n)$ such that

$$\left|\delta_N\left[f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\right]\right| = \left\|f_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}\right\|_{S_p}\sup_{m_1+\ldots+m_s>d}\Sigma_{m_1,\ldots,m_s}^{(i_1,\ldots,i_s)}(q), \tag{29}$$

where the ordered set $(M_1,\ldots,M_s)$ satisfies the following conditions:

$$M_1+\ldots+M_s>d,$$

$$\Sigma_{M_1,\ldots,M_s}^{(i_1,\ldots,i_s)}(q) = \sup_{m_1+\ldots+m_s>d}\Sigma_{m_1,\ldots,m_s}^{(i_1,\ldots,i_s)}(q).$$

This ordered set exists by virtue of Lemma 2, which is used for $M = d+1$.

The equality (26) follows from (25) and (29). □

Lemma 3. *For positive integer $m_1,\ldots,m_s$ satisfying the inequality*

$$m_1+\ldots+m_s \leqslant d, \tag{30}$$

it is true that

$$Q_N\left[K_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(x_{i_1},\ldots,x_{i_s})\right] = I\left[K_{m_1,\ldots,m_s}^{(j_1,\ldots,j_s)}(x_{i_1},\ldots,x_{i_s})\right] = 1, \tag{31}$$

where $1\leqslant i_1<\ldots<i_s\leqslant n$, $s=1,\ldots,n$, $j_r=1,\ldots,2^{m_r-1}$, $r=1,\ldots,s$.

Proof. Since each of the functions $\kappa_{m_1,j_1}(x_{i_1}),\ldots,\kappa_{m_s,j_s}(x_{i_s})$ is a Haar polynomial of one variable and the degrees of these polynomials are $m_1,\ldots,m_s$ respectively, then it follows from (14) that for $m_1,\ldots,m_s$ satisfying the condition (30), the function $K^{(j_1,\ldots,j_s)}_{m_1,\ldots,m_s}(x_{i_1},\ldots,x_{i_s})$ is a Haar polynomial of degree $m_1+\ldots+m_s\leqslant d$ of variables $x_{i_1},\ldots,x_{i_s}$. Then, by virtue of the accuracy of the cubature formula (2) for the Haar polynomials of degree at most d, the first equality in (31) holds true.

The second equality in (31) follows from the relations (14) and (11), which define the functions $K^{(j_1,\ldots,j_s)}_{m_1,\ldots,m_s}(x_{i_1},\ldots,x_{i_s})$ and $\kappa_{m_1,j_1}(x_{i_1}),\ldots,\kappa_{m_s,j_s}(x_{i_s})$. □

Lemma 4. *For positive integer l, the following inequality holds:*

$$\varSigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q)\leqslant 2^{-m_1-\ldots-m_s+ls}\left\{\sum_{j_1=1}^{2^{m_1-l}}\cdots\sum_{j_s=1}^{2^{m_s-l}}\left\{Q_N\left[K^{(j_1,\ldots,j_s)}_{m_1-l,\ldots,m_s-l}(x_{i_1},\ldots,x_{i_s})\right]\right\}^q\right\}^{\frac{1}{q}},\qquad(32)$$

where $1\leqslant i_1<\ldots<i_s\leqslant n,\ m_1,\ldots,m_s=1,2,\ldots,\ s=1,\ldots,n.$

Proof. Inequality (32) is proved by induction on l.

Applying the triangle inequality, and also taking into account the equality (12) and the positivity of the coefficients at the nodes of the cubature formula (2), we obtain:

$$\begin{gathered}\left|\sum_{k=1}^{N}C_k\,\chi_{m_1,j_1}\left(x^{(k)}_{i_1}\right)\ldots\chi_{m_s,j_s}\left(x^{(k)}_{i_s}\right)\right|\leqslant 2^{-\frac{m_1+1}{2}-\ldots-\frac{m_s+1}{2}}\times\\ \times\sum_{k=1}^{N}C_k\left|\kappa_{m_1,2j_1-1}\left(x^{(k)}_{i_1}\right)-\kappa_{m_1,2j_1}\left(x^{(k)}_{i_1}\right)\right|\ldots\left|\kappa_{m_s,2j_s-1}\left(x^{(k)}_{i_s}\right)-\kappa_{m_s,2j_s}\left(x^{(k)}_{i_s}\right)\right|.\end{gathered}\qquad(33)$$

The nonnegativity of the functions $\kappa_{m,j}(x)$ implies the inequality

$$\left|\kappa_{m_r,2j_r-1}\left(x^{(k)}_{i_r}\right)-\kappa_{m_r,2j_r}\left(x^{(k)}_{i_r}\right)\right|\leqslant\kappa_{m_r,2j_r-1}\left(x^{(k)}_{i_r}\right)+\kappa_{m_r,2j_r}\left(x^{(k)}_{i_r}\right),$$

$r=1,\ldots,s,\ \ k=1,\ldots,N$. Then, by virtue of the equalities (13) and (14), it is true that

$$\begin{gathered}\left|\kappa_{m_1,2j_1-1}\left(x^{(k)}_{i_1}\right)-\kappa_{m_1,2j_1}\left(x^{(k)}_{i_1}\right)\right|\ldots\left|\kappa_{m_s,2j_s-1}\left(x^{(k)}_{i_s}\right)-\kappa_{m_s,2j_s}\left(x^{(k)}_{i_s}\right)\right|\leqslant\\ \leqslant\left[\kappa_{m_1,2j_1-1}\left(x^{(k)}_{i_1}\right)+\kappa_{m_1,2j_1}\left(x^{(k)}_{i_1}\right)\right]\ldots\left[\kappa_{m_s,2j_s-1}\left(x^{(k)}_{i_s}\right)+\kappa_{m_s,2j_s}\left(x^{(k)}_{i_s}\right)\right]=\\ =2^s\kappa_{m_1-1,j_1}\left(x^{(k)}_{i_1}\right)\ldots\kappa_{m_s-1,j_s}\left(x^{(k)}_{i_s}\right)=2^sK^{(j_1,\ldots,j_s)}_{m_1-1,\ldots,m_s-1}\left(x^{(k)}_{i_1},\ldots,x^{(k)}_{i_s}\right).\end{gathered}$$

Combining this with (33) yields

$$\left|\sum_{k=1}^{N}C_k\,\chi_{m_1,j_1}\left(x^{(k)}_{i_1}\right)\ldots\chi_{m_s,j_s}\left(x^{(k)}_{i_s}\right)\right|\leqslant 2^{-\frac{m_1+1}{2}-\ldots-\frac{m_s+1}{2}+s}Q_N\left[K^{(j_1,\ldots,j_s)}_{m_1-1,\ldots,m_s-1}(x_{i_1},\ldots,x_{i_s})\right],$$

which implies (32) for $l=1$.

Based on the induction hypothesis that

$$\begin{gathered}\varSigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q)\leqslant 2^{-m_1-\ldots-m_s+ls-s}\times\\ \times\left\{\sum_{j_1=1}^{2^{m_1-l+1}}\cdots\sum_{j_s=1}^{2^{m_s-l+1}}\left\{Q_N\left[K^{(j_1,\ldots,j_s)}_{m_1-l+1,\ldots,m_s-l+1}(x_{i_1},\ldots,x_{i_s})\right]\right\}^q\right\}^{\frac{1}{q}},\end{gathered}\qquad(34)$$

we prove (32). The sum on the right-hand side of the inequality (34) can be written as

$$\begin{gathered}\sum_{j_1=1}^{2^{m_1-l+1}} \ldots \sum_{j_s=1}^{2^{m_s-l+1}} \left\{ Q_N \left[K^{(j_1,\ldots,j_s)}_{m_1-l+1,\ldots,m_s-l+1}(x_{i_1},\ldots,x_{i_s}) \right] \right\}^q = \\ = \sum_{j_1=1}^{2^{m_1-l}} \ldots \sum_{j_s=1}^{2^{m_s-l}} \sum_{J_1=2j_1-1}^{2j_1} \ldots \sum_{J_s=2j_s-1}^{2j_s} \left\{ Q_N \left[K^{(J_1,\ldots,J_s)}_{m_1-l+1,\ldots,m_s-l+1}(x_{i_1},\ldots,x_{i_s}) \right] \right\}^q .\end{gathered} \tag{35}$$

Using inequality

$$\sum_{i=1}^{M} a_i^q \leqslant \left\{ \sum_{i=1}^{M} a_i \right\}^q \quad (a_i \geqslant 0,\ i=1,\ldots,M,\ q>1)$$

and equality (13), we have:

$$\begin{gathered}\sum_{J_1=2j_1-1}^{2j_1} \ldots \sum_{J_s=2j_s-1}^{2j_s} \left\{ Q_N \left[K^{(J_1,\ldots,J_s)}_{m_1-l+1,\ldots,m_s-l+1}(x_{i_1},\ldots,x_{i_s}) \right] \right\}^q \leqslant \\ \leqslant \left\{ Q_N \left[\sum_{J_1=2j_1-1}^{2j_1} \ldots \sum_{J_s=2j_s-1}^{2j_s} K^{(J_1,\ldots,J_s)}_{m_1-l+1,\ldots,m_s-l+1}(x_{i_1},\ldots,x_{i_s}) \right] \right\}^q = \\ = \left\{ Q_N \left[\sum_{J_1=2j_1-1}^{2j_1} \ldots \sum_{J_s=2j_s-1}^{2j_s} \kappa_{m_1-l+1,J_1}(x_{i_1}) \ldots \kappa_{m_s-l+1,J_s}(x_{i_s}) \right] \right\}^q = \\ = \left\{ Q_N \left[\left(\kappa_{m_1-l+1,2j_1-1}(x_{i_1}) + \kappa_{m_1-l+1,2j_1}(x_{i_1}) \right) \ldots \left(\kappa_{m_s-l+1,2j_s-1}(x_{i_1}) + \kappa_{m_s-l+1,2j_s}(x_{i_s}) \right) \right] \right\}^q = \\ = \left\{ Q_N \left[2^s \kappa_{m_1-l,j_1}(x_{i_1}) \ldots \kappa_{m_s-l,j_s}(x_{i_s}) \right] \right\}^q = \left\{ 2^s Q_N \left[K^{(j_1,\ldots,j_s)}_{m_1-l,\ldots,m_s-l}(x_{i_1},\ldots,x_{i_s}) \right] \right\}^q .\end{gathered}$$

In view of the equality (35) and the last relations, it follows from (34) that the inequality (32) holds true. □

Lemma 5. *If the cubature formula (2) possesses the Haar d-property, then*

$$\sup_{m_1+\ldots+m_s>d} \Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q) \leqslant 2^{\frac{n-1}{p}} \left(2^d\right)^{-\frac{1}{p}} . \tag{36}$$

Proof. Let $(m_1,\ldots,m_s)$ be an arbitrary fixed set of indices for which the inequality $m_1+\ldots+m_s>d$ holds true. We denote by l the minimal number among all integers L satisfying the condition

$$m_1+\ldots+m_s-Ls \leqslant d. \tag{37}$$

Then the following equality holds:

$$m_1+\ldots+m_s-ls = d-r, \text{ where } r \in \{0,1,\ldots,s-1\} . \tag{38}$$

Applying Lemmas 4 and 3 (by virtue of (37), the condition of Lemma 3 for the lower indices of the Haar polynomial $K^{(j_1,\ldots,j_s)}_{m_1-l,\ldots,m_s-l}(x_{i_1},\ldots,x_{i_s})$ is satisfied) and taking into account (22) yields

$$\begin{gathered}\Sigma^{(i_1,\ldots,i_s)}_{m_1,\ldots,m_s}(q) \leqslant 2^{-m_1-\ldots-m_s+ls} \left\{ \sum_{j_1=1}^{2^{m_1-l}} \ldots \sum_{j_s=1}^{2^{m_s-l}} 1 \right\}^{\frac{1}{q}} = \\ = 2^{-m_1-\ldots-m_s+ls} \left(2^{m_1+\ldots+m_s-ls}\right)^{\frac{1}{q}} = \left(2^{m_1+\ldots+m_s-ls}\right)^{-\frac{1}{p}} .\end{gathered} \tag{39}$$

The relations (39) and (38) imply

$$\Sigma^{(i_1,\dots,i_s)}_{m_1,\dots,m_s}(q) \leqslant \left(2^{d-r}\right)^{-\frac{1}{p}} = 2^{\frac{r}{p}}\left(2^{d}\right)^{-\frac{1}{p}} \leqslant 2^{\frac{s-1}{p}}\left(2^{d}\right)^{-\frac{1}{p}} \leqslant 2^{\frac{n-1}{p}}\left(2^{d}\right)^{-\frac{1}{p}},$$

whence we obtain the inequality (36). □

Lemma 6. *If the cubature formula (2) is exact for any constant, then*

$$\sup_{m_1,\dots,m_s\in\mathbb{N}} \Sigma^{(i_1,\dots,i_s)}_{m_1,\dots,m_s}(q) \geqslant \left(2^{n+1}-n-1\right)^{-\frac{1}{p}} N^{-\frac{1}{p}}. \tag{40}$$

Proof. Consider the function

$$\varphi(C_1,\dots,C_N) = \sum_{k=1}^{N_1} {C_k}^q + 2^{1-q}\sum_{k=N_1+1}^{N_2} {C_k}^q + 2^{2(1-q)}\sum_{k=N_2+1}^{N_3} {C_k}^q + \ldots + 2^{s(1-q)}\sum_{k=N_s+1}^{N} {C_k}^q, \tag{41}$$

where the constants $N_1,\dots,N_s$ are defined in the proof of Lemma 2. By virtue of (21), the equality

$$\Sigma^{(i_1,\dots,i_s)}_{\widehat{m}_1,\dots,\widehat{m}_s}(q) = \left[\varphi\left(C_1,C_2,\dots,C_N\right)\right]^{\frac{1}{q}} \tag{42}$$

holds true.

If the cubature formula (2) satisfies the condition

$$C_1+C_2+\ldots+C_N = 1 \ (C_i>0,\ i=1,2,\dots,N),$$

which follows from the accuracy of (2) for any constant, it is easy to show that the function (41) attains its infimum, which is equal to

$$\left[N_1+2\left(N_2-N_1\right)+2^2\left(N_3-N_2\right)+\ldots+2^s\left(N-N_s\right)\right]^{1-q} =$$

$$= \left[N+\left(2^1-1\right)\left(N_2-N_1\right)+\left(2^2-1\right)\left(N_3-N_2\right)+\ldots+\left(2^s-1\right)\left(N-N_s\right)\right]^{1-q},$$

when

$$C_1 = C_2 = \ldots = C_{N_1} = \left[N_1+2\left(N_2-N_1\right)+2^2\left(N_3-N_2\right)+\ldots+2^s\left(N-N_s\right)\right]^{-1},$$

$$C_{N_1+1} = C_{N_1+2} = \ldots = C_{N_2} = 2\left[N_1+2\left(N_2-N_1\right)+2^2\left(N_3-N_2\right)+\ldots+2^s\left(N-N_s\right)\right]^{-1},$$

$$C_{N_2+1} = C_{N_2+2} = \ldots = C_{N_3} = 2^2\left[N_1+2\left(N_2-N_1\right)+2^2\left(N_3-N_2\right)+\ldots+2^s\left(N-N_s\right)\right]^{-1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$C_{N_s+1} = C_{N_s+2} = \ldots = C_{N} = 2^s\left[N_1+2\left(N_2-N_1\right)+2^2\left(N_3-N_2\right)+\ldots+2^s\left(N-N_s\right)\right]^{-1}.$$

Then, taking into account (22), we derive from (42)

$$\Sigma^{(i_1,\dots,i_s)}_{\widehat{m}_1,\dots,\widehat{m}_s}(q) \geqslant \left[N+\left(2^1-1\right)\left(N_2-N_1\right)+\left(2^2-1\right)\left(N_3-N_2\right)+\ldots+\left(2^s-1\right)\left(N-N_s\right)\right]^{-\frac{1}{p}} \geqslant$$

$$\geqslant \left[N+\left(2^1-1\right)N+\left(2^2-1\right)N+\ldots+\left(2^s-1\right)N\right]^{-\frac{1}{p}} = \left(2^{s+1}-s-1\right)^{-\frac{1}{p}} N^{-\frac{1}{p}} \geqslant$$

$$\geqslant \left(2^{n+1}-n-1\right)^{-\frac{1}{p}} N^{-\frac{1}{p}},$$

where $(\widehat{m}_1,\dots,\widehat{m}_s)$ is the ordered set chosen in the proof of Lemma 2 (in this case $M=s$, where M is the parameter from the conditions of Lemma 2). This yields the inequality (40). □

Theorem 2. *For the cubature formula (2) exact for any constants, the norm of the error functional satisfies the inequality*

$$\|\delta_N\|_{S_p^*} \geqslant \left(2^{n+1} - n - 1\right)^{-\frac{1}{p}} N^{-\frac{1}{p}}. \tag{43}$$

If the cubature formula (2) possesses the Haar d-property, then

$$|\delta_N\,[f]| \leqslant 2^{\frac{n-1}{p}} \left(2^d\right)^{-\frac{1}{p}} \|f\|_{S_p}, \tag{44}$$

$$\|\delta_N\|_{S_p^*} \leqslant 2^{\frac{n-1}{p}} \left(2^d\right)^{-\frac{1}{p}}. \tag{45}$$

Inequality (43) follows from Theorem 1 and Lemma 6, while inequalities (44), (45) follow from Theorem 1 and Lemma 5.

Remark 1. In [9] one considered the following weighted quadrature formulas possessing the Haar d-property:

$$\int_0^1 g(x) f(x)\,dx \approx \sum_{k=1}^{N} C_k f\left(x^{(k)}\right), \tag{46}$$

where $x^{(k)} \in [0,1]$ are the nodes of a formula; C_k are the coefficients of the formula at the nodes (real numbers); and $k = 1, \ldots, N$. If the weight function $g(x) \equiv 1$, then the number N of nodes of the quadrature formula (46) satisfies the inequality $N \geqslant 2^{d-1}$. The last inequality follows from a lower estimate for the number of nodes of the quadrature formula (46) possessing the Haar d-property, where $g(x)$ is an arbitrary weight function (see [9]).

Moreover, in [9] all minimal weighted quadrature formulas possessing the d-property were described. In the case of the weight function $g(x) \equiv 1$, it was proved that the minimal formula is unique: the number of its nodes is $N = 2^{d-1}$, the nodes of this formula are $x^{(k)} = 2^{-d}(2k-1)$, and the node coefficients are $C_k = 2^{-d+1}$ for $k = 1, 2, \ldots, 2^{d-1}$. The norm of the error functional of this formula satisfies the equality (see [10])

$$\|\delta_N\|_{S_p^*} = 2^{-\frac{1}{p}} N^{-\frac{1}{p}}, \tag{47}$$

which also follows from the inequalities (43) and (45) for $n = 1$; a number d related to N by $N = 2^{d-1}$.

Remark 2. In [12], one constructed the minimal cubature formulas possessing the Haar d-property for $d \geqslant 5$:

$$\int_0^1 \int_0^1 f(x_1, x_2)\,dx_1\,dx_2 \approx \sum_{k=1}^{N} C_k f\left(x_1^{(k)}, x_2^{(k)}\right), \tag{48}$$

where $\left(x_1^{(k)}, x_2^{(k)}\right) \in [0,1]^2$ are the nodes of a formula; C_k are the coefficients of the formula at the nodes (real numbers); and $k = 1, \ldots, N$. The number N of nodes of such formulas satisfies the equality

$$N = \begin{cases} 2^d - 3 \cdot 2^{\frac{d-1}{2}} + 2, & d \text{ is odd}, \\ 2^d - 2^{\frac{d}{2}+1} + 2, & d \text{ is even}, \end{cases} \tag{49}$$

where $d = 5, 6, 7, \ldots$ Then, the norm of the error functional of the minimal cubature formulas (48) possessing the Haar d-property satisfies the inequality

$$\|\delta_N\|_{S_p^*} \leqslant E_N, \tag{50}$$

$$\text{where } E_N = \begin{cases} 2^{\frac{1}{p}}\left(N+\frac{3\sqrt{2}}{2}\sqrt{N-\frac{7}{8}}+\frac{1}{4}\right)^{-\frac{1}{p}}, & d \text{ is odd,} \\ 2^{\frac{1}{p}}\left(N+2\sqrt{N-1}\right)^{-\frac{1}{p}}, & d \text{ is even.} \end{cases} \tag{51}$$

The inequality (50) follows from the estimate

$$\|\delta_N\|_{S_p^*} \leqslant 2^{\frac{1}{p}}(2^d)^{-\frac{1}{p}},$$

which was obtained in [16] for the norm of the error functional of arbitrary cubature formulas (48) having the Haar d-property. The number N of nodes of these cubature formulas is defined by (49).

The relations (50), (51) also follows from (45) for $n=2$; a number d related to N by (49).

3. Conclusions

In [8], the cubature formulas

$$\int_0^1 \cdots \int_0^1 f(x_1,\ldots,x_n)\,dx_1\ldots dx_n \approx \frac{1}{N}\sum_{k=1}^{N} f\left(x_1^{(k)},\ldots,x_n^{(k)}\right) \tag{52}$$

with nodes $\left(x_1^{(k)},\ldots,x_n^{(k)}\right) \in [0,1]^n$ $(k=1,\ldots,N)$ were considered that form P_τ-nets, i.e., nets that consist of $N=2^\nu$ nodes and satisfy the following condition: each binary parallelepiped of volume $2^{\tau-\nu}$ contains 2^τ net points $(\nu>\tau)$. For such formulas with a function f from S_p, the following upper estimate for the norm of the error functional was proved in [8]:

$$\|\delta_N\|_{S_p^*} \leqslant 2^{\frac{n-1+\tau}{p}} N^{-\frac{1}{p}}. \tag{53}$$

It is easy to see that for $n=1$ and $n=2$ P_τ-nets with an arbitrarily large number $N=2^\nu$ of nodes exist for any $\tau=0,1,2,\ldots$ Therefore, in the one- and two-dimensional cases, the constant multiplier on the right-hand side of (53) takes the least value at $\tau=0$, and estimate (53) for the cubature formulas (52) with nodes forming P_0-nets in the one-dimensional case is written as

$$\|\delta_N\|_{S_p^*} \leqslant N^{-\frac{1}{p}}, \tag{54}$$

while in the two-dimensional case this estimate is written as

$$\|\delta_N\|_{S_p^*} \leqslant 2^{\frac{1}{p}} N^{-\frac{1}{p}}. \tag{55}$$

It was proved in [8] that cubature formulas (52) with 2^d nodes forming P_0-nets have the Haar d-property. Therefore, the estimate (45), which is obtained in the present paper, is a generalization of the estimate (53) to the case of arbitrary cubature formulas possessing the Haar d-property.

Moreover, for any cubature formula (52) with a function $f \in S_p$, it was established in [8] that the norm of the error functional satisfies the lower estimate

$$\|\delta_N\|_{S_p^*} \geqslant N^{-\frac{1}{p}}.$$

Hence, the cubature formulas (52) with the nodes forming P_τ-nets have the best convergence rate of δ_N in the norm, which is equal to $N^{-\frac{1}{p}}$ as $N \to \infty$.

The relations (43), (47), (50), (51) imply that for minimal formulas possessing the Haar d-property in the one- and two-dimensional cases $\|\delta_N\|_{S_p^*} \asymp N^{-\frac{1}{p}}$ as $N \to \infty$.

Comparing the values on the right-hand sides of the relations (47) and (54), as well as (50) and (55), we conclude that the upper bounds for the $\|\delta_N\|_{S_p^*}$ in the case of minimal quadrature formulas (46) with the weight function $g(x) \equiv 1$ and the minimal cubature formulas (48) with the d-property are less than the upper bounds for this value in the inequalities (54) and (55), respectively, i.e., the upper bounds for the norm of the error functional of formulas with nodes forming the P_0-net in the one- and two-dimensional cases.

In addition, the quadrature formula (46) with the weight function $g(x) \equiv 1$ and the number $N = 2^{d-1}$ of nodes, as well as the cubature formula (48) with the number N of nodes satisfying the equality (49), being the minimal formulas of approximate integration, provide the best pointwise convergence of $\delta_N[f]$ to zero as $N \to \infty$.

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Об оценках погрешности на пространствах S_p кубатурных формул, точных для полиномов Хаара

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Аннотация. Получены верхняя и нижняя оценки нормы функционала погрешности обладающих d-свойством Хаара кубатурных формул на пространствах S_p в n-мерном случае.

Ключевые слова: d-свойство Хаара, погрешность кубатурной формулы, пространства S_p.

DOI: 10.17516/1997-1397-2020-13-4-414-421
УДК 512.7

Algebraic Geometry over Heyting Algebras

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Received 06.04.2020, received in revised form 25.04.2020, accepted 16.05.2020

Abstract. In this article, we study the algebraic geometry over Heyting algebras and we investigate the properties of being equationally Noetherian and q_ω-compact over such algebras.

Keywords: universal algebraic geometry, systems of equations, radicals, Zariski topology, Heyting algebras, equationally Noetherian algebras, q_ω-compact algebras.

Citation: M. Nouri, Algebraic Geometry over Heyting Algebras, J. Sib. Fed. Univ. Math. Phys., 2020, 13(4), 414–421. DOI: 10.17516/1997-1397-2020-13-4-414-421.

Universal algebraic geometry is a new area of modern algebra, whose subject is basically the study of equations over an arbitrary algebraic structure A (see [11]). In the classical algebraic geometry A of type $\mathcal{L}$ is a field. Many articles already published about algebraic geometry over groups, see [1, 8, 16], and [10]. O. Kharlampovich and A. Miyasnikov developed algebraic geometry over free groups to give affirmative answer for an old problem of Alfred Tarski concerning elementary theory of free groups (see [7] and also [15] for the independent solution of Z. Sela). Also in [9], a problem of Tarski about decidablity of the elementary theory of free groups is solved. Algebraic geometry over algebraic structures (universal algebraic geometry) is also developed for algebras other than groups. A systematic study of universal algebraic geometry is done in a series of articles by V. Remeslennikov, A. Myasnikov and E. Daniyarova in [2–4], and [5].

The notations of the present paper are standard and can be find in [2] or [11]. Our main aim in this article is to deal with the equational conditions in the universal algebraic geometry over Heyting algebras, i.e. different conditions relating systems of equations especially conditions about systems and sub-systems of equations over algebras. The main examples of such conditions are equational noetherian property and q_ω-compactness. We begin with a review of basic concepts of universal algebraic geometry and we describe the properties of being being equational noetherian, q_ω-compact. We will show that only finite Heyting algebras have these properties.

1. Basic notions

We need to give a brief introduction of universal algebraic geometry. Our notations here are almost the same as in the above mentioned papers, especially [11].

We begin with an algebraic language $\mathcal{L}$ and an arbitrary algebra A type $\mathcal{L}$ and then we extended the language by adding new constant symbols $a \in A$. This extended language will be denoted by $\mathcal{L}(A)$. An algebra B of type $\mathcal{L}(A)$ is called A-*algebra*, if the map $a \mapsto a^B$ is an embedding of A in B. In this notation, a^B denotes the interpretation of the constant symbol a in B.

Suppose that $X = \{x_1, \ldots, x_n\}$ is a finite set of variables. We denote the term algebra in the language $\mathcal{L}$ and variables from X by $T_{\mathcal{L}}(X)$, and similarly the term algebra in the extended

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language $\mathcal{L}(A)$ will denoted by $T_{\mathcal{L}(A)}(X)$. For the sake of simplicity, we define our notions in the coefficient free frame, i.e. in the language $\mathcal{L}$ and then we can extend all the definitions to the language $\mathcal{L}(A)$.

An *equation* is a pair (p, q) of the elements of the term algebra $T_{\mathcal{L}}(X)$. In many cases, we assume that such an equation is the same as the atomic formula $p(x_1, \dots, x_n) \approx q(x_1, \dots, x_n)$ or $p \approx q$ in short.

Any set of equations is called a *system of equations* in the language $\mathcal{L}$. A system S is called *consistent* over an algebra A, if there is an element $(a_1, \dots, a_n) \in A^n$ such that for all equations $(p \approx q) \in S$, the equality

$$p^A(a_1, \dots, a_n) = q^A(a_1, \dots, a_n)$$

holds. Otherwise, we say that S is *in-consistent* over A. Note that, p^A and q^A are the corresponding term functions on A^n. A system of equations S is called an ideal, if it corresponds to a congruence on $T_{\mathcal{L}}(X)$. For an arbitrary system of equations S, the ideal generated by S, is the smallest congruence containing S and it is denoted by $[S]$.

For an algebra A of type $\mathcal{L}$, an element $(a_1, \dots, a_n) \in A^n$ will be denoted by $\overline{a}$, sometimes. Let S be a system of equations. Then the set

$$V_A(S) = \{\overline{a} \in A^n : \forall (p \approx q) \in S,\ p^A(\overline{a}) = q^A(\overline{a})\}$$

is called an *algebraic set*. It is clear that for any non-empty family $\{S_i\}_{i \in I}$, we have

$$V_A(\bigcup_{i \in I} S_i) = \bigcap_{i \in I} V_A(S_i).$$

It is possible to define a topology on A^n using algebraic sets as elements of subbasis: define a closed set in A^n to be an arbitrary intersections of finite unions of algebraic sets. Therefore, we obtain a topology on A^n, which is called *Zariski topology*.

For any set $Y \subseteq A^n$, we define

$$\mathrm{Rad}(Y) = \{(p, q) : \forall\ \overline{a} \in Y,\ p^A(\overline{a}) = q^A(\overline{a})\}.$$

It is easy to see that $\mathrm{Rad}(Y)$ is an ideal in the term algebra. Any ideal of this type is called an *A-radical ideal* or a *radical ideal* for short. Note that any ideal in the term algebra is in fact a radical ideal. To see the reason, just note that for any ideal R in the term algebra $T_{\mathcal{L}}(X)$, if we consider the algebra $B(R) = T_{\mathcal{L}}(X)/R$, then $\mathrm{Rad}_{B(R)}(R) = R$.

It is easy to see that a set Y is algebraic if and only if $V_A(\mathrm{Rad}(Y)) = Y$. In the general case, we have $V_A(\mathrm{Rad}(Y)) = Y^{ac}$ (see [3]). The *coordinate algebra* of a set Y is the quotient algebra

$$\Gamma(Y) = \frac{T_{\mathcal{L}}(X)}{\mathrm{Rad}(Y)}.$$

An arbitrary element of $\Gamma(Y)$ is denoted by $[p]_Y$. We define a function $p^Y : Y \to A$ by the rule

$$p^Y(\overline{a}) = p^A(a_1, \dots, a_n),$$

which is a *term function* on Y, for all $a_1, \dots, a_n \in A$. The set of all such functions will be denoted by $T(Y)$ and it is naturally an algebra of type $\mathcal{L}$. It is easy to see that the map $[p]_Y \mapsto p^Y$ is a well-defined isomorphism. So, we have $\Gamma(Y) \cong T(Y)$.

For a system of equation, we can also define the radical $\mathrm{Rad}_A(S)$ to be $\mathrm{Rad}(V_A(S))$. Two systems S and S' are called equivalent over A, if they have the same set of solutions in A, i.e. $V_A(S) = V_A(S')$. So, clearly $\mathrm{Rad}_A(S)$ is the largest system which is equivalent to S. Note that $[S] \subseteq \mathrm{Rad}_A(S)$.

One of the major problems of the universal algebraic geometry is to determine the structures of algebras which appear as the coordinate algebras. There are many necessary and sufficient conditions for an algebra to be a coordinate algebra and we will give a summary of such results in the Subsection 2.4.

In this article, we are dealing with equational conditions on algebras. The first and maybe the most important condition of this type can be formulated as follows.

Definition 1. *An algebra A is called equational Noetherian, if for any system of equations S, there exists a finite subsystem $S_0 \subseteq S$, which is equivalent to S over A, i.e. $V_A(S) = V_A(S_0)$.*

If an A-algebra is equational Noetherian in the language $\mathcal{L}(A)$, then we call it A-equational Noetherian. Many examples of equational Noetherian algebras are introduced in [3]. Among them are Noetherian rings and linear groups over Noetherian rings as well as free groups. In [3], it is proved that the next four assertions are equivalent:

i- An algebra A is equational Noetherian.

ii- For any system S, there exists a finite $S_0 \subseteq [S]$, such that $V_A(S) = V_A(S_0)$.

iii- For any n, the Zariski topology on A^n is Noetherian, i.e. any descending chain of closed subsets terminates.

iv- Any chain of coordinate algebras and epimorphisims

$$\Gamma(Y_1) \to \Gamma(Y_2) \to \Gamma(Y_3) \to \cdots$$

terminates.

So, in the case of equational Noetherian algebras, any closed set in A^n is equal to a minimal finite union of *irreducible* algebraic sets which is unique up to a permutation. Note that a set is called irreducible, if it has no proper finite covering consisting of closed sets. The following theorem is proved in [3].

Theorem 1. *Let A be an equational Noetherian algebra. Then the following algebras are also equational Noetherian:*

i- any subalgebra and filter-power of A.

ii- any coordinate algebra over A.

iii- any fully residually A-algebra.

iv- any algebra belonging to the quasi-variety generated by A.

v- any algebra universally equivalent to A.

vi- any limit algebra over A.

vii- any finitely generated algebra defined by a complete atomic type in the universal theory of A or in the set of quasi-identities of A.

The most important theorem for equationally Noetherian algebras is called Unification Theorem. It describes the structure of coordinate algebras over equationally Noetherian algebras. For a proof of this theorem (see [2]).

Theorem 2. *Let A and Γ be algebras in a language $\mathcal{L}$. Suppose A is equational Noetherian and Γ is finitely generated. Then the following assertions are equivalent.*

i- Γ is the coordinate algebra of some irreducible algebraic set over A.

ii- Γ is a fully residually A-algebra. This means that for any finite subset $C \subseteq \Gamma$, there exists a homomorphism $\alpha : \Gamma \to A$, such that the restriction of α to C is injective.

iii- Γ embeds into some ultra-power of A.

iv- Γ belongs to the universal closure of A, i.e. $Th_\forall(A) \subseteq Th_\forall(\Gamma)$.

v- Γ is a limit algebra over A.

vi- Γ is defined by a complete type in $Th_\forall(A)$.

There are similar theorems for the cases where A is q_ω-compact. Note that an algebra A is called q_ω-compact, if for any system S and any equation $p \approx q$, the condition $V_A(S) \subseteq V_A(p \approx q)$ implies that $V_A(S_0) \subseteq V_A(p \approx q)$ for some finite $S_0 \subseteq S$. Clearly, every equationally Noetherian algebra is q_ω-compact.

A. Shevlyakov studied algebraic geometry over Boolean algebras, [17]. He obtained necessary and sufficient condition for a Boolean algebra to be equationally Noetherian or to be q_ω-compact. Let B be a Boolean algebra and C be a subalgebra of B. Then we can consider B as a C-algebra. Shevlyakov proved that B is C-equationally Noetherian, if and only if C is finite. Consequently only finite Boolean algebras are equationally Noetherian in the case of Diophantine algebraic geometry. He also obtained necessary and sufficient conditions for the C-Boolean algebra B to be q_ω-compact. To explain it, we need to define E_k-systems: a system of C-equations S is called E_k system over B, if $V_B(S)$ has k elements, but for any finite subsystem $S' \subsetneq S$, the algebraic set $V_B(S')$ is infinite. It is proved that B is q_ω-compact as a C-algebra, if and only if there are no any E_0 and E_1-systems over B.

In this article, we are dealing with the case of Heyting algebras, which are natural generalizations of Boolean algebras.

2. Algebraic geometry over Heyting algebras

Heyting algebras for propositional intuitionistic logic are the same as Boolean algebras for classical propositional logic. Note that in intuitionistic logic, truth is equivalent to provability. Since by the incompleteness theorem of Godel, there are sentences α in the language of arithmetic such that $\alpha \vee \neg\alpha$ is not provable, so in the case of intuitionistic logic, Boolean algebras are not useful and one must employ the more general frame of Heyting algebras. A Heyting algebra is a bounded lattice H, such that for all $a, b \in H$, there exists a maximum element x with the property $a \wedge x \leqslant b$. Let's denote that element x by $a \to b$. Then, one can see that the following identities are hold

1) $a \to a = 1$.

2) $a \wedge (a \to b) = a \wedge b$.

3) $b \wedge (a \to b) = b$.

4) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

So, let $\mathcal{L} = (\wedge, \vee, \to, 1, 0)$ be the language of bounded lattices extended by adding a new binary symbol $\to$. Then the variety of Heyting algebras is just the variety axiomatized by the identities of bounded lattices plus the above four new identities. Let $\neg a = a \to 0$. It can be shown that in any Heyting algebra, we have only one of the De Morgan's laws, namely $\neg(a \vee b) = \neg a \wedge \neg b$, but the other law $\neg(a \wedge b) = \neg a \vee \neg b$ is not valid, despite the case of Boolean algebras. It is also true that $\neg\neg\neg a = \neg a$ and $\neg a \vee \neg\neg a = 1$. Recall that a complete Heyting algebra is a Heyting algebra which is also a complete lattice.

An element a in a Heyting algebras H is called regular, if $\neg\neg a = a$. Clearly, both 0 and 1 are regular. Let H_{reg} be the set of all regular elements of H. It is easy to see that $H_{reg} = \neg H$, the set of all negated elements of H. This set is not a Heyting subalgebra in general, but it is a Boolean algebra bey the following operations

1) $a \wedge_{reg} b = a \wedge b$.

2) $\neg_{reg} a = \neg a$.

3) $a \vee_{reg} b = \neg(\neg a \wedge \neg b)$.

We will use the notation $\mathcal{L}_{reg}$ for the Boolean language $(\wedge, \vee_{reg}, \neg, 0, 1)$. This will help us to apply results of [17] for Heyting algebras.

Note that, despite Boolean algebras, the free Heyting algebra $F_H(X)$ is always infinite for any non-empty set X. For example, if $X = \{x\}$, then the free Heyting algebra over X consists of the following elements

$$0, x, \neg x, \neg\neg x, x \vee \neg x, \neg x \vee \neg\neg x, \neg\neg x \to x, (\neg\neg x \to x) \to (x \vee \neg x), \ldots$$

We focus on the case of equations with coefficients inside H (Diophantine Geometry). It is known that free groups are equationally Noetherian. Free Boolean algebras of finite rank are also equationally Noetherian since they are finite. We show that no non-trivial Heyting algebra is equationally Noetherian.

Proposition 1. *Let X be a non-empty set. Then the free Heyting algebra $F_H(X)$ is not equationally Noetherian.*

Proof. It is enough to consider the case $X = \{p\}$, because subalgebras of equationally Noetherian algebras are again equationally Noetherian. Consider the following infinite chain

$$p < \neg\neg p < \neg p \vee \neg\neg p < \cdots .$$

Let S be the system $\{x \geqslant p, x \geqslant \neg\neg p, x \geqslant \neg p \vee \neg\neg p, \ldots\}$. It is obvious that $V_F(S) = \{1\}$. But, since the above chain is infinite, so for every finite $S_0 \subseteq S$, there are infinitely many elements in $V_F(S_0)$. Hence $V_F(S) \neq V_F(S_0)$. This shows that $F = F_H(X)$ is not equationally Noetherian. □

Note that in the same time the above argument shows that non-trivial free Heyting algebras are not q_ω-compact. This is true because we have $V_F(S) \subseteq V_F(x \approx 1)$, but for any finite $S_0 \subseteq S$, the algebraic set $V_F(S_0)$ is infinite.

We now, can use the same idea to prove that infinite Heyting algebras are not equationally Noetherian. It can be also applied for infinite complete Heyting algebras to prove that they are not q_ω-compact.

Theorem 3. *Let H be a Heyting algebra and K be a subalgebra, which is infinite. Then H is not K-equationally Noetherian.*

Proof. For simplicity we discuss Diophantine case ($K = H$). The idea of the proof is taken from a similar theorem for Boolean algebras (see [6]). Let

$$b_0, b_1, b_2, \ldots$$

be an infinite set of elements in H. Let $L_0 = \{0, 1\}$ and define L_n by inductions as follows: if $L_{n-1} = \{a_0 = 0, a_1, \ldots, a_{n-1}, a_n = 1\}$, and if $0 \leqslant i \leqslant n$, then define

$$c_{i+1} = a_i \vee (a_{i+1} \wedge b_n).$$

For example, we have $L_0 = \{a_0 = 0, a_1 = 1\}$. Then we compute

$$c_1 = a_0 \vee (a_1 \wedge b_1) = b_1.$$

It is clear that $0 \leqslant b_1 \leqslant 1$. Let $L_1 = \{0, a_1, 1\}$ and rename its elements as $a_0 = 0$, $a_1 = b_1$, $a_2 = 1$. Now, to find L_2, we compute

$$c_1 = a_0 \vee (a_1 \wedge b_2) = b_1 \wedge b_2,$$

and

$$c_2 = a_1 \vee (a_2 \wedge b_2) = b_1 \vee b_2.$$

We have

$$0 \leqslant b_1 \wedge b_2 \leqslant b_1 \leqslant b_1 \vee b_2 \leqslant 1,$$

so L_2 consists of the above elements. Again rename $a_0 = 0$, $a_1 = b_1 \wedge b_2$, $a_2 = b_1$, …, and continue this process. It is clear from the construction that

$$L_0 \subset L_1 \subset L_2 \subset \cdots,$$

so the set $L = \cup_{n \geqslant 0} L_n$ is an infinite chain in H.

Now, we proved that there is an infinite chain $a_0 < a_1 < a_2 < \cdots$ in H so we can consider the following system

$$S = \{x \geqslant a_0, x \geqslant a_1, x \geqslant a_2, \ldots\}.$$

For any finite subsystem $S_0 = \{x \geqslant a_0, x \geqslant a_1, x \geqslant a_2, \ldots, a_n\}$, we have $a_{n+1} \in V_H(S_0)$, while a_{n+1} does not belong to $V_H(S)$. This proves that H is not equationally Noetherian. □

Note that if H is complete, then in the above proof we can put $a = \sup_i a_i$. Then $V_H(S) \subseteq V_H(x \geqslant a)$, but for any finite subset S_0, it is not true that $V_H(S_0) \subseteq V_H(x \geqslant a)$. This shows that if H is a complete infinite Heyting algebra, then it is not q_ω-compact. The next theorem concerns the relation between q_ω-compactness of a Heyting algebra H and H_{reg}.

Theorem 4. *Let H be q_ω-compact. Then there is no E_0 and E_1-systems in the language $\mathcal{L}_{reg}$ over the Boolean algebra H_{reg}.*

Proof. Assume that S is an $\mathcal{L}_{reg}$-system of equations with n indeterminate and denote H_{reg} for simplicity by R. Let $V_R(S) \subseteq V_R(p \approx q)$, where $p \approx q$ is an $\mathcal{L}_{reg}$-equation. Note that R^n is an algebraic set in H^n because it is just the solution set of the system $\neg\neg x_1 \approx x_1, \neg\neg x_2 \approx x_2, \ldots, \neg\neg x_n \approx x_n$. Hence we have

$$V_R(S) = V_H(S + \neg\neg x_1 \approx x_1, \ldots, \neg\neg x_n \approx x_n),$$

and

$$V_R(p \approx q) = V_H(p \approx q + \neg\neg x_1 \approx x_1, \ldots, \neg\neg x_n \approx x_n).$$

This shows that

$$V_H(S + \neg\neg x_1 \approx x_1, \dots, \neg\neg x_n \approx x_n) \subseteq V_H(p \approx q + \neg\neg x_1 \approx x_1, \dots, \neg\neg x_n \approx x_n),$$

and hence

$$V_H(S + \neg\neg x_1 \approx x_1, \dots, \neg\neg x_n \approx x_n) \subseteq V_H(p \approx q).$$

Since H is assumed to be q_ω-compact, so there is a finite subset $S_0 \subseteq S + \neg\neg x_1 \approx x_1, \dots, \neg\neg x_n \approx x_n$ such that $V_H(S_0) \subseteq V_H(p \approx q)$. Therefore, $V_R(S_0) \subseteq V_R(p \approx q)$. Let $S' = S_0 \setminus \{\neg\neg x_1 \approx x_1, \dots, \neg\neg x_n \approx x_n\}$. Then we have

$$V_R(S') \subseteq V_R(S_0) \subseteq V_R(p \approx q),$$

and this shows that the Boolean algebra R is q_ω-compact in the Boolean language $\mathcal{L}_{reg}$. Now, we can apply the result of [17] to conclude that there are no E_0 and E_1-systems in the language $\mathcal{L}_{reg}$ over H_{reg}. □

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Алгебраическая геометрия над гейтинговыми алгебрами

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Аннотация. В этой статье мы изучаем алгебраическую геометрию над гейтинговыми алгебрами и исследуем свойства быть уравновешенно нетеровыми и q_ω-компактными над такими алгебрами.

Ключевые слова: универсальная алгебраическая геометрия, системы уравнений, радикалы, топология Зарисского, алгебры Гейтинга, нетеровы алгебры уравнений, q_ω-компактные алгебры.

DOI: 10.17516/1997-1397-2020-13-4-422-430
УДК 519.2

On Estimation of Bivariate Survival Function from Random Censored Data

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Received 11.03.2020, received in revised form 08.05.2020, accepted 16.06.2020

Abstract. At present there are several approaches to estimate survival functions of vectors of lifetimes. However, some of these estimators are either inconsistent or not fully defined in the range of joint survival functions. Therefore they are not applicable in practice. In this paper three types of estimates of exponential-hazard, product-limit and relative-risk power structures for the bivariate survival function are considered when the number of summands in empirical estimates is replaced with a sequence of Poisson random variables. It is shown that proposed estimates are asymptotically equivalent.

Keywords: bivariate survival function, Poisson random variables, empirical estimates.

Citation: A.A. Abdushukurov, R.S. Muradov, On Estimation of Bivariate Survival Function from Random Censored Data, J. Sib. Fed. Univ. Math. Phys., 2020, 13(4), 422–430.
DOI: 10.17516/1997-1397-2020-13-4-422-430.

Introduction

The problem of estimation of multivariate distribution (or survival) function from incomplete data was considered from the beginning of 1980's (Campbell (1981), Campbell & Földes (1982), Hanley & Parnes (1983), Horváth (1983), Tsay, Leurgang & Crowley (1986), Burke (1988), Dabrowska (1988, 1989), Gill (1992), Huang (2000), Abdushukurov(2004) etc.) (see, [1–20]). In the special bivariate case there are the numerous examples of paired data that represent life time of individuals (twins or married couples), the failure times of components of a system and others which are subject to random censoring. At present there are several approaches to estimate survival functions of vectors of life times. However, some of these estimators are either nconsistent or not fully defined in the range of joint survival functions. Hence they are not applicable in practice. In this work we present estimators for bivariate survival function and present some sample properties of estimators. We extend some results given in [1–4] to Poisson random summation. At the end of the paper we present consistent estimators of parameters of Marshall-Olkin exponential distribution.

1. Random right censoring model

Let $\mathbb{X}=\{X_i=(X_1 i, X_2 i)\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed (i.i.d.) two-dimensional random vectors with a common continuous survival function $F(s,t)$ =

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$= P(X_{11} > s, X_{21} > t)$, $(s,t) \in \overline{R}^{+2} = [0,\infty) \times [0,\infty)$. This sequence is censored from the right by sequence $\mathbb{Y} = \{Y_i = (Y_{1i}, Y_{2i})\}_{i=1}^{\infty}$ of i.i.d. random vectors with survival function $G(s,t) = P(Y_{11} > s, Y_{21} > t)$, $(s,t) \in \overline{R}^{+2}$. Let us assume that there is the sample $\mathbb{V}^{(n)} = \{(Z_i, \Delta_i), 1 \leqslant i \leqslant n\}$, where $Z_i = (Z_{1i}, Z_{2i})$, $\Delta_i = (\delta_{1i}, \delta_{2i})$, $Z_{ki} = \min(X_{ki}, Y_{ki})$, $\delta_{ki} = I(Z_{ki} = X_{ki})$, $k = 1,2$, and $I(\cdot)$ is the indicator. The problem consist of estimating F from the sample $\mathbb{V}^{(n)}$. Let $H(s,t) = P(Z_{1i} > s, Z_{2i} > t)$, $(s,t) \in \overline{R}^{+2}$ and sequences $\mathbb{X}$ and $\mathbb{Y}$ are independent. Then $H(s,t) = F(s,t)\,G(s,t)$, $(s,t) \in \bar{R}^{+2}$. In this paper we use exponential-hazard, product-limit and relative-risk power types functionals in order to construct the corresponding estimates of three types for F. In the empirical estimates the upper index of summation n is replaced by the Poisson random variable (r.v.) μ_n with expectation $E\mu_n = n$. This arises in the insurance business as the size of group insurance payments by an insurance company to customers in connection with an insured event. Following [2], we introduce some auxiliary functionals for $(x,y) \in \bar{R}^{+2}$:

$$
\begin{aligned}
M(x,y) &= P(Z_{11} \leqslant x,\, Z_{21} > y), \quad N(x,y) = P(Z_{11} > x,\, Z_{21} \leqslant y),\\
\bar{M}(x,y) &= P(Z_{11} \leqslant x,\, Z_{21} > y,\, \delta_{11} = 1), \quad \bar{N}(x,y) = P(Z_{11} > x,\, Z_{21} \leqslant y,\, \delta_{21} = 1),\\
\Lambda_1(x,y) &= \int_0^x \frac{M(ds,y)}{H(s-,y)}, \quad \Lambda_2(x,y) = \int_0^y \frac{N(x,dt)}{H(x,t-)},\\
\bar{\Lambda}_1(x,y) &= \int_0^x \frac{\bar{M}(ds,y)}{H(s-,y)}, \quad \bar{\Lambda}_2(x,y) = \int_0^y \frac{\bar{N}(x,dt)}{H(x,t-)},\\
\Lambda(x,y) &= \Lambda_1(x,0) + \Lambda_2(x,y), \quad \bar{\Lambda}(x,y) = \bar{\Lambda}_1(x,0) + \bar{\Lambda}_2(x,y),\\
\Lambda^c(x,y) &= \Lambda_1^c(x,0) + \Lambda_2^c(x,y), \quad \bar{\Lambda}^c(x,y) = \bar{\Lambda}_1^c(x,0) + \bar{\Lambda}_2^c(x,y),
\end{aligned}
\tag{1.1}
$$

where

$$
\Lambda_1^c(x,y) = \Lambda_1(x,y) - \sum_{s \leqslant x} \Lambda_1(\triangle\, s, y), \quad \Lambda_1(\triangle\, s, y) = \Lambda_1(s,y) - \Lambda_1(s-,y),
$$
$$
\Lambda_2^c(x,y) = \Lambda_2(x,y) - \sum_{t \leqslant y} \Lambda_2(x, \triangle\, t), \quad \Lambda_2(x, \triangle\, t) = \Lambda_2(x,t) - \Lambda_2(x,t-),
$$

and similarly defined $\bar{\Lambda}_1^c$ and $\bar{\Lambda}_2^c$. To construct estimates for F we estimate functionals (1.1). Firstly, we introduce the following empirical estimates of the first four probabilities in (1.1) from the sample $\mathbb{V}^{(n)}$:

$$
\begin{aligned}
H_n(x,y) &= \frac{1}{n}\sum_{i=1}^{n} I(Z_{1i} > x,\, Z_{2i} > y),\\
M_n(x,y) &= \frac{1}{n}\sum_{i=1}^{n} I(Z_{1i} \leqslant x,\, Z_{2i} > y),\\
N_n(x,y) &= \frac{1}{n}\sum_{i=1}^{n} I(Z_{1i} > x,\, Z_{2i} \leqslant y),\\
\bar{M}_n(x,y) &= \frac{1}{n}\sum_{i=1}^{n} I(Z_{1i} \leqslant x,\, Z_{2i} > y,\, \delta_{1i} = 1),\\
\bar{N}_n(x,y) &= \frac{1}{n}\sum_{i=1}^{n} I(Z_{1i} > x,\, Z_{2i} \leqslant y,\, \delta_{2i} = 1).
\end{aligned}
\tag{1.2}
$$

Let $\{\mu_n,\, n \geqslant 1\}$ be a sequence of Poisson random variables (r.v-s.) with parameter $E\mu_n = n$, that is independent of the pair $(\mathbb{X}, \mathbb{Y})$. Along with estimates (1.2), we propose also their analogues

$H_n^*,\ M_n^*,\ N_n^*,\ \bar{M}_n^*,\ \bar{N}_n^*$ obtained from estimates (1.2) by replacing the upper limit of summation n by r.v. μ_n. However, it should be noted that these estimates have the disadvantage because they can be greater than 1. In fact, for example, for

$$H_n^*(x,y) = \frac{1}{n}\sum_{i=1}^{\mu_n} I(Z_{1i} > x,\ Z_{2i} > y),$$

we have

$$P(H_n^*(0,0) > 1) = P(\mu_n > n) = \sum_{m=n+1}^{\infty} \frac{n^m e^{-n}}{m!} > 0.$$

To avoid this disadvantage we consider the following truncated versions of estimates $H_n^*,\ M_n^*,\ N_n^*,\ \bar{M}_n^*, \bar{N}_n^*$:

$$H_n^0(x,y) = 1 - (1 - H_n^*(x,y))\, I(H_n^*(x,y) \leqslant 1) = \begin{cases} H_n^*(x,y) & \text{if } H_n^*(x,y) \leqslant 1, \\ 0 & \text{if } H_n^*(x,y) > 1, \end{cases}$$

and similarly constructed estimates $M_n^0,\ N_n^0,\ \bar{M}_n^0,\ \bar{N}_n^0$. In similar way we construct the corresponding estimates for functionals in (1.1):

$$\begin{gathered} \Lambda_{1n}(x,y) = \int_0^x \frac{M_n^0(ds,y)}{H_n^0(s-,y)}, \quad \Lambda_{2n}(x,y) = \int_0^y \frac{N_n^0(x,dt)}{H_n^0(x,t-)}, \\ \bar{\Lambda}_{1n}(x,y) = \int_0^x \frac{\bar{M}_n^0(ds,y)}{H_n^0(s-,y)}, \quad \bar{\Lambda}_{2n}(x,y) = \int_0^y \frac{\bar{N}_n^0(x,dt)}{H_n^0(x,t-)}, \qquad (1.3) \\ \Lambda_n(x,y) = \Lambda_{1n}(x,0) + \Lambda_{2n}(x,y), \quad \bar{\Lambda}_n(x,y) = \bar{\Lambda}_{1n}(x,0) + \bar{\Lambda}_{2n}(x,y). \end{gathered}$$

The relative-risk function is

$$R(x,y) = \frac{\bar{\Lambda}(x,y)}{\Lambda(x,y)}$$

and its estimator is

$$R_n(x,y) = \frac{\bar{\Lambda}_n(x,y)}{\Lambda_n(x,y)}.$$

Using estimates (1.3), we propose the following three estimates of $F(x,y)$ for exponential, product and power structures

$$\begin{aligned} F_{1n}(x,y) &= \exp\left\{-\bar{\Lambda}_n(x,y)\right\} = \exp\left\{-\left(\bar{\Lambda}_{1n}(x,0) + \bar{\Lambda}_{2n}(x,y)\right)\right\}, \\ F_{2n}(x,y) &= \prod_{s\leqslant x}\left(1 - \bar{\Lambda}_{1n}(\triangle\, s, 0)\right) \prod_{t\leqslant y}\left(1 - \bar{\Lambda}_{2n}(x, \triangle\, t)\right), \qquad (1.4) \\ F_{3n}(x,y) &= \left[H_n(x,y)\right]^{R_n(x,y)}. \end{aligned}$$

Let $\Delta_n = \left[0, Z_1^{(n)}\right] \times \left[0, Z_2^{(n)}\right] \cap \Delta$, where $Z_k^{(n)} = \max(Z_{k1}, \ldots, Z_{kn})$, $\Delta = \left[0, T_Z^{(1)}\right] \times \left[0, T_Z^{(2)}\right]$, $T_Z^{(k)} = \inf\{t \geqslant 0 : P(Z_{k1} \leqslant t) = 1\}$, $k = 1, 2$. The following theorem states the asymptotic equivalence of estimates (1.4).

Theorem 1.1. *For all* $(x,y) \in \Delta_n$:

(I)
$$0 \leqslant F_{1n}(x,y) - F_{2n}(x,y) = \mathrm{O}_p\left(\frac{1}{n}\right).$$

If the survival function G *is also continuous on* Δ_n *then*

(II)
$$|F_{1n}(x,y) - F_{3n}(x,y)| = \mathrm{O}_p\left(\left(\frac{\log n}{n}\right)^{1/2}\right).$$

One can also obtain from (I) and (II) that

$$|F_{3n}(x,y) - F_{2n}(x,y)| = \mathrm{O}_p\left(\left(\frac{\log n}{n}\right)^{1/2}\right).$$

To prove Theorem 1.1 we need the following auxiliary statements.

Lemma 1.1. *Let* $\{\mu_n, n \geqslant 1\}-$ *be a sequence of Poisson r.v-s. with expectation* n*. Then for any number* $\varepsilon > 0$ *and for* n *such that*

$$\frac{n}{\log n} \geqslant \frac{\varepsilon}{8\left(1+\dfrac{e}{3}\right)^2}, \quad e = \exp(1), \tag{1.5}$$

the inequality

$$P\left(\frac{|\mu_n - n|}{n} > \frac{1}{2}\left(\frac{\varepsilon}{2}\cdot\frac{\log n}{n}\right)^{1/2}\right) \leqslant 2n^{-c_0}, \tag{1.6}$$

is true, where $c_0 = c_0(\varepsilon) = \varepsilon/16(1+e/3)$.

Proof. Let $\gamma_1, \gamma_2, \ldots$ be a sequence of Poisson r.v.-s with expectation $E(\gamma_k) = 1$ for all $k = 1, 2, \ldots$. Then $\mu_n - n = \sum\limits_{k=1}^{n}(\gamma_k - 1) = \sum\limits_{k=1}^{n}\xi_k$, where

$$Ee^{t\xi_k} = e^{-t}Ee^{t\gamma_1} = e^{-t}e^{-1}\sum_{k=0}^{\infty}\frac{(e^t)^k}{k!} = \exp(e^t - (t+1)).$$

Using Taylor expansion of e^t, we have

$$Ee^{t\xi_k} = \exp\left(1 + t + \frac{t^2}{2} + \Psi(t) - (t+1)\right) = \exp\left(\frac{t^2}{2} + \Psi(t)\right),$$

where $\Psi(t) = \dfrac{t^3}{6}\exp(\theta t)$, $0 < \theta < 1$. For $0 \leqslant t \leqslant 1$,we have $t^3 \leqslant t^2$ and consequently $\Psi(t) \leqslant \dfrac{t^3}{6}\cdot e \leqslant e \cdot \dfrac{t^2}{6}$. From here, for $0 \leqslant t \leqslant 1$ we obtain

$$Ee^{t\xi_k} \leqslant \exp\left(\frac{t^2}{2}\left(1 + \frac{e}{3}\right)\right) = \exp\left(\frac{\lambda_k}{2}\cdot t^2\right), \quad \lambda_k = 1 + \frac{e}{3}.$$

Then using following exponential inequality for nonidentical distributed r.v.-s of Petrov ([22])

$$P\left(\left|\sum_{k=1}^{n}\xi_k\right| > u\right) \leqslant 2\exp\left(-\frac{u^2}{2}\right), \quad 0 \leqslant u \leqslant N,$$

under $0 \leqslant u = \dfrac{1}{2}\left(\dfrac{\varepsilon}{2}n\log n\right)^{\frac{1}{2}} \leqslant \lambda_k n = N$, we obtain (1.6). □

The following inequality for two-dimensional empirical estimates from [21, p. 292] is used below. Let $C = C(H) = H\left(T_Z^{(1)}, T_Z^{(2)}\right) > 0$.

Lemma 1.2 ([21])**.** *For all real* $z > 0$

$$P\left(\sup_{(x,y)\in \bar{R}^{+2}}|H_n(x,y) - H(x,y)| > zC^2\right) \leqslant \mathrm{V}_z\cdot\left(1+n^2\right)^2\exp\left(-2nz^2\cdot C^4\right), \tag{1.7}$$

where $\mathrm{V}_z = \mathrm{V}_z(H) = 4\exp\left(4zC^2 + 4z^2C^4\right)$.

Corollary 1.1. Let $z = z_0 = \left(\frac{4+\varepsilon}{2} \cdot \frac{\log n}{n}\right)^{1/2} \cdot C^{-2}$ in (2.7). Then

$$P\left(\sup_{(x,y)\in\bar{R}^{+2}} |H_n(x,y) - H(x,y)| > \left(\frac{4+\varepsilon}{2} \cdot \frac{\log n}{n}\right)^{1/2} \cdot C^{-2}\right) \leqslant q_n(\varepsilon), \tag{1.8}$$

where

$$q_n(\varepsilon) = 4\exp\left(4\left(\frac{4+\varepsilon}{2n} \cdot \log n\right)^{1/2}\left[1 + \left(\frac{4+\varepsilon}{2n} \cdot \log n\right)^{1/2}\right]\right) \cdot \left(n^2+1\right)^2 n^{-(4+\varepsilon)} = \mathrm{O}\left(n^{-\varepsilon}\right).$$

Therefore, for $\varepsilon > 1$ from (1.8) we have by Borel-Cantelli lemma that

$$\sup_{(x,y)\in\Delta_n} |H_n(x,y) - H(x,y)| \overset{a.s.}{=} \mathrm{O}\left(\left(\frac{\log n}{n}\right)^{1/2}\right). \tag{1.9}$$

In the next lemma we establish an analogue of (1.7) for an empirical estimate H_n^0. Let $q_n^0(\varepsilon)$ be obtained from $q_n(\varepsilon)$ by replacing $4+\varepsilon$ with $(4+\varepsilon)/4$.

Lemma 1.3. *Under the conditions of Lemma 1.1*

$$P\left(\sup_{(x,y)\in\Delta_n} \left|H_n^0(x,y) - H(x,y)\right| > \left(\frac{4+\varepsilon}{2} \cdot \frac{\log n}{n}\right)^{1/2} \cdot C^{-2}\right) \leqslant 2n^{-c_0(\varepsilon+4)} + q_n^0(\varepsilon). \tag{1.10}$$

Proof. For $\mu_n \leqslant n : H_n^0(x,y) = H_n^*(x,y)$ for all $(x,y) \in \bar{R}^{+2}$ and for $\mu_n > n$ we have

$$\sup_{(x,y)\in\bar{R}^{+2}} |H_n^0(x,y) - H(x,y)| \leqslant \sup_{(x,y)\in\bar{R}^{+2}} |H_n^*(x,y) - H(x,y)|.$$

Using the formula of complete probability, we obtain

$$P\left(\sup_{(x,y)\in\Delta_n} \left|H_n^0(x,y) - H(x,y)\right| > z_0 C\right) \leqslant P\left(\sup_{(x,y)\in\Delta_n} |H_n^*(x,y) - H(x,y)| > z_0 C^2\right) \leqslant$$

$$\leqslant P\left(\sup_{(x,y)\in\Delta_n} \left|H_n(x,y) - H(x,y) + \frac{1}{n}\sum_{i=n+1}^{\mu_n} I(Z_{1i} > x, Z_{2i} > y)\right| > z_0 C^2 / \mu_n > n\right) \cdot P(\mu_n > n) +$$

$$+P\left(\sup_{(x,y)\in\Delta_n} \left|H_n(x,y) - H(x,y) - \frac{1}{n}\sum_{i=n+1}^{\mu_n} I(Z_{1i} > x, Z_{2i} > y)\right| > z_0 C^2 / \mu_n > n\right) P(\mu_n > n) \leqslant$$

$$\leqslant P\left(\sup_{(x,y)\in\Delta_n} |H_n(x,y) - H(x,y)| > \frac{1}{2} z_0 C^2\right) +$$

$$+P\left(\sup_{(x,y)\in\Delta_n} \left|\frac{1}{n}\sum_{i=n\wedge\mu_n+1}^{n\vee\mu_n} I(Z_{1i} > x, Z_{2i} > y)\right| > \frac{1}{2} z_0 C^2\right) \leqslant$$

$$\leqslant q_n^0(\varepsilon) + P\left(\frac{|\mu_n - n|}{n} > \frac{1}{2} z_0 C^2\right) \leqslant 2n^{-c_0(\varepsilon+4)} + q_n^0(\varepsilon),$$

where (1.6) and (1.8) are used. □

Proof of Theorem 1.1. From inequalities (2.4.2) in [2] applied to estimates F_{1n} and F_{2n} we have

$$0 \leqslant F_{1n}(x,y) - F_{2n}(x,y) \leqslant \frac{1}{2}\sum_{i=1}^{\mu_n-1}\left[\left(q_{1i}^{(n)}(x,0)\right)^2 + \left(q_{2i}^{(n)}(x,y)\right)^2\right] =$$
$$= \frac{1}{2n^2}\left\{\sum_{i=1}^{\mu_n-1}\left[\frac{\delta_{1(i)}\, I\left(Z_1^{(i)} \leqslant x\right)}{\left(S_{1n}^{0Z}\left(Z_1^{(i)}-\right)\right)^2}\right] + \sum_{i=1}^{\mu_n-1}\left[\frac{\delta_{2(i)}\, I\left(Z_1^{(i)} \leqslant x, Z_{2i} \leqslant y\right)}{\left(H_n^0\left(x, Z_2^{(i)}-\right)\right)^2}\right]\right\} \leqslant \tag{1.11}$$
$$\leqslant \frac{\mu_n}{2n^2}\left[S_{1n}^{0Z}\left(Z_1^{(\mu_n-1)}-\right)^{-2} + \left(H_n^0\left(x, Z_2^{(\mu_n-1)}-\right)\right)^{-2}\right],$$

where $Z_k^{(1)} \leqslant \ldots \leqslant Z_k^{(n)}$ order statistics are constructed from Z_{ki}, $k = 1,2$, $\delta_{k(i)}$ corresponds to $Z_k^{(i)}$ and $S_{1n}^{0Z}(x) = H_n^0(x;0)$. It is known that for $n \to \infty$, $Z_k^{(n)} \xrightarrow{p} T_Z^{(k)}$, $k = 1,2$. We show that $Z_k^{(\mu_n)} \xrightarrow{p} T_Z^{(k)}$, $k = 1,2$ when $n \to \infty$. For $\varepsilon > 0$, $0 < \delta < 1$ and $k = 1,2$ we have

$$P\left(\left|Z_k^{(\mu_n)} - T_Z^{(k)}\right| > \varepsilon\right) \leqslant$$
$$\leqslant P\left(\left|Z_k^{(\mu_n)} - T_Z^{(k)}\right| > \varepsilon,\ \left|\frac{\mu_n}{n} - 1\right| < \delta\right) + P\left(\left|\frac{\mu_n}{n} - 1\right| \geqslant \delta\right) \leqslant$$
$$\leqslant P\left(\left|Z_k^{(\mu_n)} - T_Z^{(k)}\right| > \varepsilon,\ n(1-\delta) < \mu_n < n(1+\delta)\right) + P\left(\left|\frac{\mu_n}{n} - 1\right| \geqslant \delta\right) \leqslant$$
$$\leqslant P\left(\left|Z_k^{(n)} - T_Z^{(k)}\right| > \varepsilon\right) + P\left(\left|\frac{\mu_n}{n} - 1\right| \geqslant \delta\right).$$

For arbitrary $\eta > 0$ there are numbers n_1 and ε such that for $n \geqslant n_1$

$$P\left(\left|Z_k^{(n)} - T_Z^{(k)}\right| > \varepsilon\right) < \frac{\eta}{2},\ \ k = 1,2. \tag{1.12}$$

Since $P\left(\left|\frac{\mu_n}{n} - 1\right| \geqslant \delta\right) \to 0$ when $n \to \infty$ then for $n \geqslant n_2$

$$P\left(\left|\frac{\mu_n}{n} - 1\right| \geqslant \delta\right) < \frac{\eta}{2}. \tag{1.13}$$

Then for $n \geqslant n_0 = \max(n_1, n_2)$ we obtain from (1.12) and (1.13) that

$$P\left(\left|Z_k^{(\mu_n)} - T_Z^{(k)}\right| > \varepsilon\right) < \eta, \tag{1.14}$$

which is required result. Thus, taking into account (1.13) and (1.14), for $n \to \infty$ with probability close to 1 we have

$$\begin{gathered} Z_k^{(\mu_n-1)} \approx Z_k^{(n-1)},\ \ k = 1,2, \\ \frac{\mu_n}{n^2} = \mathrm{O}_p\left(\frac{1}{n}\right). \end{gathered} \tag{1.15}$$

Taking into account (1.15) and the following relations obtained from (1.10) for $(x,y) \in \Delta_n$

$$\frac{1}{S_{1n}^{0Z}(x)} \leqslant \frac{\left|S_{1n}^{0Z}(x) - S_1^Z(x)\right|}{S_{1n}^{0Z}(x)\, S_1^Z(x)} + \frac{1}{S_1^Z(x)} = \frac{1}{S_1^Z(x)} + \mathrm{O}_p\left(\left(\frac{\log n}{n}\right)^{1/2}\right),$$

$$\frac{1}{H_n^0(x,y)} \leqslant \frac{\left|H_n^0(x,y) - H(x,y)\right|}{H_n^0(x,y)\, H(x,y)} + \frac{1}{H(x,y)} = \frac{1}{H(x,y)} + \mathrm{O}_p\left(\left(\frac{\log n}{n}\right)^{1/2}\right),$$

we obtain the right estimate in (I). Now according to the inequality $|u-v| \leqslant |\log u - \log v|$, for $0 < u,\ v \leqslant 1,\ 0 \leqslant R_n(x,y) \leqslant 1$ and $(x,y) \in \Delta_n$ we have

$$\begin{aligned} |F_{1n}(x,y) - F_{3n}(x,y)| &\leqslant \bar{\Lambda}_n(x,y) \left| -1 + \frac{(-\log H_n^0(x,y))}{\Lambda_n(x,y)} \right| = \\ &= R_n(x,y) \left| \left(-\log H_n^0(x,y)\right) - \Lambda_n(x,y) \right| \leqslant \\ &\leqslant \left| -\log H_n^0(x,y) + \log H(x,y) \right| + \left| (-\log H(x,y)) - \Lambda_n(x,y) \right|. \end{aligned} \tag{1.16}$$

According to Lemma 1.3 and the mean value theorem for $(x,y) \in \Delta_n$ we obtain

$$\left| -\log H_n^0(x,y) + \log H(x,y) \right| = \mathrm{O}_p\left(\left(\frac{\log n}{n} \right)^{1/2} \right). \tag{1.17}$$

Taking into account continuity of G, Lemma 3.4.3, the proof of Theorem 2.4.3 and Remark 2.4.4 in [2] we obtain for $(x,y) \in \Delta_n$ that

$$|-\log H(x,y) - \Lambda_n(x,y)| = \mathrm{O}_p\left(\left(\frac{\log n}{n} \right)^{1/2} \right). \tag{1.18}$$

Now (II) follows from relations (2.16)–(2.18). □

It was shown in Theorem 2.4.3 in [2] that in the case of continuity of F and G both exponential-hazard and relative-risk power functionals coincide with the estimated survival function F. Then, taking into account Theorem 1.1, we can state that all three estimates (1.4) are consistent estimates of F (see, also [5]).

2. Estimation of parameters of Marshall-Olkin exponential distribution

Let us consider survival function $F(s,t) = P(X_{11} > s, X_{21} > t)$, $(s,t) \in \overline{R}^{+2}$ of Marshall-Olkin exponential form with unknown parameters $\lambda_1, \lambda_2, \lambda_{12}$:

$$F(s,t) = \exp\left(-\lambda_1 s - \lambda_2 t - \lambda_{12} \max(s,t)\right), \quad (s,t) \in \overline{R}^{+2}. \tag{2.1}$$

Then corresponding cumulative hazard function is

$$\Lambda(s,t) = -\log F(s,t) = \lambda_1 s + \lambda_2 t + \lambda_{12} \max(s,t). \tag{2.2}$$

Nonparametric estimator of $\Lambda(s,t)$ from (2.4) is $\overline{\Lambda}_n(s,t) = -\log F_{1n}(s,t) = \overline{\Lambda}_{1n}(s,0) + \overline{\Lambda}_{2n}(s,t)$. It is easy to verify from (2.2) that we have the system of equations for $s > 0$

$$\begin{cases} \Lambda(s,0) = \lambda_1 s + \lambda_{12} s, \\ \Lambda(0,s) = \lambda_2 s + \lambda_{12} s, \\ \Lambda(s,s) = \lambda_1 s + \lambda_2 s + \lambda_{12} s. \end{cases} \tag{2.3}$$

From (2.3) we find expressions for unknown parameters λ_1, λ_2 and λ_{12} for a fixed point $s = s_0 > 0$:

$$\begin{cases} \lambda_1 = \dfrac{1}{s_0} \left(\Lambda(s_0,s_0) - \Lambda(0,s_0) \right), \\ \lambda_2 = \dfrac{1}{s_0} \left(\Lambda(s_0,s_0) - \Lambda(s_0,0) \right), \\ \lambda_{12} = \dfrac{1}{s_0} \left(\Lambda(s_0,0) + \Lambda(0,s_0) - \Lambda(s_0,s_0) \right). \end{cases} \tag{2.4}$$

Now we obtain estimators of parameters from (2.4) by replacing Λ with $\overline{\Lambda}_n$:

$$\begin{cases} \lambda_1^{(n)} = \dfrac{1}{s_0}\left(\overline{\Lambda}_n\left(s_0, s_0\right) - \overline{\Lambda}_n\left(0, s_0\right)\right), \\ \lambda_2^{(n)} = \dfrac{1}{s_0}\left(\overline{\Lambda}_n\left(s_0, s_0\right) - \overline{\Lambda}_n\left(s_0, 0\right)\right), \\ \lambda_{12}^{(n)} = \dfrac{1}{s_0}\left(\overline{\Lambda}_n\left(s_0, 0\right) + \overline{\Lambda}_n\left(0, s_0\right) - \overline{\Lambda}_n\left(s_0, s_0\right)\right). \end{cases} \tag{2.5}$$

It follows from Theorem 1.1 that $\overline{\Lambda}_n(s,t)$ is consistent estimator of $\Lambda(s,t)$. Consequently, relations (2.5) give consistent estimators of corresponding parameters (2.4) of distribution (2.1).

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Об оценивании двумерной функции выживания по случайно цензурированным данным

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Аннотация. В настоящее время существует несколько подходов к оценке функций выживания векторов времени жизни. Однако некоторые из этих оценок либо являются несостоятельными, либо не полностью определены в области функций совместного выживания и поэтому не применимы на практике. В работе авторами предложены состоятельные оценки совместной функции выживания экспоненциальной, множительной и степенной структур при случайном пуассоновском объёме выборки. Показано, что эти оценки асимптотически эквивалентны.

Ключевые слова: двумерная функция выживания, пуассоновские случайные величины, эмпирические оценки.

DOI: 10.17516/1997-1397-2020-13-4-431-438
УДК 537.9+539.216.2

Magnetic and Structure Properties of CoPt-In_2O_3 Nanocomposite Films

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Received 06.04.2020, received in revised form 06.06.2020, accepted 06.07.2020

Abstract. The structural and magnetic properties of CoPt-In_2O_3 nanocomposite films formed by vacuum annealing of the In/(Co_3O_4 + Pt)/MgO film system in the temperature range of 100–800 °C have been investigated. The synthesized nanocomposite films contain ferromagnetic CoPt grains with an average size of 5 nm enclosed in an In_2O_3 matrix, and have a magnetization of 600 emu/cm^3, and a coercivity of 150 Oe at room temperature. The initiation 200 °C and finishing 800 °C temperatures of synthesis were determined, as well as the change in the phase composition of the In/(Co_3O_4 + Pt)/MgO film during vacuum annealing.

Keywords: thin films, ferromagnetic nanocomposites, CoPt alloy, In_2O_3 oxide.

Citation: L.E.Bykova, V.G.Myagkov, V.S.Zhigalov, A.A.Matsynin, D.A.Velikanov, G.N.Bondarenko, G.S.Patrin, Magnetic and Structure Properties of CoPt-In_2O_3 Nanocomposite Films, J. Sib. Fed. Univ. Math. Phys., 2020, 13(4), 431–438. DOI: 10.17516/1997-1397-2020-13-4-431-438.

Introduction

In recent years, composite nanomaterials have been the subject of numerous studies due to their novel functional properties that differ from the properties of their components [1]. Composite ferromagnetic films containing nanoclusters of transition-metal Co, Fe, or Ni in a dielectric or semiconductor matrix obtained by different physical and chemical methods, including the sol-gel method, spray pyrolysis, the microemulsion method, magnetron sputtering, pulsed laser deposition, ion implantation, and joint deposition have been intensively studied [2–9]. The synthesis of these nanocomposites often passes under equilibrium conditions, but lately there has been a surge in nonequilibrium processing of ferromagnetic composites using methods like pulsed laser irradiation [10], pulsed laser deposition [11], ion implantation [12, 13], and the ball-milling process [14] and thermite synthesis of materials. Nanocomposites obtained under nonequilibrium

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conditions often have metastable phases and possess unusual magnetic and physicochemical properties. Recently, a simple and effective method of solid state synthesis of magnetic nanogranular thin films has been proposed, based on initiating thermite reactions between 3d-metal oxide films (Fe_2O_3, Co_3O_4) and In, Zr, Zn, Al metals, whose oxides are wide-gap semiconductors or dielectrics [15–19]. Such an approach makes it possible to obtain thin single-layer and multilayer nanogranular films with a well-controlled size and distribution of magnetic granules over the thickness of the film [19]. CoPt and FePt alloy films have attracted a great deal of attention because of their strong perpendicular magnetic anisotropy, which is important for many practical applications. To date, there have been a small number of studies on the synthesis and investigation of nanocomposites containing CoPt and FePt nanoparticles in oxide matrices [20–26]. These investigations are important for applications involving the synthesis of nanocomposites with the desired magnetic, structural, and transport properties.

In this work, we report the results of the synthesis and investigation of the structure and magnetic properties of CoPt-In_2O_3 nanocomposite films. The films were synthesized by a solid-state reaction in the In/(Co_3O_4 + Pt)/MgO film system with annealing in a vacuum at 10^{-6} Torr in the temperature range of $100 - 800\,^{\circ}$C. The main synthesis parameters, including the initiation temperature and the phase composition of the reagents and reaction products, were determined.

Experimental procedures

Fig. 1 shows the scheme for synthesizing CoPt-In_2O_3 nanocomposite films. First, we prepared the CoPt(111) ferromagnetic films using the technique described in [20]. This began with the magnetron sputtering of Pt films with a thickness of ~ 50 nm in a vacuum at a residual pressure of 10^{-6} Torr onto a MgO(001) substrate heated to a temperature of $\sim 250\,^{\circ}$C, which ensured epitaxial growth of the Pt(111) plane relative to the substrate surface. Next was the thermal deposition of a polycrystalline Co film with a thickness of ~ 70 nm in a vacuum at a residual pressure of 10^{-6} Torr onto the Pt film at room temperature to prevent a reaction between the layers (the chosen thicknesses of the reacting layers were $\sim$70 nm for Co and ~ 50 nm for Pt, which provided an equiatomic composition), followed by the annealing of the obtained Co/Pt(111)/MgO bilayer samples in a vacuum at 10^{-6} Torr at a temperature of $650\,^{\circ}$C for 90 min. After annealing the Co/Pt(111)/MgO samples, the magnetically hard $L1_0$-CoPt(111) phase forms in the Co/Pt(111) film structure based on the oriented Pt(111) layer [20, 27].

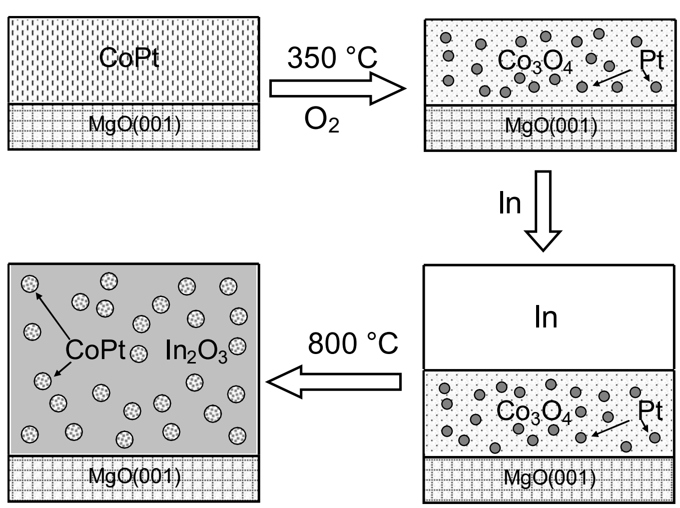

Fig. 1. Schematic of the formation of the CoPt-In_2O_3 nanocomposite films

Then, the $L1_0$-CoPt/MgO films were oxidized in air at a temperature of $\sim 350\,^{\circ}$C for 3 h. The oxidation yielded a Co_3O_4 + Pt film structure containing Pt nanoclusters dispersed in a Co_3O_4 matrix. It should be noted that in the method used, the Co was oxidized, while the Pt remained unoxidized.

The CoPt-In_2O_3 nanocomposite films were obtained by annealing the initial In/(Co_3O_4 + Pt)/MgO(001) samples in a vacuum at 10^{-6} Torr in the temperature range of $100-800\,^{\circ}$C with a step size of $100\,^{\circ}$C and exposure at each temperature for 40 min. Film magnetization was measured after each annealing. The formations of the Co and CoPt magnetic phases were detected by the occurrence of magnetization. Through these measurements, the temperatures of initiation and end of the CoPt-In_2O_3 nanocomposite synthesis were determined.

The thicknesses of the reacting layers were determined by X-ray fluorescence analysis. The saturation magnetization M_s was measured with a torque magnetometer in a maximum magnetic field of 17 kOe. Hysteresis loops in the CoPt-In_2O_3 film plane and perpendicular to it were measured on a vibrating sample magnetometer in magnetic fields up to 20 kOe. The phase composition was investigated by X-ray diffraction using a DRON-4-07 diffractometer in CuK_{α} radiation ($\alpha = 0.15418$ nm). The analysis of the intensity of the X-ray diffraction reflections were made using the ICDD PDF 4+ crystallographic database [28].

Results and discussion

Cobalt reduction and the formation of the CoPt ferromagnetic grains were investigated by measuring the saturation magnetization of the initial In/(Co_3O_4 + Pt)/MgO(001) samples as a function of the annealing temperature $M_s(T)$ (Fig. 2). It can be seen from the $M_s(T)$ dependence that, below $200\,^{\circ}$C, Co reduction processes do not occur in the investigated In/(Co_3O_4 + Pt) structure and its magnetization is therefore close to zero. The magnetization sharply increases at $T > 400\,^{\circ}$C and reaches a maximum at $T > 700\,^{\circ}$C. The $M_s(T)$ (Fig. 2) dependence includes three portions: near $T_1 \sim 200\,^{\circ}$C, near $T_2 \sim 400\,^{\circ}$C and near $T_3 \sim 700\,^{\circ}$C. It is well known [17] that T_1 is close to the temperature $\sim 200\,^{\circ}$C of Co reduction from the Co_3O_4 oxide in the In/Co_3O_4 film system. At the same time, it is well-known [27] that the $L1_0$-CoPt phase starts forming at a temperature of $\sim 375\,^{\circ}$C in Pt/Co films. We can conclude that, at $T_2 \sim 400\,^{\circ}$C, the reaction of the Co reduction from the Co_3O_4 oxide with the formation of the CoPt and In_2O_3 phases continues. At temperatures above $400\,^{\circ}$C, the magnetization of the film sharply grows, which indicates the continuation of the solid-state reaction in the In/(Co_3O_4 + Pt)/MgO(001) film with the formation of the CoPt and In_2O_3 phases. Annealing at $T > 700\,^{\circ}$C facilitates the occurrence of the maximum number of CoPt grains.

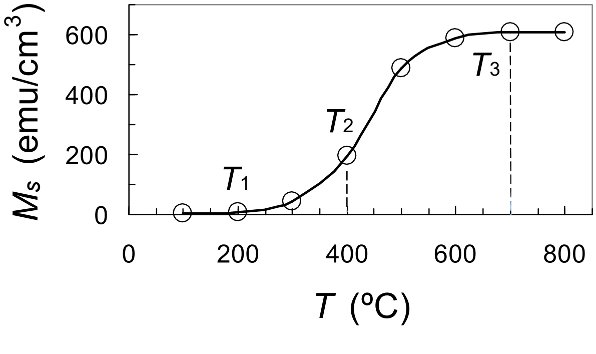

Fig. 2. Dependence of the saturation magnetization M_s on the annealing temperature T of the In/(Co_3O_4 + Pt)/MgO film

X-ray measurements performed after the oxidation of the $L1_0$-CoPt/MgO films in air at a temperature of $\sim 350\,^\circ$C for 3 h and the deposition of the In layer showed that the obtained system consists of the Co_3O_4 (the space group Fd-3m, lattice constant a = 8.0837 Å, PDF Card # 00-042-1467), Pt (the space group Fm-3m, lattice constant a = 3.9231 Å, PDF Card # 00-004-0802), and In (the space group I4/mmm, lattice constants: a = 3.252 Å, c = 4.9466 Å, PDF Card # 04-004-7737) phases (Fig. 3 a). Annealing at a temperature of 400 °C (Fig. 3 b) led to the formation of a small amount of the ordered $L1_0$-CoPt tetragonal phase in the reaction products, which is confirmed by the presence of the (001) superstructural reflection (the space group P4/mmm, lattice constant a = 2.677 Å, c = 3.685 Å, PDF Card # 04-003-4871). The In_2O_3 reflections are also present in the diffraction pattern (the space group Ia-3, lattice constant a = 10.118 Å, PDF Card # 00-006-0416). When annealing at temperatures below 400 °C reflections from the reduced cobalt were not observed because of its high dispersion.

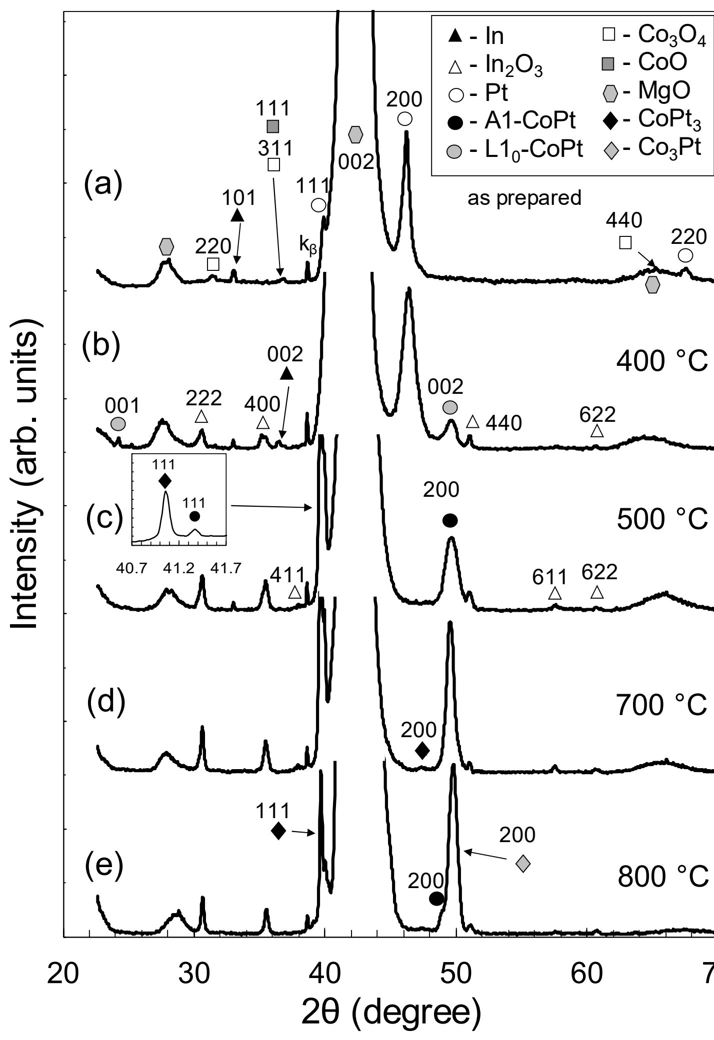

Fig. 3. X-ray diffraction patterns of the In/(Co_3O_4 + Pt)/MgO film after annealing in a vacuum in the temperature range of 100 – 800 °C

When the sample was heated to 500 °C (Fig. 3 c), the reflections from the Pt phase disappear and reflections from the disordered A1-CoPt (the space group Fm-3m, lattice constant a = 3.768 Å, PDF Card # 04-001-0115) and $CoPt_3$ (the space group Pm-3m, lattice constant a = 3.831 Å, PDF Card # 04-004-5243) phases appear. When the sample was annealed to 700 °C (Fig. 3 d), the intensity of the diffraction reflections increased, which is related to reaction relaxation processes, including the increase of the size of the CoPt grains and the improvement of the crystal quality in the insulating In_2O_3 matrix, but no new phases were formed. Annealing at $T = 800$ °C (Fig. 3 e) led to the formation of the Co_3Pt (the space group Fm-3m, lattice constant a = 3.668 Å, PDF Card # 01-071-7411) phase.

The CoPt grain size was estimated from the width of the Co_3Pt (200) reflections (Fig. 3 e) by the Scherrer formula $d = k\lambda/\beta\cos\theta$, where d is the mean crystal grain size, β is the diffraction maximum width measured at half the maximum, λ is the X-ray radiation wavelength (0.15418 nm), θ is the diffraction angle corresponding to the maximum of the peak, and $k = 0.9$. The obtained calculated size of the crystal grains of CoPt was $\sim$ 5 nm.

X-ray diffraction allows us to conclude that after annealing the film contains CoPt (A1-CoPt + $CoPt_3$ + Co_3Pt) alloy nanograins surrounded by In_2O_3. The synthesis of the nanocomposite includes the following successive solid-state reactions:

1. 200 °C $\rightarrow$ 8In + $3Co_3O_4$ = 9Co + $4In_2O_3$,
2. 400 °C $\rightarrow$ Co + Pt = $L1_0$-CoPt,
3. 500 - 700 °C $\rightarrow$ Co + Pt = A1-CoPt and A1-CoPt + 2Pt = $CoPt_3$,
4. 800 °C $\rightarrow$ A1-CoPt + 2Co = Co_3Pt.

When annealing above 400 °C, the transition of the cubic CoPt phase to the tetragonal $L1_0$-CoPt phase does not occur and the formed films are low-coercive. Recently, we synthesized high-coercive CoPt-Al_2O_3 films under the same synthesis conditions (an equiatomic composition Co:Pt = 50:50 on an MgO(001) substrate, vacuum annealing) [20, 27]. It's possible this difference between the synthesis of CoPt-In_2O_3 and CoPt-Al_2O_3 nanocomposite films is due to the fact that in In/(Co_3O_4 + Pt)/MgO(001) films the cobalt is restored before ($\sim$ 200 °C) the formation of the $L1_0$-CoPt phase ($\sim$ 400 °C) and the formed In_2O_3 phase prevents the transition of the cubic CoPt phase to the tetragonal $L1_0$-CoPt phase. In the synthesis of CoPt-Al_2O_3 films, in Al/(Co_3O_4 + Pt)/MgO(001) films, the formation of the $L1_0$-CoPt phase occurs at $\sim$ 375 °C and the Co is restored from the Co_3O_4 oxide at $\sim$ 490 °C [20, 27].

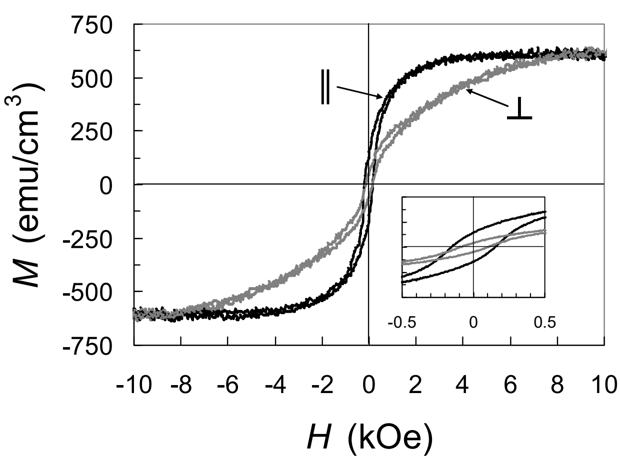

Fig. 4. Hysteresis loops in the CoPt-In_2O_3 nanocomposite film plane and the perpendicular plane

Fig. 4 presents the hysteresis loops measured in the CoPt-In_2O_3 film plane and the perpendicular plane. They have a coercivity of $H_c \sim 150$ Oe, and a saturation magnetization of $M_s \sim 600$ emu/cm^3. The relatively large ratio $M_r/M_s < 0.3$ between the remnant magnetization M_r and saturation magnetization M_s (Fig. 4) shows that the CoPt nanoparticles consist of randomly oriented grains with a cubic magnetocrystalline anisotropy [29].

Conclusion

The main results of our investigations are as follows. The low-coercivity CoPt-In_2O_3 nanocomposite films were obtained by annealing the In/(Co_3O_4 + Pt)/MgO(001) samples in a vacuum at 10^{-6} Torr in the temperature range of 100 – 800 °C with a step size of 100 °C and exposure at each temperature for 40 min. Comprehensive structural and magnetic investigations unambiguously indicate that after annealing the film contains CoPt (A1-CoPt + $CoPt_3$ + Co_3Pt) alloy nanograins by the In_2O_3 layer, with an average size of 5 nm. The synthesized CoPt-In_2O_3 film nanocomposites had a magnetization of about 600 emu/cm^3 and a coercivity of about 150 Oe at room-temperature. The initiation 200 °C and finishing 800 °C temperatures of synthesis and the phase composition of the reaction products were determined. It has been suggested that the formed In_2O_3 phase prevents the transition of the cubic CoPt phase to the tetragonal $L1_0$-CoPt phase and, as a result of the synthesis, low-coercive films were formed. Thus, the solid-state method is promising for synthesizing ferromagnetic nanocomposite thin films consisting of ferromagnetic nanoparticles.

This study was supported by the Russian Foundation for Basic Research, Government of Krasnoyarsk Territory, Krasnoyarsk Regional Fund of Science to the research projects no. 19-43-240003.

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Магнитные и структурные свойства нанокомпозитных пленок CoPt-In_2O_3

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Аннотация. Исследованы структурные и магнитные свойства нанокомпозитных пленок CoPt-In_2O_3, полученных вакуумным отжигом пленочной системы In/(Co_3O_4 + Pt)/MgO в интервале температур 100 – 800 °C. Синтезированные нанокомпозитные пленки содержали ферромагнитные CoPt-кластеры со средним размером 5 nm, заключенные в матрицу In_2O_3, и имели намагниченность 600 emu/cm^3, коэрцитивную силу 150 Oe при комнатной температуре. Определены температуры начала 200 °C и окончания 800 °C синтеза, а также изменение фазового состава пленки In/(Co_3O_4 + Pt)/MgO при вакуумном отжиге.

Ключевые слова: тонкие пленки, ферромагнитные нанокомпозиты, сплав CoPt, оксид In_2O_3.

Journal of Siberian Federal University. Mathematics & Physics 2020, 13(4), 439–450

DOI: 10.17516/1997-1397-2020-13-4-439-450
УДК 536.423.1

To the Question of Analytical Estimate of Evaporation Time of the Drop, Crossing Through the Heat Media

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Received 01.02.2020, received in revised form 13.03.2020, accepted 20.05.2020

Abstract. Due to the kinetic approach the modelling description of the drop evaporation is offer. The main equation of the theory received due to the conservation law of dissipative functions of the vapor – liquid system. The diapason of drop size it's finding when its stability. It's comparison of the results with the famous classical is given. The numerical estimate of the linear size of small disperse phase when take place usually evaporation (i.e. the Knudsen's number is a small $Kn = \dfrac{l}{R} \ll 1$, where l is a free length path of the molecule and R is an drop radius) are given.

Keywords: dissipative function, evaporation, free length path.

Citation: S.O. Gladkov, To the Question of Analytical Estimate of Evaporation Time of the Drop, Crossing Through the Heat Media, J. Sib. Fed. Univ. Math. Phys., 2020, 13(4), 439–450.
DOI: 10.17516/1997-1397-2020-13-4-439-450.

The task that will be discussed in this article is not new, and has about a century of backstory. It must be said that for many physical tasks devoted to the study of the properties of fine environments (fogs, steam, smoke, dust, etc.), it is characteristic that their solution in the vast majority of cases has an empirical and experimental character. Although the number of theoretical works in this direction has been growing quite rapidly in recent years, the conclusion of the main equations is usually based on the dependents obtained purely experimentally. In this paper, we will move away from the well-established stereotype of problem-solving in this direction, and use the general principles of the theory of non-equilibrium processes, using as the basic method of describing the dissipative function $\dot{Q} = T\dot{S}$, where T is an equilibrium temperature, S is an entropy, and the "point" under the letter as usually shows the differentiation in the time.

1. The conclusion of the main equation

Let's write the balance equation taking into account the interaction of gas phase molecules and molecules in a drop at the edge of their contact in the form of the next amount of dissipative functions

$$T\frac{d}{dt}\int_{V_1} s_1 dV + T\frac{d}{dt}\int_{V-V_1} s_2 dV + \frac{d}{dt}\int_{\sigma} \alpha d\sigma = 0, \tag{1}$$

where s_1 is an entropy of drop in the unite of its volume, s_2 is an entropy is a unit of volume surrounding the gas phase drop, including molecules of the already evaporated drop matter, V_1

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variable drop volume, $V = V_1 + V_2 = const$ fullvolume occupied by drop and gas, σ the surface area of the drop. By performing a simple time-by-time differentiation, we find

$$T\dot{S}_1 + Ts_1\dot{V}_1 + T\dot{S}_2 - Ts_2\dot{V}_1 + \alpha\dot{\sigma}_1 = 0. \tag{2}$$

Introducing here the hidden warmth of steam formation

$$\Delta Q_V = T\left(s_1 - s_2\right). \tag{3}$$

Getting out (2)

$$T\dot{S}_1 + T\dot{S}_2 + \Delta Q_V\dot{V}_1 + \alpha\dot{\sigma}_1 = 0. \tag{4}$$

Our task now is to calculate the first two components that are part of the equation (4). According to the definition of entropy in the language of the distribution function (see [1]), we have

$$S_1 = -\frac{1}{Z_1}\int n_1 \ln n_1 d^3p, \tag{5}$$

where n_1 is an nonequilibrium function of the distribution of fluid molecules by pulses, and the rationing multiplier

$$Z_1 = \int \bar{n}_1 d^3p, \tag{6}$$

where the equilibrium distribution function

$$\bar{n}_1 = \exp\left(-\frac{\varepsilon_1(p) - \mu_1(P,T)}{T}\right). \tag{7}$$

Note here that Boltzmann's constant in (5) and in (7) and beyond we will believe an equal unit. Kinetic energy of molecules in liquid is $\varepsilon_1(p) = \dfrac{p^2}{2m_1}$, where $\mu_1(P,T)$ is the chemical potential of a liquid molecules in drop. Similarly

$$S_2 = -\frac{1}{Z_2}\int n_2 \ln n_2 d^3p, \tag{8}$$

where $Z_2 = \int \bar{n}_2 d^3p$,

$$\bar{n}_2 = \exp\left(-\frac{\varepsilon_2(p) - \mu_2(P,T)}{T}\right), \tag{9}$$

where $\varepsilon_2(p) = \dfrac{p^2}{2m_2}$ is the kinetic energy of gas molecules, and $\mu_2(P,T)$ their chemical potential. Differentiating (5) and (8) on time, we have, lowering the permanent term

$$\dot{S}_1 = -\frac{1}{Z_1}\int \dot{n}_1 \ln n_1 d^3p, \quad \dot{S}_2 = -\frac{1}{Z_2}\int \dot{n}_2 \ln n_2 d^3p. \tag{10}$$

In the accordance with Boltzmann's kinetic equation, we have the right to write down that

$$\dot{n}_1 = L_1(n_1, n_2), \quad \dot{n}_2 = L_2(n_2, n_1), \tag{11}$$

where $L_1(n_1, n_2)$ and $L_2(n_2, n_1)$ respectively, the integrals of the collisions of liquid and gas molecules at the border of their contact. Therefore, with the account of expressions (10) and (11) the equation (4) will take the form

$$-\frac{T}{Z_1}\int L_1(n_1, n_2)\ln n_1 d^3p - \frac{T}{Z_2}\int L_2(n_2, n_1)\ln n_2 d^3p + \Delta Q_V\dot{V}_1 + \alpha\dot{\sigma}_1 = 0. \tag{12}$$

The solution of kinetic equations we will look for in the so-called "tau- approximation", according to which the integrals of collisions are replaced by the approximation of expression

$$L_1 \approx -\frac{\delta n_1}{\tau_{12}}, \quad L_2 \approx -\frac{\delta n_2}{\tau_{21}}, \tag{13}$$

where τ_{12} the relaxation time of the liquid molecules when they are scattered on gas molecules, and τ_{21} the relaxation time of the gas molecules when they are scattered on liquid molecules. It is clear that these times are different. We will now find amendments $\delta n_{1,2}$ to the distribution function due to the interaction. According to the kinetic equation, we have

$$\dot{n}_1 = \frac{\partial n_1}{\partial t} + \mathbf{v} \cdot \nabla n_1 + \mathbf{F} \cdot \frac{\partial n_1}{\partial \mathbf{p}} = -\frac{n_1 - \bar{n}_1}{\tau_{12}}. \tag{14}$$

As we are looking for a stationary solution $\dfrac{\partial n_1}{\partial t} = \dfrac{\partial n_2}{\partial t} = 0$. To the rum of that, it should be considered that strength $\mathbf{F} = 0$. As a result

$$\mathbf{v} \cdot \nabla n_1 = -\frac{n_1 - \bar{n}_1}{\tau_{12}}. \tag{15}$$

And similarly

$$\mathbf{v} \cdot \nabla n_2 = -\frac{n_2 - \bar{n}_2}{\tau_{21}}. \tag{16}$$

We will look for solutions to equations (15) and (16) by the method of successive approximations, that is, let's put that

$$n_1 = \bar{n}_1 + \delta n_1, \quad n_2 = \bar{n}_2 + \delta n_2. \tag{17}$$

That's why we get

$$\begin{aligned} \mathbf{l}_{12} \cdot \delta n_1 + \delta n_1 &= -\mathbf{l}_{12} \cdot \nabla \bar{n}_1, \\ \mathbf{l}_{21} \cdot \delta n_2 + \delta n_2 &= -\mathbf{l}_{21} \cdot \nabla \bar{n}_2. \end{aligned} \tag{18}$$

where free-range vectors are introduced $\mathbf{l}_{12} = \mathbf{v}\tau_{12}$, $\mathbf{l}_{21} = \mathbf{v}\tau_{21}$. The solution of equations (18) is convenient to look for by decomposition of the desired functions in the integral Fourier. Indeed, have for arbitrary (yet) function

$$f(\mathbf{r}) = \int_{-\infty}^{\infty} \exp{(i\mathbf{k}\mathbf{r})} f_{\mathbf{k}} \frac{d^3\mathbf{k}}{(2\pi)^3}, \tag{19}$$

where by one-dimensional integral we mean three-dimensional integral, $f_{\mathbf{k}}$ is the Fourier image of the function f. Substituting (19) in any of the equations (18), easy find

$$\int (1 + i\mathbf{k}\mathbf{l})\, \delta n_{\mathbf{k}} \frac{d^3\mathbf{k}}{(2\pi)^3} = -\mathbf{l} \cdot \nabla \int \bar{n}_{\mathbf{k}} \exp{(i\mathbf{k}\mathbf{r})} \frac{d^3\mathbf{k}}{(2\pi)^3}.$$

From where

$$\delta n_{\mathbf{k}} = -i \frac{(\mathbf{k} \cdot \mathbf{l}) \bar{n}_{\mathbf{k}}}{1 + i\mathbf{k} \cdot \mathbf{l}}, \tag{20}$$

where $\bar{n}_{\mathbf{k}}$ is Fourier image of the equilibrium function of molecule distribution $\bar{n}(\mathbf{r})$. Substituting now the solution (20) in the definition (19), find the amendment of interest to the equilibrium function of distribution

$$\delta n = -\frac{i}{(2\pi)^3} \int \frac{(\mathbf{k} \cdot \mathbf{l}) \bar{n}_{\mathbf{k}}}{1 + i\mathbf{k} \cdot \mathbf{l}} \exp{(i\mathbf{k}\mathbf{r})} d^3\mathbf{k}. \tag{21}$$

Here and beyond we simplify the recording of the Integral Fourier, lowering the limits of integration. To calculate the resulting integral, it is convenient to use the next artificial technique. Let's imagine the function $\dfrac{1}{1+i\mathbf{k}\cdot\mathbf{l}}$ in the form of an integral

$$\frac{1}{1+i\mathbf{k}\cdot\mathbf{l}} = \int_0^\infty \exp\left(-x(1+i\mathbf{k}\cdot\mathbf{l})\right)dx. \tag{22}$$

Then from (21) it follows

$$\delta n = -\frac{i}{(2\pi)^3}\int_0^\infty \exp\left(-x\right)dx \int (\mathbf{k}\cdot\mathbf{l})\,\bar{n}_{\mathbf{k}}\exp\left(i\mathbf{k}(\mathbf{r}-x\mathbf{l})\right)d^3\mathbf{k}. \tag{23}$$

Next, as

$$\bar{n}_{\mathbf{k}} = \int \bar{n}(\mathbf{r}')exp(-i\mathbf{k}\mathbf{r}')d^3\mathbf{r}', \tag{24}$$

then substituting (24) in the solution (23), will have as a result of a simple regrouping of multipliers

$$\delta n = -\frac{i}{(2\pi)^3}\int_0^\infty \exp\left(-x\right)dx \int \bar{n}(\mathbf{r}')d^3\mathbf{r}' \int (\mathbf{k}\cdot\mathbf{l})\exp\left(i\mathbf{k}(\mathbf{r}-\mathbf{r}'-x\mathbf{l})\right)d^3\mathbf{k}. \tag{25}$$

To calculate the internal integral, let's use the following technique. Let's write it down as

$$\int \exp\left(i\mathbf{k}(\mathbf{R}-\mathbf{l}x)\right)(\mathbf{k}\mathbf{l})d^3\mathbf{k} = i\frac{\partial}{\partial x}\int \exp\left(i\mathbf{k}(\mathbf{R}-\mathbf{l}x)\right)d^3\mathbf{k} = i(2\pi)^3\frac{\partial}{\partial x}\delta(\mathbf{R}-\mathbf{l}x),$$

where radius–vector $\mathbf{R} = \mathbf{r}-\mathbf{r}'$. As a result, from (25) it follows

$$\delta n = \int_0^\infty \exp\left(-x\right)dx\frac{\partial}{\partial x}\int \bar{n}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}'-\mathbf{l}x)d^3\mathbf{r}' = \int_0^\infty \exp\left(-x\right)dx\frac{\partial}{\partial x}\bar{n}(\mathbf{r}-\mathbf{l}x)dx.$$

We will take the resulting integral by means of integration piece by piece. In fact,

$$\delta n = \int_0^\infty \exp\left(-x\right)dx\frac{\partial}{\partial x}\bar{n}(\mathbf{r}-\mathbf{l}x)dx = \int_0^\infty \exp\left(-x\right)\bar{n}(\mathbf{r}-\mathbf{l}x)dx - \bar{n}(\mathbf{r}). \tag{26}$$

Remembering now the operator of the broadcast, namely the rule

$$\bar{n}(\mathbf{r}-\mathbf{l}x) = \exp\left(-x\mathbf{l}\cdot\nabla\right)\bar{n}(\mathbf{r}).$$

Find out (26)

$$\delta n = \int_0^\infty \exp\left(-x(1+\mathbf{l}\cdot\nabla)\right)\bar{n}(\mathbf{r}) - \bar{n}(\mathbf{r}). \tag{27}$$

Therefore, for the amendments we are in, we get such solutions to equations (18)

$$\begin{cases} \delta n_1 = \displaystyle\int_0^\infty \exp\left(-x(1+\mathbf{l}_{12}\cdot\nabla)\right)\bar{n}_1(\mathbf{r}) - \bar{n}_1(\mathbf{r}), \\ \delta n_2 = \displaystyle\int_0^\infty \exp\left(-x(1+\mathbf{l}_{21}\cdot\nabla)\right)\bar{n}_2(\mathbf{r}) - \bar{n}_2(\mathbf{r}). \end{cases} \tag{28}$$

And hence, according to (12) and (13) find

$$\frac{T}{Z_1}\int \frac{\delta n_1}{\tau_{12}}\ln\bar{n}_1 d^3p + \frac{T}{Z_2}\int \frac{\delta n_2}{\tau_{21}}\ln\bar{n}_2 d^3 + \Delta Q_V\dot{V}_1 + \alpha\dot{\sigma}_1 = 0, \tag{29}$$

where the amendments δn_1, δn_2 accurate solutions (28), albeit in tau approximation. By determining the equilibrium functions of distribution (7) and (9) of (29) the dissipative balance equation follows

$$-\frac{T}{Z_1}\int\frac{\varepsilon_1-\mu_1}{\tau_{12}}\delta n_1 d^3p-\frac{T}{Z_2}\int\frac{\varepsilon_2-\mu_2}{\tau_{21}}\delta n_2 d^3p+\Delta Q_V\dot{V}_1+\alpha\dot{\sigma}_1=0. \tag{30}$$

Note that the last (30) is also convenient to present as $\int \alpha dS=\bar{\varepsilon}_1 N_1=\bar{\varepsilon}_1\int c_1 dV_1$, where $\bar{\varepsilon}_1$ some medium energy coming from one particle of liquid, c_1 is their concentration. In the accordance with (28) the solution can be written down in the form of an endless series of

$$\delta n=\int_0^\infty \exp\left(-x(1+\mathbf{l}\cdot\nabla)\right)\bar{n}(\mathbf{r})dx-\bar{n}=$$
$$=\int_0^\infty \exp(-x)\left(1-x\mathbf{l}\cdot\nabla+\frac{x^2}{2}(\mathbf{l}\cdot\nabla)^2-\frac{x^3}{3!}(\mathbf{l}\cdot\nabla)^3+\ldots\right)\bar{n}(\mathbf{r})dx-\bar{n}.$$

Integrating here each of the material on, we come to the next decision (see [3])

$$\begin{aligned}\delta n&=\left[1-\mathbf{l}\cdot\nabla+(\mathbf{l}\cdot\nabla)^2-(\mathbf{l}\cdot\nabla)^3+\ldots\right]\bar{n}-\bar{n}=\\&=\left[-\mathbf{l}\cdot\nabla+(\mathbf{l}\cdot\nabla)^2-(\mathbf{l}\cdot\nabla)^3+(\mathbf{l}\cdot\nabla)^4\ldots\right]\bar{n},\end{aligned} \tag{31}$$

where the shortness of the decision record (28) is presented with a single designation δn and $\mathbf{l}$, i.e. $\delta n=\{\delta n_1,\delta n_2\}$ and $\mathbf{l}=\{\mathbf{l}_{12},\mathbf{l}_{21}\}$. If you now put the solution (31) in the balance equation (30), $(\mathbf{l}\cdot\nabla)$ then thanks to the integration of momentum all odd degrees will disappear, and instead (30) we get

$$\begin{aligned}&-\frac{T}{Z_1}\int\frac{\varepsilon_1-\mu_1}{\tau_{12}}\left[(\mathbf{l}_{12}\cdot\nabla)^2+(\mathbf{l}_{12}\cdot\nabla)^4+(\mathbf{l}_{12}\cdot\nabla)^6\ldots\right]\bar{n}d^3p-\\&-\frac{T}{Z_2}\int\frac{\varepsilon_2-\mu_2}{\tau_{21}}\left[(\mathbf{l}_{21}\cdot\nabla)^2+(\mathbf{l}_{21}\cdot\nabla)^4+(\mathbf{l}_{21}\cdot\nabla)^6\ldots\right]\bar{n}d^3p+\Delta Q_V\dot{V}_1+\alpha\dot{\sigma}_1=0.\end{aligned} \tag{32}$$

Leaving in (32) only square length of free run components, and given the clear kind of equilibrium function of molecule distribution (7), (9), as a result of elementary differentiation come to such an equation

$$\begin{aligned}&-\frac{T-\varepsilon_1-\mu_1}{T}\frac{l_{12}^2}{\tau_{12}}\left(\Delta\mu_1+\frac{(\nabla\mu_1)^2}{T}\right)-\\&-\frac{T-\mu_2}{T}\frac{l_{21}^2}{\tau_{21}}\left(\Delta\mu_2+\frac{(\nabla\mu_2)^2}{T}\right)+\Delta Q_V\dot{V}_1+\alpha\dot{S}\Big|_{r=R}=0.\end{aligned} \tag{33}$$

Since at the border of the two phases in the absence of chemical reactions must be met the condition of continuity of entropy, it is quite clear that there is equality

$$\Delta Q_V=T\left(s_1-s_2\right)\big|_{r=R}=0. \tag{34}$$

As we can see, this condition is true if the temperature is constant. However, it is quite clear that the equality of entropy at the border of the contact of the drop and gas mixture does not mean the equality of their specific heat-intensiveness, because from the point of view of mathematics equality (34) should be recorded in a slightly different form, namely how

$$s_1\big|_{r=R-0}=s_2\big|_{r=R+0}. \tag{35}$$

That is, the limits are taken to the left and right of the contact boundary. Therefore, it is quite clear that due to the lumpy smoothness of entropy (35) follows and condition for temperature derivatives from entropy, which just characterizes the heat intensity of both phases. Formally, this means that there is equality

$$c_1|_{r=R-0} = c_2|_{r=R+0} + \Delta c, \tag{36}$$

where the heat capacity supplement Δc means the final spike in heat capacity at the edge of the section of the two environments, and the isobaric heat intensity is introduced here in accordance with the generally accepted definition [2] $c_i = T\left(\frac{\partial s_i}{\partial T}\right)_P$, where index is $i = 1, 2$.

As for the physical side of the equation (33), it should be stressed immediately that as soon as we introduce the concept of variable entropy, we automatically move on to taking into account the dissipative properties of the matter. That is, in an nonequilibrium case, which is described by the equation (33), has a condition of increasing entropy (H – Boltzmann's famous theorem). As it becomes clear now, taking into account the interaction between molecules of both phases, that is, the transfer of energy from water molecules to gas molecules and vice versa leads to the destruction of the weak surface tension of the drop. To analytically describe this process, it is necessary to focus on the remarkable property of any natural physical phenomenon, like the hierarchy of relaxation times [4].

Indeed, by the order of magnitude, the free path of molecules in the liquid l_{12} is much less than the free path of gas molecules l_{21}, that is, inequality is performed $l_{12} \ll l_{21}$.

This means that in terms of the hierarchy of times by virtue of the condition $\tau_{12} \ll \tau_{21}$, which actually follows from a condition $\bar{n}_1 \gg \bar{n}_2$, where $\bar{n}_1, \bar{n}_2$ accordingly the average concentrations of liquid and gas molecules, the basic evaporation process belongs to the first composed (33), and it is this important fact that allows us to neglect the second term.

Otherwise. The first process, as the fastest, has already occurred and the drop has begun to evaporate, and the second has not yet had time to begin. This does not mean, however, that it does not contribute to the evaporation process: in a later period of time, this contribution will appear. So, given the continuity of entropy at the contact boundary (35) and with all that said, we get this equation from (33)

$$-\frac{T - \bar{\varepsilon}_1 - \mu_1}{T}\frac{l_{12}^2}{\tau_{12}}\left(\Delta\mu_1 + \frac{(\nabla\mu_1)^2}{T}\right) + \alpha\dot{S}\Big|_{r=R} = 0. \tag{37}$$

Note also that for the chemical potentials of both phases at the border there is a condition of equilibrium

$$\mu_1|_{r=R} = \mu_2|_{r=R}\,. \tag{38}$$

Because $\dot{S} = 8\pi\alpha R\dot{R}$, of (37) find

$$8\pi\alpha R\dot{R} = \frac{\mu_1 - \bar{\varepsilon}_1 - T}{T}\frac{l_{12}^2}{\tau_{12}}\left(\Delta\mu_1 + \frac{1}{T}\left(\frac{\partial\mu_1}{\partial R}\right)^2\right). \tag{39}$$

Because the distribution of heterogeneous chemical potential in contact between the two media (see [5]) describing due to the equation

$$\Delta\mu + \frac{\mu}{\delta^2} - \frac{\xi\mu^3}{\delta^2 T^2} = 0, \tag{40}$$

where δ is the length of heterogeneity that satisfies inequality $\delta \ll l_{min}$, where $l_{min} = = \min\{l_{12}, l_{21}\}$, and ξ is some coefficient leading to a correct decision (41), then in one-dimensional case out of (40) we will get

$$\mu(r) = \frac{\mu_1 + \mu_2}{2} - \frac{\mu_1 - \mu_2}{2} \operatorname{th}\left(\frac{\delta r}{\delta}\right). \tag{41}$$

Therefore, at the border of contact we have

$$\left.\frac{\partial \mu}{\partial r}\right|_{r=R} = \frac{\mu_2 - \mu_1}{2\delta}. \tag{42}$$

Thus, the equation (42) takes on the form of

$$8\pi\alpha R\dot{R} = -\frac{\mu_1 + \bar{\varepsilon}_1 - T}{T} \frac{l_{12}^2 \mu_1}{\tau_{12}\delta^2} \left(\frac{\mu_1}{4T}\left(1 - \frac{\mu_2}{\mu_1}\right)^2 + \frac{\xi\mu_1^2}{T^2} - 1\right). \tag{43}$$

Where do we get a direct integration, taking into account the initial conditions $R(0) = R_0$

$$R = \sqrt{R_0^2 - D_T t}, \tag{44}$$

where the diffusion coefficient is

$$D_T = \frac{\mu_1 + \bar{\varepsilon}_1 - T}{4\pi\alpha T} \frac{l_{12}^2 \mu_1}{\tau_{12}\delta^2} \left(\frac{\mu_1}{4T}\left(1 - \frac{\mu_2}{\mu_1}\right)^2 + \frac{\xi\mu_1^2}{T^2} - 1\right). \tag{45}$$

And hence the time of evaporation of the liquid drop is from here from the condition of equality of zero subdivided expression, that is,

$$t_{evap} = \frac{R_0^2}{D_T}. \tag{46}$$

As to the time of relaxation τ_{12} it is easy to show that it can be calculated by formula

$$\frac{1}{\tau_{12}} = \frac{2r_2^2 \bar{n}_1}{3\pi\sqrt{2\pi}} \frac{m_1^2 \bar{\mu}_1}{(m_1 + m_2)^3} \sqrt{\frac{m_2}{T}}, \tag{47}$$

where r_2 is a radius of a molecule of the gas, $\bar{\mu}_1$ their medium chemical potential, m_1 is the mass of molecule of the water, m_2 is the mass of the molecule of the gas, $\bar{n}_1$ is the middle concentration of molecules of the water. In order of magnitude (47) it follows that $\tau_{12} \approx 10^{-10}\, s$. A similar formula has a place for relaxation time τ_{21}. It comes from a formula (47) formally replacing indices $''1''$ with $''2''$. In the order of magnitude $\tau_{21} \approx 10^{-8}\, s$. Calculating the evaporation time of the formula (45) also requires substitution of the chemical potential of gas and liquid. Based on the general definition of the average energy of a large particle statistical system, namely $\Omega = \mu(P, T)N$, where N is a number of particles in the system, for its differential we have

$$d\Omega = \left(\frac{\partial \mu}{\partial T}\right)_P N dT + \left(\frac{\partial \mu}{\partial P}\right)_T N dP + \mu dN. \tag{48}$$

According to [2], for example, in the variable (T, P, N) the Helmholtz's energy differential is

$$d\Phi = -S dT + V dP + \mu dN. \tag{49}$$

From the comparison (48) and (49) we see that

$$S=-N\left(\frac{\partial\mu}{\partial T}\right)_P, \quad V=N\left(\frac{\partial\mu}{\partial P}\right)_T. \tag{50}$$

It is known from [2] the one that entropy per particle can be calculated as

$$s=\frac{S}{N}=-\frac{1}{Z}\int n\ln n d^3\mathbf{p}, \tag{51}$$

where the rationing multiplier $Z=\int \bar{n}d^3\mathbf{p}$, and

$$\bar{n}=\exp\left(-\frac{\varepsilon(p)-\mu}{T}\right) \tag{52}$$

is the equilibrium Maxwell distribution function, $\mathbf{p}$ momentum of molecule. Neglecting in (51) molecule scattering processes, we have

$$S=-\frac{N}{Z}\int \bar{n}\ln\bar{n}d^3\mathbf{p}=\frac{N}{ZT}\int(\varepsilon-\mu)\exp\left(-\frac{\varepsilon-\mu}{T}\right)d^3\mathbf{p}. \tag{53}$$

The chemical potentials in the exhibitor indicators under integral in (53) and in the normal multiplier will be reduced, and as a result of simple calculation we will come to such an answer

$$S=N\left(\frac{3}{2}-\frac{\mu}{T}\right). \tag{54}$$

Remembering now the definition (50), we get the following differential equation to determine of chemical potential μ

$$\left(\frac{\partial\mu}{\partial T}\right)_P=\frac{\mu}{T}-\frac{3}{2}. \tag{55}$$

Simple integration leads us to the next result

$$\mu(P,T)=C(P)T-\frac{3}{2}T\ln T, \tag{56}$$

where the dependence $C(P)$ we can easy find due to the second ratio in expr. (50), i.e.

$$V=N\left(\frac{\partial\mu}{\partial P}\right)_T=NT\frac{dC}{dP}. \tag{57}$$

Since the Clapeyron-Mendeleev equation $PV=NT$ is in place for the ideal gas, we immediately get that

$$C(P)=A+\ln P, \tag{58}$$

where A is an constant of integration. Assuming that $A=1$ and substituting expr. (58) to the (56), find the dependency we're going to find

$$\mu(P,T)=T+T\ln\left(\frac{P}{P_0}\right)-\frac{3}{2}T\ln\left(\frac{T}{T_0}\right), \tag{59}$$

where T_0, P_0 are the temperature and the pressure at normal condition, i.e. $T_0=300\ K$, $P_0=1\ atm=10^5\ Pa$. That is, for the gas phase, the chemical potential is determined by (59) as

$$\mu_2=T+T\ln\left(\frac{P_2}{P_0}\right)-\frac{3}{2}T\ln\left(\frac{T}{T_0}\right). \tag{60}$$

As for the drop of water, it is very problematic to use the gas approximation for it, and in this case it is necessary to apply the equation of the state of Van-der-Waals. As a result, the chemical potential can also be calculated analytically, but now we will not stop there, and move on to the assessment of the time of evaporation, considering for simplicity that $\mu_1 \sim \mu_2$. Note, by the way, that this ratio is quite correct. To estimate the evaporation time according to the general expression (43), we will select the following values of the parameters included in it

$$\alpha = 70\frac{erg}{cm^2}, \quad \mu_1 \sim \mu_2 = 6 \cdot 10^{-14}\, erg, \quad T = 300\ K = 4 \cdot 10^{-14}\, erg,$$
$$R_0 = 5 \cdot 10^{-1}\, cm, \quad \tau_{12} = 10^{-10}\, c, \quad \delta \approx 10^{-6}\, cm.$$

In the result

$$t_{evap} = \tau_{12}\frac{4\pi\alpha R_0^2\delta^2}{\left(\frac{\mu_1}{4T}\left(1-\frac{\mu_2}{\mu_1}\right)^2 + \frac{\xi\mu_1^2}{T^2} - 1\right)\left(\frac{\mu_1+\bar{\varepsilon}_1}{T} - 1\right)\mu_1 l_{12}^2} \approx$$
$$\approx 10^{-10}\frac{4\pi \cdot 70 \cdot 25 \cdot 10^{-2} \cdot 10^{-12}}{3 \cdot 6 \cdot 10^{-14} \cdot 10^{-10}} = \frac{2 \cdot 70 \cdot 25}{3} \approx 1.15 \cdot 10^3\ s = 20\ min. \tag{61}$$

That is, a drop of water with a diameter of five millimeters evaporates in about twenty minutes. And then there's. Looking at the equation (39), we clearly see an equation such as a thermal conductivity equation with a temperature-conductivity factor χ, or a diffusion-type equation with a diffusion factor D, which is determined by the ratio of the right side of the equation (39), i.e.

$$D \sim \chi \sim \frac{l_{12}^2}{\tau_{12}} = \frac{v_{1T}^2\tau_{12}^2}{\tau_{12}} = v_{1T}^2\tau_{12}. \tag{62}$$

This remarkable result is evidence that the evaporation process is purely dissipative and in isotherm conditions is determined by the heterogeneity of chemical potential at the border of contact between liquid and gas. In light of what has been said, it can be argued that according to (62) the described evaporation effect is nothing more than isotherm diffusion. In fact, the assessment (61) of the task of analytical description of the drop evaporation process can be considered solved.

The theoretical approach described above is worth comparing with the approach outlined, for example, in the Fuchs's classic monograph [6]. It is worth noting that this monograph is entirely based on the interpretation of purely empirical dependencies, that is, dependencies obtained experimentally. However, the formulas in it allow us to draw some parallel with the theoretical analysis given a little above. If we enter the Sherwood number according to the formula (see [6, 7])

$$Sh = \frac{I_f}{2\pi RD(c_0 - c_\infty)}, \tag{63}$$

where I_f is a speed of evaporation, having the dimension $\frac{g}{s}$, D is the diffusion coefficient with the dimension $\frac{sm^2}{s}$, c_0 is the concentration of steam in close proximity to the drop (its dimension is $\frac{g}{cm^3}$), c_∞ is concentration of steam on infinity with the same dimension, in the case of a stationary drop, the Sherwood's number is exactly 2. Using empirical dependence (63) we will find the dependence of the radius of the evaporating drop from time to time. Assuming that

$I_f = \dot{m}$, where the mass of the drop is $m = \rho_k V = \dfrac{4\pi}{3}\rho_k R^3$, and accounting that $Sh = 2$, we find from expr. (63),

$$4\pi\rho_k R^2 \dot{R} = 4\pi R D(c_0 - c_\infty).$$

Or

$$R\dot{R} = \frac{D(c_0 - c_\infty)}{\rho_k}. \qquad (64)$$

Where does the solution come from immediately

$$R(t) = \sqrt{R_0^2 - D_{eff}t}, \qquad (65)$$

where the effective diffusion coefficient is

$$D_{eff} = \frac{D(c_0 - c_\infty)}{\rho_k}. \qquad (66)$$

Comparing (65) with our decision (44) we see their full identity. According to the formula (44), the rate of evaporation behaves like

$$v_{vap} = |\dot{R}(t)| = \frac{D_T}{2\sqrt{R_0^2 - D_T t}}. \qquad (67)$$

It's the right place to go, that when the drop size is reduced, its evaporation rate increases dramatically, which is experimental observe (see an example papers [8, 9]). Dependencies (44) and (67) are illustrated by drawings in Figs. 1, 2.

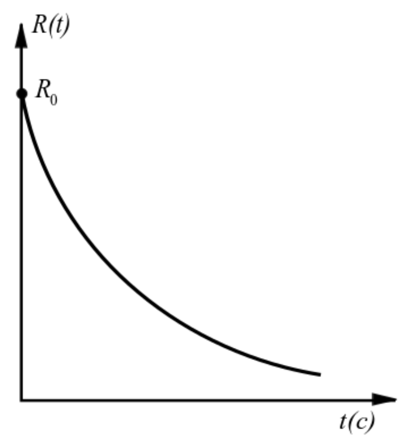

Fig. 1. Schematic representation of the time as dependence of the radius drop

However, our diffusion coefficient (45) and the empirical formula (63) according to (66) are quite different from each other qualitatively. Although in order of magnitude they both give the correct value of the time of evaporation of the stationary drop at condition that in the formulae (66) difference $c_0 - c_\infty$ choose equal $1\dfrac{g}{cm^3}$, and the diffusion coefficient put equal as in our theory value $D = 5 \cdot 10^{-5}\dfrac{cm^2}{s}$. This is, in principle, understandable, since the rigorous analytical solution to the problem, based on the equation of preserving the amount of dissipative function (1) and the experimentally obtained dependence (63), is based on different physical assumptions on which the authors rely.

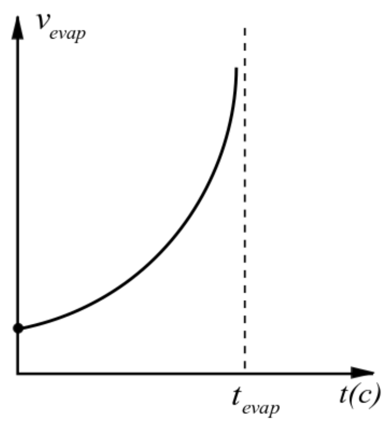

Fig. 2. The time as dependence of the evaporation rate drop

Conclusion

In the conclusion, it is worth noting three important points.

1. The theory of evaporation of droplets of fine-dispersed environment, based on the condition of preservation of dissipative function, has been built (dissipated energy cannot disappear without a trace, but passes into something).

2. Suggested description of the dynamics of the drop in a high-temperature environment, taking into account its evaporation.

3. The numerical estimates of the optimal size of the drops and their initial speed in the jet are.

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К вопросу аналитической оценки времени испарения капли, проходящей сквозь горячую среду

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Аннотация. С помощью кинетического подхода предложено модельное описание процесса испарения капли, движущейся в раскаленной среде. Основное уравнение теории получено благодаря использованию закона сохранения диссипативных функций системы пар – жидкость. Найден диапазон размеров капли, при которых она устойчива. Дано сравнение полученных результатов с известными классическими. Приведены численные оценки размеров мелкодисперсной фазы, при которых имеет место обычное испарение (то есть выполняется условие на число Кнудсена $Kn = \frac{l}{R} \ll 1$, где l — длина свободного пробега молекулы, а R — радиус капли).

Ключевые слова: диссипативная функция, испарение, длина свободного пробега.

DOI: 10.17516/1997-1397-2020-13-4-451-458
УДК 537.622.4

Structural Changes of Co Caused a Change of the Solution pH During Chemical Deposition

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Received 10.04.2020, received in revised form 13.05.2020, accepted 25.06.2020

Abstract. The phase transformations of the Co lattice are discussed, which determine the anomalous changes in the magnetic properties of chemically deposited Co-P films obtained at various pH values. The coercivity of the H_c films obtained at low pH values exceeds 1 kOe and decreases to several units Oe in the films obtained at high pH values. It is shown that the observed changes in the magnetic properties of Co-P films are caused by the transition of the cobalt crystal lattice to the nanocrystalline state.

Keywords: Co-P films, chemical reduction of metals, induced magnetic anisotropy, nanocrystalline material.

Citation: S.A. Podorozhnyak, A.V. Chzhan, V.K. Maltsev, I.N. Krayuhin, G.S. Patrin, I.Yu. Sakash, Structural Changes of Co Caused a Change of the Solution pH During Chemical Deposition, J. Sib. Fed. Univ. Math. Phys., 2020, 13(4), 451–458. DOI: 10.17516/1997-1397-2020-13-4-451-458.

The most famous and broadly used technology of metal's chemical deposition from salt solutions is the application of anticorrosion coating upon metallic surfaces, as well as producing the connecting stripes on printed-circuit boards for the needs of radio electronics. Besides

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these industries, using the chemical technology is very perspective for the purposes of producing thin magnetic films which are necessary for developing such devices as memory cells for magnetic/thermomagnetic recording and data storage, as well as creating highly sensitive signal transmitters. The most interesting, from an applied science's point of view, is a possibility of producing materials with various characteristics that is easily performed by changing conditions of chemical reactions, such as chemical composition and acidity of the deposition's environment, a temperature of the deposition and so on. There upon, the magnetic films produced by Co-P alloying are especially interesting. This alloy, because of high values of some cobalt parameters, is most usable in terms of practical application. Therefore, the specificity of its producing is a subject of scientific interest that resulted in a number of publications. The most of them are focused on the influence of solution acidity upon the films' structural and magnetic properties. Using some additional reagents, able to change solutions' pH, it is possible to improve the morphology of film's surface, to change its structure [1–5] and to produce, depending on the pH value, either highcoercivity or lowcoercivity specimens [6–8]. However, because of the experiment's multi centricity and the complexity of describing the redox processes, the technology of metals' chemical deposition from water solutions is not properly studied and developed. This is the main factor barring broad use of chemical deposition for the purposes of producing magnetic films.

This work, based upon experimental data, demonstrates that, within the given range of the process solution's concentrations, growth of the solution's pH follows to a polymorphic phase transition of the Co crystal lattice from the *hcp* structure to the *fcc* one, that results in anomalous changes of its magnetic properties: coercive force and magnetic anisotropy. Basing on the analysis of redox processes of the cobalt deposition, the procedure of revealing the pH influence upon the granular microstructure is proposed.

1. The technology of specimen producing and the procedure of measurement

The process solution is the water one of cobalt sulphate ($CoSO_4 \cdot 7H_2O$) with the concentration of 15 g/l, the solution of sodium hypophosphite ($NaH_2PO_2 \cdot H_2O$) of 10 g/l, and the one of sodium citrate ($Na_3C_6H_5O_7$) of 25 g/l. The required pH value is reached by adding alkaline reagents such as sodium hydro carbonate $\mathrm{NaHCO_3}$ or caustic soda NaOH of different concentrations. The value of solutions' acidity is measured by the pH-150MI apparatus to an accuracy of $\pm$ 0.05. The deposition goes under the temperature of 100^0C and the magnetic-field strength of H=3 kOe on a cover glass faceplate, previously cleaned, sensitized and activated by the methods, standard for the technology of dielectrics' chemical metallization.

The films' microstructure is analyzed by the methods of transmission electron microscopy, including revealing the specimens' elemental composition, using a TEM HT-7700 (Hitachi) with a X-Flash 6T/60 (Bruker) energy dispersive detector. The values of coercive force and saturation intensity are revealed by, consequently, the meridional Kerr effect and a SQUID magnetometer (room temperature). Measuring NMR-signals is performed by a standard spin echo apparatus within the range of 150–250 MHz.

2. Research results

Fig. 1 shows dependence of the created films' induced anisotropy constant $\mathrm{K_U}$ and the coercivity $\mathrm{H_C}$ upon solutions' acidity.

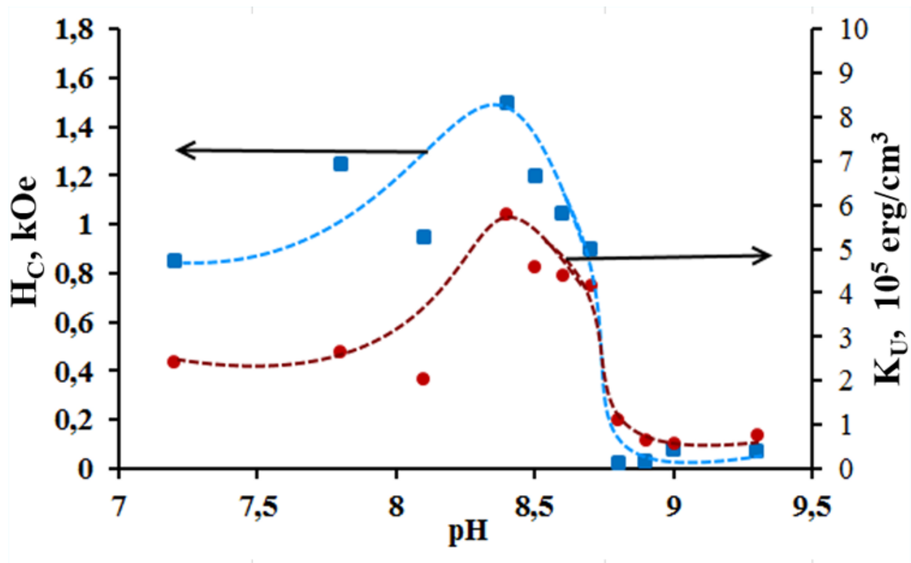

Fig. 1. Dependence of H_C and K_U on the solution's acidity

Within the range of low pH (7.2÷ 8.5) K_U value is $\sim 2.5 \times 10^5$ erg/cm^3. Growing pH follows to rising anisotropy, and within the range of pH$\sim$ 8.5 K_U is $\sim 6 \times 10^5$ erg/cm^3. The following growth of acidity results in a step-like drop of K_U by an order, to 5×10^4 erg/cm^3. The value of $H_C$with pH growing to $\sim$ 8.5 rises from $\sim$ 1 kOe to $\sim$ 1.5 kOe, and later, with the following pH growth, H_C step-likely drops down to few Oes.

As it follows from the data of specimens' X-ray diffraction analysis, obtained for pH $<$ 8.5 (Fig. 2a), the highly coercivity is characterized by existing a *hcp*-phase of cobalt. Relatively equal heights of major peaks reflect presence of the polycrystalline structure with no preferred direction of the crystallite growth; that may also be derived from the analysis of electron microscopic (TEM) micrographs of the surfaces, represented by Fig. 2b. Fig. 2c demonstrates a typical electron diffraction pattern of a highcoercive polycrystalline film.

The transition to lowly coercive conditions with pH $>$ 8.5 corresponds with the film structure changes, visible at transmission electron microscopic (TEM) micrographs of the surface, represented by Fig. 3b. X-ray photographs of such specimens show diffused reflexes, typical for nanocrystalline materials (Fig. 3a). Layers, dividing denser structural formations, are visible at TEM micrographs as well as it is in the case of highly coercive Co-P films. These electron diffraction patterns (Fig. 3c) bear witness to the presence of a nanocrystalline structure.

Basing on the analysis of TEM micrographs of the highly anisotropic polycrystalline films and electron diffraction patterns of the lowly anisotropic polycrystalline ones, the dependence of the size of granules (of which the films consist) on pH values is revealed (Fig. 4).

Because the X-ray diffraction analysis is poor for obtaining data on the Co atomic environment in the films with small grains the spin echo NMR method has been used for these purposes [9]. It is well-known that the amplitude of the spin echo NMR signal is determined by the specimen's magnetic susceptibility χ, that is inversely proportional to the value of the magnetic anisotropy's field ($A_e \sim 1/H$). Thus the NMR technology has been used for researching lowly anisotropic specimens. The integral spin echo spectrum of the studied specimens is a broad curve (within the range of 185–230 MHz) having a diffused maximum, located near 200 MHz, the signal intensiveness dropping to zero at the low-frequency left part of the curve and, at the same time,

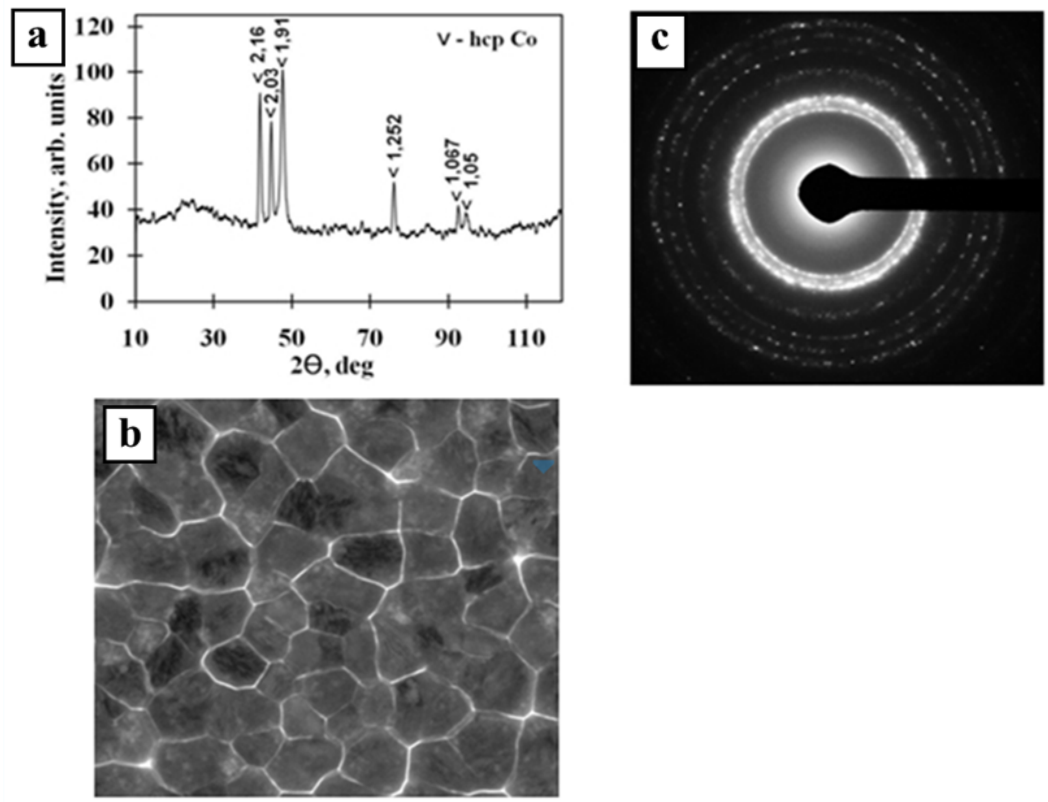

Fig. 2. Research data on the highly anisotropic specimens: a — roentgenogram, b — electron microscopic (TEM) micrograph, c — electron diffraction pattern

certain absorption features at the right part of the spectrum (Fig. 5). This absorption maximum corresponds with incomplete filling of the first coordination sphere (11 instead of 12) in the nearest environment of Co atoms within close packing of the *fcc*-phase. This effect may be a result of cobalt lattice defects, diamagnetic substitution of a cobalt atom by one of phosphorus, edge (surface) effects and uncompensated ties on the surface of crystallites. Additional absorption maxima (216 MHz and 227 MHz) correspond to the high-frequency absorption of Co nuclei for *fcc* and *hcp* structures consequently. This peak looks more intensive for the A-specimen and more diffused for the B one, that is caused by a greater degree of disordering within the nearest atomic environment of the *fcc* lattice. There is an additional peak (227 MHz) for the B-specimen that corresponds with the influence of the *hcp* lattice.

3. Discussion

It is well-known that there is close interrelation between the cobalt particles' sizes and their crystal structure. Small particles($d < 10$ nm) have cubic *fcc* structure (β-phase), whereas large ones ($d > 40$ nm) have *hcp* structure (α-phase); the intermediary field has a mixture of both [10]. Such transformations of crystal structures are caused by a various degree of dependence of the α- and β-phase free energy on cobalt particles' diameters. The structural changes of cobalt

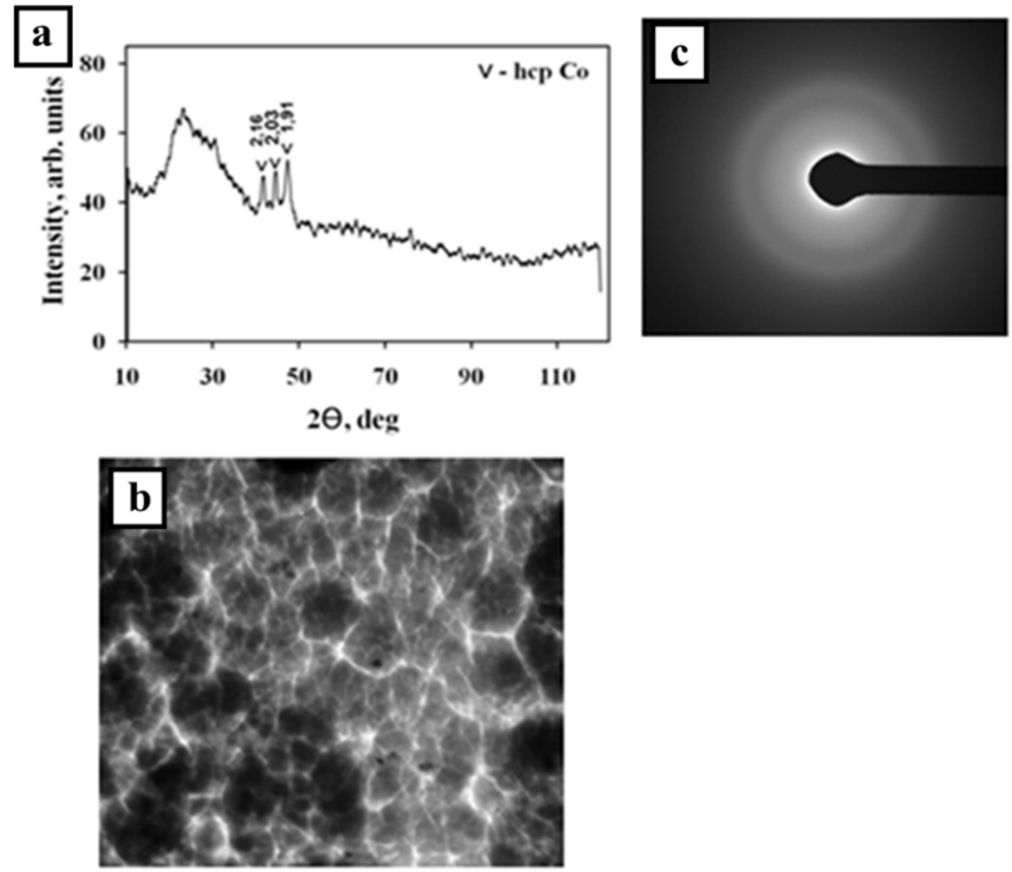

Fig. 3. Research data on the lowly anisotropic specimens a — roentgenogram, b — electron microscopic (TEM) micrograph, c — electron diffraction pattern

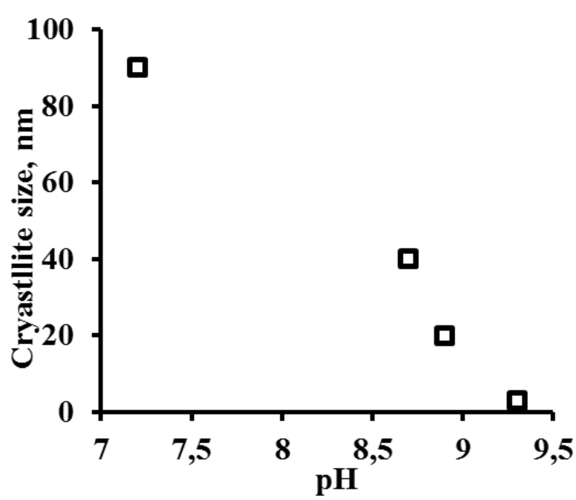

Fig. 4. Dependence of crystallites sizes on solution's pH

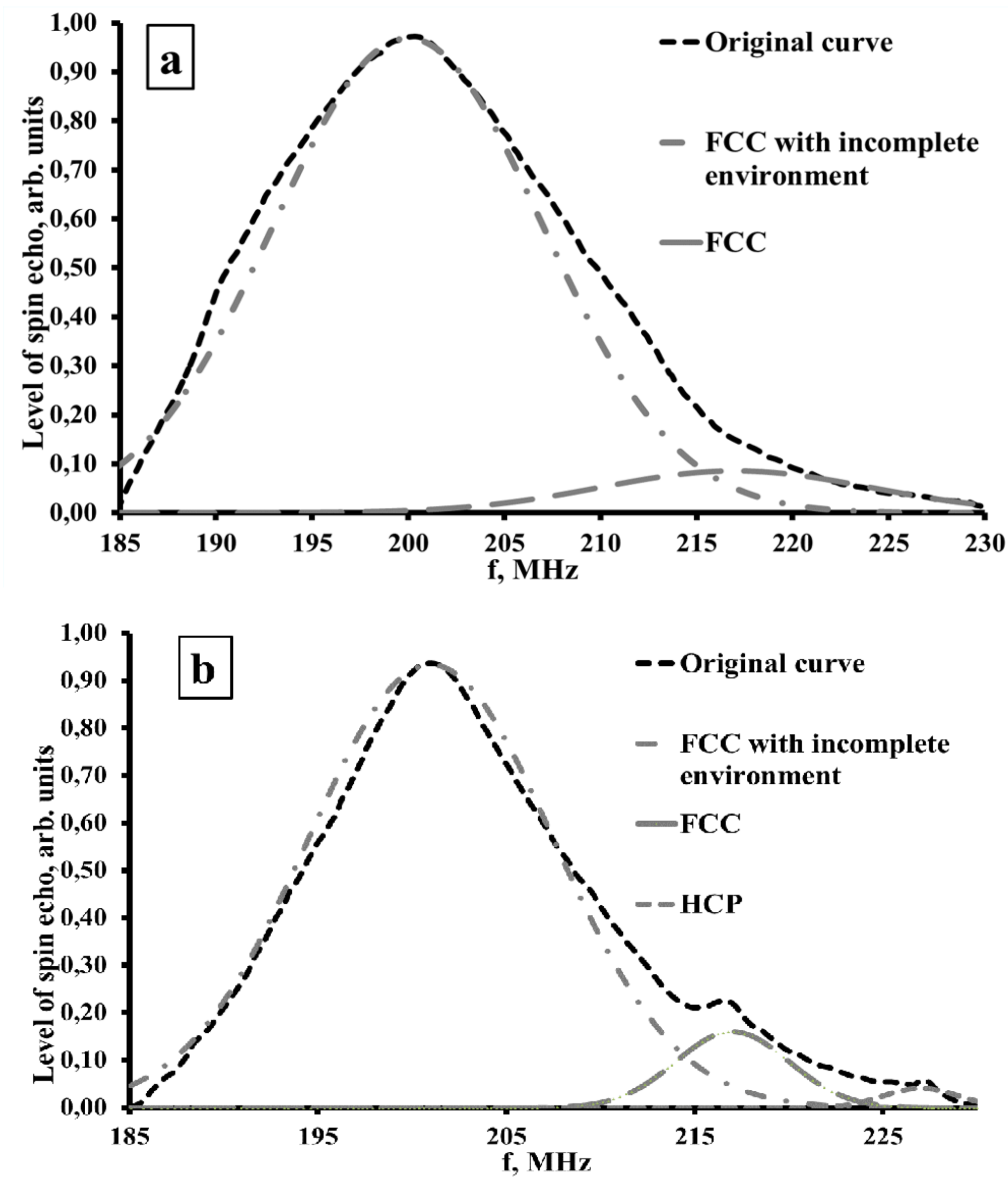

Fig. 5. Spectrum of spin echo specimens, created by pH= 8.9 (a) and 9.15 (b)

particles correspond with visible structural variations of the films studied. If pH is low, large Co particles emerge (up to 70 nm); their stable phase is α-phase. Growing acidity results in the emergence of smaller Co particles (up to 5 nm) whose stable phase is β-phase. As it follows from the aforementioned data, the films, created within the environment with low pH values, mostly have the *hcp* structure with large crystallites; the latter are diminishing together with growing acidity and, if pH becomes more than 8.5, the structure of the substance is becoming transformed into the *fcc*-modification. In the beginning of this transformation we can see a mixture of the phases with the dominated influence of *hcp*-cobalt upon magnetic properties of the substance. The phase transition ends at pH $\approx$ 8.7 when the substance's magnetic properties become to be determined by cobalt's β-phase. This transition is diffused because of sufficient dispersion of the particles' sizes in the films that results in their phase heterogeneity. The following acidity growth leads to subsequent depressing α-phase of cobalt; the substance mostly transforms into a *fcc* lattice with an incomplete environment.

4. Conclusion

As it follows from the data obtained, sharp changes of the induced magnetic anisotropy and the coercive force of Co-P films in the conditions of growing acidity correspond with Co crystal lattice's modifications. The films, created in the environment with low pH, have a structure with large Co crystallites of *hcp* lattice. The acidity growth leads to decrease of the crystallites' typical sizes and, as a result, to the polymorphic phase transition with emerging *fcc* structure.

We are grateful to the Center for the common use of Federal Research Center KSC SB RAS.

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Структурные изменения кобальта, вызванные изменением pH при химическом осаждении

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Аннотация. Обсуждаются фазовые превращения решетки Co, которые определяют аномальные изменения магнитных свойств химически осажденных пленок Co-P, полученных при различных значениях pH. Коэрцитивная сила пленок, полученных при низких значениях pH, превышает 1 кЭ и снижается до нескольких единиц Э в пленках, полученных при высоких значениях pH. Показано, что наблюдаемые изменения магнитных свойств пленок Co-P вызваны переходом кристаллической решетки кобальта в нанокристаллическое состояние.

Ключевые слова: пленки Co-P, химическое восстановление металлов, наведенная магнитная анизотропия, нанокристаллический материал.

DOI: 10.17516/1997-1397-2020-13-4-459-465
УДК 539.231; 539.2526

The Study of the Fine Structure of Ti-Al Coatings on the Surface of Ti, Obtained by Mechanical Alloying

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Received 27.12.2019, received in revised form 27.04.2020, accepted 10.06.2020

Abstract. The work is devoted to the study of structural-phase transformations in composite coatings (Ti-Al)+Ti during mechanical alloying. The data on the structural-phase states of (Ti-Al)-Ti coatings after mechanical alloying have been obtained, confirming the mechanism of formation of the modified layer due to deformation compaction of powder particles on the titanium surface under mechanical action. As a result of mechanochemical fusion, a $TiAl_3$ phase with a bcc lattice ($I4/mmm$ structure) was detected, which corresponds to the stable state of the $TiAl_3$ alloy. Under conditions of mechanical alloying of the structure, $I4/mmm$ transforms into the $L1_2$ structure, which corresponds to the metastable state of $TiAl_3$.

Keywords: structural phase transformations, composite coatings, mechanical alloying, solid phase processes.

Citation: Z.B. Sagdoldina, B.K. Rakhadilov, R.B. Abylkalykova, L.I. Kveglis, P.O. Shalaev, R.S. Kozhanova, The Study of the Fine Structure of Ti-Al Coatings on the Surface of Ti, Obtained by Mechanical Alloying, J. Sib. Fed. Univ. Math. Phys., 2020, 13(4), 459–465.
DOI: 10.17516/1997-1397-2020-13-4-459-465.

Introduction

Currently, it is known that intermetallics represent a unique class of materials that retain an ordered structure up to the melting point, i.e., the melting and ordering temperatures coincide. Long-range order provides a stronger interatomic bond.

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Of the large number of known intermetallic compounds, the greatest attention of both experimenters and theoreticians is attracted by alloys based on titanium and aluminum.

Alloys of the Ti-Al system retain their structure and strength at high temperatures, have good anticorrosion and antifriction properties, which are significantly superior to conventional metals. In addition, alloys of the Ti-Al system have: high melting point, low density, high modulus of elasticity, resistance to oxidation and fire, high specific heat resistance [1, 2].

Titanium aluminides are considered promising structural materials for high-temperature applications in modern industries, such as aerospace, automotive, shipbuilding and others.

High values of specific strength of Ti-Al compounds in comparison with nickel superalloys make titanium aluminides very promising for the production of components of modern aircraft engines and turbines, however, their corrosion resistance remains lower than desired. In addition, a balance between the mechanical properties of titanium aluminides and their resistance to external factors cannot always be achieved.

Recently, processes activated by mechanical action (mechanochemical synthesis, mechanical activation, mechanical fusion) have become the subject of intensive research in connection with their promising application in various industries, since they provide the creation of new non-traditional, environmentally friendly and less costly technologies compared to existing methods of coating metal surfaces, such as chemical and physical vapor deposition, self-propagating juice temperature synthesis, thermal spraying, sol-gel method, etc.

The use of mechanical alloying to obtain coatings on a metal surface is a new area of surface treatment. The idea of this method is to use the impact energy of a moving ball to coat metal surfaces. This method, due to the solid phase state of the process, has practically no restrictions on the combinations of the deposited and base metal, does not require special preparation of the surface of the samples, and has relatively low energy costs for coating [3].

Currently, there are more than dozens of models of mechanochemical interactions; nevertheless, up to now, an empirical approach has been used in the development of functional materials, since existing models cannot explain the entire set of experimental results. This is due to the fact that mechanochemical fusion (MF) is a complex process, since the dispersion, phase composition, defective structure, and mechanical properties of the reaction mixture continuously change during its mechanical processing. In the process of MF, the number of parameters involved is very large (time, size of grinding media, the ratio of the mass of balls to the mass of powder, temperature, surrounding atmosphere, amplitude and frequency of oscillations) [4]. The variety of types of equipment leads to a huge variety of possible machining modes. And therefore, the identification of the main regularities of the MF process, which serve to predict the state of the final product of machining, is still an unresolved problem.

The aim of the work is to study the structural-phase transformations in composite coatings (Ti-Al) + Ti during mechanical alloying.

1. Methodology and discussion of experimental results

The sample for the study was obtained by mechanical alloying on a vibrating unit SVU2. A mixture of Ti + Al powder under the influence of ball impacts was deposited (welded) on the surface of a substrate of technical pure titanium VT1-0 with dimensions of $7 \times 7 \times 2$ mm. The coating thickness was about 25–30 microns. Powder fraction size: Ti — 45 microns; Al — 5 microns.

X-ray phase studies of the samples were carried out on a DRON-6 diffractometer using CuKα-radiation. The image capture was carried out in the following modes: tube voltage $U = 40$ kV;

tube current $I = 20$ mA; exposure time 3 s; capture step 0.02°. Processing and analysis of experimental data was carried out using the PDWIN4.0 software package (NPP «Burevestnik» St. Petersburg), using the attached database.

The phase composition and structural parameters of the samples were studied on an XRD-6000 diffractometer using CuKα-radiation. An analysis of the phase composition was carried out using PDF4+ databases. The capture was carried out in the following modes: tube voltage $U = 40$ kV; tube current $I = 30$ mA; exposure time 1 s; capture step 0.02°.

The study of the microstructure and analysis of the chemical composition of the samples was carried out on a JCXA-733 «Superprobe» electron probe microanalyzer with an INCA Energy SEM 300 energy-dispersive microanalysis attachment, and on a JSM-6390 scanning electron microscope with an electronic probe attachment for local microanalysis.

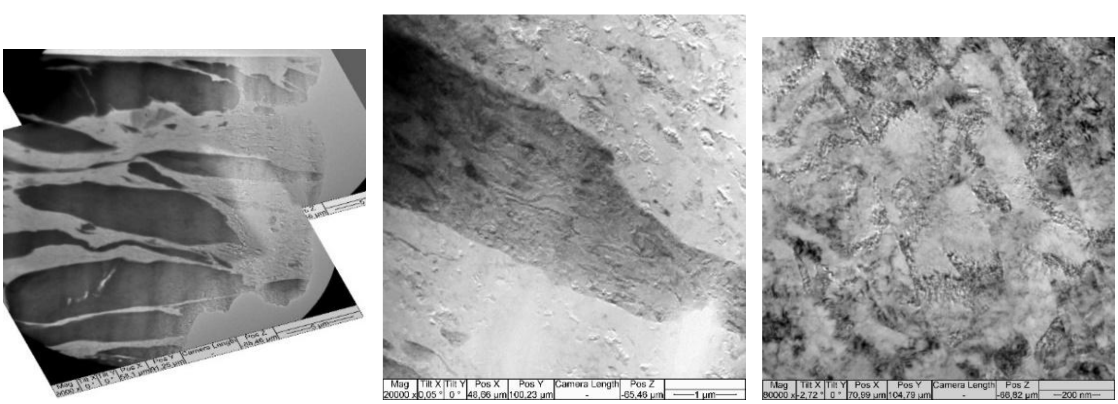
In a vibrating installation SVU2, particles of Ti-Al powder are cold-welded to the surface of Ti under the influence of ball impacts. The Ti-Al powder particles were subjected to mechanical grinding and repeatedly repeated deformation, and densification on the Ti surface. Intensive energy supply by spheres accelerated chemical reactions and solid-phase diffusion, both in the coating and at the interface, which led to strong adhesion of a metal matrix with particles of Ti-Al powder.

As a result, a coating was formed on the surface — a layer of composite material having a nano- and microstructure, which are characterized by very high adhesion.

Figs. 1 and 2 show the results of Ti-Al coatings on the titanium surface obtained inside the vibration chamber in the light and dark field modes.

Fig. 1 a–c shows that in the cross section of the coating there are dark patches in the light matrix. This is evidenced by the distribution of elements over the thickness of the coatings. Under the action of impacts, Al + Ti particles are driven into the Ti matrix, as a result of which a coating is formed.

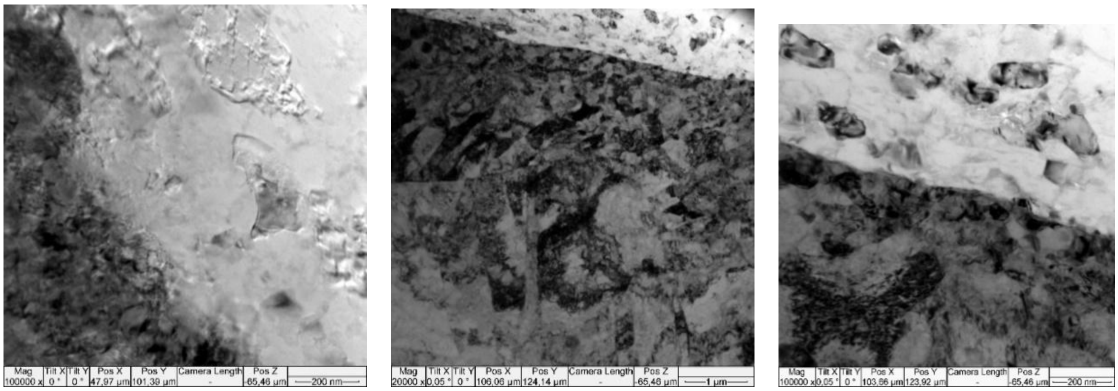
This is more clearly evidenced by the images of the contact area between the dark and light parts of the coating, the substrate with the coating adhering at the top with different increases, and the image of the transition coating-substrate layer shown in Fig. 2 a–c.



a)   b)   c)

Fig. 1. Cross section of the coating: a) — general view (there are dark areas in the light matrix), b) — and c) — an enlarged image of a dark area of the coating

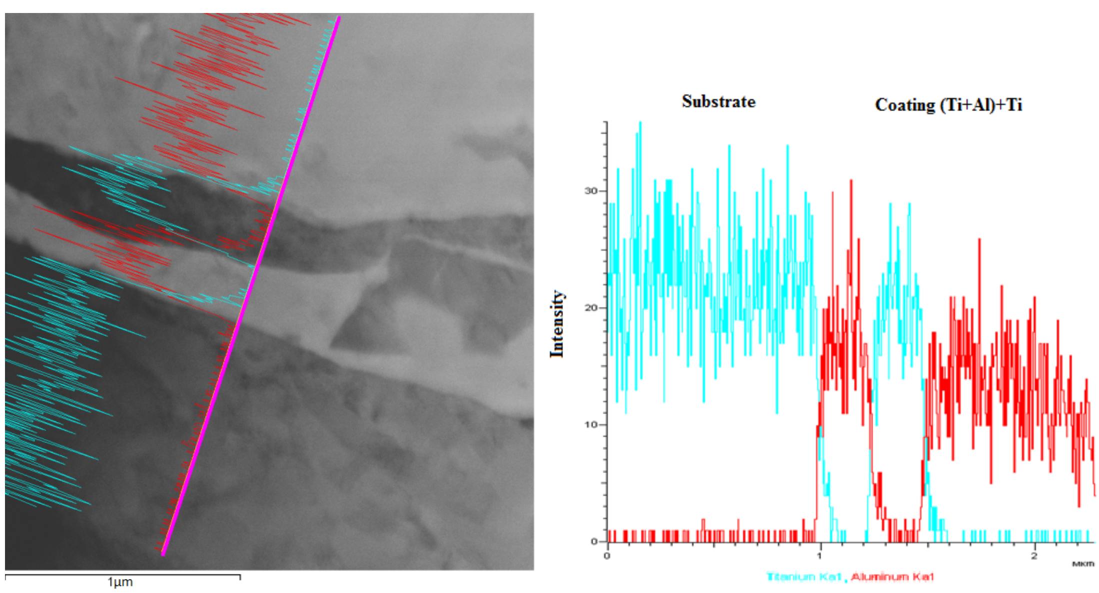
Fig. 3 shows the images of the cross section (a) and the concentration profile of the cross section of the substrate (b). It is seen that during the formation of coatings, a process of



a) b) c)

Fig. 2. (a) The contact area of the dark and light areas of the coating. (b) Images of the substrate with a coating adhering on top at different magnifications. (c) The transition layer is coating-substrate

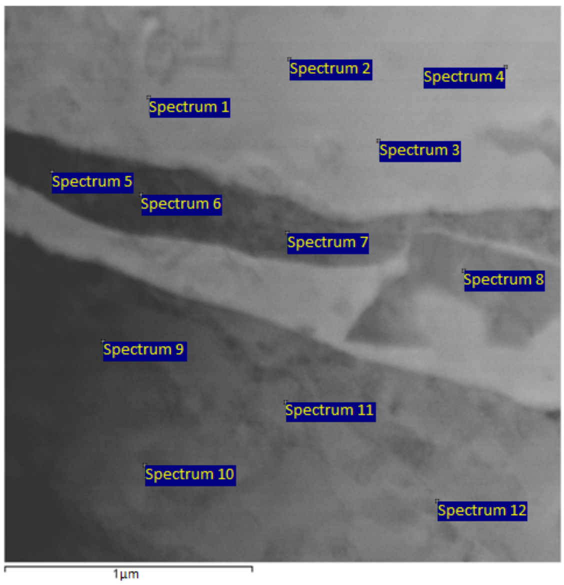
conglomeration of particles of Ti and Al powders occurs, particles of a soft element, in our case Al, envelop Ti particles, forming a plastic matrix on the substrate surface. Under the influence of ball impacts in the surface layer, a lamellar structure is formed from flattened particles of powder components. A detailed image indicates the viscous behavior of the material in MF. The cellular structure in some areas, observed at high magnifications, confirms the flow of the material during processing (Figs. 1 and 2).



a) b)

Fig. 3. Ti-Al coating on the surface of Ti: a) — cross section, b) — concentration profile

Spectra of energy dispersive analysis were taken from the cross section. The results of the decoding of the spectra are shown in Fig. 4 a, b.



a)

Spectrum	In stat.	Al	Si	Ti
Specrtum 1	Yes	98.56	0.81	0.63
Specrtum 2	Yes	97.98	1.41	0.60
Specrtum 3	Yes	98.34	0.90	0.75
Specrtum 4	Yes	95.65	3.19	1.16
Specrtum 5	Yes	5.99	1.65	92.36
Specrtum 6	Yes	5.04	2.00	92.96
Specrtum 7	Yes	6.60	2.19	91.22
Specrtum 8	Yes	0.83	8.62	90.55
Specrtum 9	Yes	1.14	1.53	97.34
Specrtum 10	Yes	0.86	2.49	96.66
Specrtum 11	Yes	1.35	1.85	96.80
Specrtum 12	Yes	1.49	3.59	94.92
Average		34.49	2.52	62.99
Standard deviation		46.69	2.09	45.99
Max		98.56	8.62	97.34
Min		0.83	0.81	0.60

b)

Fig. 4. Interpretation of spectra from the cross section shown in Fig. 1

Deciphering the microdiffraction pattern from the substrate surface (Fig. 5 a, b) showed that, as a result of mechanochemical fusion, a $TiAl_3$ phase with a bcc lattice (structure $I4/mmm$) was detected. This corresponds to the stable state of the $TiAl_3$ alloy. The presence of reflections of the atomically ordered structure of $L1_2$ was found, which corresponds to the metastable state of $TiAl_3$. This can be explained by the fact that under mechanical alloying the structure $I4/mmm$ transforms into the structure $L1_2$.

The processes of structural-phase transitions under extreme conditions of mechanochemical fusion proceed according to the principle of maximum entropy production. As a result, the entropy of the resulting structures can be negative. This is possible due to the switching of chemical bonds in the process of mechanochemical reactions that occur in time 120 min, and the switching time of chemical bonds is from $10^{-10} - 10^{-13}$ s.

It was shown in [5–12] that the formation of cubic Al_3Ti ($cP4$) was detected in thin films deposited from vapors [5], mechanically doped [6–9] and rapidly solidified [10] samples. Tetragonal Al3Ti (Al_3Ti ($tI16$) forms a metastable phase in the temperature range 495–800°C upon heating of mechanically doped cubic Al_3Ti ($cP4$) [11]. Above 800°C, Al_3Ti ($tI16$) transforms into the equilibrium structure of Al_3Ti ($tI8$). Another form of Al_3Ti, is Al_3Ti ($tI64$), which is regarded as the superstructure of Al_3Ti ($tI8$), which was observed in diffusion pairs [12]. A recent study of phase equilibria in Al-Ti [10] using bulk alloy samples has not confirmed the stability of this structure. Therefore, the authors of [1] consider Al_3Ti ($tI64$) as a metastable phase, possibly stabilized by the action of voltage.

Indeed, the atomic volumes of Al_3Ti phases for various modifications are presented in [1], where it was shown that the volume of the metastable phase with structure $L1_2$ is much smaller than the volume of the stable phase $I4/mmm$, i.e. the phase with the $L1_2$ structure has a higher

a)   b)

Fig. 5. (a) — Microdiffraction pattern of the surface of a titanium substrate treated with particles of Al + Ti powder. (b) — Scheme of deciphering the diffraction pattern in which superstructural reflexes are present

specific strength compared to the phase with the $I4/mmm$ structure. In this way, the possibility of hardening the surface layers (Ti-Al) + Ti by mechanochemical fusion was found in the work. Hardening is associated with the polymorphic transformation of the stable phase $I4/mmm$ into the metastable phase $L1_2$.

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Исследование тонкой структуры Ti-Al покрытий на поверхности Ti, полученных методом механического сплавления

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Аннотация. Работа посвящена исследованию структурно-фазовых превращений в композиционных покрытиях (Ti-Al)+Ti при механическом сплавлении. Получены данные о структурно-фазовых состояниях (Ti-Al)-Ti покрытий после механического сплавления, подтверждающие механизм формирования модифицированного слоя за счет деформационного уплотнения частиц порошка на поверхности титана под механическим воздействием. В результате механохимического сплавления обнаружена фаза $TiAl_3$ с ОЦК-решеткой (структура $I4/mmm$), что соответствует стабильному состоянию сплава $TiAl_3$. В условиях механического сплавления структура $I4/mmm$ переходит в структуру $L1_2$, что соответствует метастабильному состоянию $TiAl_3$.

Ключевые слова: структурно-фазовые превращения, композиционные покрытия, механическое сплавление, твердофазные процессы.

Journal of Siberian Federal University. Mathematics & Physics 2020, 13(4), 466–479

DOI: 10.17516/1997-1397-2020-13-4-466-479
УДК 517.9

L^p Regularity of the Solution of the Heat Equation with Discontinuous Coefficients

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Received 16.02.2020, received in revised form 23.04.2020, accepted 06.06.2020

Abstract. In this paper, we consider the transmission problem for the heat equation on a bounded plane sector in L^p-Sobolev spaces. By Applying the theory of the sums of operators of Da Prato-Grisvard and Dore-Venni, we prove that the solution can be splited into a regular part in L^p-Sobolev space and an explicit singular part.

Keywords: transmission heat equation, sums of linear operators, singular behavior, non-smooth domains.

Citation: S. Kouicem, W. Chikouche, L^p Regularity of the Solution of the Heat Equation with Discontinuous Coefficients, J. Sib. Fed. Univ. Math. Phys., 2020, 13(4), 466–479.
DOI: 10.17516/1997-1397-2020-13-4-466-479.

1. Introduction and preliminaries

Let $\mathcal{B} = G\times]0,T[$, where G is a bounded plane sector constitued of two plane sectors G_1, G_2 with respective opening ω_1 and ω_2, separated by an interface Σ.

$$\begin{aligned} G_1 &= \{(r\cos\theta, r\sin\theta); -\omega_1 < \theta < 0, 0 < r < 1\}, \\ G_2 &= \{(r\cos\theta, r\sin\theta); 0 < \theta < \omega_2, 0 < r < 1\}, \\ \Sigma &= \{(r,0); 0 < r < 1\}. \end{aligned}$$

In this paper we study the regularity of the solution of the following transmission problem for the heat equation

$$\partial_t u_i - \Delta u_i = g_i \quad \text{in } \mathcal{B}_i = G_i\times]0,T[;\ i=1,2, \tag{1}$$
$$u_1 = u_2 \quad \text{on } \Sigma\times[0,T], \tag{2}$$
$$\alpha_1\frac{\partial u_1}{\partial n_1} + \alpha_2\frac{\partial u_2}{\partial n_2} = 0 \quad \text{on } \Sigma\times[0,T], \tag{3}$$
$$u_i = 0 \quad \text{on } (\partial G_i\setminus\Sigma)\times[0,T];\ i=1,2, \tag{4}$$
$$u_i(.,0) = 0 \quad \text{in } G_i;\ i=1,2, \tag{5}$$

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where n_i denotes the unit normal vector to $\Sigma\times[0,T]$ directed outside $\mathcal{B}_i$, u_i means the restriction of u to $\mathcal{B}_i$, $g\in L^p(\mathcal{B})$ and α_1,α_2 are two positive real numbers such that $\alpha_1\neq\alpha_2$.

Problem (1)–(5) is a particular case of the abstract Cauchy problem

$$\partial_t u(t)+Au(t) = f(t) \quad 0\leqslant t\leqslant T, \tag{6}$$

$$u(0) = 0, \tag{7}$$

where $f\in L^p(]0,T[;X)$, also considered as a special case of the more general operator equation

$$(A+B)u=f, \tag{8}$$

in which B is the derivative operator defined on the interval $[0,T]$ with values in a Banach function space.

The study of the abstract equation (8), where A and B are two closed linear operators with dense domains acting in a complex Banach space X, is based on the theory of sums of operators in Banach spaces. In [2], G. Da Prato and P. Grisvard proved under appropriate assumptions on the resolvents of A and B that the sum operator $A+B$ is closable. As an application, problem (6)–(7) has a strong solution that is a solution in $L^p(]0,T[;X)$. In their famous paper [6], G. Dore and A. Venni showed under appropriate assumptions on the imaginary powers of A and B and if the space X is U.M.D. that $A+B$ is closed. As an application, problem (6)–(7) has the L^p maximal regularity property. We refer to [1, 4, 5, 7–12] for some applications of the theory of sums of operators.

By analogy with Grisvard [8, 9] who studied the heat equation in plane polygonal domains in L^p-Sobolev spaces and De Coster-Nicaise [5] who extended his results to the weighted L^p-Sobolev spaces setting, we show that the solution u of problem (1)–(5) is decomposable into a regular part having the optimal regularity $L^p(]0,T[;PW^{2,p}(G)\cap W_0^{1,p}(G))\cap W^{1,p}(]0,T[;L^p(G))$ and a finite sum of explicit singular functions. For the sake of simplicity, we restrict ourselves to the case of two sectors G_i, $i=1,2$ with a common interface Σ. The result of this paper can be easily extended to the case of more than two sectors using the results from [13].

The paper is organised as follows:

In Section 2, we present the main results of the theory of the sums of operators of Da Prato-Grisvard [2] and Dore-Venni [6]. Applying this theory requires some results concerning a transmission problem with complex parameter z. This will be recalled from [1] in Section 3 and extended for z in a larger part of the complex plane. In Section 4 we apply the strategy of Da Prato-Grisvard to show existence and uniqueness of a strong solution u of problem (1)–(5) which admits a decomposition in regular and singular parts. Optimal regularity of the regular part is obtained in Section 5 by applying a Dore-Venni result.

Let us finish this introduction with some notation used in the whole paper: if D is an open subset of $\mathbb{R}^N$ ($N=1$ or 2), we denote by $L^p(D)$, ($p>1$) the Lebesgue spaces, and by $W^{s,p}(D)$, $s\geqslant 0$, the standard Sobolev spaces built on. The space $W_0^{1,p}(D)$ is defined as usual by $W_0^{1,p}(D):=\{v\in W^{1,p}(D); v=0 \text{ on } \partial D\}$. When $p=2$, we use the common notation $H_0^1(D)$ instead of $W_0^{1,p}(D)$.

For any separable Banach space X provided with the norm $\|\cdot\|_X$, we denote by $L^p(]0,T[;X)$ the space of measurable functions v from $]0,T[$ in X such that

$$\|v\|_{L^p(]0,T[;X)}=\left(\int_0^T\|v(\cdot,t)\|_X^p dt\right)^{\frac{1}{p}}<+\infty,$$

and by $W^{1,p}(]0,T[;X)$ the Sobolev space of functions v in $L^p(]0,T[;X)$ such that $\partial_t v$ belongs to $L^p(]0,T[;X)$.

2. Sums of linear operators

For an operator P we denote by $\sigma(P)$ and $\rho(P)$ respectively its spectrum and its resolvent set.

2.1. The first strategy

We recall some results on sums of operators of Da prato-Grisvard taken from [9].

Let E be a complex Banach space and A, B two closed linear operators with dense domains $D(A)$ and $D(B)$ respectively. Their sum is defined by

$$Lx = Ax + Bx,$$

for every $x \in D(L) = D(A) \cap D(B)$.

Assumptions

$\mathbf{H}_1$ There exist positive numbers M_A, M_B, R, θ_A, θ_B such that $\theta_A + \theta_B > \pi$ and $\rho(-A)$ contains the truncated sector

$$S_A = \{\lambda; |\lambda| \geqslant R, |\arg\lambda| \leqslant \theta_A\},$$

while $\rho(-B)$ contains the truncated sector

$$S_B = \{\lambda; |\lambda| \geqslant R, |\arg\lambda| \leqslant \theta_B\},$$

and

$$\|(A+\lambda)^{-1}\| \leqslant \frac{M_A}{|\lambda|}, \qquad \forall\lambda \in S_A,$$

$$\|(B+\lambda)^{-1}\| \leqslant \frac{M_B}{|\lambda|}, \qquad \forall\lambda \in S_B.$$

$\mathbf{H}_2$ $\sigma(-A) \cap \sigma(B) = \emptyset$.

$\mathbf{H}_3$ The resolvents of A and B commute, i.e.

$$(A+\lambda)^{-1}(B+\mu)^{-1} = (B+\mu)^{-1}(A+\lambda)^{-1},$$

for every $\lambda \in \rho(-A)$ and every $\mu \in \rho(-B)$.

Theorem 2.1 (Da Prato-Grisvard [2])**.** *Under the assumptions $\boldsymbol{H}_1$, $\boldsymbol{H}_2$, $\boldsymbol{H}_3$, the closure $\overline{L}$ of L is invertible.*

An explicit construction of the inverse of $\overline{L}$ is given by the Dunford integral

$$(\overline{L})^{-1} = \frac{1}{2\pi i}\int_\gamma (A+\lambda I)^{-1}(\lambda I - B)^{-1} d\lambda,$$

where the path γ separates $\sigma(-A)$ and $\sigma(B)$ and joins $\infty e^{-i\theta_\gamma}$ to $\infty e^{i\theta_\gamma}$ where θ_γ is chosen so that $\pi - \theta_B < \theta_\gamma < \theta_A$.

The unique solution $v \in D(\overline{L})$ of the equation

$$\overline{L}v = (\overline{A+B})v = f,$$

is called the strong solution of $Lv = f$. This means the existence of a sequence $(v_n) \subset D(L)$ such that $v_n \longrightarrow v$ and $Lv_n \longrightarrow f$ in E.

2.2. The second strategy

Here we just recall from [6] an application of the theory of sums of operators of Dore-Venni. Let X be an U.M.D. complex Banach space and $A : D(A) \to X$ a closed linear operator with dense domain in X satisfying the following assumptions

$\mathbf{H}_4$ $\rho(A) \supset]-\infty, 0]$ and there exists $M_A > 0$ such that

$$\|(A+t)^{-1}\| \leqslant \frac{M_A}{t+1} \qquad \forall t \geqslant 0.$$

$\mathbf{H}_5$ $A^{is} \in \mathcal{L}(X)$ for all $s \in \mathbb{R}$ and there exist $K > 0$, τ_A such that $0 \leqslant \tau_A < \dfrac{\pi}{2}$ and

$$\|A^{is}\| \leqslant K e^{|s|\tau_A} \qquad \forall s \in \mathbb{R},$$

where A^{is} are the complex powers of A.

If we also call A the operator induced on $E = L^p(]0,T[;X)$ by the equality $(Au)(t) = A(u(t))$, it is obvious that A has the same properties in E as in X. The application of Theorems 2.1 and 3.1 of [6] gives

Theorem 2.2 (Dore-Venni [6]). *Under the assumptions $\boldsymbol{H}_4$ and $\boldsymbol{H}_5$, the Cauchy problem (6)–(7) has the L^p maximal regularity property, that is for each $f \in L^p(]0,T[;X)$, $1 < p < \infty$, it has a unique solution $u \in W^{1,p}(]0,T[;X) \cap L^p(]0,T[;D(A))$.*

3. Results on the transmission problem

We consider the following Helmholtz transmission problem with complex parameter z

$$\begin{aligned} -\Delta u_i + z u_i &= f_i \ \text{ in } G_i;\ i = 1,2, && (9)\\ u_i &= 0 \ \text{ on } \partial G_i \setminus \Sigma;\ i = 1,2, && (10)\\ u_1 &= u_2 \ \text{ on } \Sigma, && (11)\\ \alpha_1 \frac{\partial u_1}{\partial n_1} + \alpha_2 \frac{\partial u_2}{\partial n_2} &= 0 \ \text{ on } \Sigma, && (12) \end{aligned}$$

where $f \in L^p(G)$, $1 < p < +\infty$, n_i here denotes the normal vector to Σ directed outside G_i and u_i is still the restriction of u to G_i.

Problem (9)–(12) admits the equivalent variational formulation:
Find $u \in H_0^1(G)$ such that

$$a_z(u,v) = \int_G \alpha\, f\overline{v}\, dx \quad \forall v \in H_0^1(G), \tag{13}$$

where

$$a_z(u,v) = \int_G \alpha \left\{ \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \frac{\partial \overline{v}}{\partial x_i} + z\, u\overline{v} \right\} dx,$$

$$\alpha(x) = \alpha_i > 0 \ \text{ for } \ x \in G_i; \quad i = 1,2, \ \text{with } \alpha_1 \neq \alpha_2.$$

In what follows, we use the positive constant C to denote a generic constant and may take different values in different places.

Lemma 3.1. *Let $\theta_A \in]0,\pi[$. Then problem* (13) *admits a unique solution $u \in H_0^1(G)$ for any $f \in L^p(G)$ and any $z \in \mathbb{C}$ with $|\arg z| \leqslant \theta_A$.*

Proof. Since $H_0^1(G) \hookrightarrow L^{p'}(G)$ (p' being the conjugate exponent of p), we can easily see that the antilinear form

$$\begin{aligned} K : H_0^1(G) &\longrightarrow \mathbb{C} \\ v &\longmapsto \int_G \alpha\, f\, \overline{v}\, dx, \end{aligned}$$

is continuous on $H_0^1(G)$, with the estimate

$$|K(v)| \leqslant C\|f\|_{L^p(G)}\, \|v\|_{H_0^1(G)}, \ \forall v \in H_0^1(G). \tag{14}$$

It is clear that the sesquilinear form a_z is continuous on $H_0^1(G) \times H_0^1(G)$. It is also coercive observing that, for all $z \in \mathbb{C}$ with $|\arg z| \leqslant \theta_A$, there exists $\theta \in [0, 2\pi]$ such that $\cos\theta > 0$ and $\Re(ze^{i\theta}) \geqslant 0$, which implies thanks to Poincaré's inequality

$$\Re(e^{i\theta}\, a_z(v,v)) \geqslant \cos\theta \int_G \alpha|\nabla v|^2 dx \geqslant C\|v\|_{H_0^1(G)},$$

for all $v \in H_0^1(G)$. We conclude using the Lax-Milgram lemma. □

For $R > 0$ and $\theta_A \in \left]\frac{\pi}{2}, \pi\right[$ fixed, we define the sets S^+ and S_A as follows

$$S^+ = \{z \in \mathbb{C} /\, \Re(z) \geqslant 0\},$$

$$S_A = \{z \in \mathbb{C} /\, |z| \geqslant R \text{ and } |\arg z| \leqslant \theta_A\}.$$

Lemma 3.2. *Let $R > 0$ and $\theta_A \in \left]\frac{\pi}{2}, \pi\right[$ be fixed. Let $z \in S^+ \cup S_A$ and $u \in H_0^1(G)$ be the solution of* (13), *then u satisfies the estimate*

$$\|u\|_{L^p(G)} \leqslant C\|f\|_{L^p(G)}. \tag{15}$$

Proof. We proceed as in [3, Lemma 2.4]. By Applying (13) with $v = u$, we obtain

$$\int_G \alpha\{|\nabla u|^2 + z|u|^2\}\, dx = \int_G \alpha\, f\, \overline{u}\, dx. \tag{16}$$

Taking the real and the imaginary parts of (16) respectively, we obtain using (14)

$$\int_G \alpha|\nabla u|^2\, dx + \Re(z) \int_G \alpha|u|^2\, dx \leqslant C\, \|f\|_{L^p(G)}\|u\|_{H_0^1(G)}, \tag{17}$$

and

$$|\Im(z)| \int_G \alpha|u|^2\, dx \leqslant C\, \|f\|_{L^p(G)}\|u\|_{H_0^1(G)}. \tag{18}$$

Case 1 : $\Re(z) \geqslant 0$. Due to (17) and Poincaré's inequality, we deduce that

$$\|u\|_{H_0^1(G)} \leqslant C\|f\|_{L^p(G)},$$

which gives (15) since $H_0^1(G) \hookrightarrow L^p(G)$ for all $1 < p < \infty$.

Case 2 : $\Re(z) < 0$. In this case $z \in S_A$, then $\Re(z) = \rho\cos\theta$, $\Im(z) = \rho\sin\theta$ with $\rho \geqslant R$ and $|\Im(z)| > \rho\sin\theta_A$.

Consequently, from (18) we obtain

$$\|u\|^2_{L^2(G)} \leqslant C\frac{1}{\rho}\|f\|_{L^p(G)}\|u\|_{H_0^1(G)}. \tag{19}$$

As $\Re(z) < 0$, the estimate (17) gives

$$\|\nabla u\|^2_{L^2(G)} \leqslant C(\|f\|_{L^p(G)}\|u\|_{H^1_0(G)} - \Re(z)\,\|u\|^2_{L^2(G)}).$$

Due to (19) and Poincaré's inequality this gives

$$\|u\|_{H^1_0(G)} \leqslant C\left(1 - \Re(z)\frac{1}{\rho}\right)\|f\|_{L^p(G)}.$$

We conclude using the inequality $-\Re(z) \leqslant \rho$. □

For all $1 < p < \infty$, we consider the operator A_p defined by

$$D_{A_p} = \left\{u \in H^1_0(G); \Delta u_i \in L^p(G_i), u_1 = u_2 \text{ and } \alpha_1\frac{\partial u_1}{\partial n_1} + \alpha_2\frac{\partial u_2}{\partial n_2} = 0 \text{ on } \Sigma\right\},$$

$$\begin{array}{rcl} A_p : D_{A_p} \subset L^p(G) & \longrightarrow & L^p(G) \\ u = (u_1, u_2) & \longmapsto & (-\Delta u_1, -\Delta u_2). \end{array}$$

Note that $L^p(G) \equiv L^p(G_1) \times L^p(G_2)$ and $u_i = u_{|G_i}$; $i = 1, 2$.

Theorem 3.1 ([1]). *Let $f \in L^p(G), z \in \mathbb{C}$ with $\Re(z) \geqslant 0$ and $u \in H^1_0(G)$ be the solution of (13), then u satisfies the estimates*

$$\Re(z)\,\|u\|_{L^p(G)} \leqslant \|f\|_{L^p(G)},$$

$$|\Im(z)|\,\|u\|_{L^p(G)} \leqslant \frac{p}{2}\|f\|_{L^p(G)},$$

and

$$|z|\,\|u\|_{L^p(G)} \leqslant C\|f\|_{L^p(G)}. \tag{20}$$

Corollary 3.1. *$-A_p$ is the infinitesimal generator of a C_0 semigroup of contraction $T(t)$ for $t \geqslant 0$.*

As in [3], we can prove the estimate (20) for z in a larger part of the complex plane.

Corollary 3.2. *There exists $\theta_A \in \left]\frac{\pi}{2}, \pi\right[$ such that, for all $f \in L^p(G)$, all $z \in \mathbb{C}$ with $|\arg z| \leqslant \theta_A$ and $u \in H^1_0(G)$ solution of* (13)*, we have*

$$|z|\|u\|_{L^p(G)} \leqslant C\|f\|_{L^p(G)}. \tag{21}$$

Proof. By Theorem 3.1, there exists a positif constant c such that, for all $\sigma > 0$ and $\tau \in \mathbb{R}^*$

$$|\sigma + i\tau|\|(A_p + (\sigma + i\tau)I)^{-1}\| \leqslant c,$$

hence, thanks to Corollary 3.1, we can apply Theorem II-5.2 in [14] to deduce the existence of $\delta \in \left]0, \frac{\pi}{2}\right[$ and $M > 0$ such that

$$\rho(-A_p) \supset \Gamma := \left\{z \in \mathbb{C}/|\arg z| < \frac{\pi}{2} + \delta\right\} \cup \{0\}, \tag{22}$$

and, for all $z \in \Gamma$,

$$|z|\|(A_p + zI)^{-1}\| \leqslant M. \tag{23}$$

This proves (21) with $\theta_A = \delta' + \frac{\pi}{2}$, where $0 < \delta' < \delta$. □

In order to present the singular behavior of the variational solution of the transmission problem (9)–(12), we need the following notations.
For $s > 0$,

$$PW^{s,p}(G) := \{u \in H^1(G);\ u_i \in W^{s,p}(G_i),\ i = 1, 2\},$$

is the space of piecewise $W^{s,p}$ functions on G.
Let $S^{(m)}$ be the function defined by

$$S^{(m)} = \eta r^{\lambda_m} t_m(\theta), \tag{24}$$

where η is a radial cut-off function such that $\eta \equiv 1$ in a small ball centered at the origin and $\eta \equiv 0$ outside a larger ball of radius strictly less than 1, λ_m is a nonnegative real number and λ_m^2, t_m are respectively the eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$\begin{cases} -t_m''(\theta) = \lambda_m^2 t_m(\theta) \quad \text{for} \quad \theta \in [-\omega_1, \omega_2],\ \ \theta \neq 0, \\ t_m(0^-) = t_m(0^+), \\ \alpha_2 t_m'(0^-) = \alpha_1 t_m'(0^+), \\ t_m(-\omega_1) = t_m(\omega_2) = 0. \end{cases}$$

Theorem 3.2. *If $\lambda_m \neq \dfrac{2}{p'}$ for all $m \in \mathbb{N}^*$, then there exists $\theta_A \in \left]\dfrac{\pi}{2}, \pi\right[$ such that, for all $f \in L^p(G)$, all $z \in S^+ \cup S_A$, the unique solution $u \in H_0^1(G)$ of problem (9)–(12) admits the decomposition*

$$u = u_R + \sum_{0<\lambda_m<\frac{2}{p'}} c_m \psi_m(z), \tag{25}$$

where

$$\psi_m(z) = \begin{cases} e^{-r\sqrt{z}} S^{(m)} & \text{if } \dfrac{2}{p'} - 1 < \lambda_m < \dfrac{2}{p'}, \\ e^{-r\sqrt{z}}(1 + r\sqrt{z}) S^{(m)} & \text{if } \lambda_m \leqslant \dfrac{2}{p'} - 1, \end{cases} \tag{26}$$

$u_R \in PW^{2,p}(G)$ satisfies

$$\|u_R\|_{PW^{2,p}(G)} + |z|^{\frac{1}{2}} \|u_R\|_{W^{1,p}(G)} + |z| \|u_R\|_{L^p(G)} \leqslant C \|f\|_{L^p(G)};\ \ z \neq 0, \tag{27}$$

and c_m satisfies

$$|c_m| \leqslant C |z|^{\frac{\lambda_m}{2} - \frac{1}{p'}} \|f\|_{L^p(G)};\ \ z \neq 0, \tag{28}$$

$$\sum_{0<\lambda_m<\frac{2}{p'}} |c_m| \big(1 + |z|^{\frac{1}{p'} - \frac{\lambda_m}{2}}\big) \leqslant C \|f\|_{L^p(G)}. \tag{29}$$

Proof. The proof of this Theorem stays as in [1] until the inequality (28), thanks to Theorem 3.1 and Corollary 3.2. It remains to prove the estimate (29).

From the decomposition (25), we have

$$\sum_{0<\lambda_m<\frac{2}{p'}} c_m \psi_m(z) = u - u_R.$$

By using the estimates (15) and (27), we deduce that

$$\Big\| \sum_{0<\lambda_m<\frac{2}{p'}} c_m \psi_m(z) \Big\|_{L^p(G)} \leqslant C \|f\|_{L^p(G)}.$$

As the space V_S=span$\left\{\psi_m(z); 0 < \lambda_m < \dfrac{2}{p'}\right\}$ being of finite dimension, we have

$$\sum_{0<\lambda_m<\frac{2}{p'}} |c_m| \leqslant C\| \sum_{0<\lambda_m<\frac{2}{p'}} c_m\psi_m(z)\|_{L^p(G)} \leqslant C\|f\|_{L^p(G)}.$$

From the previous estimate and inequality (28), we deduce that we have (29). □

With the notation introduced above, we can write

$$u = (A_p + z)^{-1} f,$$

consequently the decomposition (25) implies a similar decomposition of the resolvent of A_p. Namely we may write

$$(A_p + z)^{-1} = R(z) + \sum_{0<\lambda_m<\frac{2}{p'}} T_m(z) \otimes \psi_m(z), \tag{30}$$

where $R(z)$ is the continuous linear operator from $L^p(G)$ into $PW^{2,p}(G)$ defined by

$$R(z)f := u_R,$$

and $T_m(z)$ is the continuous linear functional on $L^p(G)$ defined by

$$< T_m(z), f >:= c_m.$$

Recall that

$$(T_m(z) \otimes \psi_m(z))(f) =< T_m(z), f > \psi_m(z).$$

The estimates (27) and (29) imply that

$$\|R(z)\|_{L^p(G)\to PW^{2,p}(G)} + |z|^{\frac{1}{2}} \|R(z)\|_{L^p(G)\to W^{1,p}(G)} + |z| \, \|R(z)\|_{L^p(G)\to L^p(G)} \leqslant C,$$

and

$$\|T_m(z)\|_{L^{p'}(G)} \leqslant C \frac{1}{1 + |z|^{\frac{1}{p'} - \frac{\lambda_m}{2}}}, \tag{31}$$

for all $z \in S^+ \cup S_A$.

4. Application of the first strategy

In order to apply Theorem 2.1 to problem (1)–(5), we write it as the sum of linear operators on the Banach space $E = L^p(I; L^p(G))$ (where $I =]0, T[$) by setting

$$Au = -\{\Delta u_i\}_{i=1,2},$$

for $u \in D(A) = L^p(I; D(A_p))$,

$$Bu = \partial_t u = \{\partial_t u_i\}_{i=1,2},$$

for $u \in D(B) = \{v \in W^{1,p}(I; L^p(G)); v(.,0) = 0\}$.

Proposition 4.1. *The closure $\overline{L}$ of $L = A + B$ is invertible, i.e. for all $g \in L^p(I; L^p(G))$, there exists a unique strong solution $u \in L^p(I; L^p(G))$ of $(A + B)u = g$. In addition u is explicitly given by*

$$u = \frac{1}{2\pi i} \int_\gamma (A + z)^{-1}(z - B)^{-1} g \, dz. \tag{32}$$

Proof. The properties of A are those of its realization A_p. Thanks to (22) and (23), hypothesis $\mathbf{H}_1$ is fulfilled for the operator A with some θ_A in $\left]\frac{\pi}{2},\pi\right[$, while from [2, p. 330–331], it is fulfilled for the operator B for all $\theta_B < \frac{\pi}{2}$. So there exists $M > 0$ such that

$$\|(B+\lambda I)^{-1}\| \leqslant \frac{M}{|\lambda|}, \tag{33}$$

for all $\lambda \in S_B = \{\lambda \in \mathbb{C}/|\arg(\lambda)| \leqslant \theta_{\mathrm{B}}\}$.

Hence, we conclude that $\mathbf{H}_1$ is satisfied with θ_A in $\left]\frac{\pi}{2},\pi\right[$ and $\theta_B = \frac{\pi}{2} - \delta_B$ with $0 < \delta_B < \delta < \theta_A - \frac{\pi}{2}$.

On the other hand, it's clear that $]-\infty, 0] \subset \rho(B)$. Thus the assumption $\mathbf{H}_2$ is fulfilled since A has a discrete spectrum that contains only strictly positive eigenvalues (see [1, p. 20–21]).

The commutativity assumption $\mathbf{H}_3$ follows from the fact that the variables are separate in these two operators.

Hence we can apply Theorem 2.1 to conclude. □

For each t, we can write

$$[(A+zI)^{-1}h](t) = (A_p+z)^{-1}(h(t)).$$

Using the decomposition (30), the representation formula (32) can be split as follows

$$u = u_R + \sum_{0<\lambda_m<\frac{2}{p'}} u_m, \tag{34}$$

where

$$u_R = \frac{1}{2\pi i}\int_\gamma R(z)[(z-B)^{-1}g]dz, \tag{35}$$

$$u_m = \frac{1}{2\pi i}\int_\gamma \langle T_m(z),(z-B)^{-1}g\rangle \psi_m(z)dz. \tag{36}$$

Summing up, we have prove the following theorem.

Theorem 4.1. *Suppose that $\lambda_m \neq \frac{2}{p'}$, then for all $g \in L^p(I;L^p(G))$, the problem (1)–(5) has a unique strong solution u which is in the form*

$$u = u_R + \sum_{0<\lambda_m<\frac{2}{p'}} u_m,$$

where u_R (resp. u_m) is given by (35)(resp. (36)).

Theorem 4.2. *Let $p \geqslant 2$, suppose that $\lambda_m \neq \frac{2}{p'}$. Denote $\sigma_m = \frac{1}{p'} - \frac{\lambda_m}{2}$, then for all $g \in L^p(I;L^p(G))$, there exist $q_m \in W^{\sigma_m,p}(I)$ and E_m such that u_m defined by (36) can be written as*

$$u_m = (E_m *_t q_m)S^{(m)}, \tag{37}$$

the symbol $*_t$ means the convolution product in t. Moreover we have

$$q_m = \frac{1}{2\pi i}\int_\gamma \langle T_m(z),(z-B)^{-1}g\rangle dz, \tag{38}$$

$$E_m(r,t) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} e^{i\xi t} e^{-r\sqrt{i\xi}}\, d\xi & \textit{for}\ \ \lambda_m > 1-\dfrac{2}{p}, \\ \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} e^{i\xi t}(1+r\sqrt{i\xi})\, e^{-r\sqrt{i\xi}}\, d\xi & \textit{for}\ \ \lambda_m \leqslant 1-\dfrac{2}{p}, \end{cases} \tag{39}$$

and the mappings

$$\begin{aligned} U_1 : L^p(I;L^p(G)) &\longrightarrow W^{\sigma_m,p}(I) \\ g &\longmapsto q_m, \end{aligned}$$

$$\begin{aligned} U_2 : W^{\sigma_m,p}(I) &\longrightarrow L^p(I;L^p(G_i)) \\ q_m &\longmapsto (\frac{\partial}{\partial t} - \Delta)u_{m,i}, \end{aligned}$$

are continuous.

Proof. We proceed as in [9, Proposition 6.2] and [5, Proposition 2.2]. First we consider the extension of g to $G\times\mathbb{R}$, defined by

$$\tilde{g}(x,t) = \begin{cases} g(x,t) & \text{if } \ t\in[0,T], \\ 0 & \text{if } \ t\notin[0,T], \end{cases}$$

and denote by $\tilde{u}_z = (zI - B_\infty)^{-1}\tilde{g}$, the solution of

$$\begin{cases} z\tilde{u} - \partial_t\tilde{u} = \tilde{g} & \text{in } G\times\mathbb{R}, \\ \tilde{u}(.,0) = 0 & \text{in } \ G, \end{cases}$$

where B_∞ is the operator, defined by

$$B_\infty u = \partial_t u \ \text{ for } \ u\in D(B_\infty) = \{v\in W^{1,p}(]0,\infty[;L^p(G)); v(\cdot,0)=0\}.$$

Observe that, by uniqueness of the solution of the Cauchy problem, we have $\tilde{u}_z|_{G\times[0,T]} = = (zI-B)^{-1}g$.

Consider the functions

$$\tilde{u}_m(x,t) = \frac{1}{2\pi i}\int_\gamma \langle T_m(z), (zI-B_\infty)^{-1}\tilde{g}\rangle \psi_m(z)\, dz, \tag{40}$$

$$\tilde{q}_m(t) = \frac{1}{2\pi i}\int_\gamma \langle T_m(z), (zI-B_\infty)^{-1}\tilde{g}(.,t)\rangle\, dz. \tag{41}$$

We assume that $\tilde{g}\in\mathcal{D}(G\times\mathbb{R})$, a dense subspace of $L^p(G\times\mathbb{R})$. Then we can apply partial Fourier transform in t, to (40) and (41). By Fubini's theorem, we obtain

$$\begin{aligned} \mathfrak{F}_t\tilde{u}_m(x,\xi) &= \frac{1}{2\pi i}\int_\gamma \Big\langle T_m(z), \mathfrak{F}_t\Big((zI-B_\infty)^{-1}\tilde{g}\Big)(.,\xi)\Big\rangle \psi_m(z)\, dz = \\ &= \frac{1}{2\pi i}\int_\gamma \Big\langle T_m(z), \frac{\mathfrak{F}_t\,\tilde{g}(.,\xi)}{z-i\xi}\Big\rangle \psi_m(z)\, dz, \end{aligned}$$

and

$$\mathfrak{F}_t\,\tilde{q}_m(\xi) = \frac{1}{2\pi i}\int_\gamma \Big\langle T_m(z), \frac{\mathfrak{F}_t\,\tilde{g}(.,\xi)}{z-i\xi}\Big\rangle\, dz.$$

The decaly at infinity of $T_m(z)$ and $\psi_m(z)$ due to (31) and (26) allows us to apply Cauchy's foumula. We get

$$\mathfrak{F}_t \tilde{u}_m(x,\xi) = -\langle T_m(i\xi), \mathfrak{F}_t\, \tilde{g}(.,\xi)\rangle\, \psi_m(i\xi), \tag{42}$$

and

$$\mathfrak{F}_t \tilde{q}_m(\xi) = -\langle T_m(i\xi), \mathfrak{F}_t\, \tilde{g}(.,\xi)\rangle.$$

According to (26), the identity (42) can be written as follows

$$\mathfrak{F}_t \tilde{u}_m(x,\xi) = \begin{cases} -\langle T_m(i\xi), \mathfrak{F}_t\, \tilde{g}(.,\xi)\rangle\, e^{-r\sqrt{i\xi}}\, S^{(m)} & \text{if} \quad \lambda_m > 1 - \dfrac{2}{p}, \\ -\langle T_m(i\xi), \mathfrak{F}_t\, \tilde{g}(.,\xi)\rangle\, e^{-r\sqrt{i\xi}}\, (1 + r\sqrt{i\xi})\, S^{(m)} & \text{if } \lambda_m \leqslant 1 - \dfrac{2}{p}. \end{cases} \tag{43}$$

Now we consider the function E_m defined by

$$\mathfrak{F}_t E_m(x,\xi) = \begin{cases} e^{-r\sqrt{i\xi}} & \text{if } \lambda_m > 1 - \dfrac{2}{p}, \\ e^{-r\sqrt{i\xi}}\, (1 + r\sqrt{i\xi}) & \text{if } \lambda_m \leqslant 1 - \dfrac{2}{p}. \end{cases}$$

It is clear that E_m is given by (39).

Then (43) can be seen as the Fourier transform of a convolution in t. We have

$$\tilde{u}_m = (E_m *_t \tilde{q}_m) S^{(m)}.$$

This identity is easily extended from $\tilde{g} \in \mathcal{D}(G \times \mathbb{R})$ to any $\tilde{g} \in L^p(G \times \mathbb{R})$. The identity (37) follows by observing that $\tilde{u}_m|_{G\times[0,T]} = u_m$ and $\tilde{q}_m|_{[0,T]} = \dfrac{1}{2\pi i} \int\limits_\gamma \langle T_m(z), (z - B)^{-1} g\rangle dz = q_m$. Let us underline that we differ from [5] in the definition of A, the operator B being the same but in Sobolev spaces instead of the weighted L^p-Sobolev spaces. This comes from the fact that α depends only on the plane variables thus the interface has no effect on the variable t. Thus, the continuity of the operators U_1 and U_2 can be shown as done in [5, Theorem 2.3 and Theorem 3.2]. □

5. Application of the second strategy

Now we are able to prove the regularity of u_R.

From [1, Section 5.2], the following estimate for $R(z)$ (defined in Section 3) is derived thanks to an interpolation argument.

$$\|R(z)\|_{L^p(G)\to PW^{s,p}(G)} = O\left(\frac{1}{|z|^{1-\frac{s}{2}}}\right) \ \forall s < 2.$$

With the help of (33), this yields

$$u_R \in L^p(I; PW^{s,p}(G)) \ \text{ for every s} < 2, \tag{44}$$

with the estimate

$$\|u_R\|_{L^p(I;PW^{s,p}(G))} \leqslant C(s)\, \|g\|_{L^p(I;L^p(G))}. \tag{45}$$

Going back to (34), we have

$$\partial_t u_{R,i} - \Delta u_{R,i} = g_i - \sum_{0<\lambda_m<\frac{2}{p'}} (\partial_t u_{m,i} - \Delta u_{m,i}) := g_{R,i}; \quad i = 1, 2, \tag{46}$$

the function $g_R \in L^p(I; L^p(G))$ by Theorem 4.2.

We shall apply Theorem 2.2 to study the equation (46) with $X = L^p(G)$, the space E and the operator A are defined exactly as in Section 4.

By Theorem 3.1 and Lemma 3.2, the assumption $\mathbf{H}_4$ is fulfilled. It remains to check $\mathbf{H}_5$. Thanks to [1] there exists $\tau_A < \dfrac{\pi}{2}$ such that

$$\|A_p^{is}\| = 0(e^{|s|\tau_A}).$$

Accordingly Theorem 2.2 may be applied, then we have the existence and the uniqueness of $w_R \in W^{1,p}\big(]0,T[; L^p(G)\big) \cap L^p\big(]0,T[; D(A_p)\big)$ solution of

$$\begin{cases} \partial_t w_R + A w_R = g_R, \\ w_R(.,0) = 0. \end{cases}$$

w_R do not coincide necessarily with u_R, so we will prove that $w_R = u_R$.
First we show that u_R is a strong solution of

$$\begin{aligned}
\partial_t u_{R,i} - \Delta u_{R,i} &= g_i - \sum_{0<\lambda_m<\frac{2}{p'}} (\partial_t u_{m,i} - \Delta u_{m,i}) \text{ in } \mathcal{B}_i;\ i = 1,2, && (47)\\
u_{R,1} &= u_{R,2} \quad \text{on } \Sigma \times [0,T], && (48)\\
a_1 \frac{\partial u_{R,1}}{\partial n_1} + a_2 \frac{\partial u_{R,2}}{\partial n_2} &= 0 \quad \text{on } \Sigma \times [0,T], && (49)\\
u_{R,i} &= 0 \quad \text{on } (\partial G_i \setminus \Sigma) \times [0,T];\ i = 1,2, && (50)\\
u_{R,i}(.,0) &= 0 \quad \text{in } G_i;\ i = 1,2. && (51)
\end{aligned}$$

Due to Proposition 4.1, u is a strong solution of

$$\begin{aligned}
\partial_t u_i - \Delta u_i &= g_i \quad \text{in } \mathcal{B}_i;\ i = 1,2,\\
u_1 &= u_2 \quad \text{on } \Sigma \times [0,T],\\
a_1 \frac{\partial u_1}{\partial n_1} + a_2 \frac{\partial u_2}{\partial n_2} &= 0 \quad \text{on } \Sigma \times [0,T],\\
u_i &= 0 \quad \text{on } (\partial G_i \setminus \Sigma) \times [0,T];\ i = 1,2,\\
u_i(.,0) &= 0 \quad \text{in } G_i;\ i = 1,2,
\end{aligned}$$

i.e. there exists $(u_n) \subset D(A) \cap D(B)$ and $(g_n) \subset E$ such that $(A+B)u_n = g_n$, $u_n \longrightarrow u$ and $g_n \longrightarrow g$ in E.

Moreover, as in Section 4, for every n, we have

$$u_n = u_{n,R} + \sum_{0<\lambda_m<\frac{2}{p'}} u_{n,m}.$$

Thanks to Theorem 4.2, we have

$$\partial_t u_{n,m,i} - \Delta u_{n,m,i} \longrightarrow \partial_t u_{m,i} - \Delta u_{m,i},$$

then

$$\partial_t u_{n,R,i} - \Delta u_{n,R,i} \longrightarrow g_i - \sum_{0<\lambda_m<\frac{2}{p'}} (\partial_t u_{m,i} - \Delta u_{m,i}).$$

From the estimate (45) it follows that $u_{n,R} \longrightarrow u_R$ in E.

It is obvious that w_R is a strong solution of (47)–(51). Consequently, by applying the first strategy to (47)–(51), we have by uniqueness of the strong solution that $w_R = u_R$.

This implies that $u_R \in L^p\big(I; D(A_p)\big) \cap W^{1,p}\big(I; L^p(G)\big)$. With the help of (44), this yields

$$u_R \in L^p(I; D(A_p) \cap PW^{s,p}(G)).$$

Then, from [1, Lemma 5.4], we deduce that

$$u_R \in L^p(I; PW^{2,p}(G) \cap W_0^{1,p}(G)) \cap W^{1,p}(I; L^p(G)).$$

Summing up, we have proved the following Theorem.

Theorem 5.1. *Let $p \geqslant 2$, suppose that $\lambda_m \neq \dfrac{2}{p'}$ $\forall m \in \mathbb{N}^*$. Then for every $g \in L^p(I; L^p(G))$, there exists a unique solution $u \in L^p(I; L^p(G))$ to the transmission problem* (1)*–*(5)*. Moreover u admits the decomposition*

$$u = u_R + \sum_{0<\lambda_m<\frac{2}{p'}} (E_m *_t q_m) S^{(m)},$$

with $u_R \in L^p\big(I; PW^{2,p}(G) \cap W_0^{1,p}(G)\big) \cap W^{1,p}\big(I; L^p(G)\big)$, where $E_m\big($resp. $q_m \in W^{-\frac{\lambda_m}{2}+1-\frac{1}{p}}(I)\big)$ is defined by (39) (*resp.* (38)) *and the singular functions $S^{(m)}$ are given by* (24).

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L^p-регулярность решения уравнения теплопроводности с разрывными коэффициентами

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Аннотация. В этой статье мы рассмотрим задачу прохождения для уравнения теплопроводности на ограниченном плоском секторе в пространствах L^p-Соболева. Применяя теорию сумм операторов Да Прато-Грисварда и Доре-Венни, мы доказываем, что решение можно разбить на регулярную часть в пространстве L^p-Соболева и явную особую часть.

Ключевые слова: уравнение теплопередачи, суммы линейных операторов, сингулярное поведение, негладкие области.

DOI: 10.17516/1997-1397-2020-13-4-480-491
УДК 519.21

Rate of the Almost Sure Convergence of a Generalized Regression Estimate Based on Truncated and Functional Data

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Received 06.02.2020, received in revised form 25.04.2020, accepted 26.05.2020

Abstract. In this paper, a nonparametric estimation of a generalized regression function is proposed. The real response random variable (r.v.) is subject to left-truncation by another r.v. while the covariate takes its values in an infinite dimensional space. Under standard assumptions, the pointwise and the uniform almost sure convergences, of the proposed estimator, are established.

Keywords: functional data, truncated data, almost sure convergence, local linear estimator.

Citation: H. Boudada, S. Leulmi, S. Kharfouchi, Rate of the Almost Sure Convergence of a Generalized Regression Estimate Based on Truncated and Functional Data, J. Sib. Fed. Univ. Math. Phys., 2020, 13(4), 480–491. DOI: 10.17516/1997-1397-2020-13-4-480-491.

1. Introduction and preliminaries

The investigation of the link between a scalar variable of interest Y and a functional covariate X is among the most famous nonparametric statistical works in the last two decades. We mention [1] who proposed a new version of the estimator of the regression operator $m(x) = E(Y/X = x)$, in the case of independent and identically distributed (i.i.d.) observations, and studied its almost complete convergence. They used the so called local linear method.

In the case of complete data, many works followed this last method. For example, in [9] the uniform almost-complete convergence of the local linear conditional quantile estimator was established, while in [8] the case of a generalized regression function with functional dependent data was considered. The asymptotic normality of the local linear estimator of the conditional density for functional time series data was studied in [12] and both the pointwise and the uniform almost complete convergences, of a generalized regression estimate, were investigated in [7]. All these studies were carried in the case of complete data, however in practice, one or more truncation variables may interfere with the variable of interest and prevent its observation in a complete manner. In this setting of truncation model, one can find many works such as that

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of [5] where a kernel conditional quantile estimator was proposed and its strong uniform almost sure convergence established. Similarly, [2] studied the almost complete convergence rate and the asymptotic normality of a family of nonparametric estimators for the ψ-regression model. But, as far as we know, the local linear method has not been investigated for truncated data.

Hence, our goal is to propose a generalized regression estimator, when the response variable is subject to left-truncation, and to establish both its pointwise and its uniform almost sure convergences.

To this end, this article is ordered as follows. In Section 2, we recall some basic knowledge of the left -truncation model and we construct our local linear estimator. Section 3 is devoted to prove its pointwise almost sure convergence. Finally, its uniform convergence is established in Section 4.

To make things more easier for readers, we give the definition of the almost complete convergence:
Let $(W_n)_{n\in\mathbb{N}^*}$ be a sequence of real random variables r.r.v.. We say that $(W_n)_{n\in\mathbb{N}^*}$ converges almost completely to some r.r.v. W, and we note $W_n \longrightarrow^{a.co.} W$, if and only if $\forall\epsilon > 0$, $\sum\limits_{n=1}^{\infty} P(|W_n - W| > \epsilon) < \infty$. Moreover, let $(v_n)_{n\in\mathbb{N}^*}$ be a sequence of positive real numbers going to zero; we say that the rate of the almost complete convergence of $(W_n)_{n\in\mathbb{N}^*}$ to W is of order (v_n) and we note $W_n - W = O_{a.co.}(v_n)$, if and only if $\exists\epsilon_0 > 0$, $\sum\limits_{n=1}^{\infty} P(|W_n - W| > \epsilon_0 v_n) < \infty$. It is clear, from Borel Cantelli lemma, that this convergence is stronger than the almost-sure one (a.s.).

2. Estimation

Let (X_i, Y_i) for $i = 1, \dots, N$, be N identical and independent couples distributed as (X, Y) which takes its values in $\mathcal{F} \times \mathbb{R}$, where $\mathcal{F}$ is a semi metric space endowed with a semi metric d. The unknown distribution function (d.f.) of Y is denoted by F.

Let T be another r.v. which has unknown d.f. G and $(T_i)_{i=1,\dots,N}$ be a sample of i.i.d. random variables that are distributed as T. T is supposed independent of (X, Y). N is unknown but deterministic. In the left truncation model, the lifetime Y_i and the truncation r.v. T_i are both observable only when $Y_i \geqslant T_i$. We denote $(Y_i, T_i), i = 1, 2, \dots, n\,(n \leqslant N)$ the actual observed sample of size n which, as a consequence of truncation, is a binomial r.v. with parameters N and $\mu = \mathbb{P}(Y \geqslant T)$. It is clear that if $\mu = 0$, no data can be observed, and therefore, we suppose throughout this article that $\mu > 0$.

By the strong law of large numbers, we have

$$\widehat{\mu}_n := \frac{n}{N} \to \mu,\ \mathbb{P} - p.s.$$

We point out that if the original data $(Y_i, T_i), i = 1, 2, \dots, N$ are i.i.d., the observed data (Y_i, T_i), $i = 1, 2, \dots, n$ are still i.i.d. (see [6]).

Under random left truncation model, following [10], the d.f.s of Y and T are expressed respectively as,

$$F^*(y) = \mu^{-1}\int_{-\infty}^{y} G(u)dF(u) \quad \text{and} \quad G^*(t) = \mu^{-1}\int_{-\infty}^{\infty} G(t\wedge u)dF(u),$$

where $t \wedge u = \min(t, u)$ and are estimated by their empirical estimators,

$$F_n^*(y) = n^{-1}\sum_{i=1}^{n} 1_{\{Y_i\leqslant y\}} \quad \text{and} \quad G_n^*(t) = n^{-1}\sum_{i=1}^{n} 1_{\{T_i\leqslant t\}}.$$

Define
$$C(y) := G^*(y) - F^*(y) = \mu^{-1}G(y)(1 - F(y)),$$
the empirical estimator of $C(y)$ is defined by
$$C_n(y) = n^{-1}\sum_{i=1}^{n} 1_{\{T_i \leqslant y \leqslant Y_i\}}.$$
The nonparametric maximum likelihood estimators of F and G are given respectively by
$$F_n(y) = 1 - \prod_{i/Y_i \leqslant y}\left[\frac{nC_n(Y_i) - 1}{nC_n(Y_i)}\right] \quad \text{and} \quad G_n(y) = \prod_{i/T_i > y}\left[\frac{nC_n(T_i) - 1}{nC_n(T_i)}\right].$$
According to [4], μ can be estimated by
$$\mu_n = C_n^{-1}(y)G_n(y)(1 - F_n(y)),$$
which is independent of y.

Our results will be stated with respect to the conditional probability $\mathbf{P}(.)$ related to the n-sample instead of the probability measure $\mathbb{P}(.)$ related to the N-sample. We donate by $\mathbf{E}$ and $\mathbb{E}$ the respective expectation operators of $\mathbf{P}(.)$ and $\mathbb{P}(.)$.

For any d.f. L, let $a_L = \inf\{y : L(y) > 0\}$ and $b_L = \sup\{y : L(y) < 1\}$ be its two endpoints. The asymptotic properties of F_n, G_n and μ_n are obtained only if $a_G \leqslant a_F$ and $b_G \leqslant b_F$. We take two real numbers c and d such that $[c, d] \subset [a_F, b_F]$, we are going to use this inclusion in the uniform consistency of the distribution law $G(.)$ of the truncated r.v. T which is stated over a compact set (see Remark 6 in [11]).

Hence, based on the idea of the Nadaraya-Watson kernel smoother, the estimator of the general regression function $m_\varphi(x)$ defined, for all $x \in \mathcal{F}$, by $m_\varphi(x) = E(\varphi(Y)/X = x)$, where φ is a known real-valued borel function, is defined by
$$\widehat{m}_\varphi(x) = \frac{\sum_{i=1}^{n}\varphi(Y_i)K\left(h^{-1}d(X_i, x)\right)G_n^{-1}(Y_i)}{\sum_{i=1}^{n}K\left(h^{-1}d(X_i, x)\right)G_n^{-1}(Y_i)},$$
where K is a standard univariate kernel function and the bandwidth $h := h_n$ is a sequence of strictly positive real numbers which plays a smoothing parameter role.

Note that all the sums containing $G_n^{-1}(Y_i)$ are taken for i such that $G_n(Y_i) \neq 0$.

Following [1] and [7], the local linear estimator of m_φ in the case of truncated data is obtained as the solution for a of the following minimization problem
$$\min_{(a,b)\in\mathbb{R}^2}\sum_{i=1}^{n}\left(\varphi(Y_i) - a - b\beta(X_i, x)\right)^2 K(h^{-1}d(X_i, x))G_n^{-1}(Y_i),$$
where $\beta(., .)$ is a known operator from $\mathcal{F} \times \mathcal{F}$ into $\mathbb{R}$ such that, $\forall x \in \mathcal{F},\ \beta(x, x) = 0$.

By a simple calculus, one can derive the following explicit estimator
$$\widehat{m}_\varphi(x) = \frac{\sum_{i,j=1}^{n}W_{ij}(x)\varphi(Y_j)}{\sum_{i,j=1}^{n}W_{ij}(x)} \quad \left(\frac{0}{0} := 0\right),$$
where
$$W_{ij}(x) = \Delta_{ij}(x)G_n^{-1}(Y_i)G_n^{-1}(Y_j),$$
with
$$\Delta_{ij}(x) := \beta(X_i, x)\left(\beta(X_i, x) - \beta(X_j, x)\right)K(h^{-1}d(X_i, x))K(h^{-1}d(X_j, x)).$$

3. Pointwise almost sure corvengence

For any positive real h, let $B(x,h) := \{y \in \mathcal{F} \diagup d(x,y) \leqslant h\}$ be a closed ball in $\mathcal{F}$ of center x and radius h, $\Phi_x(h,h') := P(h \leqslant d(x,X) \leqslant h')$ and $\Phi_x(h) := \Phi_x(0,h)$.

To establish the asymptotic behaviour of our estimator $\widehat{m}_\varphi(x)$ for a fixed point x in $\mathcal{F}$, we use the following assumptions:

(H1) For any $h > 0; \Phi_x(h) > 0$.

(H2) There exists $b > 0$ such that for all $x_1, x_2 \in B(x,h)$; $|m_\varphi(x_1) - m_\varphi(x_2)| \leqslant C_x d^b(x_1,x_2)$ where C_x is a positive constant depending on x.

(H3) The function $\beta(.,.)$ is such that

$$\exists\, 0 < M_1 < M_2, \forall x' \in \mathcal{F}, M_1 d(x,x') \leqslant |\beta(x,x')| \leqslant M_2 d(x,x').$$

(H4) The kernel K is a positive and differentiable function on its support $[0,1]$.

(H5) The bandwidth h satisfies $\lim\limits_{n\to\infty} h = 0$ and $\lim\limits_{n\to\infty} \left(\sqrt{\dfrac{\ln n}{n\Phi_x(h)}}\right) = 0$.

(H6) There exists an integer n_0, such that

$$\forall n > n_0,\ \frac{1}{\Phi_x(h)} \int_0^1 \Phi_x(zh,h) \frac{d}{dz}\left(z^2 K(z)\right) > 0.$$

(H7) $h \int\limits_{B(x,h)} \beta(u,x) d\mathbf{P}_X(u) = o\left(\int\limits_{B(x,h)} \beta^2(u,x) d\mathbf{P}_X(u)\right)$, where $d\mathbf{P}_X$ is the distribution of X.

(H8) $\forall m \geqslant 2$; $\sigma_m : x \longmapsto \mathbf{E}(|\varphi^m(Y)|/X)$ is a continuous operator on $\mathcal{F}$.

Remark 3.1. *Hypotheses (H1)–(H5) are standard in the nonparametric functional regression setting. The rest of the hypotheses have already been used in the literature, we refer for (H6) and (H7) to [1] and for (H8) to [7].*

Theorem 3.1. *Assume that assumptions (H1)–(H8) are satisfied, then*

$$\widehat{m}_{\varphi(x)} - m_{\varphi(x)} = O(h^b) + O_{a.s.}\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right).$$

We remark that to prove our theorem we need to define the following pseudo-estimators

$$r_l(x) = \frac{\mu_n^2}{n(n-1)\mathbf{E}\left(\Delta_{12}(x)\right)} \sum_{i\neq j} G_n^{-1}(Y_i) G_n^{-1}(Y_j) \Delta_{ij}(x) \varphi^l(Y_j)$$

and

$$\tilde{m}_l(x) = \frac{\mu^2}{n(n-1)\mathbf{E}\left(\Delta_{12}(x)\right)} \sum_{i\neq j} G^{-1}(Y_i) G^{-1}(Y_j) \Delta_{ij}(x) \varphi^l(Y_j), \quad for \quad l = 0,1.$$

Consider the following decomposition

$$\widehat{m}_\varphi(x) - m_\varphi(x) = \frac{r_1(x)}{r_0(x)} - m_\varphi(x) =$$

$$= \frac{1}{r_0(x)}\{r_1(x) - \tilde{m}_1(x)\} + \frac{1}{r_0(x)}\{\tilde{m}_1(x) - \mathbf{E}(\tilde{m}_1(x))\} + \frac{1}{r_0(x)}\{\mathbf{E}(\tilde{m}_1(x)) - m_\varphi(x)\} +$$

$$+ \frac{m_\varphi(x)}{r_0(x)}\{(\tilde{m}_0(x) - r_0(x)) + (\mathbf{E}(\tilde{m}_0(x)) - \tilde{m}_0(x)) + (-\mathbf{E}(\tilde{m}_0(x)) + 1)\}. \quad (1)$$

Moreover, we note for any $x \in \mathcal{F}$ and for all $i = 1, \ldots, n$

$$K_i(x) := K\left(h^{-1}d(X_i, x)\right) \quad and \quad \beta_i(x) := \beta(X_i, x).$$

To make things easier, we introduce the following lemmas.

Lemma 1. *Under the assumptions (H1)–(H8), we have*

$$|r_l(x) - \tilde{m}_l(x)| = O_{a.s.}\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right).$$

Proof. For $l = 0, 1$

$$\begin{aligned}
|r_l(x) - \tilde{m}_l(x)| &= \Big|\frac{\mu_n^2}{n(n-1)\mathbf{E}\left(\Delta_{12}(x)\right)} \sum_{i\neq j} G_n^{-1}(Y_i)G_n^{-1}(Y_j)\Delta_{ij}(x)\varphi^l(Y_j) \; - \\
&- \frac{\mu^2}{n(n-1)\mathbf{E}\left(\Delta_{12}(x)\right)} \sum_{i\neq j} G^{-1}(Y_i)G^{-1}(Y_j)\Delta_{ij}(x)\varphi^l(Y_j)\Big| \leqslant \\
&\leqslant \left[\frac{|\mu_n^2 - \mu^2|}{G_n^2(a_F)} + \mu^2\left(\frac{\sup_{y\in[c,d]}|G_n^2(y) - G^2(y)|}{G^2(a_F)G_n^2(a_F)}\right)\right] \times \\
&\times \sum_{i\neq j}\Big|\frac{\Delta_{ij}(x)\varphi^l(Y_j)}{n(n-1)\mathbf{E}\left(\Delta_{12}(x)\right)}\Big|.
\end{aligned}$$

From Theorem 3.2 of [4] we have $|\mu_n - \mu| = O_{a.s}(n^{-1/2})$, while Remark 6 of [11] gives $|G_n(a_F) - G(a_F)| = O_{a.s}(n^{-1/2})$ which are negligible with respect to $O\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right)$. The rest of the proof is completed in [7]. Thus, we have $|r_l(x) - \tilde{m}_l(x)| = O_{a.s.}\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right)$. □

Lemma 2. *Under the assupmtions (H1), (H2) and (H4), we obtain*

$$|\boldsymbol{E}(\tilde{m}_1(x)) - m_\varphi(x)| = O(h^b).$$

Proof. We have

$$\begin{aligned}
\mathbf{E}(\tilde{m}_1(x)) &= \mathbf{E}\left(\frac{\mu^2}{n(n-1)\mathbf{E}\left(\Delta_{12}(x)\right)} \sum_{i\neq j} G^{-1}(Y_i)G^{-1}(Y_j)\Delta_{ij}(x)\varphi(Y_j)\right) = \\
&= \frac{\mu^2}{\mathbf{E}\left(\Delta_{12}(x)\right)}\mathbf{E}\left(\frac{1}{G(Y_1)G(Y_2)}\Delta_{12}(x)\varphi(Y_2)\right) = \\
&= \frac{\mu^2}{\mathbf{E}\left(\Delta_{12}(x)\right)}\mathbf{E}\left[\mathbb{E}\left(\Delta_{12}(x)\varphi(Y_2)\frac{1_{\{Y_1\geqslant T_1\}}1_{\{Y_2\geqslant T_2\}}}{\mu^2 G(Y_1)G(Y_2)}/\sigma(X_1, Y_1, X_2, Y_2)\right)\right] = \\
&= \frac{1}{\mathbf{E}\left(\Delta_{12}(x)\right)}\mathbf{E}\left(\Delta_{12}(x)m_\varphi(X_2)\right).
\end{aligned}$$

So we can write, under assumption (H4)

$$\begin{aligned}
|m_\varphi(x) - \mathbf{E}(\tilde{m}_1(x))| &= \frac{1}{|\mathbf{E}\left(\Delta_{12}(x)\right)|}|\mathbf{E}\left(\Delta_{12}(x)\left(m_\varphi(x) - m_\varphi(X_2)\right)\right)| \leqslant \\
&\leqslant \sup_{x'\in B(x,h)} |m_\varphi(x) - m_\varphi(x')|.
\end{aligned}$$

Using (H2), we obtain $|\mathbf{E}(\tilde{m}_1(x)) - m_\varphi(x)| = O(h^b)$. □

Lemma 3. *i) Under the assumptions (H1)–(H8), we get*

$$\tilde{m}_1(x) - \boldsymbol{E}(\tilde{m}_1(x)) = O_{a.co}\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right).$$

ii) Under the assumptions (H1), (H3)–(H7), we obtain

$$\tilde{m}_0(x) - 1 = O_{a.co}\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right) \quad \textit{and}$$

$$\exists \vartheta > 0, \quad \textit{such that} \quad \sum_{n=1}^{\infty} \boldsymbol{P}\big(\tilde{m}_0(x) < \vartheta\big) < \infty.$$

Proof. Remark that

$$\tilde{m}_1(x) = Q(x)\left[M_{2,1}(x)M_{4,0}(x) - M_{3,1}(x)M_{3,0}(x)\right], \tag{2}$$

where for $p = 2, 3, 4$ and $l = 0, 1$

$$Q(x) = \frac{n^2h^2\Phi_x^2(h)}{n(n-1)\mathbf{E}\left(\Delta_{12}(x)\right)} \tag{3}$$

and

$$M_{p,l}(x) = \frac{1}{n\Phi_x(h)}\sum_{i=1}^{n}\frac{\mu K_i(x)\beta_i^{p-2}(x)\varphi^l(Y_i)}{h^{p-2}G(Y_i)}. \tag{4}$$

So, we have

$$\begin{aligned} \tilde{m}_1(x) - \mathbf{E}\left(\tilde{m}_1(x)\right) = Q(x)\left\{M_{2,1}(x)M_{4,0}(x) - \mathbf{E}\left(M_{2,1}(x)M_{4,0}(x)\right)\right\} - \\ - Q(x)\left\{M_{3,1}(x)M_{3,0}(x) - \mathbf{E}\left(M_{3,1}(x)M_{3,0}(x)\right)\right\}. \end{aligned} \tag{5}$$

Notice that $Q(x) = O(1)$, see the proof of Lemma (4.4) of [1].

We need to prove that for $p = 2, 3, 4$ and $l = 0, 1$

$$\mathbf{E}(M_{p,l}(x)) = O(1);\ M_{p,l}(x) - \mathbf{E}(M_{p,l}(x)) = O_{a.co}\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right),$$

$$\mathbf{E}(M_{2,1}(x))\mathbf{E}(M_{4,0}(x)) - \mathbf{E}(M_{2,1}(x)M_{4,0}(x)) = O\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right),$$

$$\mathbf{E}(M_{3,1}(x))\mathbf{E}(M_{3,0}(x)) - \mathbf{E}(M_{3,1}(x)M_{3,0}(x)) = O\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right).$$

- Using assumptions (H1)–(H4), we can easily have for $p = 2, 3, 4$ and $l = 0, 1$

$$\begin{aligned} \mathbf{E}(M_{p,l}(x)) &= \mathbf{E}\left(\frac{1}{\Phi_x(h)}\sum_{i=1}^{n}\frac{\mu K_i(x)\beta_i^{p-2}(x)\varphi^l(Y_i)}{h^{p-2}G(Y_i)}\right) = \\ &= \mu h^{2-p}\Phi_x^{-1}(h)\mathbf{E}\left[\mathbb{E}\left(K_1(x)\beta_1^{p-2}(x)\varphi^l(Y_1)\frac{1_{\{Y_1\geqslant T_1\}}}{\mu G(Y_1)}/\sigma(X_1, Y_1)\right)\right] = \\ &= h^{2-p}\Phi_x^{-1}(h)\mathbf{E}\left(K_1(x)\beta_1^{p-2}(x)m_\varphi^l(X_1)\right). \end{aligned}$$

Lemma A.1 (i) in [1] and the condition (H2) allow us to get $\mathbf{E}(M_{p,l}(x)) = O(1)$.

- Treatment of the term $M_{p,l}(x) - \mathbf{E}(M_{p,l}(x))$. We put

$$M_{p,l}(x) - \mathbf{E}(M_{p,l}(x)) = \frac{1}{n}\sum_{i=1}^{n} Z_i^{(p,l)}(x),$$

where

$$Z_i^{(p,l)}(x) = \frac{1}{h^{p-2}\Phi_x(h)}\left\{\frac{\mu K_i(x)\beta_i^{p-2}(x)\varphi^l(Y_i)}{G(Y_i)} - \mathbf{E}\left(\frac{\mu K_i(x)\beta_i^{p-2}(x)\varphi^l(Y_i)}{G(Y_i)}\right)\right\}.$$

The main point is to evaluate asymptotically the mth-order moment of the r.r.v. $Z_i^{(p,l)}(x)$. By using Lemma A.1 (i) in [1] , we have

$$\begin{aligned}\mathbf{E}\left|\left\{Z_i^{(p,l)}(x)\right\}^m\right| &= h^{(-p+2)m}\Phi_x^{-m}(h)\mathbf{E}\left|\sum_{k=0}^{m} C_m^k(-1)^{m-k}\left(\frac{\mu K_i(x)\beta_i^{p-2}(x)\varphi^l(Y_i)}{G(Y_i)}\right)^k \times\right.\\ &\quad\left.\times\left(\mathbf{E}\left[\frac{\mu K_i(x)\beta_i^{p-2}(x)\varphi^l(Y_i)}{G(Y_i)}\right]\right)^{m-k}\right| =\\ &= O\left(\Phi_x^{(-m+1)}(h)\right).\end{aligned}$$

Finally, it suffices to apply Corollary A.8 (ii) in [3] with $a_n^2 = \Phi_x^{(-1)}(h)$ to get, for $p \in \{2,3,4\}$ and $l \in \{0,1\}$

$$M_{p,l}(x) - \mathbf{E}(M_{p,l}(x)) = O_{a.co}\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right).$$

- Moving to study the term $\mathbf{E}(M_{2,1}(x))\mathbf{E}(M_{4,0}(x)) - \mathbf{E}(M_{2,1}(x)M_{4,0}(x))$, we have

$$\begin{aligned}&\mathbf{E}(M_{2,1}(x))\mathbf{E}(M_{4,0}(x)) - \mathbf{E}(M_{2,1}(x)M_{4,0}(x)) =\\ &\qquad= n^{-1}h^{-2}\Phi_x^{-2}(h)\mathbf{E}\left(K_1(x)\beta_1^2(x)\right)\mathbf{E}\left(K_1(x)\varphi(Y_1)\right) + O((n\Phi_x(h))^{-1}),\end{aligned}$$

by using similar arguments as previously, we get

$$\mathbf{E}(M_{2,1}(x))\mathbf{E}(M_{4,0}(x)) - \mathbf{E}(M_{2,1}(x)M_{4,0}(x)) = O((n\Phi_x(h))^{-1}),$$

which is, under (H5), negligible with respect to $O\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right)$.

- By similar arguments, one can prove that

$$\mathbf{E}(M_{3,1}(x))\mathbf{E}(M_{3,0}(x)) - \mathbf{E}(M_{3,1}(x)M_{3,0}(x)) = O\left(\sqrt{\frac{\ln n}{n\Phi_x(h)}}\right).$$

For the second part of the lemma, it's easy to find that $\mathbf{E}\left(\tilde{m}_0(x)\right) = 1$ and this leads us to get the last result.

Theorem 3.1 is proved. □

4. Uniform almost sure convergence

In this section, we will investigate the uniform almost sure convergence of $\widehat{m}_\varphi$ on some subset $\mathcal{S}_\mathcal{F}$ of $\mathcal{F}$, such that $\mathcal{S}_\mathcal{F} \subset \bigcup_{k=1}^{d_n} B(x_k, r_n)$, where $x_k \in \mathcal{S}_\mathcal{F}$ and r_n (respectively d_n) is a sequence of positive real (respectively integer) numbers. For this, we need the following assumptions.

(U1) There exist a differentiable function Φ and strictly positive constants C, C_1 and C_2 such that

$$\forall x \in \mathcal{S}_\mathcal{F},\, \forall h > 0;\quad 0 < C_1\Phi(h) \leqslant \Phi_x(h) \leqslant C_2\Phi(h) < \infty$$

and

$$\exists\, \eta_0 > 0,\, \forall \eta < \eta_0,\, \Phi'(\eta) < C,$$

where Φ' denotes the first derivative of Φ with $\Phi(0) = 0$.

(U2) The generalized regression function m_φ satisfies

$$\exists\, C > 0,\, \exists\, b > 0,\, \forall x \in \mathcal{S}_\mathcal{F},\, x' \in B(x,h),\ |m_\varphi(x) - m_\varphi(x')| \leqslant Cd^b(x,x').$$

(U3) The function $\beta(.,.)$ satisfies (H3) uniformly on x and the following Lipschitz's condition

$$\exists\, C > 0,\, \forall x_1 \in \mathcal{S}_\mathcal{F},\, x_2 \in \mathcal{S}_\mathcal{F},\, x \in \mathcal{F},\ |\beta(x,x_1) - \beta(x,x_2)| \leqslant Cd(x_1,x_2).$$

(U4) The kernel K fulfils (H4) and is Lipschitzian on $[0,1]$.

(U5) $\lim_{n\longrightarrow\infty} h = 0$, and for $r_n = O\left(\dfrac{\ln n}{n}\right)$, we have for n large enough

$$\frac{(\ln n)^2}{n\Phi(h)} < \ln d_n < \frac{n\Phi(h)}{\ln n}$$

and

$$\sum_{n=1}^{\infty} d_n^{(1-\beta)} < \infty \quad \text{for some} \quad \beta > 1.$$

(U6) The bandwidth h satisfies $\exists\, n_0 \in \mathbb{N}$, $\exists\, C > 0$, such that

$$\forall n > n_0, \forall x \in \mathcal{S}_\mathcal{F}, \frac{1}{\Phi_x(h)} \int_0^1 \Phi_x(zh,h) \frac{d}{dz}(z^2K(z)) > C > 0$$

and

$$h \int_{B(x,h)} \beta(u,x) d\mathbf{P}_X(u) = o\left(\int_{B(x,h)} \beta^2(u,x) d\mathbf{P}_X(u)\right)$$

uniformly on x.

(U7) $\exists\, C > 0$ such that $\forall m \geqslant 2 : \mathbf{E}(|\varphi^m(Y)|/X = x) < \upsilon_m(x) < C < \infty$ with $\upsilon_m(.)$ continuous on $\mathcal{S}_\mathcal{F}$.

Remark 4.1. *These hypothesis are the uniform version of the assumed conditions in the point-wise case and have already been used in the literature (see [7]).*

Theorem 4.1. *Under assumptions (U1)–(U7), we have*

$$\sup_{x \in \mathcal{S}_\mathcal{F}} |\widehat{m}_\varphi(x) - m_\varphi(x)| = O(h^b) + O_{a.s}\left(\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right).$$

The proof of Theorem 4.1 is based on the same decomposition (1) and on the following lemmas

Lemma 4. *Under the assumptions (U1)–(U7), we get*

$$\sup_{x\in\mathcal{S}_{\mathcal{F}}} |r_l(x) - \tilde{m}_l(x)| = O_{a.s.}\left(\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right).$$

Proof. By following the same steps as the proof of Lemma 1 and using Lemma 2.2 in [7] we get our result. □

Lemma 5. *Under the assumptions (U1), (U2) and(U4), we obtain that*

$$\sup_{x\in\mathcal{S}_{\mathcal{F}}} |\boldsymbol{E}(\tilde{m}_1(x)) - m_\varphi(x)| = O(h^b).$$

Proof. Poof of Lemma 5 is similar to that of Lemma 2. □

Lemma 6. *i) Under the assumptions (U1)–(U7), we have*

$$\sup_{x\in\mathcal{S}_{\mathcal{F}}} |\tilde{m}_1(x) - \boldsymbol{E}(\tilde{m}_1(x))| = O_{a.co}\left(\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right).$$

ii) If assumptions (U1), (U3)-(U6) are satisfied, we get

$$\sup_{x\in\mathcal{S}_{\mathcal{F}}} |\tilde{m}_0(x) - 1| = O_{a.co}\left(\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right) \qquad \textit{and}$$

$$\exists\vartheta > 0, \quad \textit{such that} \quad \sum_{n=1}^{\infty} \boldsymbol{P}\left(\inf_{x\in\mathcal{S}_{\mathcal{F}}} \tilde{m}_0(x) < \vartheta\right) < \infty.$$

Proof. By considering the same decompositions and notations (2)–(5), following the same steps as in the proof of Lemma 3 and using Lemma 6 (i) in [7] instead of Lemma A.1 (i) in [1], we get under assumptions (U1)–(U4) and (U6)

$$\sup_{x\in\mathcal{S}_{\mathcal{F}}} Q(x) = O(1) \quad \text{and} \quad \sup_{x\in\mathcal{S}_{\mathcal{F}}} \mathbf{E}(M_{p,l}(x)) = O(1)$$

uniformly on x, for $p = 2, 3, 4$ and $l = 0, 1$,

$$\sup_{x\in\mathcal{S}_{\mathcal{F}}} |\mathbf{E}(M_{2,1}(x))\mathbf{E}(M_{4,0}(x)) - \mathbf{E}(M_{2,1}(x)M_{4,0}(x))| = O\left(\frac{1}{n\Phi(h)}\right)$$

and

$$\sup_{x\in\mathcal{S}_{\mathcal{F}}} |\mathbf{E}(M_{3,1}(x))\mathbf{E}(M_{3,0}(x)) - \mathbf{E}(M_{3,1}(x)M_{3,0}(x))| = O\left(\frac{1}{n\Phi(h)}\right),$$

which is, using hypothesis (U5), equals to $O\left(\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right)$.

Now we prove that

$$\sup_{x\in\mathcal{S}_{\mathcal{F}}} |M_{p,l}(x) - \mathbf{E}(M_{p,l}(x))| = O_{a.co}\left(\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right).$$

To this end, we need the following decomposition.
Let be $j(x) = \arg \min_{j\in\{1,2,\ldots,d_n\}} d(x, x_j)$; we have

$$\begin{aligned}
\sup_{x\in\mathcal{S}_\mathcal{F}} |M_{p,l}(x) - \mathbf{E}(M_{p,l}(x))| &\leqslant \sup_{x\in\mathcal{S}_\mathcal{F}} |M_{p,l}(x) - M_{p,l}(x_{j(x)})| + \\
&+ \sup_{x\in\mathcal{S}_\mathcal{F}} |M_{p,l}(x_{j(x)}) - \mathbf{E}(M_{p,l}(x_{j(x)}))| + \\
&+ \sup_{x\in\mathcal{S}_\mathcal{F}} |\mathbf{E}(M_{p,l}(x_{j(x)})) - \mathbf{E}(M_{p,l}(x))| \\
&:= D_1^{p,l} + D_2^{p,l} + D_3^{p,l}.
\end{aligned}$$

Using (U1), (U3) and (U4), we get

$$D_1^{p,l} \leqslant \frac{Cr_n}{nh\Phi(h)} \sup_{x\in\mathcal{S}_\mathcal{F}} \sum_{i=1}^{n} |\varphi^l(Y_i)| 1_{B(x,h)\cup B(x_{j(x)},h)}(X_i).$$

Taking

$$Z_i = \frac{Cr_n}{h\Phi(h)} |\varphi^l(Y_i)| \sup_{x\in\mathcal{S}_\mathcal{F}} 1_{B(x,h)\cup B(x_{j(x)},h)}(X_i);$$

The assumption (U7) allows us to write

$$\mathbf{E}|Z_1^m| \leqslant \frac{Cr_n^m}{h^m\Phi^{m-1}(h)}. \tag{6}$$

Using Corollary A.8 (ii) in [3] with $a_n^2 = \dfrac{r_n}{h\Phi(h)}$, we get

$$\frac{1}{n}\sum_{i=1}^{n} Z_i = \mathbf{E}(Z_1) + O_{a.co}\left(\sqrt{\frac{r_n \ln n}{nh\Phi(h)}}\right).$$

Applying (6) again (for m = 1), one gets

$$D_1^{p,l} = O\left(\frac{r_n}{h}\right) + O_{a.co}\left(\sqrt{\frac{r_n \ln n}{nh\Phi(h)}}\right).$$

Combining this last result with assumption (U5) and the second part of the assumption (U1), we obtain

$$D_1^{p,l} = O_{a.co}\left(\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right). \tag{7}$$

For the term $D_3^{p,l}$, since

$$D_3^{p,l} \leqslant \mathbf{E}\left(\sup_{x\in\mathcal{S}_\mathcal{F}} |M_{p,l}(x) - M_{p,l}(x_{j(x)})|\right)$$

thus

$$D_3^{p,l} = O_{a.co}\left(\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right). \tag{8}$$

And finally for the term $D_2^{p,l}$, we have For all $\eta > 0$

$$\begin{aligned}
\mathbf{P}\left(D_2^{p,l} > \eta\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right) &= \mathbf{P}\left(\max_{j\in\{1,\ldots,d_n\}} |M_{p,l}(x_{j(x)}) - \mathbf{E}(M_{p,l}(x_{j(x)}))| > \eta\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right) \leqslant \\
&\leqslant d_n \times \max_{j\in\{1,\ldots,d_n\}} \mathbf{P}\left(|M_{p,l}(x_{j(x)}) - \mathbf{E}(M_{p,l}(x_{j(x)}))| > \eta\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right).
\end{aligned}$$

Taking for $p = 2, 3, 4$

$$\Upsilon_{p,i} = \frac{1}{h^{p-2}\Phi_x(h)}\left[\frac{\mu K_i(x_{j(x)})\beta_i^{p-2}(x_{j(x)})\varphi^l(Y_i)}{G(Y_i)} - \mathbf{E}\left(\frac{\mu K_i(x_{j(x)})\beta_i^{p-2}(x_{j(x)})\varphi^l(Y_i)}{G(Y_i)}\right)\right].$$

Using the binomial Theorem and hypothesis (U1), (U2) and (U7), we obtain for $p = 2, 3, 4$

$$\mathbf{E}|\Upsilon_{p,i}|^m = O(\Phi^{-m+1}(h)).$$

So, we can apply a Bernstein-type inequality as done in the Corollary A.8 (i) in [3], to obtain

$$\mathbf{P}\left(\frac{1}{n}\left|\sum_{i=1}^{n}\Upsilon_{p,i}\right| > \eta\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right) \leqslant 2\exp\left(-C\eta^2\ln d_n\right).$$

Thus, by choosing β such that $C\eta^2 = \beta$, we get

$$\mathbf{P}\left(D_2^{p,l} > \eta\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right) \leqslant Cd_n^{1-\beta}.$$

Then, hypothesis (U5) allows us to write

$$D_2^{p,l} = O_{a.co}\left(\sqrt{\frac{\ln d_n}{n\Phi(h)}}\right). \tag{9}$$

Finally, the result of lemma (6) follows from the relations (7), (8) and (9).

The second part of the lemma (6) can be directly deduced from the proof of the first one such that $\mathbf{E}(\tilde{m}_0(x)) = 1$. For the last part, it comes straightforward that

$$\inf_{x\in\mathcal{S}_\mathcal{F}}\tilde{m}_0(x) < \frac{1}{2} \Rightarrow \exists x \in \mathcal{S}_\mathcal{F} \text{ such that}$$
$$1 - \tilde{m}_0(x) > \frac{1}{2} \Rightarrow \sup_{x\in\mathcal{S}_\mathcal{F}}|1-\tilde{m}_0(x)| > \frac{1}{2} \Rightarrow \sum_{n=1}^{\infty}\mathbf{P}\left(\inf_{x\in\mathcal{S}_\mathcal{F}}\tilde{m}_0(x) < \frac{1}{2}\right) < \infty.$$

□

Theorem 4.1 is proved.

The authors would like to thank the Editor and the anonymous reviewer for their valuable comments.

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Скорость почти надежной сходимости обобщенной регрессионной оценки на основе усеченных и функциональных данных

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Аннотация. В этой статье предлагается непараметрическая оценка обобщенной функции регрессии. Случайная переменная реального ответа (r.v.) подвергается усечению влево другим r.v., в то время как ковариата принимает свои значения в бесконечномерном пространстве. При стандартных предположениях устанавливаются точечные и равномерные почти наверняка сходимости предлагаемой оценки.

Ключевые слова: функциональные данные, усеченные данные, почти уверенная сходимость, локальная линейная оценка.

Journal of Siberian Federal University. Mathematics & Physics 2020, 13(4), 492–502

DOI: 10.17516/1997-1397-2020-13-4-492-502
УДК 514.7

Generalized Contractions to Coupled Fixed Point Theorems in Partially Ordered Metric Spaces

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Received 06.04.2020, received in revised form 25.04.2020, accepted 16.05.2020

Abstract. The purpose of this paper is to establish some coupled fixed point theorems for a self mapping satisfying certain rational type contractions along with strict mixed monotone property in a metric space endowed with partial order. Also, we have given the result of existence and uniqueness of a coupled fixed point for the mapping. This result generalize and extend several well known results in the literature.

Keywords: partially ordered metric spaces, rational contractions, coupled fixed point, monotone property.

Citation: N. Seshagiri Rao, K. Kalyani, Generalized Contractions to Coupled Fixed Point Theorems in Partially Ordered Metric Spaces, J. Sib. Fed. Univ. Math. Phys., 2020, 13(4), 492–502.
DOI: 10.17516/1997-1397-2020-13-4-492-502.

The existence and uniqueness of a fixed point was given by Banach [1] in 1922, which was acclaimed as Banach contraction principle and plays an important role in the development of various results connected with Fixed point Theory and Approximation Theory. The Banach's fixed point theorem or the contraction principle concerns certain mappings of a complete metric space into itself. It lays down conditions; sufficient for the existence and uniqueness of a fixed point. Besides, this famous classical theorem gives an iteration process through which we can obtain better approximation to the fixed point. Banach's fixed point theorem has rendered a key role in solving systems of linear algebraic equations involving iteration process. Iteration procedures are used in nearly every branch of applied mathematics, convergence proof and also in estimating the process of errors, very often by an application of Banach's fixed point theorem.

After that several mathematicians contributed to the growth of this area of knowledge and extensively reported in their work by taking various conditions on mappings as well as on spaces (see [2–11]). Also, numerous generalizations of this theorem have been obtained by weakening its hypotheses on various spaces like rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi-metric spaces, probabilistic metric spaces, D-metric spaces, G-metric spaces, F-metric spaces, cone metric spaces, and so on. More work on fixed points, common fixed points results in cone metric spaces, partially ordered metric spaces and others spaces can see from [12–24]. Recently, The existence and uniqueness of coupled fixed points on ordered sets have been investigated by many authors with various conditions on the mappings, readers may refer to [25–42] and references therein.

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In this paper, we proved some coupled fixed point results in the frame work of partially ordered metric space satisfying a generalized contractive condition of rational type with strict mixed monotone property of the mapping. Also, we presented the existence and uniqueness of a coupled fixed point result for the mapping. These results generalized many well known results in partially ordered metric space.

1. Preliminaries

Definition 1. *Let $(X, \leqslant)$ be a partially ordered set. A self mapping $f : X \to X$ is said to be strictly increasing if $f(x) < f(y)$, for all $x, y \in X$ with $x < y$ and is also said to be strictly decreasing if $f(x) > f(y)$, for all $x, y \in X$ with $x < y$.*

Definition 2. *Let $(X, \leqslant)$ be a partially ordered set and f is a self mapping defined over X is said to be strict mixed monotone property, if $f(x, y)$ is strictly increasing in x and strictly decreasing in y as well.*

$$\text{i.e., for any } x_1, x_2 \in X \text{ with } x_1 < x_2 \Rightarrow f(x_1, y) < f(x_2, y) \text{ and also}$$
$$\text{for any } y_1, y_2 \in X \text{ with } y_1 < y_2 \Rightarrow f(x, y_1) > f(x, y_2).$$

Definition 3. *Let $(X, \leqslant)$ be a partially ordered set and $f : X \times X \to X$ be a mapping. A point $(x, y) \in X \times X$ is said to be a coupled fixed point to f, if $f(x, y) = x$ and $f(y, x) = y$.*

Definition 4. *The triple $(X, d, \leqslant)$ is called partially ordered metric space if $(X, \leqslant)$ is a partially ordered set together with (X, d) is a metric space.*

Definition 5. *If (X, d) is a complete metric space, then triple $(X, d, \leqslant)$ is called a partially ordered complete metric space.*

Definition 6. *A partially ordered metric space $(X, d, \leqslant)$ is called an ordered complete (OC), if for each convergent sequence $\{x_n\}_{n=0}^{\infty} \subset X$, the following one of the condition holds*

- *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ then $x_n \leqslant x$, for all $n \in \mathbb{N}$ that is, $x = \sup\{x_n\}$ or*
- *if $\{x_n\}$ is a non-increasing sequence in X such that $x_n \to x$ then $x \leqslant x_n$, for all $n \in \mathbb{N}$ that is, $x = \inf\{x_n\}$.*

2. Main results

In this section, we prove some coupled fixed point theorems for a self mapping satisfying certain rational contraction condition in ordered metric space.

Theorem 1. *Let $(X, d, \leqslant)$ be a complete partially ordered metric space. Suppose that a self mapping $f : X \times X \to X$ has a strict mixed monotone property on X satisfying the following condition*

$$\begin{aligned} d(f(x,y), f(\mu,\upsilon)) \leqslant{} & \alpha \frac{d(x, f(x,y))\,[1 + d(\mu, f(\mu,\upsilon))]}{1 + d(x,\mu)} + \beta \frac{d(x, f(x,y))\; d(\mu, f(\mu,\upsilon))}{d(x,\mu)} \\ & + \gamma[d(x, f(x,y)) + d(\mu, f(\mu,\upsilon))] + \delta[d(x, f(\mu,\upsilon)) + d(\mu, f(x,y))] \\ & + \lambda d(x,\mu) \end{aligned} \tag{1}$$

for all $x, y, \mu, \upsilon \in X$ with $x \geqslant \mu$ and $y \leqslant \upsilon$, where $\alpha, \beta, \gamma, \delta, \lambda \in [0, 1)$ with $0 \leqslant \alpha + \beta + 2(\gamma + \delta) + {} + \lambda < 1$. Suppose that either f is continuous or X has an ordered complete property (OC) then f has a coupled fixed point $(x, y) \in X \times X$, if there exists two points $x_0, y_0 \in X$ with $x_0 < f(x_0, y_0)$ and $y_0 > f(y_0, x_0)$.

Proof. Suppose f is a continuous map on X. Let $x_0, y_0 \in X$ such that $x_0 < f(x_0, y_0)$ and $y_0 > f(y_0, x_0)$ then, define two sequences $\{x_n\}, \{y_n\}$ in X as follows

$$x_{n+1} = f(x_n, y_n) \text{ and } y_{n+1} = f(y_n, x_n) \text{ for all } n \geqslant 0. \tag{2}$$

Next, we have to show that for all $n \geqslant 0$,

$$x_n < x_{n+1} \tag{3}$$

and

$$y_n > y_{n+1} \tag{4}$$

for this, we use the method of mathematical induction. Suppose $n = 0$, since $x_0 < f(x_0, y_0)$ and $y_0 > f(y_0, x_0)$ and from (2), we have $x_0 < f(x_0, y_0) = x_1$ and $y_0 > f(y_0, x_0) = y_1$ and hence the inequalities (3) and (4) hold for $n = 0$. Suppose that the inequalities (3) and (4) hold for all $n > 0$ and by using the strict mixed monotone property of f, we get

$$x_{n+1} = f(x_n, y_n) < f(x_{n+1}, y_n) < f(x_{n+1}, y_{n+1}) = x_{n+2} \tag{5}$$

and

$$y_{n+1} = f(y_n, x_n) > f(y_{n+1}, x_n) > f(y_{n+1}, x_{n+1}) = y_{n+2}. \tag{6}$$

Thus, the inequalities (3) and (4) hold for all $n \geqslant 0$ and we obtain that

$$x_0 < x_1 < x_2 < x_3 < \cdots < x_n < x_{n+1} < \dots \tag{7}$$

and

$$y_0 > y_1 > y_2 > y_3 > \cdots > y_n > y_{n+1} > \dots. \tag{8}$$

We know that $x_n < x_{n+1},\ y_n > y_{n+1}$ for all n then, by (1) and use of (2), we get

$$\begin{aligned} d(x_{n+1}, x_n) =& d(f(x_n, y_n), f(x_{n-1}, y_{n-1})) \\ &\leqslant \alpha\, \frac{d(x_n, f(x_n, y_n))\,[1 + d(x_{n-1}, f(x_{n-1}, y_{n-1}))]}{1 + d(x_n, x_{n-1})} \\ &+ \beta\, \frac{d(x_n, f(x_n, y_n))\; d(x_{n-1}, f(x_{n-1}, y_{n-1}))}{d(x_n, x_{n-1})} \\ &+ \gamma[d(x_n, f(x_n, y_n)) + d(x_{n-1}, f(x_{n-1}, y_{n-1}))] \\ &+ \delta[d(x_n, f(x_{n-1}, y_{n-1})) + d(x_{n-1}, f(x_n, y_n))] + \lambda d(x_n, x_{n-1}) \end{aligned}$$

which implies that

$$\begin{aligned} d(x_{n+1}, x_n) \leqslant& \alpha\, \frac{d(x_n, x_{n+1})\,[1 + d(x_{n-1}, x_n)]}{1 + d(x_n, x_{n-1})} + \beta\, \frac{d(x_n, x_{n+1})\; d(x_{n-1}, x_n)}{d(x_n, x_{n-1})} \\ &+ \gamma[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + \delta[d(x_n, x_n) + d(x_{n-1}, x_{n+1})] \\ &+ \lambda d(x_n, x_{n-1}). \end{aligned}$$

Finally, we arrive at

$$d(x_{n+1}, x_n) \leqslant \left(\frac{\gamma + \delta + \lambda}{1 - \alpha - \beta - \gamma - \delta} \right) d(x_n, x_{n-1}). \tag{9}$$

Similarly by following above, we get

$$d(y_{n+1}, y_n) \leqslant \left(\frac{\gamma + \delta + \lambda}{1 - \alpha - \beta - \gamma - \delta} \right) d(y_n, y_{n-1}). \tag{10}$$

So, from equations (9) and (10), we have

$$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \leqslant \left(\frac{\gamma + \delta + \lambda}{1 - \alpha - \beta - \gamma - \delta} \right) [d(x_n, x_{n-1}) + d(y_n, y_{n-1})].$$

Now, let us define a sequence $\{S_n\}$ in X as $\{S_n\} = \{d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\}$. Therefore, by induction we get

$$0 \leqslant S_n \leqslant kS_{n-1} \leqslant k^2 S_{n-2} \leqslant k^3 S_{n-3} \leqslant \ldots \leqslant k^n S_0,$$

where $k = \dfrac{\gamma + \delta + \lambda}{1 - \alpha - \beta - \gamma - \delta} < 1$ and hence, we obtain

$$\lim_{n \to +\infty} S_n = \lim_{n \to +\infty} [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] = 0.$$

Consequently, we get $\lim\limits_{n \to +\infty} d(x_n, x_{n+1}) = 0$ and $\lim\limits_{n \to +\infty} d(y_n, y_{n+1}) = 0$. By using triangular inequality for $m \geqslant n$, we get

$$d(x_m, x_n) \leqslant d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n)$$

and

$$d(y_m, y_n) \leqslant d(y_m, y_{m-1}) + d(y_{m-1}, y_{m-2}) + \cdots + d(y_{n+1}, y_n).$$

Therefore,

$$\begin{aligned} d(x_m, x_n) + d(y_m, y_n) &\leqslant S_{m-1} + S_{m-2} + \cdots + S_n \\ &\leqslant \left(k^{m-1} + k^{m-2} + \cdots + k^n\right) S_0 \\ &\leqslant \frac{k^n}{1-k} S_0. \end{aligned}$$

Letting limit as $n, m \to \infty$ in the above inequality, we obtain that $d(x_m, x_n) + d(y_m, y_n) \to 0$. Consequently, the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and by completeness of X, there exists a point $(x, y) \in X \times X$ such that $x_n \to x$ and $y_n \to y$. And also from the continuity of f, we have

$$x = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} f(x_n, y_n) = f(\lim_{n \to \infty} x_n, \lim_{n \to \infty} y_n) = f(x, y),$$

and

$$y = \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} f(y_n, x_n) = f(\lim_{n \to \infty} y_n, \lim_{n \to \infty} x_n) = f(y, x).$$

Therefore, we have $x = f(x, y)$ and $y = f(y, x)$, i.e., f has a coupled fixed point in $X \times X$.

Another way, suppose X has an ordered complete property (OC). From above discussion there is an increasing Cauchy sequence $\{x_n\}$ in X converges to $x \in X$. Then from (OC) property of X, we have $x = \sup\{x_n\}$, i.e., $x_n \leqslant x$, for all $n \in \mathbb{N}$. Therefore, we conclude that $x_n < x$, for all n otherwise there exists a number $n_0 \in \mathbb{N}$ such that $x_{n_0} = x$, and hence $x < x_{n_0} \leqslant x_{n_0+1} = x$ which is wrong. Thus, from the strict monotone increasing of f over the first variable, we get

$$f(x_n, y_n) < f(x, y_n). \tag{11}$$

Similarly, from above there is a decreasing Cauchy sequence $\{y_n\}$ in X, which converges to a point $y \in X$. Thus, by (OC) property of X, we have $y = \inf\{y_n\}$, i.e., $y_n \geqslant y$, for all $n \in \mathbb{N}$.

As from similar argument above, we have $y_n > y$, for all $n \in \mathbb{N}$. Also, from the strict monotone decreasing of f on the second variable, we get

$$f(x, y_n) < f(x, y). \tag{12}$$

Therefore, from equations (11) and (12), we obtain

$$f(x_n, y_n) < f(x, y) \Rightarrow x_{n+1} < f(x, y), \text{ for all } n \in \mathbb{N}. \tag{13}$$

Since $x_n < x_{n+1} < f(x, y)$, for all $n \in \mathbb{N}$ and $x = \sup\{x_n\}$, then we obtain $x \leqslant f(x, y)$. Now, let $z_0 = x$ and $z_{n+1} = f(z_n, y_n)$ then, by similar argument above the sequence $\{z_n\}$ is a nondecreasing Cauchy sequence, since $z_0 \leqslant f(z_0, y_0)$ and converges to a point z in X, implies that $z = \sup\{z_n\}$.

Since for all $n \in \mathbb{N}$, $x_n \leqslant x = z_0 \leqslant f(z_0, y_0) \leqslant z_n \leqslant z$ then from (1), we have

$$\begin{aligned} d(x_{n+1}, z_{n+1}) &= d(f(x_n, y_n), f(z_n, y_n)) \\ &\leqslant \alpha \, \frac{d(x_n, f(x_n, y_n)) \, [1 + d(z_n, f(z_n, y_n))]}{1 + d(x_n, z_n)} \\ &\quad + \beta \, \frac{d(x_n, f(x_n, y_n)) \; d(z_n, f(z_n, y_n))}{d(x_n, z_n)} \\ &\quad + \gamma[d(x_n, f(x_n, y_n)) + d(z_n, f(z_n, y_n))] \\ &\quad + \delta[d(x_n, f(z_n, y_n)) + d(z_n, f(x_n, y_n))] + \lambda d(x_n, z_n). \end{aligned}$$

On taking limit as $n \to \infty$ in the above inequality, we get

$$d(x, z) \leqslant (2\delta + \lambda) d(x, z),$$

but $2\delta + \lambda < 1$, then we obtain that $d(x, z) = 0$. Hence $x = z = \sup\{x_n\}$, implies that $x \leqslant f(x, y) \leqslant x$. Thus, $x = f(x, y)$. Again following the similar above argument, we obtain that $y = f(y, x)$. Hence, f has a coupled fixed point in $X \times X$. □

For the existence and uniqueness of a coupled fixed point of f over a complete partial ordered metric space X, we furnish the following partial order relation.

$$(\mu, \upsilon) \leqslant (x, y) \Leftrightarrow x \geqslant \mu, \; y \leqslant \upsilon, \text{ for any } (x, y), (\mu, \upsilon) \in X \times X.$$

Theorem 2. *Along the hypothesis stated in Theorem 1 and suppose that for every $(x, y), (r, s) \in X \times X$, there exists $(u, v) \in X \times X$ such that $(f(u, v), f(v, u))$ is comparable to $(f(x, y), f(y, x))$ and $(f(r, s), f(s, r))$, then f has a unique coupled fixed point in $X \times X$.*

Proof. As we know from Theorem 1, the set of coupled fixed points of f is non empty. Suppose that (x, y) and (r, s) are two coupled fixed points of the mapping f, then $x = f(x, y)$, $y = f(y, x)$, $r = f(r, s)$ and $s = f(s, r)$. Now, we have to show that $x = r$, $y = s$ for the uniqueness of a coupled fixed point of f.

From hypotheses, there exists $(u, v) \in X \times X$ such that $(f(u, v), f(v, u))$ is comparable to $(f(x, y), f(y, x))$ and $(f(r, s), f(s, r))$. Put $u = u_0$, $v = v_0$ then choose $u_1, v_1 \in X$ such that $u_1 = f(u_0, v_0)$ and $v_1 = f(v_0, u_0)$. Thus, following the proof of Theorem 1, we construct two sequences $\{u_n\}$, $\{v_n\}$ from $u_{n+1} = f(u_n, v_n)$ and $v_{n+1} = f(v_n, u_n)$ for all $n \in \mathbb{N}$. Similarly, define the sequences $\{x_n\}$, $\{y_n\}$, $\{r_n\}$ and $\{s_n\}$ by setting $x = x_0$, $y = y_0$, $r = r_0$ and $s = s_0$. As form Theorem 1, we have $x_n \to x = f(x, y)$, $y_n \to y = f(y, x)$, $r_n \to r = f(r, s)$ and $s_n \to s = f(s, r)$ for all $n \geqslant 1$. But $(f(x, y), f(y, x)) = (x, y)$ and $(f(u_0, v_0), f(v_0, u_0)) = (u_1, v_1)$ are comparable and then we have $x \geqslant u_1$ and $y_{\leqslant} v_1$. Next to show that (x, y) and (u_n, v_n) are comparable, i.e., to show that $x \geqslant u_n$ and $y_{\leqslant} v_n$ for all $n \in \mathbb{N}$. Suppose the inequalities hold for some $n \geqslant 0$, then from strict mixed monotone property of f, we have $u_{n+1} = f(u_n, v_n) \leqslant f(x, y) = x$ and $v_{n+1} = f(v_n, u_n) \geqslant f(y, x) = y$. Therefore, we have $x \geqslant u_n$ and $y_{\leqslant} v_n$ for all $n \in \mathbb{N}$.

Again from (1), we have

$$\begin{aligned} d(x, u_{n+1}) &= d(f(x, y), f(u_n, v_n)) \\ &\leqslant \alpha \, \frac{d(x, f(x, y)) \, [1 + d(u_n, f(u_n, v_n))]}{1 + d(x, u_n)} + \beta \, \frac{d(x, f(x, y)) \; d(u_n, f(u_n, v_n))}{d(x, u_n)} \\ &\quad + \gamma [d(x, f(x, y)) + d(u_n, f(u_n, v_n))] + \delta [d(x, f(u_n, v_n)) + d(u_n, f(x, y))] + \lambda d(x, u_n) \end{aligned}$$

which implies that

$$d(x, u_{n+1}) \leqslant \left(\frac{\gamma + \delta + \lambda}{1 - \gamma - \delta} \right) d(x, u_n).$$

Similarly, we can obtain

$$d(y, v_{n+1}) \leqslant \left(\frac{\gamma + \delta + \lambda}{1 - \gamma - \delta} \right) d(y, v_n).$$

Suppose $D = \dfrac{\gamma + \delta + \lambda}{1 - \gamma - \delta} < 1$, then from above equations, we have

$$\begin{aligned} d(x, u_{n+1}) + d(y, v_{n+1}) &\leqslant D \, [d(x, u_n) + d(y, v_n)] \\ &\leqslant D^2 \, [d(x, u_{n-1}) + d(y, v_{n-1})] \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots \\ &\leqslant D^n \, [d(x, u_0) + d(y, v_0)] \, . \end{aligned}$$

Taking limit as $n \to +\infty$ to the above inequality, we get $\lim\limits_{n \to +\infty} d(x, u_{n+1}) + d(y, v_{n+1}) = 0$. Consequently, we obtain $\lim\limits_{n \to +\infty} d(x, u_{n+1}) = 0$ and $\lim\limits_{n \to +\infty} d(y, v_{n+1}) = 0$. Similarly, one can prove that $\lim\limits_{n \to \infty} d(r, u_n) = 0$ and $\lim\limits_{n \to \infty} d(s, v_n) = 0$.

Further form triangular inequality, we obtain that

$$d(x, r) \leqslant d(x, u_n) + d(u_n, r) \text{ and } d(y, s) \leqslant d(y, v_n) + d(v_n, s).$$

On taking limit as $n \to \infty$ to the above inequalities, we obtain that $d(x, r) = 0 = d(y, s)$, implies that $x = r$ and $y = s$. Hence, f has a unique coupled fixed point in $X \times X$. This completes the proof. □

Theorem 3. *Along the hypotheses stated in Theorem 1 and if x_0, y_0 are comparable then f has a coupled fixed point in $X \times X$.*

Proof. Suppose (x, y) is a coupled fixed point of f, then from Theorem 1, there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \to x$ and $y_n \to y$ in X.

Assume that $x_0 \leqslant y_0$, then we have to show that $x_n \leqslant y_n$, for all $n \geqslant 0$. Suppose it hods for some $n \geqslant 0$. So, by the strict mixed monotone property of f, we get $x_{n+1} = f(x_n, y_n) \leqslant f(y_n, x_n) = y_{n+1}$. Then, from the contraction condition (1), we get

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &= d(f(x_n, y_n), f(y_n, x_n)) \\ &\leqslant \alpha \, \frac{d(x_n, f(x_n, y_n)) \, [1 + d(y_n, f(y_n, x_n))]}{1 + d(x_n, y_n)} \\ &\quad + \beta \, \frac{d(x_n, f(x_n, y_n)) \; d(y_n, f(y_n, x_n))}{d(x_n, y_n)} \\ &\quad + \gamma \, [d(x_n, f(x_n, y_n)) + d(y_n, f(y_n, x_n))] \\ &\quad + \delta \, [d(x_n, f(y_n, x_n)) + d(y_n, f(x_n, y_n))] + \lambda d(x_n, y_n). \end{aligned}$$

On taking limit as $n \to \infty$, we get

$$d(x,y) \leqslant (2\delta + \lambda)\ d(x,y)$$

which is a contradiction, since $2\delta + \lambda < 1$. Thus, $d(x,y) = 0$. Therefore, we have $f(x,y) = x = y = f(y,x)$. Similarly, we can also show that $f(x,y) = x = y = f(y,x)$ by considering $y_0 \leqslant x_0$. Hence, (x,y) is a coupled fixed point of f in $X \times X$. □

Remarks:

1. If $\alpha = \gamma = \delta = 0$, in above Theorems, we obtain Theorems 2.1 and Theorem 2.2 of Ciric et al. [30].
2. If $\alpha = 0$ in above Theorems, we can get Theorem 2.1–Theorem 2.3 of Chandok et al. [38].
3. Banach [1] type contraction in partially ordered metric spaces can get by taking $\alpha = \beta = \gamma = \delta = 0$.
4. Kannan [7] type contraction for coupled fixed point theorem in partially ordered metric spaces can get by putting $\alpha = \beta = \delta = \lambda = 0$ in above Theorem 2.1–Theorem 2.3.
5. Chatterjee [3] type contraction for coupled fixed point theorem in partially ordered metric spaces can obtain by giving $\alpha = \beta = \gamma = \lambda = 0$ in above Theorem 2.1–Theorem 2.3.
6. Singh and Chatterjee [9] type contraction for coupled fixed point theorem in partially ordered metric spaces can get by giving $\alpha = \gamma = 0$ in above Theorem 2.1–Theorem 2.3.

3. Applications

In this section, we state some applications of the main result to a self mapping involving an integral type contractions.

Let us consider the set of all functions χ defined on $[0,\infty)$ satisfying the following conditions:

1. Each χ is Lebesgue integrable mapping on each compact subset of $[0,\infty)$.
2. For any $\epsilon > 0$, we have $\int_0^{\epsilon} \chi(t)dt > 0$.

Theorem 4. *Let $(X, d, \leqslant)$ be a complete partially ordered metric space. Suppose that a self mapping $f : X \times X \to X$ has a strict mixed monotone property on X satisfying the following condition*

$$\begin{aligned} \int_0^{(d(f(x,y),f(\mu,v))} \varphi(t)dt \leqslant{}& \alpha \int_0^{\frac{d(x,f(x,y))[1+d(\mu,f(\mu,v))]}{1+d(x,\mu)}} \varphi(t)dt + \beta \int_0^{\frac{d(x,f(x,y))\ d(\mu,f(\mu,v))}{d(x,\mu)}} \varphi(t)dt \\ &+ \gamma \int_0^{d(x,f(x,y))+d(\mu,f(\mu,v))} \varphi(t)dt + \delta \int_0^{d(x,f(\mu,v))+d(\mu,f(x,y))} \varphi(t)dt \\ &+ \lambda \int_0^{d(x,\mu)} \varphi(t)dt \end{aligned} \tag{14}$$

for all $x, y, \mu, v \in X$ with $x \geqslant \mu$ and $y \leqslant v$, $\varphi(t)$ is a function satisfies the above conditions defined on $[0,\infty)$ and $\alpha, \beta, \gamma, \delta, \lambda \in [0,1)$ with $0 \leqslant \alpha + \beta + 2(\gamma + \delta) + \lambda < 1$. Suppose that either f is continuous or X has an ordered complete property (OC) then f has a coupled fixed point $(x,y) \in X \times X$, if there exists two points $x_0, y_0 \in X$ with $x_0 < f(x_0,y_0)$ and $y_0 > f(y_0,x_0)$.

Similarly, we can obtain the following coupled fixed point results in partially ordered metric space, by taking $\gamma = \delta = 0$, $\beta = 0$, $\beta = \gamma = 0$ and $\beta = \delta = 0$ in Theorem 4.

Theorem 5. *Let $(X, d, \leqslant)$ be a complete partially ordered metric space. Suppose that a self mapping $f : X \times X \to X$ has a strict mixed monotone property on X satisfying the following condition*

$$\int_0^{(d(f(x,y),f(\mu,\upsilon))} \varphi(t)dt \leqslant \alpha \int_0^{\frac{d(x,f(x,y))[1+d(\mu,f(\mu,\upsilon))]}{1+d(x,\mu)}} \varphi(t)dt + \beta \int_0^{\frac{d(x,f(x,y))\ d(\mu,f(\mu,\upsilon))}{d(x,\mu)}} \varphi(t)dt + \lambda \int_0^{d(x,\mu)} \varphi(t)dt \tag{15}$$

for all $x, y, \mu, \upsilon \in X$ with $x \geqslant \mu$ and $y \leqslant \upsilon$, $\varphi(t)$ is a function satisfies the above conditions defined on $[0, \infty)$ and $\alpha, \beta, \lambda \in [0, 1)$ with $0 \leqslant \alpha + \beta + \lambda < 1$. Suppose that either f is continuous or X has an ordered complete property (OC) then f has a coupled fixed point $(x, y) \in X \times X$, if there exists two points $x_0, y_0 \in X$ with $x_0 < f(x_0, y_0)$ and $y_0 > f(y_0, x_0)$.

Theorem 6. *Let $(X, d, \leqslant)$ be a complete partially ordered metric space. Suppose that a self mapping $f : X \times X \to X$ has a strict mixed monotone property on X satisfying the following condition*

$$\int_0^{(d(f(x,y),f(\mu,\upsilon))} \varphi(t)dt \leqslant \alpha \int_0^{\frac{d(x,f(x,y))[1+d(\mu,f(\mu,\upsilon))]}{1+d(x,\mu)}} \varphi(t)dt + \gamma \int_0^{d(x,f(x,y))+d(\mu,f(\mu,\upsilon))} \varphi(t)dt + \delta \int_0^{d(x,f(\mu,\upsilon))+d(\mu,f(x,y))} \varphi(t)dt + \lambda \int_0^{d(x,\mu)} \varphi(t)dt \tag{16}$$

for all $x, y, \mu, \upsilon \in X$ with $x \geqslant \mu$ and $y \leqslant \upsilon$, $\varphi(t)$ is a function satisfies the above conditions defined on $[0, \infty)$ and $\alpha, \gamma, \delta, \lambda \in [0, 1)$ with $0 \leqslant \alpha + 2(\gamma + \delta) + \lambda < 1$. Suppose that either f is continuous or X has an ordered complete property (OC) then f has a coupled fixed point $(x, y) \in X \times X$, if there exists two points $x_0, y_0 \in X$ with $x_0 < f(x_0, y_0)$ and $y_0 > f(y_0, x_0)$.

Theorem 7. *Let $(X, d, \leqslant)$ be a complete partially ordered metric space. Suppose that a self mapping $f : X \times X \to X$ has a strict mixed monotone property on X satisfying the following condition*

$$\int_0^{(d(f(x,y),f(\mu,\upsilon))} \varphi(t)dt \leqslant \alpha \int_0^{\frac{d(x,f(x,y))[1+d(\mu,f(\mu,\upsilon))]}{1+d(x,\mu)}} \varphi(t)dt + \delta \int_0^{d(x,f(\mu,\upsilon))+d(\mu,f(x,y))} \varphi(t)dt + \lambda \int_0^{d(x,\mu)} \varphi(t)dt \tag{17}$$

for all $x, y, \mu, \upsilon \in X$ with $x \geqslant \mu$ and $y \leqslant \upsilon$, $\varphi(t)$ is a function satisfies the above conditions defined on $[0, \infty)$ and $\alpha, \delta, \lambda \in [0, 1)$ with $0 \leqslant \alpha + 2\delta + \lambda < 1$. Suppose that either f is continuous or X has an ordered complete property (OC) then f has a coupled fixed point $(x, y) \in X \times X$, if there exists two points $x_0, y_0 \in X$ with $x_0 < f(x_0, y_0)$ and $y_0 > f(y_0, x_0)$.

Theorem 8. *Let $(X, d, \leqslant)$ be a complete partially ordered metric space. Suppose that a self mapping $f : X \times X \to X$ has a strict mixed monotone property on X satisfying the following condition*

$$\int_0^{(d(f(x,y),f(\mu,\upsilon))} \varphi(t)dt \leqslant \alpha \int_0^{\frac{d(x,f(x,y))[1+d(\mu,f(\mu,\upsilon))]}{1+d(x,\mu)}} \varphi(t)dt + \gamma \int_0^{d(x,f(x,y))+d(\mu,f(\mu,\upsilon))} \varphi(t)dt + \lambda \int_0^{d(x,\mu)} \varphi(t)dt \tag{18}$$

for all $x, y, \mu, \upsilon \in X$ with $x \geqslant \mu$ and $y \leqslant \upsilon$, $\varphi(t)$ is a function satisfies the above conditions defined on $[0, \infty)$ *and* $\alpha, \gamma, \lambda \in [0, 1)$ *with* $0 \leqslant \alpha + 2\gamma + \lambda < 1$. *Suppose that either f is continuous or X has an ordered complete property (OC) then f has a coupled fixed point $(x, y) \in X \times X$, if there exists two points $x_0, y_0 \in X$ with $x_0 < f(x_0, y_0)$ and $y_0 > f(y_0, x_0)$.*

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Обобщенные сжатия для связанных теорем о неподвижных точках в частично упорядоченных метрических пространствах

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Аннотация. Цель этой статьи — установить некоторые связанные теоремы о неподвижной точке для самопредставления, удовлетворяющего определенным рациональным сокращениям типов наряду со строго смешанной монотонной собственностью в метрическом пространстве, снабженном частичным порядком. Также мы дали результат существования и единственности связанной неподвижной точки для отображения. Этот результат обобщает и расширяет несколько хорошо известных в литературе результатов.

Ключевые слова: частично упорядоченные метрические пространства, рациональные сокращения, связанная фиксированная точка, монотонная собственность.

Journal of Siberian Federal University. Mathematics & Physics 2020, 13(4), 503–514

DOI: 10.17516/1997-1397-2020-13-4-503-514
УДК 517.55+517.33

On New Decomposition Theorems in some Analytic Function Spaces in Bounded Pseudoconvex Domains

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Received 06.04.2020, received in revised form 23.05.2020, accepted 06.07.2020

Abstract. We provide new sharp decomposition theorems for multifunctional Bergman spaces in the unit ball and bounded pseudoconvex domains with smooth boundary expanding known results from the unit ball.

Namely we prove that $\prod_{j=1}^{m} ||f_j||_{X_j} \asymp ||f_1 \dots f_m||_{A^p_\alpha}$ for various (X_j) spaces of analytic functions in bounded pseudoconvex domains with smooth boundary where f, f_j, $j = 1, \dots, m$ are analytic functions and where A^p_α, $0 < p < \infty$, $\alpha > -1$ is a Bergman space. This in particular also extend in various directions a known theorem on atomic decomposition of Bergman A^p_α spaces.

Keywords: pseudoconvex domains, unit ball, Bergman spaces, decomposition theorems, Hardy type spaces.

Citation: R.F. Shamoyan, E.B. Tomashevskaya, On New Decomposition Theorems in some Analytic Function Spaces in Bounded Pseudoconvex Domains, J. Sib. Fed. Univ. Math. Phys., 2020, 13(4), 503–514. DOI: 10.17516/1997-1397-2020-13-4-503-514.

Introduction and preliminaries

The problem we consider is well-known for functional spaces in $\mathbb{R}^n$ (the problem of equivalent norms) (see, for example, [1]).

Let $X, (X_j)$ be a function space in a fixed domain D in $\mathbb{C}^n$ (normed or quazinormed) we wish to find equivalent expression for $||f_1 \dots f_m||_X$; $m \in \mathbb{N}$. Note these are closely connected with spaces on product domains since

$$f(z_1, \dots, z_m) = \prod_{j=1}^{m} f_j(z_j), \quad ||f||_X = \prod_{j=1}^{m} ||f_j||_{X_j}, \quad z_j \in D;\ j = 1, \dots, m.$$

These our results also extend some well-known assertions on atomic decomposition of Bergman A^p_α type spaces as we will see below. For $m = 1$ Hardy space case (see, for example, [2–4]).

To study such group of functions it is natural, for example, to ask about structure of each $\{f_j\}_{j=1}^m$ of this group.

This can be done for example if we turn to the following question find conditions on $\{f_1, \dots, f_m\}$, so that $||f_1, \dots, f_m||_X \asymp \prod_{j=1}^{m} ||f_j||_{X_j}$ sharp (R) decomposition is valid. In this case for example we have if for some positive constant c;

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$$\prod_{j=1}^{m} ||f_j||_{X_j} \leqslant c||f_1, \dots, f_m||_X;$$

then we have each f_j, $f_j \in X_j; j = 1, \dots, m$, where X_j is a new normed (or quazinormed) function space in D domain and we can easily now provide properties of f_j based on facts of already known one functional function space theory. (For example to use known theorems for each $f_j \in X_j; j = 1, \dots, m$ on atomic decompositions). This idea was used for Bergman spaces in the unit ball then in bounded pseudoconvex domains with smooth boundary in recent papers [5] and [6]. In this paper we extend these results in various directions using modification of known proof.

We refer to [5, 6] for a complete and not difficult proof of a basic known "purely A^p_α" case then in this paper show in details how to modify it to get new results. The old known proof is simple and very flexible as it turns out and we can easily get, as we can see below, various new interesting results from it directly. This remark is leading us to provide only some sketchy arguments sometimes below of proofs when we deal with new theorems , since the core of all proofs is basically the same in all our theorems. Here is partially the transparent proof of the classical case of the Bergman space A^p_α case in the unit ball B_n of $\mathbb{C}^n$. The case of A^p_α Bergman space in more general bounded pseudoconvex domain can be seen in our recent paper [6].

We define A^p_α space as usual

$$A^p_\alpha = \Big\{ f \in H(B) : ||f||^p_{A^p_\alpha} = \int_B \Big| f(z) \Big|^p (1 - |z|)^\alpha dv(z) < \infty \Big\},$$

dv is a Lebeques measure on B, f_j is analytic in B, $0 < p < \infty, \alpha > -1$, $j = 1, \dots, m$ and where $H(B)$ is a class of all analytic functions in the unit ball B.

We see in [6] that $||f_1 \dots f_m||_{A^p_\tau} \asymp \prod\limits_{j=1}^{m} ||f_j||_{A^p_\alpha}$ is valid under certain integral (A) condition if $p \leqslant 1$ and if $\tau = \tau(p, \alpha_1, \dots, \alpha_m, m)$.

We denote constants as usual by $C, C_1, C_2, \dots$

Note from our discussion above the only interesting part is to show that

$$\prod_{j=1}^{m} ||f_j||_{A^p_{\alpha_j}(B_n)} \leqslant c_1 ||f_1 \dots f_m||_{A^p_\tau(B_n)}, \tag{S}$$

since the reverse follows directly from the uniform estimate (see [6, 7]).

$$|f(z)|(1 - |z|)^{\frac{\alpha_j + n + 1}{p}} \leqslant c||f||_{A^p_{\alpha_j}};\ 0 < p < \infty,\ \alpha_j > -1,\ j = 1, \dots, m$$

and ordinary induction. This also lead easily to the fact that τ can be calculated

$$\tau = (n + 1)(m - 1) + \sum_{j=1}^{m} \alpha_j;\ \alpha_j > -1;\ 0 < p < \infty.$$

Note similar very simple proof based only on various known uniform estimates can be used in all our proofs below in similar inequalities for various spaces. So we mainly concentrate on reverse to (S) estimates. Let further H^p be a usual Hardy H^p space in B_n (see [7, 8]).

Note further if α_0 is large enough and if

$$\prod_{i=1}^{m} f_i(\omega_i) = c_\alpha \int_B \frac{\prod\limits_{i=1}^{m} f_i(z)(1 - |z|)^\alpha dv(z)}{\prod\limits_{i=1}^{m} (1 - \tilde{z}\omega_i)^{\frac{n+1+\alpha}{m}}},\ \ \omega_j \in B,\ j = 1, \dots, m,\ \alpha > \alpha_0,$$

then we have easily using directly well-known estimates (see [7]) from last equality for $p \leqslant 1$ (we refer to [6] for details in more general situation).

$$I = \prod_{i=1}^{m} \int_B |f_k(z_k)|^p (1-|z_k|^2)^{\alpha_k} dv(z_k) = \int_B \cdots \int_B \prod_{k=1}^{m} |f_k(z_k)|^p \cdot \prod_{k=1}^{m} (1-|z_k|^2)^{\alpha_k} \cdot \prod_{k=1}^{m} dv(z_k) \leqslant$$

$$\leqslant c \int_B \prod_{k=1}^{m} |f_k(z)|^p (1-|z_k|^2)^{\tau} dv(z) \times \int_B \cdots \int_B \frac{\prod_{k=1}^{m} (1-|z_k|^2)^{\alpha_k} \prod_{k=1}^{m} dv(z_k)}{\prod_{k=1}^{m} |1-\tilde{z} z_k|^{(\frac{n+1+\alpha}{m})p}};$$

where $\alpha_k > -1$, $k = 1, \ldots, m$, $\tau = (n+1+\alpha)p - (n+1) > -1$, $\alpha > \alpha_0 = \alpha_0(p, \vec{\alpha}, n, m)$. And hence we have finnally

$$c_1 \int_B \prod_{k=1}^{m} |f_k(z)|^p (1-|z_k|^2)^{\tau_1} dv(z) < \infty,$$

where $0 < p \leqslant 1$; $\tau_1 = (m-1)(n+1) + \sum_{k=1}^{m} \alpha_k$; and $\alpha_k > -1$, $\tau_1 > -1, \alpha > \alpha_0$.

This result is valid also for $p > 1$ (see [5]). We will repeat this type simple argument several times below.

The same more general problem which we consider in bounded pseudoconvex domain D is the following. To find equivalent expressions for $||f_1 \ldots f_m||_X; f_j \in H(D)$, $j = 1, \ldots, m$. Can we also say that each fi can be decomposed into "atoms" (BMOA atoms, Bloch atoms, Hardy atoms, Bergman atoms (see [2–4, 6, 7, 9, 10])) if

$$\int_D \Big| \prod_{j=1}^{m} f_j(z) \Big|^p \delta^{\tau}(z) dv(z) < \infty, \quad 0 < p < \infty, \quad \tau > -1; \quad \delta(z) = dist(z, \partial D)$$

and dv is a Lebeques measure on D. Only for $m = 1$ A^p_α Bergman class the answer is well known in the unit ball and in bounded pseudoconvex domains (see [5–7, 11]).

For $m > 1$ the answer is known only partially each (f_j) can be decomposed into $A^p_{\alpha_j}$ atoms for some α_j see [5,6]. For $m = 1$ Hardy space and other spaces (see [2–4,7]) and references there.

We extend these known results in various directions below. It is easy to note that in our proof at least one f_j must be decomposed into A^p_α atoms.

Let us remark the following typical for this paper fact in bounded pseudoconvex D domains an extension of a classical result namely the following result is valid (note same result with the same proof even can be provided with the same proofs in unbounded tube domains over symmetric cones). This will be studied in our next papers. Let H^p and A^p_α, $0 < p < \infty$, $\alpha > -1$ be Bergman and Hardy space in D domain (see [6, 8, 12, 13]) and definitions below.

Note since proofs are rather simple some arguments have sketchy forms and can be easily recovered by readers (see [6, 13]).

We denote by C_β Bergman representation constant below.

Theorem 1. *Let $f_i \in A^{p_i}_{\alpha_i}$, $i = 1, \ldots, k$; $f_i \in H^{p_i}$; $i = k+1, \ldots, m$, $p_i \leqslant 1$, $i = 1, \ldots, m$, $\alpha_j > -1$, $j = 1, \ldots, k$, $\tau = n(m-k) + (n+1)(k-1) + \sum_{j=1}^{k} \alpha_j$, then*

$$\int_D \prod_{j=1}^{m} \Big| f_j \Big|^{p_j} \delta(z)^{n(m-k) + \left(\sum_{j=1}^{k} \alpha_j\right) + (n+1)(k-1)} dv(z) \leqslant C \prod_{j=k+1}^{m} \Big|\Big| f_i \Big|\Big|^{p_i}_{H^{p_i}} \prod_{j=1}^{k} \Big|\Big| f_i \Big|\Big|^{p_i}_{A^{p_i}_{\alpha_i}}; \qquad (\widetilde{A})$$

and for cases when $p_i = p$, $j = 1, \ldots, m$ the reverse is also true and we have a new sharp result

$$I(\vec{f}) = \int_D \prod_{j=1}^{m} \left|f_j(z)\right|^p \delta(z)^{n(m-k)+\left(\sum\limits_{j=1}^{k}\alpha_j\right)+(n+1)(k-1)} dv(z) \asymp \prod_{j=k+1}^{m} \left|\left|f_i\right|\right|_{H^p}^p \prod_{j=1}^{k} \left|\left|f_j\right|\right|_{A^p_{\alpha_j}}^p; \qquad (\widetilde{\widetilde{A}})$$

if

$$\prod_{j=1}^{m} f_i(w_i) = C_\beta \int_D \Big(\prod_{j=1}^{m} f_j(z)\Big) \times \prod_{j=1}^{m} K_{\frac{\beta+n+1}{m}}(z, w_j)\delta^\beta(w)dv(w);$$

$$\beta > \beta_0, w_j \in D_j, j = 1, \ldots, k,$$
$$w_j \in D, j = k+1, \ldots, m. \qquad (T)$$

This Theorem 1 is probably true also for $p_i > 1$ (see [5,6] for proof in this case based only on Holder inequality) we give also very similar same type result for analytic $A^{\infty,p}_\alpha$ weighted Hardy class below.

Remark 1.

1) Note for $m = 1$ (T) integral condition vanishes (see [12]) and we have an obvious relation. and hence f_1 can be decomposed into atoms, $f_1 \in A^p_\alpha$ (see [6,7]).

2) Our result as a root has the following simple estimate in the unit disk which can be easily checked.

$$\int_U \prod_{i=1}^{k} \left|f_i(z)\right|^{p_i} \delta(z)^{k-1} dv(z) \leqslant C \prod_{i=1}^{k} \left|\left|f_i\right|\right|_{H^{p_i}}^{p_i}, \ \ p_i \in (0, \infty),\ f_i \in H^{p_i},\ i = 1, \ldots, k,\ k \in \mathbb{N}.$$

Remark 2.

Note for $m > 1$ we can hence using $(\widetilde{\widetilde{A}})$ decompose if $I(\vec{f}) < \infty$ each function (f_j) to H^p atoms and (or) A^p_α atoms using well-known one functional results. Note for $m = 1$ (T) vanish and we obtain A^p_α atomic decomposition classical result.

We refer to [14] for other new interesting sharp results in mulifunctional Bergman spaces.

1. Main results

We provide our main results in this section. Throughout this paper $H(D)$ denotes the space of all holomorphic functions on an open set $D \subset \mathbb{C}^n$.

We follow notation from [11]. Let D be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, let $d(z) = dist(z, \partial D)$.

Then there is a neighborhood U of $\bar{D}$ and $\rho \in C^\infty(U)$ such that $D = \{z \in U : \rho(z) > 0\}$, $|\bigtriangledown \rho(z)| \geqslant c > 0$ for $z \in \partial D$, $0 < \rho(z) < 1$ for $z \in D$ and $-\rho$ is strictly plurisubharmonic in a neighborhood U_0 of ∂D. Note that $d(z) \asymp \rho(z)$, $z \in D$. Then there is an $r_0 > 0$ such that the domains $D_r = \{z \in D : \rho(z) > r\}$ are also smoothly bounded strictly pseudoconvex domains for all $0 \geqslant r \geqslant r_0$. Let $d\sigma_r$ be the normalized surface measure on ∂D_r and dv the Lebesgue measure on D. The following mixed norm spaces were investigated in [11]. For $0 < p < \infty$, $0 < q \leqslant \infty$, $\delta > 0$ and $k = 0, 1, 2, \ldots$ set

$$||f||_{p,q,\delta;k} = \left(\sum_{|\alpha|\leqslant k} \int_0^{r_0} \left(r^\delta \int_{\partial D_r} |D^\alpha f|^p d\sigma_r\right)^{q/p} \frac{dr}{r}\right)^{1/q}, \ 0 < q < \infty$$

and weighted Hardy space ($A^{p,\infty}_0 = H^p$)

$$||f||_{p,\infty,\delta;k} = \sup_{0<r<r_0} \sum_{|\alpha|\leqslant k} \left(r^\delta \int_{\partial D_r} |D^\alpha f|^p d\sigma_r\right)^{1/q}, \ 0 < q < \infty,$$

where D^{α} is a derivative of f (see [11]) The corresponding spaces $A^{p,q}_{\delta;k} = A^{p,q}_{\delta;k}(D) = \{f \in H(D) : ||f||_{p,q,\delta;k<\infty}\}$ are complete quasi normed spaces, for $p, q \geqslant 1$ they are Banach spaces. We mostly deal with the case $k = 0$, when we write simply $A^{p,q}_{\delta}$ and $||f||_{p,q,\delta}$. We also consider this spaces for $p = \infty$ and $k = 0$, the corresponding space is denoted by $A^{\infty,p}_{\delta} = A^{\infty,p}_{\delta}(D)$ and consists of all $f \in H(D)$ such that

$$||f||_{\infty,p,\delta} = \left(\int_0^{r_0} (\sup_{\partial D_r} |f|)^p r^{\delta p-1} dr\right)^{1/p} < \infty.$$

Also, for $\delta > -1$, the space $A^{\infty}_{\delta} = A^{\infty}_{\delta}(D)$ consist of all $f \in H(D)$ such that

$$||f||_{A^{\infty}_{\delta}} = \sup_{z\in D} |f(z)|\rho(z)^{\delta} < \infty,$$

and the weighted Bergman space $A^p_{\delta} = A^p_{\delta}(D) = A^{p,p}_{\delta+1}(D)$ consists of all $f \in H(D)$ such that

$$||f||_{A^p_{\delta}} = \left(\int_D |f(z)|^p \rho^{\delta}(z) dv(z)\right)^{1/p} < \infty.$$

We denote by K_{β} the weighted Bergman kernel on D (see [6, 12]).

Since $|f(z)|^p$ is subharmonic (even plurisubharmonic) for a holomorphic f, we have $A^p_s(D) \subset A^{\infty}_t(D)$ for $0 < p < \infty$, $sp > n$ and $t = s$. Also, $A^p_s(D) \subset A^1_s(D)$ for $0 < p \leqslant 1$ and $A^p_s(D) \subset A^1_t(D)$ for $p > 1$ and t sufficiently large. Therefore we have an integral representation

$$f(z) = C_{\beta} \int_D f(\xi) K(z,\xi) \rho^t(\xi) dv(\xi), \quad f \in A^1_t(D),\ z \in D, \qquad (*)$$

where $K(z,\xi)$ is a kernel of type t, that is a smooth function on $D \times D$ such that $|K(z,\xi)| \leqslant C|\tilde{\Phi}(z,\xi)|^{-(n+1+t)}$, where $\tilde{\Phi}(z,\xi)$ is so called Henkin-Ramirez function for D. Note that (*) holds for functions in any space X that embeds into A^1_t. We review some facts on $\tilde{\Phi}$ and refer reader to [15] for details. This function is C^{∞} in $U \times U$, where U is a neighborhood of $\overline{D}$, it is holomorphic in z, and $\tilde{\Phi}(\zeta,\zeta) = \rho(\zeta)$ for $\zeta \in U$. Moreover, on $\overline{D} \times \overline{D}$ it vanishes only on the diagonal (ζ,ζ), $\zeta \in \partial D$. Locally, it is up to a non vanishing smooth multiplicative factor equal to the Levi polynomial of ρ. From now on the work with a fixed Henkin-Ramirez function $\tilde{\Phi}$.

The proof of the following theorem is very similar to the proof of the Theorem 1.

Theorem 2. *Let $f_i \in A^{\infty}_{\beta_i}$, $i = 1, \dots, k$ and $f_i \in A^{p_i}_{\alpha_i}$, $i = k+1, \dots, m$.*
Let $\beta_j \geqslant 0$, $j = 1, \dots, m$, let also $p_i \leqslant 1$, let $\alpha_j > -1$; $j = 1, \dots, m$; then we have

$$\int_D \prod_{j=1}^m \left|f_j\right|^{p_i} \cdot \delta(z)^{\sum_{j=1}^{k}\left(\beta_j p_j\right) + (n+1)(m-k-1) + \sum_{i=1}^{m}\alpha_i} dv(z) \leqslant C \prod_{i=k+1}^{m} \left|\left|f_i\right|\right|^{p_i}_{A^{p_i}_{\alpha_i}} \times \prod_{i=1}^{k} \left|\left|f_i\right|\right|^{p_i}_{A^{\infty}_{\beta_i}}; \qquad (K)$$

and if $p_i = p$, $i = 1, \dots, m$ we have a sharp result (the reverse of (K) is valid) if

$$\prod_{i=1}^m f_i(w_i) = C_{\beta} \int_D \prod_{j=1}^m f_j(z) \times K_{\frac{\beta+n+1}{m}}(z, w_j) \delta^{\beta}(z) dv(z);\ \beta > \beta_0;\ w_j \in D,\ j = 1, \dots, m.$$

The same type results with very similar proof is valid not only for A^{∞}_{β} but also for weighted Hardy space

$$A^{p,\infty}_{\alpha} = \left\{ f \in H(D) : \sup_{\varepsilon>0} \left(\int_{\partial D_{\varepsilon}} \left|f(\xi)\right|^p \tilde{\sigma}(\xi)\right)^{\frac{1}{p}} \times \varepsilon^{\alpha} < \infty;\ \alpha \geqslant 0;\ 0 < p < \infty \right\}$$

where $\partial D_{\varepsilon} = \{z : \rho(z) = \varepsilon\}$, $\tilde{\sigma}(\xi)$ is a Lebeques measure on ∂D_{ε} (see [11] for these analytic Hardy type spaces).

Theorem 2 can be also viewed similarly (as Theorem 1) as another direct extension of a known theorem on atomic decomposition of classical Bergman space A^p_α in the D domain. Indeed we can easily see that (see [7, 12]) for $m = 1$ in the ball (T) integral condition vanishes and we have A^p_α known atomic decomposition result. For $m > 1$ taking into account known atomic decomposition theorems (see [9]) for A^p_α and A^∞_β in D, each $f_j, j = 1, \dots, m$ from Theorem 2 can be decomposed into A^∞_β or A^p_α atoms.

The same type result is valid for some Herz type spaces in bounded pseudoconvex domains and BMOA type spaces in the unit ball instead of A^∞_β .

We refer to [16] for some interesting results in such type analytic function spaces.

Namely, let $B(z, r)$ be a Kobayashi ball in $D, z \in D, r > 0$ (see [11]).

Let also B^{pq}_α, $\widetilde{B}^{pq}_\alpha$, $p, q \in (0, \infty)$, $\alpha > -1$, be Herz type spaces in pseudoconvex D domain

$$B^{p,q}_\alpha(D) = \left\{ f \in H(D) : \int_D \left(\int_{B(z,r)} \Big|f(w)\Big|^q \delta^\alpha(w) dv(w) \right)^{\frac{p}{q}} dv(z) < \infty \right\},$$

$$\widetilde{B}^{p,q}_\alpha(D) = \left\{ f \in H(D) : \sum_{k \geqslant 0} \left(\int_{B(a_k,r)} \Big|f(w)\Big|^q \times \delta^\alpha(w) dv(w) \right)^{\frac{p}{q}} < \infty \right\},$$

where $\{a_k\}$ is known r-lattice in D (see [13]).

Let also

$$BMOA^p_{s,\beta,t}(B_n) = \left\{ f \in H(B_n) : \sup_{w \in B} \int_B \frac{|f(z)|^p \times (1 - |z|)^s dv(z)}{|1 - \bar{w}z|^\beta} (1 - |w|)^t < \infty \right\},$$

be BMOA type space in the unit ball (see also [7, 17, 18]), where $0 < p, q < \infty$; $s > -1$, $\beta, t \geqslant 0$.

Uniform estimates for BMOA in the unit ball can be seen in [7], for $B^{p,q}_\alpha$ and $\widetilde{B}^{p,q}_\alpha$ Herz type spaces they can be easily obtained also based on elementary known estimates (see [12, 13])

$$\Big|f(z)\Big|^p \leqslant \widetilde{C}\left(\int_{B(z,r)} \Big|f(w)\Big|^p dv(w) \right) \cdot \delta^{-(n+1)}(z); \; z \in D, \, 0 < p < \infty.$$

As a result we immediately have that

$$\int_D \prod_{k=1}^m \Big|f_k\Big|^{p_i} \times \delta(z)^s dv(z) \leqslant C \prod_{k=1}^t \Big|\Big|f_k\Big|\Big|^{p_k}_{A^{p_k}_{\beta_k}} \times \prod_{k=t+1}^m \Big|\Big|f_k\Big|\Big|^{\widetilde{l_k}}_{B^{\widetilde{p_k}, \widetilde{q_k}}_{\alpha_k}}, \qquad (A_3)$$

for some $s = s\Big(\vec{p}, n, m, \vec{\alpha}, \vec{\beta}, \vec{\tilde{p}}, \vec{\tilde{q}}\Big)$ and the same type estimate obviously is valid for $\widetilde{B}^{p,q}_\alpha(D)$ and $BMOA^p_{s,\beta,t}(B_n)$ (We simply replace $\prod\limits_{k=t+1}^m \Big|\Big|f_k\Big|\Big|_{B^{\widetilde{p}_k, \widetilde{q}_k}}$ by quazinorms of these spaces).

For particular values of parameters we under integral condition (T) can again show similarly that this (A_3) estimate is sharp, so each f_k can be decomposed into BMOA and $\widetilde{B}^{p,q}_\alpha\Big(B^{p,q}_\alpha\Big)$ atoms if only $\prod\limits_{i=1}^m |f_i|^p \in L^1_s(D)$, $0 < p \leqslant 1$ for some s.

These results in details will be given in another our paper.

Proof of Theorem 1.

The $(\tilde{\tilde{A}})$ estimate follows from two known uniform estimates directly

$$\Big|f(z)\Big|\Big(1 - |z|\Big)^{\frac{n}{p_i}} \leqslant C\Big|\Big|f\Big|\Big|^{p_i}_{H^{p_i}}, \; z \in D$$

and

$$\Big|f(z)\Big|\Big(1 - |z|\Big)^{\frac{n+1+\alpha_i}{p_i}} \leqslant C_1\Big|\Big|f\Big|\Big|^{p_i}_{A^{p_i}_{\alpha_i}}, \; z \in D, \; \alpha_i > -1, \; 0 < p < \infty$$

(see [6, 7, 11, 13, 19]). The (A_3) can be shown similarly. To get the reverse estimate we must use first that for $p \leqslant 1$ (see [6]) we have

$$\left(\int_D \Big|f(w)\Big| \cdot \prod_{j=1}^{m} \Big|K_\tau(z_j, w)\Big| \delta^\alpha(w) dv(w)\right)^p \leqslant C \int_D \Big|f(w)\Big|^p \cdot \Big|K_\tau(z, w)\Big|^p \delta^{\alpha p + (n+1)(p-1)}(w) dv(w),$$

$\tau > 0$, $\alpha > -1$, $p \leqslant 1$ and also the following known lemma (see [6, 12, 13, 19, 20]). (Forelly-Rudin type estimates).

Lemma 1. *Let* $\tilde{\alpha}$, $\beta > -1$, $s > 0$, $y \in D$, $0 < t < t_0 = t_0(\lambda, r)$ *then*

$$\int\limits_{\{x: r(x) = t\}} \Big|K_\alpha(x, y)\Big|^s d\sigma(x) \asymp \Big[r(y) + t\Big]^{n-q}, \ n < q,$$

and

$$\sup_{w \in D} \Big|K_\alpha(z, w)\Big| \delta^v(z) \leqslant C \delta^{-\tilde{\alpha} + v}(z); \qquad (S)$$

$v \geqslant 0$, $v - \tilde{\alpha} < 0$ *and*

$$\int_D \Big|K_\alpha(x, y)\Big|^s \Big(r(x)\Big)^\beta dv(x) \asymp \Big(r(y)\Big)^{n-q+\beta+1}, \ \ n - q + \beta + 1 < 0,$$

and $r(y) \asymp \delta(y)$, $y \in D$; $q = \alpha s$.

Indeed using (T) and mentioned estimates we have the following chain of inequalities

$$\prod_{j=1}^{k} \left(\int_D \Big|f_j(w_j)\Big|^p \delta^{\alpha_j}(w_j) dv(w_j)\right) \cdot \prod_{j=k+1}^{m} \sup_{\varepsilon > 0} \int_{\partial D_\varepsilon} \Big|f_j(\xi)\Big|^p d\sigma(\xi) =$$

$$= C \prod_{j=1}^{k} \Big|\Big|f_i\Big|\Big|^{p_i}_{A^{p_i}_{\alpha_i}} \cdot \prod_{j=k+1}^{m} \Big|\Big|f_j\Big|\Big|^{p_i}_{H^{p_i}} \leqslant$$

$$\leqslant C \int\limits_X \int\limits_D \prod_{j=1}^{m} \Big|f_s(z)\Big|^p \cdot \Big|K_s(z, w_j)\Big|^p \cdot \delta^{\beta p + (n+1)(p-1)}(z) dv(z) d\tilde{v}(x),$$

where

$$s = \frac{\beta + n + 1}{m};$$

$$\int_X d\tilde{v}(x) = \Big(\prod_{j=1}^{k} \int_D \delta^{\alpha_j}(w_j) dv(w_j)\Big) \cdot \Big(\prod_{j=k+1}^{m} \sup_{\varepsilon > 0} \int_{\partial D_\varepsilon} d\sigma(\xi)\Big).$$

Applying Lemma 1 we have after small calculations that

$$I \leqslant \tilde{C} \int_D \prod_{j=1}^{m} \Big(\Big|f_j(w)\Big|^p \delta^\tau(w) dv(w)\Big).$$

Theorem 1 is proved.

The proof of Theorem 2 is almost the same. We omit easy details.

Put

$$BMOA^p_{t,v,s} = \left\{f \in H(B) : \sup_{z \in B} \int_B \frac{\Big|f(w)\Big|^p (1 - |w|)^t}{|1 - \bar{z}w|^v} dv(w) \times (1 - |z|)^s < \infty\right\},$$

$v, s \geqslant 0$, $p > 0$, $t > -1$.

For $p \geqslant 1$ this is a Banach space and complete metric space for $p < 1$. Obviously based on properties of r-lattices (the same result with same proof is valid in pseudoconvex domains) based on vital estimate from below of Bergman Kernel on Bergman ball (see [6,13]),

$$\Big|\Big|f\Big|\Big|_{BMOA^p_{t,vs}} \geqslant C \sup_{z\in B}\Big|f(z)\Big|(1-|z|)^{s+t-v+n+1},$$

$v, s \geqslant 0$, $p > 0$, $t > -1$.

This uniform estimate leads immediately to next theorem.

Theorem 3. *Let $f_i \in A^p_{\alpha_i}, i = 1, \ldots, m$ and $f_j \in BMOA^p_{t_j,s_j,v_j}$, $j = m+1, \ldots, m+k$.*

Let $0 < p \leqslant 1$, $s_j \geqslant 0$ and also $t_j > -1$, $v_j \geqslant 0$, $j = m+1, \ldots, m+k$, $\alpha_k > -1$, $k = 1, \ldots, m$; let $v_j - s_j - t_j < n+1$,

$$\frac{\beta+n+1}{m+1}p < t_j + n + 1 < \frac{\beta+n+1}{m+1}p + v_j - s_j,$$

$j = m+1, \ldots, m+k$, $\beta > \beta_0$, $n \in \mathbf{N}$, $m > 1$, $m \in \mathbf{N}$.

Then for $\delta(z) = 1 - |z|$, $z \in B$, we have

$$\int_B \prod_{j=1}^{m+k}\Big|f_j(z)\Big|^p \delta(z)^\tau dv(z) \asymp \prod_{k=1}^{m}\Big|\Big|f_k\Big|\Big|^p_{A^p_{\alpha_k}} \times \prod_{j=m+1}^{m+k}\Big|\Big|f_j\Big|\Big|^p_{BMOA^p_{t_j,s_j,v_j}}$$

if

$$\prod_{j=1}^{m+k} f_j(z_j) = C_\beta \int_B \prod_{j=1}^{m+k} f_j(w)\frac{1}{(1-z\overline{w})^{\frac{\beta+n+1}{m+k}}}\delta^\beta(w)dv(w),$$

$\beta > \beta_0$, $z_j \in B$, $j = 1, \ldots, m+k$; β_0 is large enough

$$\tau = (m-1)(n+1) + \sum_{k=1}^{m}\alpha_k + \sum_{j=m+1}^{m+k}(t_j + s_j - v_j) + (n+1)k.$$

Remark 3. A third group with $\prod_{j=1}^{k}\Big|\Big|f_j\Big|\Big|^p_{H^p}$ can also be added in mentioned relation of Theorem 3 with similar proof. One part of theorem (estimate from above) can be even given with group of more general $\prod_{i=1}^{m}\Big|\Big|f_i\Big|\Big|^{p_i}_{A^{p_i}_{\alpha_i}}$ form with almost same proof.

Proof of Theorem 3.

Proof of Theorem 3 we have as in previous theorems. The proof is based on uniform estimate for BMOA we provided above, arguments of proof of previous theorem and the following Lemma.

Lemma A (See [21]). *Let $s > -1$, $r, t \geqslant 0$, $r + t - s < n + 1$ then*

$$\int_B \frac{(1-|z|)^s dv(z)}{|1-\bar{z}w|^r|1-\bar{z}w_1|^t} \leqslant \frac{\tilde{C}}{|1-w\bar{w}_1|^{r+t-s-n-1}}, \quad w, w_1 \in B; \quad r-s, t-s < n+1,$$

for some constant $\tilde{C} > 0$.

We omit easy details leaving some calculations with indexes to interested readers. Even more for other restriction to parameters this theorem is valid in bounded pseudoconvex domain with smooth boundary D. We'll discuss in other our papers this in more detail.

We will formulate that interesting result also below. The proof (in BMOA spaces in bounded pseudoconvex domains) is the same as in theorem above, but is based on new Lemma (see [17]).

We will below provide proofs of unit ball case in D the proof is practically the same.

Lemma B. *Let* $t-s<n+1<r-s,\ s>-1,\ r,t\geqslant 0,\ r+t-s>n+1$

$$\int_D \Big(\delta^s(z)|K_r(z,w)|\cdot|K_t(z,w_1)|\Big)dv(z)\leqslant \frac{C}{(\delta(w))^{r-s-n-1}}|K_t(w,w_1)|,$$

where $w,w_1\in D$.

The rest is the simple repetition of arguments of previous theorems.

We first consider model case of the unit ball. The proof of general case is the same. We have (when one BMOA is in chain) the general case with several functions from BMOA is the same.

$$\Big|\prod_{j=1}^{m+1} f_j(w_j)\Big|^p = C_\beta\int_B \frac{\Big|\prod\limits_{j=1}^{m+1} f_j(z)\Big|^p\times(1-|z|)^{\beta p+(n+1)(p-1)}dv(z)}{\Big|\prod\limits_{j=1}^{m+1} 1-\bar{z}w_j\Big|^{\frac{\beta+n+1}{m+1}}}.$$

Then using Lemma A and well-known Forell-Rudin type estimate (see [7])

$$J\leqslant C_1\int\limits_X\int\limits_B \frac{\prod\limits_{j=1}^{m+1}\Big|f_j(z)\Big|^p\times(1-|z|)^{\beta p+(n+1)(p-1)}dv(z)}{\prod\limits_{j=1}^{m+1}\Big|1-\bar{z}w_j\Big|^{\frac{\beta+n+1}{m+1}}},$$

where

$$\int_X=\sup_{\tilde{w}}\int_B\cdots\int_B\prod_{j=1}^{m}(1-|w_j|)^{\alpha_j}dv(w_j)\times\frac{(1-|w_{m+1}|)^t\cdot(1-(\tilde{w}))^s}{|1-w_{m+1}\tilde{w}|^v}dv(w_{m+1}).$$

Hence

$$J\leqslant C\int_B\prod_{j=1}^{m+1}\Big|f_j(z)\Big|^p(1-|z|)^\tau dv(z);\qquad \tau=(1+n)(m-1)+\sum_{k=1}^{m}\alpha_k+s+t-v+n+1;$$

as easy calculation with indexes shows.

Indeed, we have of our Theorem 3 that $k=\left(\dfrac{\beta+n+1}{m+1}\right)p$;

$$\sup_{\tilde{w}\in B}\int_B\frac{(1-|w|)^t dv(w)(1-|\tilde{w}|)^s}{|1-w\tilde{w}|^v|1-zw|^k}\leqslant\sup_{w\tilde{\in}B}\int_B\frac{(1-|w|)^t dv(w)}{|1-w\tilde{w}|^{v-s}\cdot|1-z\bar{w}|^k}\leqslant$$

$$\leqslant\sup_{w\in B}\frac{1}{|1-wz|^{v-s+k-t-(n+1)}}\leqslant\frac{\tilde{\tilde{C}}}{(1-|z|)^r};\qquad r=v-s+k-t-(n+1),$$

if $t>-1$, $v-s-t<n+1$, $k-t<n+1$, $v-s+k-t-(n+1)>0$.

This finished the proof of our theorem for the case of the unit ball.

Now we consider the case of pseudoconvex domains, the proof is a repetition of unit ball case so we again fix our attention to the unit ball case in C^n.

We have the following chain of estimates now based on Lemma from [6] (see also above). The only change for general D pseudoconvex domain is to replace $(1-|z|)^\alpha$ be $\delta(z)$ and $\dfrac{1}{|1-zw|^{\alpha+n+1}}$ by $K_\alpha(z,w)$.

$$J=\int\limits_X\prod_{j=1}^{m}\Big|f_j\Big|^p\leqslant C\int\limits_X\int\limits_B\frac{\prod\limits_{j=1}^{m}\Big|f_j\Big|^p\times(\delta(z))^{\beta p+(n+1)(p-1)}}{\prod\limits_{j=1}^{m}\Big|1-\bar{z}w_j\Big|^{\frac{\beta+n+1}{m}p}}dv(z),$$

where

$$\int_X = \sup_{w\tilde{\in}B}\int_B\cdots\int_B \frac{\prod_{j=1}^{m-1}(1-|w_j|)^{\alpha_j}\times(1-|w_m|)^t\cdot(1-|\tilde{w}|)^s dv(w_j)}{|1-\bar{w}_m\tilde{w}|^v}.$$

Taking into account Lemma B we obtain

$$J\leqslant C\int_B\prod_{j=1}^m\Big|f_j(z)\Big|(1-|z|)^\tau dv(z),$$

where τ was defined in our previous theorem.

Indeed, for $w_1,\dots,w_{m-1}$ variables we must use Forelly-Rudin estimates

$$\int_B\frac{(1-|w_j|)^{\alpha_j}dv(w_j)}{\Big|1-zw_j\Big|^{\frac{\beta+n+1}{m}}}\leqslant C\cdot(1-|z|)^{\alpha_j-\frac{\beta+n+1}{m}+n+1},\quad z\in B.$$

These estimates are valid also in bounded pseudoconvex domains (see [6,13]).

Then by Lemma B we have

$$M=\sup_{\tilde{w}\in B}\left(\int_B\frac{(1-|w_m|)^t}{|1-\bar{w}_m\tilde{w}|^v|1-zw_m|^{\frac{\beta+n+1}{m}p}}dv(w_m)\right)(1-|\tilde{w}|)^s\leqslant\sup_{\tilde{w}\in B}\frac{C}{|1-z\tilde{w}|^{v-s}\cdot|1-|z||^u}$$

for $v-s<n+1$, $\dfrac{\beta+n+1}{m}p-t>n+1$, $-s+v+\dfrac{\beta+n+1}{m}p-t-(n+1)>0$, where $u=\dfrac{\beta+n+1}{m}p-t-(n+1)$;

$$M\leqslant\tilde{C}\,(1-|z|)^{-v+s-u}.$$

Our last general Theorem is the following.

Let

$$\left(BMOA^p_{\tau,v,s}\right)=\left\{f\in H(D):\sup_{w\in D}\int_D\Big|f(z)\Big|^p\delta(z)^t\cdot\Big|K_v(z,w)\Big|dv(z)\cdot(\delta^s(w))<\infty\right\},$$

$0<p<\infty$, $v>0$, $t>-1$, $s\geqslant 0$ is a BMOA-type space in a bounded pseudoconvex domain with smooth boundary in C^n. BMOA type spaces in such domains were studied in [17].

Theorem 4. *Let $p\leqslant 1$, let $v_j-s_j<n+1$, $\dfrac{\beta+n+1}{m}p-t_j>n+1$, $-s_j+v_j+\dfrac{\beta+n+1}{m}p-$ $-t_j-(n+1)>0$, $j=m+1,\dots,k$, $\beta>\beta_0$, $n\in\mathbf{N}$, $m>1$, $m\in\mathbf{N}$ and $\alpha_j>-1$, $j=1,\dots,m$ then if $s_j>0$, $t_j>-1$, $v_j>0$, $j=m+1,\dots,m+k$ then the assertion of previous theorem is valid if we replace $\dfrac{1}{(1-zw)^\tau}$ by $K_\tau(z,w)$ for $\tau>0$ in pseudoconvex domains for $BMOA^p_{\tau,v,s}(D)$ spaces and for Bergman A^p_α spaces in same type domains.*

Hence each (f_j) can be represented as A^p_α or BMOA atoms (see [7]) if $\prod_{i=1}^{m+k}|f_i|^p\in L^1_u$, for some parameter u and $m,k\in\mathbb{N}$. These results again coincide for $m=1$ with known results on atomic decomposition of A^p_α Bergman class theorems (see [6]).

Remark 4. Let us stress in all these assertion is vital in main estimate to keep at least one component $\Big|\Big|f_i\Big|\Big|_{A^{p_i}_{\alpha_i}}$ (Bergman space component) in right side of the main estimates.

Concerning groups without $\prod_{i=1}^m||f_i||_{A^{p_i}_{\alpha_i}}$ like $\prod_{i=1}^k\Big|\Big|f_i\Big|\Big|^p_{H^p}\times\prod_{i=k+1}^m\Big|\Big|f_i\Big|\Big|_{BMOA}$ our methods don't work other approached here must be invented, based maybe on other integral representations.

The $p > 1$ case can be probably covered similarly we refer to [6] for "pure" A^p_α case with $\prod_{i=1}^{m} ||f_i||_{A^p_{\alpha_i}}$ groups ($p > 1$) based purely on Holders inequality.

Our methods also covers cases when at least one component is our product is Herz-type spaces. This will be treated in our other papers, so we can similarly also consider the following products

$\prod_{i=1}^{m} \left|\left|f_i\right|\right|_{B^{p,q}_{\alpha_j}} \times \prod_{i=m+1}^{N} \left|\left|f_i\right|\right|^p_{H^p}$ or $\prod_{i=1}^{m} \left|\left|f_i\right|\right|^p_{\tilde{B}^{p,q}_{\alpha}} \times \prod_{i=m+1}^{N} \left|\left|f_i\right|\right|^p_{BMOA}$

with some restrictions on indexes

or

$\prod_{i=1}^{m} \left|\left|f_i\right|\right|_{B^{p,q}_{\alpha_j}} \times \prod_{i=m+1}^{N} \left|\left|f_i\right|\right|_{\tilde{A}^p_{\beta_j}}$ or $\prod_{j=1}^{m} \left|\left|f_j\right|\right|^p_{\tilde{B}^{p,q}_{\alpha_j}} \times \prod_{j=m+1}^{N} \left|\left|f_j\right|\right|_{\tilde{A}^p_{\beta_j}}$

with some restrictions on indexes.

These cases will be considered in our other papers, though methods of this and those papers will be rather similar.

Note all results of this paper have direct analogues also in analytic spaces in unbounded tubular domains over symmetric cones. Proofs of such type results can be obtained by simple substitution of our estimates we used in our proofs in pseudoconvex domains to parallel known estimates in tube domains and on some parallel known related facts on Bergman represenation formula in tubular domains (see for example [22] and references there).

The only additional condition is on Bergman Kernel in tube domains over symmetric cones is Lemma B, which is probably also valid.in tube domains also.

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О новых теоремах разложения в некоторых пространствах аналитических функций в ограниченных псевдовыпуклых областях

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Аннотация. Мы даем новые точные теоремы разложения для многофункциональных пространств Бергмана в единичном шаре и ограниченных псевдовыпуклых областей с гладкой границей, расширяющей известные результаты из единичного шара.

А именно мы докажем, что $\prod_{j=1}^{m} ||f_j||_{X_j} \asymp ||f_1 \dots f_m||_{A^p_\alpha}$ для различных (X_j) пространства аналитических функций в ограниченных псевдовыпуклых областях с гладкой границей, где f, f_j, $j = 1, \dots, m$ — аналитические функции, а A^p_α, $0 < p < \infty$, $\alpha > -1$ — пространство Бергмана. Это, в частности, также расширяет в разных направлениях известную теорему об атомном разложении пространств A^p_α Бергмана.

Ключевые слова: псевдовыпуклые области, единичный шар, пространства Бергмана, классы типа Харди, теоремы декомпозиции.