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УДК 532.13

Rheological Properties of PVDF Solutions

Bair B. Damdinov*
Siberian Federal University
Krasnoyarsk, Russian Federation
Buryat State University named after D. Banzarov
Ulan-Ude, Russian Federation

Victoria A. Danilova†
Buryat State University named after D. Banzarov,
Ulan-Ude, Russian Federation

Andrey V. Minakov‡
Maxim I. Pryazhnikov§
Siberian Federal University
Krasnoyarsk, Russian Federation
Kutateladze Institute of Thermophysics SB RAS
Novosibirsk, Russian Federation

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Abstract. The rheological properties of polyvinylidene fluoride (PVDF) solutions in N-methylpyrrolidone were studied using the rheometric method. It was shown that the viscosity of polymer solutions decreases non-linearly with increasing temperature. The viscosity of the N-methylpyrrolidone used as solvent remains practically unchanged. It was shown that solutions exhibit Newtonian behaviour at concentrations less than 7 wt.%. At higher concentrations, solutions exhibit properties of pseudoplastic fluid.

Keywords: liquids, structure, viscosity, rheological properties, Newtonian and non-Newtonian behaviour, polyvinylidene fluoride (PVDF), N-methylpyrrolidone, materials performance, temperature dependences.

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Introduction

The paper deals with the study of fundamental problem of fluid structure engineering. The basic and essential feature of the present paper is the research object — a suspension of functional piezo polymer and piezo polymer-based nano-composite suspensions. The ferroelectric, piezoelectric polymer PVDF is considered. Its spontaneous electric polarization varies between 3 and 7 $\mu c/cm^2$. One of the simple and effective methods of modifying the properties of functional materials is mechanical processing (compression/stretching) which leads to the accumulation of mechanical stresses in the material, crystal lattice distortions and deformations of chemical bonds. This approach is called the strain engineering [1,2]. Many studies have demonstrated the

*Bdamdinov@sfu-kras.ru https://orcid.org/0000-0003-0914-0096
†newironwoman@gmail.com
‡AMinakov@sfu-kras.ru
§Arrivent@yandex.ru https://orcid.org/0000-0001-9143-7950

effectiveness of this approach in changing optical, magnetic, chemical (catalytic) and electrical properties. Low-symmetry ferroelectrics occupy an exceptional place among various available materials. By analogy with magnetic materials, they have two important properties: orientation of a spontaneous polarization vector (the total dipole moment of the system) can be changed by the direction of the external electric field and the ferroelectric-paraelectric phase transition at the Curie temperature. Unlike quantum dots and superparamagnetic particles, ferroelectrics show significant degradation of their functional properties which is explained by a violation of the balance of short-range and long-range forces of the system in the framework of the Landau-Devonshire-Ginzburg theory of ferroelectric effect [3].

It was experimentally shown in 2004 that the spontaneous polarization of epitaxially grown barium titanate film with a certain difference in the parameters of an elementary crystal cell can be 250 % above the spontaneous polarization of a bulk crystal [4]. Theoretical studies showed that this increase can be induced by mechanical deformation, in particular, by biaxial compression of the barium titanate film [5]. The strain engineering concept which consists in selecting systems with certain difference in the lattice parameters can significantly enhance the functional characteristics of low-dimensional ferroelectrics. However, this is only possible for thin films and two-dimensional systems. In the case of nanoparticles the situation is essentially different. Methods of chemical synthesis of monodisperse nanoparticles do not yet allow producing particles in the "stressed, deformed" state due to the passive role of ligands covering the surface [6]. Therefore, it is of interest to stabilize such polymer particles in viscous media and thus to create ensembles of particles periodically located in space and stabilized by interaction with the surrounding polymer molecules. This in turn can significantly affect the low-frequency rheological characteristics of nanosuspensions and be evidence of "solidification". Recently, there was great interest in the study of fluids that exhibit non-Newtonian behaviour [7–12]. Previously, it was shown that PVDF solutions mainly exhibit Newtonian behaviour [13–16]. In this work, a deviation from the Newtonian behaviour of PVDF solutions in N-methylpyrrolidone (NMP) was found at certain weight concentrations.

1. Experiment

1.1. Materials under study

Polyvinylidenefluoride(PVDF)

Polyvinylidene fluoride or polyvinylidene difluoride (PVDF) is a highly reactive thermoplastic fluoropolymer obtained by polymerization of vinylidene fluoride [17] (see Fig. 1). The chemical formula of polyvinylidene fluoride is $(C_2H_2F_2)_n$.

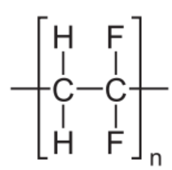

Fig. 1. The structural formula of the PVDF

The PVDF is a crystalline polymer of white or translucent colour with a molecular weight of over 100,000. The PVDF is a special plastic used in areas where the highest purity as well as resistance to solvents, acids, and hydrocarbons is required. Compared to other fluoropolymers, such as polytetrafluoroethylene (Teflon), PVDF has low density (1.78 g/cm^3). The polymer has high mechanical strength, wear and weather resistance as well as resistance to ionizing and ultraviolet radiation [18]. Besides, it also exhibits high chemical resistance and compatibility with thermoplastic materials. In industry, PVDF solutions are used to produce fluoroplastic membranes.

***N*-methylpyrrolidone**

N-methylpyrrolidone (NMP) is an organic compound comprising of five-membered lactams (see Fig. 2). It is a colourless liquid but impure samples may look yellow. It also belongs to the class of dipolar aprotic solvents, such as dimethylformamide and dimethyl sulfoxide. It is mixed with water, and it is the most common organic solvent used in the petrochemical and plastic industries due to its volatility and ability to dissolve various materials including polymers.

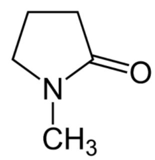

Fig. 2. The structural formula of the NMP

Its chemical formula is C_5H_9NO, molar weight is 99.133 $g{\cdot}mol^{-1}$, and density is 1.028 g/cm^3. The NMP is used for the extraction of certain hydrocarbons formed in the processing of petrochemical products, such as the reduction of 1,3-butadiene and acetylene. It is also used for the absorption of hydrogen sulfide from acid gases and hydrodesulfurization plants. N-methylpyrrolidone is used for dissolving a wide range of polymers.

Typically, polymers dissolved in NMP are used to treat the surface of electrodes or produce polymer electrolytes. The results of measurements of viscosity of PVDF solutions in NMP are presented in the paper. These solutions are pseudoplastic at high concentrations of the solution and the presence of PVDF. Viscosity was defined as a function of the shear rate. The PVDF and NMP materials and reagents of the biotechnological class were purchased from the Sigma-Aldrich Chemistry Products catalog. The average molecular weight of PVDF was 534,000.

1.2. Preparation of solutions

To prepare polymer solutions the following features were taken into account: the ability to form stable suspension at the stage of polymer dispersion in the NMP, solubility in the NMP and the viscosity of the resulting solution. To prepare a 10% PVDF solution 10 ml of N-methylpyrrolidone was poured into a vessel at ambient temperature. Then a polymer sample (1 g) was added, left for 24 hours, and after that it was dispersed for an hour. The resulting solution was homogeneous, transparent, and it did not contain undissolved particles. Solutions with concentrations of 0.1, 0.2, 0.4, 1, 2, 3, 5, and 7 wt.% were prepared by the same procedure.

1.3. Rheological measurements

The rheological properties of polymer solutions were studied with the use of Anton Paar MCR 52 rotary rheometer with a plane-plane unit (see Fig. 3). The solution was placed into a gap between two round plates 20 mm in diameter. The width of the gap remained constant and it is equal to 1 mm.

The temperature dependence of viscosity was obtained. Temperature was varied between 15 to 40$^\circ C$. The relationship between viscosity and shear rate was also established. The shear rate was varied between 1 to 200 s^{-1} at constant temperature of 20$^\circ C$.

Fig. 3. General view of the Anton Paar MCR 52 rheometer

2. Results and discussion

Fig. 4 shows viscosity of solutions versus shear rate.

It is apparent that the shape of the curves for concentrations less than 7 wt.% is typical for Newtonian fluid, i.e., the viscosity does not depend on the velocity gradient. For concentrations of 7 and 10 wt.% the viscosity decreases with increasing shear rate. This behaviour is typical for non-Newtonian (pseudoplastic) fluids. The rheology of PVDF solutions in dimethyl acetate was studied before [13]. It was shown that the viscosity of the solution practically did not change with an increase in the strain rate (Newtonian behaviour) up to concentrations of 15 wt.%. Thus, the present results are in qualitative agreement with the results obtained before [13].

The rheological behaviour of solutions with concentrations of 7 and 10 % is described by the Power Law model:

$$\mu = k\dot{\gamma}^{n-1},$$

where k is a consistency index (Pa$\cdot s^n$), $\dot{\gamma}$ is the shear rate (s^{-1}), and n is the flow behaviour index. Consistency indexes for the concentrations of 7 % and 10 % are k =215.1 mPa$\cdot s^n$ and k =1245 mPa$\cdot s^n$, respectively. Flow indexes for the concentrations of 7 % and 10 % are n =0.966 and n =0.954, respectively.

A similar behaviour of viscosity was observed for suspensions of nanoparticles [11] (see Fig. 5). Like polymer solutions, suspensions are Newtonian fluids at low concentrations of nanoparticles. When concentration of nanoparticles in the suspension increases the non-Newtonian properties emerge. However, unlike the solutions considered above, the rheology of nanosuspensions is not always described by the power-law model. In some cases, yield shear stresses τ_0 occur in nanosuspensions and the rheology is better described by the Herschel-Bulkley model:

$$\mu = (k\dot{\gamma}^n + \tau_0)/\dot{\gamma}\,.$$

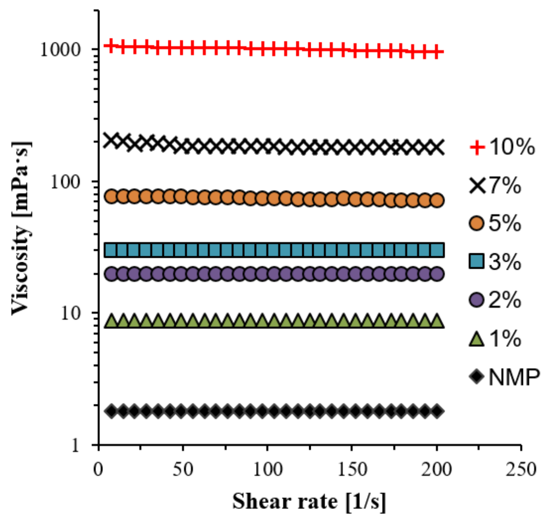

Fig. 4. Viscosity coefficient of the solution versus shear rate at different weight concentrations of PVDF in NMP

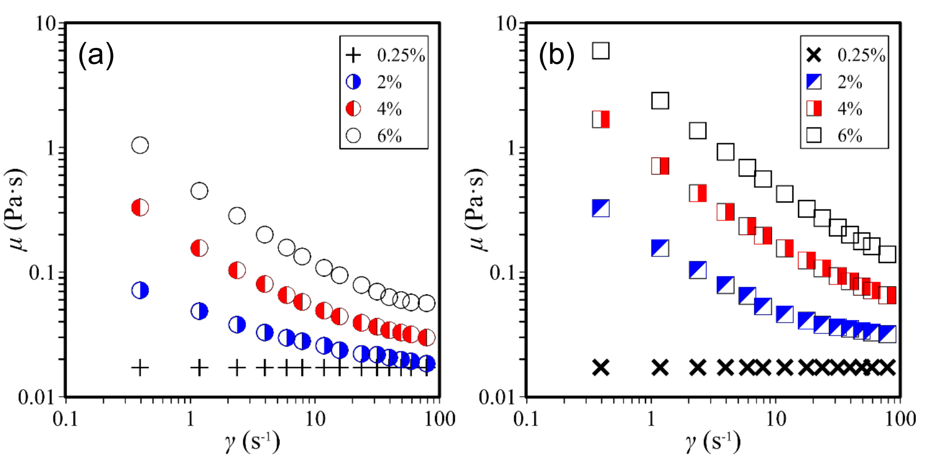

Fig. 5. Viscosity coefficient versus shear rate for ethylene glycol-based nanosuspensions with 150 nm particles of Al_2O_3 (a) and TiO_2 (b) [11]

The volume not occupied by molecules (free volume) is very small in fluids. Then even small molecules from the faster-moving layer can not penetrate the slower-moving layer. As a result, the exchange of the momentum between the layers does not result from the collisions of molecules but because fast-moving molecules entrain the slow-moving molecules.

Polymer molecules are fully oriented and straightened at high shear rates. Then the transfer of momentum from the faster-moving layer to the slower-moving layer should occur in the same way as in fluids since the size of the free volume is small to accommodate a long molecule. Therefore, the viscosity at high shear rates will be low. It approaches the value characteristic for fluids consisting of monomeric rather than polymer molecules [14]. Fig. 6 shows the relationship between the viscosity coefficient and the weight concentration of solutions at three different shear rates (69.6, 131, and 200 s^{-1}). It is apparent that viscosity coefficient does not depend on shear rate. The viscosity coefficient increases with increasing concentration. A deviation from the

Newtonian behaviour begins at concentrations of more than 7 wt.%.

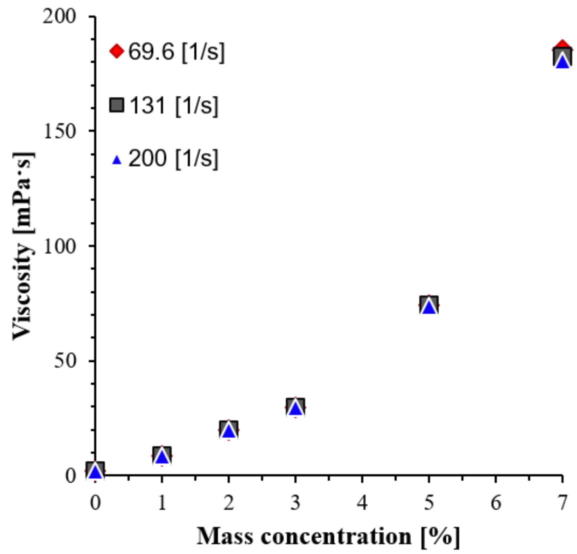

Fig. 6. Relationship between the viscosity coefficient and the weight concentration of solutions at three different shear rates (69.6, 131, and 200 s^{-1})

For solutions with concentrations of up to 7 %, the relative viscosity coefficient was calculated (the viscosity coefficient of the solution referred to the NMP viscosity coefficient). The relationship between the relative viscosity coefficient and concentration is well described by a quadratic correlation (coefficient of determination $R^2 = 0.996$):

$$\mu_{rel}(C) = 1 + a \cdot C + b \cdot C^2,$$

where $a = 2.11$, b =1.16. The relationship between viscosity coefficient of the nanosuspension and particle concentration is generally also nonlinear. For example, the viscosity coefficient of water-based nanofluids with Al_2O_3 (150 nm) particles is described by following equation

$$\mu_{rel}(C) = 1 + 1.52 \cdot C + 4.61 \cdot C^2.$$

Temperature dependences of the viscosity coefficient and relative viscosity coefficient of PVDF solutions in NMP at different weight concentrations are shown in Fig. 7. It was found that the relative viscosity coefficient is independent of temperature:

$$\mu_{rel}(C, t) = \mu_{NMP}(t) \cdot \mu_{rel}(C).$$

At high concentrations (above 7 %) these fluids exhibit non-Newtonian behaviour. Similar phenomenon was observed for suspensions of nanoparticles [12]. So, it was shown that at low concentrations of particles the relative viscosity of nanosuspensions does not depend on temperature.

It should be noted that the viscosity of solutions, like the viscosity of low-molecular fluids, decreases with increasing temperature. This is because the average distances between the molecules increase with the increase of temperature, and the mutual attraction between molecules weakens. For example, the temperature dependences of the viscosity coefficient of PVDF solutions in dimethyl acetate were studied at concentrations above 14 wt.% [15]. Like in the present work, Newtonian behaviour is exhibited by solutions at temperatures up to $30°C$ while at the temperature of $50°C$ the phase transition and destruction of the pseudo structure was detected.

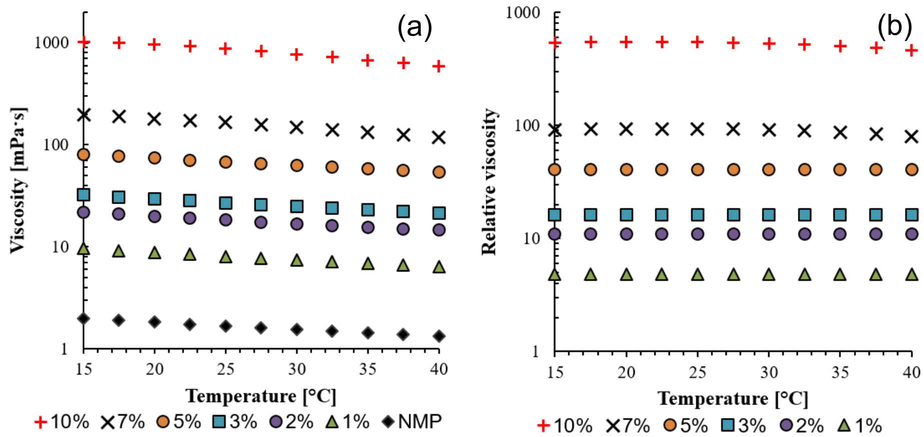

Fig. 7. Relationship between viscosity coefficient (a) and relative viscosity coefficient (b) of PVDF solutions in NMP and temperature at different weight concentrations

Conclusions

The rheological properties of PVDF solutions in N-methylpyrrolidone were studied. It was shown that the viscosity coefficient of polymer solutions decreases with increasing temperature though this relation ship is not linear. The viscosity coefficient of the pure N-methylpyrrolidone solution remains almost constant. As shown experimentally, solutions with concentrations of PVDF up to 7 wt.% behave as Newtonian fluids, i.e., their viscosity does not depend on shear rate. For higher concentrations of PVDF, the pseudoplastic behaviour was observed. It was demonstrated that addition of polymers allows one to modify the behaviour of solutions.

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Реологические свойства растворов ПВДФ

Баир Б. Дамдинов
Сибирский федеральный университет
Красноярск, Российская Федерация
Бурятский государственный университет им. Д. Банзарова
Улан-Удэ, Российская Федерация
Виктория А. Данилова
Бурятский государственный университет им. Д. Банзарова
Улан-Удэ, Российская Федерация
Андрей В. Минаков
Максим И. Пряжников
Сибирский федеральный университет
Красноярск, Российская Федерация
Институт теплофизики им. Кутателадзе СО РАН
Новосибирск, Российская Федерация

Аннотация. Реологические свойства растворов ПВДФ в н-метилпирролидоне исследованы реометрическим методом. Было показано, что вязкость растворов полимеров нелинейно уменьшается с увеличением температуры. Вязкость растворителя — н-метилпирролидона — остается практически неизменной. Показано, что при массовых концентрациях менее 7 % растворы проявляют ньютоновское поведение. При более высоких концентрациях растворы проявляют псевдопластические свойства.

Ключевые слова: жидкости, структура, вязкость, реологические свойства, ньютоновское и неньютоновское поведение, полиминилденфторид (ПВДФ), н-метилпирролидон, свойства материалов, температурные зависимости.

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УДК 519.2

Estimating the Mean of Heavy-tailed Distribution under Random Truncation

Ben Dahmane Khanssa*
Benatia Fateh†
Brahimi Brahim‡
Laboratory of Applied Mathematics
Mohamed Khider University
Biskra, Algeria

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Abstract. Inspired by L.Peng's work on estimating the mean of heavy-tailed distribution in the case of completed data. we propose an alternative estimator and study its asymptotic normality when it comes to the right truncated random variable. A simulation study is executed to evaluate the finite sample behavior on the proposed estimator.

Keywords: random truncation, Hill estimator, Lynden-Bell estimator, heavy-tailed distributions.

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1. Introduction and motivation

Let $(\mathbf{X}_1, \ldots, \mathbf{X}_N)$ be independent copies of a non-negative random variable (rv) $\mathbf{X}$ with cumulative distribution (cdf) $\mathbf{F}$, defined over some probability space $(\Omega, \mathcal{A}, \mathcal{P})$, suppose that $\mathbf{X}$ is right truncated by sequences of independent copies $(\mathbf{Y}_1, \ldots, \mathbf{Y}_N)$ of (rv) $\mathbf{Y}$ with cdf $\mathbf{G}$, throughout the paper, we assume that $\mathbf{F}$ and $\mathbf{G}$ are heavy-tailed in other words that $\overline{\mathbf{F}} = 1 - \mathbf{F}$ and $\overline{\mathbf{G}} = 1 - \mathbf{G}$ are regularly varying ($\mathcal{RV}$) at infinity with respective negative indices $-1/\gamma_1$ and $-1/\gamma_2$; we will use the notation: $\overline{\mathbf{F}} \in \mathcal{RV}(-1/\gamma_1)$ and $\overline{\mathbf{G}} \in \mathcal{RV}(-1/\gamma_2)$ that is for any $x > 0$.

$$\lim_{t\to\infty} \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)} = x^{-\frac{1}{\gamma_1}} \quad \text{and} \quad \lim_{t\to\infty} \frac{\overline{\mathbf{G}}(tx)}{\overline{\mathbf{G}}(t)} = x^{-\frac{1}{\gamma_2}} \,. \tag{1}$$

The statistical literature on such problems of extremes [4] and [13] events is very extensive, one of those problems is for the estimation of the mean $\mathbf{E}(X)$, this problem was already treated by [11] and [3] in the case of complete data, nevertheless in numerous survival practical applications, it happens that one is not able to observe a subject entire lifetime. The subject may leave the study may survive to the closing data, or may enter the study at some time after its lifetime has started, the most current forms of such incomplete data are censorship and truncation. As we mention our aim is to propose an asymptotically normal estimator for the mean of X:

$$\mu = \mathbf{E}(X) = \int_0^\infty \overline{\mathbf{F}}(x)dx. \tag{2}$$

*khanssa.bendahmane@univ-biskra.dz https://orcid.org/0000-0003-0256-0127
†fatahbenatia@hotmail.com
‡brah.brahim@gmail.com

Whose existence requires that $\gamma_1 < 1$, The sample mean for censored data is obtained and equal to:

$$\widetilde{\mu}_n = \sum_{i=2}^{n} \frac{\delta_{[i:n]}}{n-i+1} \prod_{j=1}^{i-1} \left(\frac{n-j}{n-j+1} \right)^{\delta_{[j:n]}} Z_{i,n}, \tag{3}$$

the asymptotic normality of $\widetilde{\mu}_n$ is established by [14]. The model studied here is based on the random right truncated ($\mathcal{RRT}$) data, in the sense that the rv of interest $\mathbf{X}_i$ and the truncated rv $\mathbf{Y}_i$ are observable only when $\mathbf{X}_i \leqslant \mathbf{Y}_i$, whereas nothing is observed if $\mathbf{X}_i > \mathbf{Y}_i$. We denote (X_i, Y_i), $i = 1; n$ to be observed data as copies of a couple of rv's (X, Y) corresponding to the truncated sample $(\mathbf{X}_i, \mathbf{Y}_i)_{1 \leqslant i \leqslant N}$, where $n = n_N$ is a sequence of discrete rv's by the weak law of large numbers, we have

$$\frac{n}{N} \longrightarrow p = \mathbf{P}(\mathbf{X} \leqslant \mathbf{Y}) \quad \text{as} \quad N \to \infty.$$

We shall assume that $p > 0$, otherwise nothing will be observed. The joint $\mathbf{P}$-distribution of on observed (X, Y) is given by:

$$H(x, y) = \mathbf{P}(X \leqslant x, Y \leqslant y) = \mathbf{P}(\mathbf{X} \leqslant x, \mathbf{Y} \leqslant y \mid \mathbf{X} \leqslant \mathbf{Y} = p^{-1} \int_0^y \mathbf{F}(\min(x, z)) d\mathbf{G}(z).$$

The marginal distributions of the rv's X and Y respectively denoted by F and G are defined by:

$$F(x) = p^{-1} \int_0^x \overline{\mathbf{G}}(z) d\mathbf{F}(z) \quad \text{and} \quad G(y) = p^{-1} \int_0^y \mathbf{F}(z) d\mathbf{G}(z),$$

$$\overline{F}(x) = -p^{-1} \int_x^\infty \overline{\mathbf{G}}(z) d\overline{\mathbf{F}}(z) \quad \text{and} \quad \overline{G}(y) = -p^{-1} \int_y^\infty \mathbf{F}(z) d\overline{\mathbf{G}}(z).$$

For randomly truncated data; the truncation product-limit estimate is the maximum likelihood estimate (MLE) for non-parametric models the well-known non-parametric estimator of F in $\mathcal{RRT}$ model, proposed by [10] :

$$\mathbf{F}_n^{(\mathbf{LB})}(x) = \prod_{i: X_i > x} \exp\left(1 - \frac{1}{nC_n(X_i)}\right). \tag{4}$$

Where $C_n(x) = n^{-1} \sum_{i=1}^{n} \mathbf{1}(X_i \leqslant x \leqslant Y_i)$ the empirical counterparts of $C(z) = P(X \leqslant z \leqslant Y)$. Since F and G are heavy-tailed their right endpoints are infinite and thus are equal. As we mentioned this problem has been studied by [11] in the case of sets of complete data from heavy-tailed distributions with a range of $\gamma_1 \in (1/2, 1)$ throughout this paper we restrict ourselves on the case where γ_1 belongs to the following range:

$$\mathcal{R} = \left\{ \gamma_1, \gamma_2 > 0 : \frac{\gamma_2}{1 + 2\gamma_2} < \gamma_1 < 1 \right\}. \tag{5}$$

To ensure that the mean is finite and since we have applied both conditions of [15] paper:

$$I_1 = \int_1^\infty \frac{\varphi^2(x)}{\mathbf{G}(x)} d\mathbf{F}(x), \qquad I_2 = \int_1^\infty \frac{d\mathbf{F}(x)}{\mathbf{G}(x)}. \tag{6}$$

We find those conditions may be infinite when we deal with heavy-tailed distributions. Assumed that both of X and Y are $Pareto(\gamma_1)$ and $Pareto(\gamma_2)$ respectively:

$$1 - \mathbf{F}(x) = \overline{\mathbf{F}}(x) = x^{-\frac{1}{\gamma_1}}, \qquad 1 - \mathbf{G}(x) = \overline{\mathbf{G}}(x) = x^{-\frac{1}{\gamma_2}} \quad \text{with} \quad \gamma_1 > 0, \ \gamma_2 > 0 \quad \text{and} \quad x \geqslant 1.$$

We figure out that the central limit theorem (CTL) established by [15] cannot be applied in the previous range when $I_1 = I_2 = \infty$. It is worth to mention that in the case of non truncation we have $\gamma_1 = \gamma$ and $\gamma_2 = \infty$ so $\mathcal{R}$ abbreviate to Peng's range. To define our new estimator we introduce an integer sequences $k = k_n$ representing a fraction of extreme order statistics satisfying the following conditions:

$$1 < k < n, \quad k \longrightarrow \infty \text{ and } k/n \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{7}$$

So by decomposing μ as the sum of two terms

$$\mu = \int_0^t \overline{\mathbf{F}}(x)dx + \int_t^\infty \overline{\mathbf{F}}(x)dx = \mu_1 + \mu_2. \tag{8}$$

Then we can estimate μ_i, $i = \overline{1,2}$ separately, after integration μ_1 by parts and after changing variables in μ_2 we may write:

$$\mu_1 = t\overline{\mathbf{F}}(t) + \int_0^t xd\mathbf{F}(x) \quad \text{and} \quad \mu_2 = t\overline{\mathbf{F}}(t)\int_1^\infty \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)}dx.$$

By replacing t by $X_{n-k,n}$ where $X_{1,n} < \cdots < X_{n,n}$ denote the order statistics pertaining to $X_1, \ldots, X_n$; and $\mathbf{F}$ by $\mathbf{F}_n^{(\mathbf{LB})}$ we get that:

$$\widehat{\mu}_1 = X_{n-k,n}\overline{\mathbf{F}_n}^{(\mathbf{LB})}(X_{n-k,n}) + \int_0^{X_{n-k,n}} xd\mathbf{F}_n^{(\mathbf{LB})}(x),$$

hence from [16] we may write:

$$\widehat{\mu}_1 = X_{n-k,n}\overline{\mathbf{F}_n}^{(\mathbf{LB})}(X_{n-k,n}) + \frac{1}{n}\sum_{i=1}^{n-k}\frac{\mathbf{F}_n^{(\mathbf{LB})}(X_{i,n})}{C_n(X_{i,n})}X_{i,n}. \tag{9}$$

Back to μ_2 building on the Karamata Theorem [9, page 363] we may write:

$$\mu_2 \sim \frac{\gamma_1}{1-\gamma_1}t\overline{\mathbf{F}}(t) \text{ as } n \longrightarrow \infty, \qquad 0 < \gamma_1 < 1. \tag{10}$$

Notice to estimate (10) it is based on estimator of tail index γ_1, in view of the history of the estimation of γ_1. In [8] introduced an estimator of γ_1 under random truncation. In [1] established the asymptotic normality of this estimator under the tail dependence and the second order conditions of regular variation, throughout this paper we use the estimation of [1]. So that yield us to an estimator to μ_2 :

$$\widehat{\mu}_2 = \frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1}X_{n-k,n}\overline{\mathbf{F}_n}^{(\mathbf{LB})}(X_{n-k,n}), \tag{11}$$

finally with (9) and (11), we build our estimator $\widehat{\mu}$ for the mean (2) as follow:

$$\hat{\mu} = X_{n-k,n}\ \overline{\mathbf{F}}_n(X_{n-k,n})\frac{1}{1-\hat{\gamma}_1} + \frac{1}{n}\sum_{i=1}^{n-k}\frac{\mathbf{F}_n^{LB}(X_{i,n})}{C_n(X_{i,n})}X_{i,n}.$$

The rest of this paper is organized as follows. In the second section, we state our main result. This is followed by a simulation study of our proposed estimator where we discuss its behavior with a finite sample.

2. The main results

In extreme value analysis and in the second-order frame work (see, e.g. [9]), weak approximation are achieved. Consequently, it seems quite natural to suppose that df's $\mathbf{F}$ and $\mathbf{G}$ satisfy the well-known second-order condition of regular variation we express in terms of the tail quantile functions. That is we assume that for $x > 0$. we have

$$\lim_{t\to\infty} \frac{U_{\mathbf{F}}(tx)/U_{\mathbf{F}}(t) - x^{\gamma_1}}{\mathbf{A_F}(t)} = x^{\gamma_1} \frac{x^{\tau_1} - 1}{\tau_1} \tag{12}$$

and

$$\lim_{t\to\infty} \frac{U_{\mathbf{G}}(tx)/U_{\mathbf{G}}(t) - x^{\gamma_2}}{\mathbf{A_G}(t)} = x^{\gamma_2} \frac{x^{\tau_2} - 1}{\tau_2}, \tag{13}$$

where $\tau_1,\tau_2 < 0$ are the second-order parameters and $\mathbf{A_F}, \mathbf{A_G}$ are functions tending to zero and not changing signs near infinity with regularly varying absolute values at infinity with indices τ_1,τ_2 respectively.

Theorem 2.1. *Assume that (12 and 13) hold and $\sqrt{k}\mathbf{A}_\circ(n/k) = O(1)$ for $\gamma_2/(1+2\gamma_2) < \gamma_1 < 1$. Let $k = k_n$ denote an intermediate integer sequences satisfying (7), then $\hat{\mu} \to \mu$ in probability:*

$$\frac{\sqrt{k}(\widehat{\mu} - \mu)}{\overline{\overline{\mathbf{F}}}(X_{n-k,n})X_{n-k,n}} =$$

$$= \mathbf{c}_1\mathbf{W}(1) + \int_0^1 \left\{\mathbf{c}_2 s^{-\frac{2\gamma_1}{\gamma}+\frac{\gamma}{\gamma_2}+1} + \mathbf{c}_3 s^{-\gamma_1+\frac{\gamma}{\gamma_2}+1} + \mathbf{c}_4\log(s) + \mathbf{c}_5\right\} s^{-\frac{\gamma}{\gamma_2}-1}\mathbf{W}(s)ds +$$

$$+ \frac{(\gamma_1 + \tau_1 - 1)(1-\gamma_1) + (1-\tau_1)}{(1-\tau_1)(\gamma_1+\tau_1-1)(1-\gamma_1)}\sqrt{k}\mathbf{A}_\circ(n/k).$$

Corollary 2.1. *Under the assumptions of Theorem 2.1 we suppose that $\sqrt{k}\mathbf{A}_\circ(n/k) \to \lambda$,*

$$\frac{\sqrt{k}(\widehat{\mu} - \mu)}{\overline{\overline{\mathbf{F}}}(X_{n-k,n})X_{n-k,n}} \to \mathcal{N}\left(\lambda\frac{(\gamma_1 + \tau_1 - 1)(1-\gamma_1) + (1-\tau_1)}{(1-\tau_1)(\gamma_1+\tau_1-1)(1-\gamma_1)}, \sigma^2\right) \quad as \;\; n \to \infty.$$

Where

$$\sigma^2 := \frac{p(1-p)\left[p(1-p) + 2\gamma_1^2\right]}{(1-\gamma_1)^2} + \frac{p^3\gamma_1}{1-\gamma_1} + \frac{2p^2(1-p)}{(1-\gamma_1)(-\gamma_1+2)} +$$

$$+ \frac{-2p^4}{(-2+p)(-4+3p)} + \frac{3p^5\gamma_1}{(-2+p)(-2+\gamma_1 p+3p)} + \frac{-2\gamma_1 p^3(1-p)}{(-2+p)(-\gamma_1+2)} +$$

$$+ 3p^5\gamma_1^2(\frac{p}{2} - \frac{1}{4-p})^2 - 2p^3\gamma_1^2(1-p)\frac{3p-2}{6}(\frac{p}{1+p})^2 +$$

$$+ \frac{p^2\gamma_1(p-1)(1-\gamma_1) - p^2\gamma_1^3}{(-1+p)(-2+p)(1-\gamma_1)}\left[\frac{\gamma_1(-p^3+4-6p) + p^2(\gamma_1-2) + 2}{(-1-p) + \gamma_1(-p-2)}\right] +$$

$$+ \frac{1-2p}{p^2} + \frac{-2p^2(1-p)^2(1-\gamma_1) + \gamma_1^2 p}{(1-\gamma_1)(\gamma_1+2)(-\gamma_1+p+1)^2} +$$

$$+ \frac{2p^2(1-p)(1-\gamma_1) + \gamma_1^2 p}{(1-\gamma_1)^2}\left(\left(\frac{p}{p^2-1}\right)^2 + \left(\frac{1}{1-p}\right)^2\right)$$

and

$$p = \frac{\gamma_2}{\gamma_1 + \gamma_2}.$$

3. Simulation study

The main purpose of this section is to study the execution of our new estimator $\widehat{\mu}$ for that we generate the data as follows:

- The interset and the truncated variable: we generate two sets of truncated and truncation data both pulled for the first hand from Fréchet model:

$$\overline{\mathbf{F}}(x) = 1 - \exp(-x^{\frac{1}{\gamma_1}}), \quad \overline{\mathbf{G}}(x) = 1 - \exp(-x^{\frac{1}{\gamma_2}}), \qquad x \geqslant 0$$

  and the other hand from Burr model:

$$\overline{\mathbf{F}}(x) = (1 + x^{\frac{1}{\delta}})^{-\frac{\delta}{\gamma_1}}, \quad \overline{\mathbf{G}}(x) = (1 + x^{\frac{1}{\delta}})^{-\frac{\delta}{\gamma_2}}, \qquad x \geqslant 0 \quad \text{and} \quad \delta, \gamma_1, \gamma_2 > 0.$$

- The observed data: for the proportion of observed data is equal to $p = \gamma_2/\gamma_1 + \gamma_2$ we take $p = 70\,\%$, $80\,\%$ and $90\,\%$ we fix $\delta = 1/4$ and choose the values 0.6, 0.7 and 0.8 for γ_1. For each couple (γ_1, p); we solve the equation $p = \gamma_2/\gamma_1 + \gamma_2$ to get the pertaining γ_2-value.
- We vary the common size N of both samples $(\mathbf{X}_1, \ldots, \mathbf{X}_N)$ and $(\mathbf{Y}_1, \ldots, \mathbf{Y}_N)$.
- We apply the algorithm of [12] page 137, to select the optimal numbers of upper order statistics (k^*) used in the computation of $\hat{\gamma}_1$.

The performance of this new estimator named by $\hat{\mu}$ is evaluated in terms of absolute bias (A-bias) root mean squared error (RMSE) which are summarized in tables for Burr model in Tables: 1 for $\gamma_1 = 0.6$, 2 for $\gamma_1 = 0.7$, 3 for $\gamma_1 = 0.8$ and for Fréchet models Tables: 4 for $\gamma_1 = 0.6$, 5 for $\gamma_1 = 0.7$, 6 for $\gamma_1 = 0.8$ adding two forms of graphical representation; we consider two truncated schema of Burr truncated by Burr the first for $\gamma_1 = 0.6$ and the second for $\gamma_1 = 0.8$ we represent the Biases and the RMSE of our estimator as functions of k (number of the longest order statistics).

After examining all tables and figures, and as expected, the sample size affects the estimate in the sense that a larger N gives a better estimate. It is noticeable that the estimation accuracy of estimator decreases when the truncation percentage increase and it is quite expected. Moreover the estimator performs best for the larger value of the tail index larger than 0.5 especially when truncation proportion is high.

4. Appendix

4.1. Proof of Theorem 2.1

We begin by setting $U_i = \overline{F}(X_i)$ and define the corresponding uniform tail process by $\alpha_n(s) = \sqrt{k}(U_n(s) - s)$, for $0 \leqslant s \leqslant 1$ where $U_n(s) = 1/k \sum_{i=1}^{n} \mathbf{1}\left(\mathbf{U}_i \leqslant k\frac{s}{n}\right)$. The weighted weak approximation to $\alpha_n(s)$ given in terms of either a sequence of wiener processes (see, eg., [6] and [5]) or a single Wienner process as in Proposition 3.1 of [7], will be very crucial to our proof procedure.

In the sequel, we use the latter representation which says that: there exists a Wiener process $\mathbf{W}$, such that for every $0 \leqslant \eta \leqslant 1$

$$\sup_{0<s\leqslant 1} \mid \alpha_n(s) - \mathbf{W}(s) \mid \to \mathbf{0}, \text{ as } n \to \infty. \tag{14}$$

Observe that $\widehat{\mu} - \mu = (\widehat{\mu}_1 - \mu_1) + (\widehat{\mu}_2 - \mu_2)$ and starting by:

$$\widehat{\mu}_1 - \mu_1 = \int_0^{X_{n-k;n}} \overline{\mathbf{F}_n}(x)dx - \int_0^t \overline{\mathbf{F}}(x)dx,$$

Table 1. Bias and RMSE of the mean estimator based on samples of Burr models with $\gamma_1 = 0.6$

$\gamma_1 = 0.6 \longrightarrow \mu = 2.371$					
$p = 0.7$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.002	0.130	27	2.374	198
400	0.069	0.858	31	2.440	278
500	0.072	0.257	39	2.300	355
1000	0.001	0.048	40	2.372	681
$p = 0.8$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.008	0.180	10	2.380	244
400	0.008	0.119	16	2.379	318
500	0.001	0.174	27	2.372	399
1000	0.001	0.106	25	2.372	811
$p = 0.9$					
N	A-bias	RMSE	k^*	$\hat{\mu}$	n
300	0.005	0.040	4	2.406	268
400	0.006	0.028	7	2.406	361
500	0.003	0.067	8	2.374	445
1000	0.003	0.097	12	2.374	886

Table 2. Bias and RMSE of the mean estimator based on samples of Burr models with $\gamma_1 = 0.7$

$\gamma_1 = 0.7 \longrightarrow \mu = 3.218$					
$p = 0.7$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.016	0.634	25	3.234	215
400	0.008	0.067	34	3.227	290
500	0.008	0.063	58	3.226	3362
1000	0.004	0.023	88	3.222	701
$p = 0.8$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.021	0.178	18	3.239	246
400	0.002	0.306	23	3.221	319
500	0.002	0.367	39	3.220	403
1000	0.001	0.193	52	3.219	788
$p = 0.9$					
N	A-bias	RMSE	k^*	$\hat{\mu}$	n
300	0.005	0.028	19	3.223	268
400	0.000	0.134	21	3.218	368
500	0.008	0.246	25	3.226	458
1000	0.002	0.049	37	3.220	896

Table 3. Bias and RMSE of the mean estimator based on samples of Burr models with $\gamma_1 = 0.8$

$\gamma_1 = 0.8 \longrightarrow \mu = 4.896$					
$p = 0.7$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.000	0.152	73	4.896	207
400	0.029	0.070	75	4.925	278
500	0.065	0.631	147	4.961	348
1000	0.013	0.302	228	4.919	697
$p = 0.8$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.106	0.613	55	5.002	239
400	0.014	0.446	14	4.910	315
500	0.001	0.321	146	4.897	404
1000	0.030	0.039	173	4.926	810
$p = 0.9$					
N	A-bias	RMSE	k^*	$\hat{\mu}$	n
300	0.094	0.962	67	4.990	275
400	0.058	0.240	86	4.954	359
500	0.029	0.171	67	4.925	451
1000	0.006	0.041	187	4.902	894

Table 4. Bias and RMSE of the mean estimator based on samples of Frechét models with $\gamma_1 = 0.6$

$\gamma_1 = 0.6 \longrightarrow \mu = 2.218$					
$p = 0.7$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.155	0.537	28	2.373	170
400	0.153	0.186	25	2.371	217
500	0.004	0.065	32	2.222	284
1000	0.002	0.010	43	2.220	568
$p = 0.8$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.259	0.263	17	2.475	178
400	0.031	0.598	40	2.249	241
500	0.066	0.222	33	2.284	293
1000	0.074	0.076	31	2.307	569
$p = 0.9$					
N	A-bias	RMSE	k^*	$\hat{\mu}$	n
300	0.010	0.084	5	2.228	180
400	0.009	0.185	11	2.218	231
500	0.004	0.052	19	2.222	314
1000	0.008	0.106	23	2.227	594

we consider the following decomposition:

$$\widehat{\mu}_1 - \mu_1 = T_{n_1}(x) + T_{n_2}(x).$$

Where:

$$T_{n_1}(x) = \int_0^{X_{n-k;n}} \left(\overline{\mathbf{F}}_n(x) - \overline{\mathbf{F}}(x)\right) dx \quad \text{and} \quad T_{n_2}(x) = \int_{X_{n-k;n}}^{t} \overline{\mathbf{F}}(x) dx.$$

Table 5. Bias and RMSE of the mean estimator based on samples of Frechét models with $\gamma_1 = 0.7$

$\gamma_1 = 0.7 \longrightarrow \mu = 2.992$					
$p = 0.7$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.085	0.213	23	3.076	168
400	0.080	0.356	57	3.072	227
500	0.025	0.365	49	3.016	278
1000	0.020	0.385	58	3.011	564
$p = 0.8$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.031	0.171	30	3.022	169
400	0.000	0.063	26	2.992	250
500	0.016	0.352	44	3.007	274
1000	0.001	0.122	48	2.993	598
$p = 0.9$					
N	A-bias	RMSE	k^*	$\hat{\mu}$	n
300	0.001	0.213	22	2.993	193
400	0.082	0.206	25	3.074	225
500	0.086	0.189	29	3.078	306
1000	0.000	0.257	40	2.992	584

Table 6. Bias and RMSE of the mean estimator based on samples of Frechét models with $\gamma_1 = 0.8$

$\gamma_1 = 0.8 \longrightarrow \mu = 4.591$					
$p = 0.7$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.084	0.720	15	4.675	164
400	0.185	0.604	42	4.776	225
500	0.001	0.037	52	4.591	297
1000	0.063	0.674	109	4.654	540
$p = 0.8$					
N	A-bias	RMSE	k*	$\hat{\mu}$	n
300	0.267	0.282	12	4.857	173
400	0.131	0.147	29	4.722	222
500	0.044	0.045	41	4.635	306
1000	0.011	0.331	68	4.690	597
$p = 0.9$					
N	A-bias	RMSE	k^*	$\hat{\mu}$	n
300	0.222	0.301	37	4.813	172
400	0.128	0.283	72	4.719	256
500	0.057	0.576	70	4.648	302
1000	0.001	0.382	133	4.592	604

It follows after changing variables that:

$$T_{n_1}(x) = X_{n-k,n} \int_0^1 \frac{\overline{\mathbf{F}}(a_k x)}{\overline{\mathbf{F}}(a_k x)} \overline{\mathbf{F}}_n(xX_{n-k,n}) - \overline{\mathbf{F}}(xX_{n-k,n}) dx,$$

$$T_{n_2}(x) = -X_{n-k,n} \int_1^{\frac{t}{X_{n-k,n}}} \overline{\mathbf{F}}(xX_{n-k,n}) dx.$$

In order to established the result of theorem we apply the results of [2], we have:

$$\sqrt{k}\frac{\overline{\mathbf{F}}_n(xX_{n-k,n}) - \overline{\mathbf{F}}(xX_{n-k,n})}{\overline{\mathbf{F}}(a_k x)} = x^{\frac{1}{\gamma}} \frac{\gamma}{\gamma_1} W(x^{-\frac{1}{\gamma_1}}) + \frac{\gamma}{\gamma_1 + \gamma_2} x^{\frac{1}{\gamma_1}} \int_0^1 s^{-\frac{\gamma}{\gamma_2}-1} \mathbf{W}(x^{-\frac{1}{\gamma_1}} s) ds.$$

After some elementary but tedious manipulations of integral calculus (change of variables and integration by parts) and by making use of the uniform inequality of the second-order regularly varying functions $\overline{\mathbf{F}}$, to $T_{n_1}(x)$ becomes:

$$\sqrt{k}\frac{T_{n_1}(x)}{X_{n-k,n}\overline{\mathbf{F}}(a_k)} = \int_0^1 (-\gamma s^{-\frac{2\gamma_1}{\gamma}} + \frac{\gamma\gamma_1}{(\gamma_1+\gamma_2)(\gamma_1+1)} s^{-\frac{\gamma}{\gamma_2}-1} + \\ + \frac{\gamma\gamma_1}{(\gamma_1+\gamma_2)(\gamma_1+1)} s^{-\gamma_1}) \mathbf{W}(s) ds + o_{\mathbf{p}}(1). \tag{15}$$

Next we move to$T_{n_2}(x)$ which we may write it as follow after changing variables:

$$\frac{\sqrt{k}T_{n_2}(x)}{X_{n-k,n}\overline{\mathbf{F}}(X_{n-k,n})} = \int_1^{\frac{t}{X_{n-k,n}}} \sqrt{k}\frac{\overline{\mathbf{F}}(xX_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} - x^{-\frac{1}{\gamma_1}} dx + \int_1^{\frac{t}{X_{n-k,n}}} x^{-\frac{1}{\gamma_1}} dx = \mathbf{I}_1 + \mathbf{I}_2.$$

For $\mathbf{I}_1$ we apply the results of [2]

$$\sqrt{k}\frac{\overline{\mathbf{F}}(xX_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} - x^{-\frac{1}{\gamma_1}} = x^{-\frac{1}{\gamma_1}} \frac{x^{-\frac{\tau_1}{\gamma_1}} - 1}{\gamma_1 \tau_1} \sqrt{k}\mathbf{A}_\circ(n/k) + o_p\left(x^{-\frac{1}{\gamma_1}+(1-\eta)/\gamma \pm \varepsilon}\right).$$

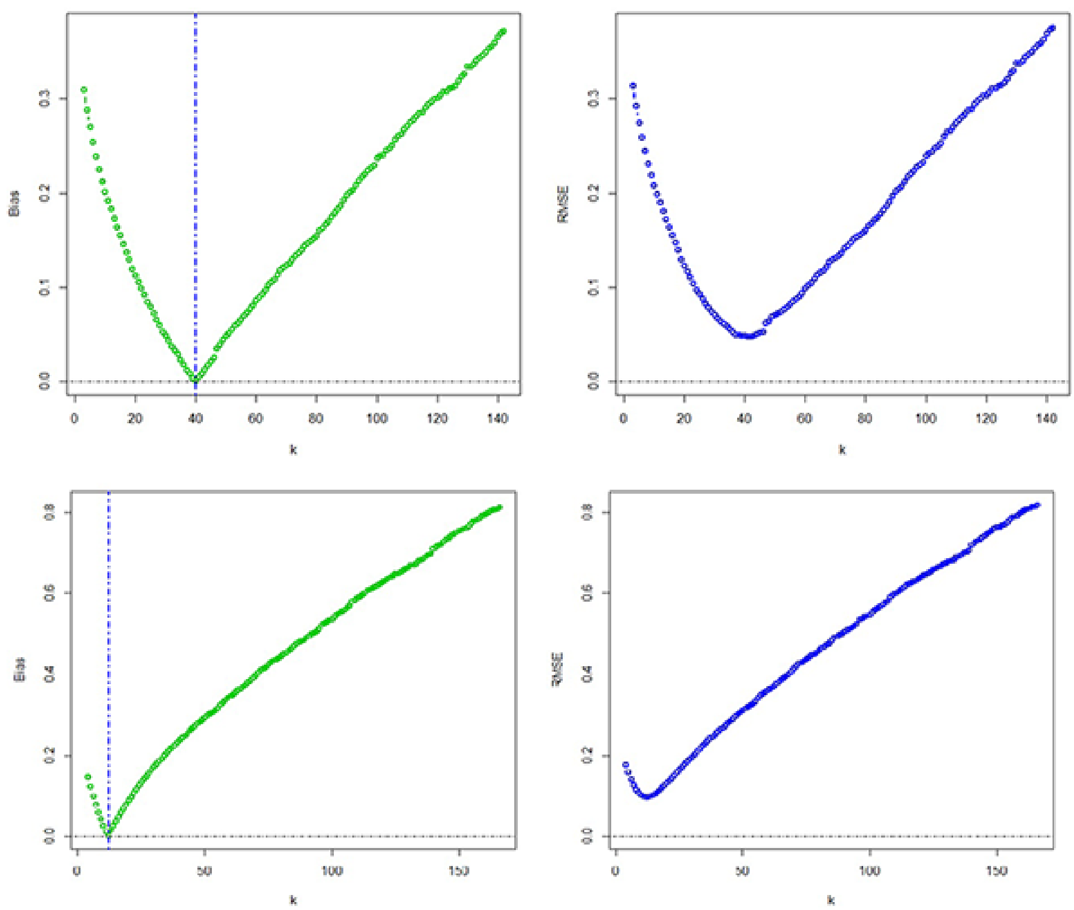

Fig. 1. Absolute Bias (left panel) and RMSE (right panel) of $\widehat{\mu}$ based on samples of size 1000 from Burr distribution truncated by another Burr model with $p = 0.7$ (top) and $p = 0.9$ (bottom) and $\gamma_1 = 0.6$

This implies, almost surely, that

$$\int_1^{\frac{t}{X_{n-k,n}}} \sqrt{k}\frac{\overline{\mathbf{F}}(xX_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} - x^{-\frac{1}{\gamma_1}}\,dx = \int_1^{\frac{t}{X_{n-k,n}}} x^{-\frac{1}{\gamma_1}}\,\frac{x^{-\frac{\tau_1}{\gamma_1}}-1}{\gamma_1\tau_1}\sqrt{k}\mathbf{A}_\circ(n/k)dx.$$

Which is equal after simple calculus and by using the mean value theorem we get $\mathbf{I}_1 = o_{\mathbf{p}}(1)$, for the second step by similar argument and using the fact that from Theorem 2.1 of [1] we have $\sqrt{k}\left(\dfrac{X_{n-k,n}}{t} - 1\right) - \gamma\mathbf{W}(1) = o_{\mathbf{P}}(1)$ we get $\mathbf{I}_2 = -\gamma\mathbf{W}(1) + o_{\mathbf{p}}(1)$, that yield to:

$$\frac{\sqrt{k}T_{n_2}(x)}{X_{n-k,n}\overline{\mathbf{F}}(X_{n-k,n})} = -\gamma\mathbf{W}(1) + o_{\mathbf{p}}(1). \qquad (16)$$

The two approximation 15 and 16 together give:

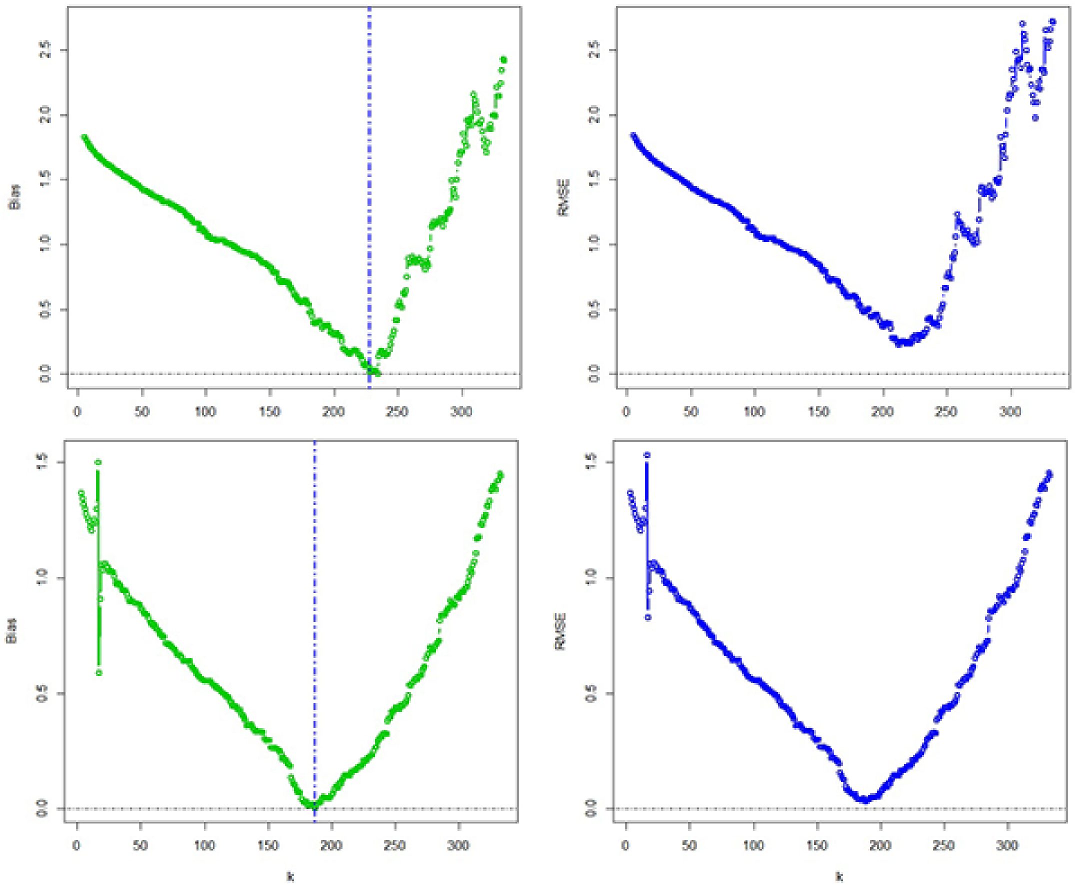

Fig. 2. Absolute Bias (left panel) and RMSE (right panel) of $\widehat{\mu}$ based on samples of size 1000 from Burr distribution truncated by another Burr model with $p = 0.7$ (top) and $p = 0.9$ (bottom) and $\gamma_1 = 0.8$

$$\sqrt{k}\frac{\widehat{\mu}_1 - \mu_1}{X_{n-k,n}\overline{\mathbf{F}}(X_{n-k,n})} = \int_0^1 \Big(-\gamma s^{-\frac{2\gamma_1}{\gamma}} + \frac{\gamma\gamma_1}{(\gamma_1+\gamma_2)(\gamma_1+1)} s^{-\frac{\gamma}{\gamma_2}-1} + \\ + \frac{\gamma\gamma_1}{(\gamma_1+\gamma_2)(\gamma_1+1)} s^{-\gamma_1}\Big)\mathbf{W}(s)ds - \gamma\mathbf{W}(1) + o_{\mathbf{p}}(1). \tag{17}$$

Let us now treat term $\dfrac{\sqrt{k}(\widehat{\mu}_2 - \mu_2)}{t\,\overline{\mathbf{F}}(t)}$. Consider the following forms of μ_2 and $\widehat{\mu}_2$:

$$\widehat{\mu}_2 = \frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} X_{n-k,n}\overline{\mathbf{F}}_n(X_{n-k,n}) \quad \text{and} \quad \mu_2 = \int_t^\infty \overline{\mathbf{F}}(x)dx,$$

$$\widehat{\mu}_2 - \mu_2 = \frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} X_{n-k,n}\overline{\mathbf{F}}_n(X_{n-k,n}) - \int_t^\infty \overline{\mathbf{F}}(x)dx.$$

After changing variables we can obtain:

$$\mu_2 = \int_1^\infty t\overline{\mathbf{F}}(tx)dx = t\overline{\mathbf{F}}(t)\int_1^\infty \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)}dx$$

and

$$\widehat{\mu}_2 = \frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} X_{n-k,n} \overline{\mathbf{F}}_n(X_{n-k,n}) \frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})}$$

so the previous equation leads to

$$\widehat{\mu}_2 - \mu_2 = \frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} X_{n-k,n} \overline{\mathbf{F}}_n(X_{n-k,n}) \frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} - t\overline{\mathbf{F}}(t) \int_1^\infty \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)} dx$$

if we devise this equation by $t\overline{\mathbf{F}}(t)$ we can get:

$$\frac{\sqrt{k}\widehat{\mu}_2 - \mu_2}{t\overline{\mathbf{F}}(t)} = \sqrt{k} \frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} X_{n-k,n} \frac{\overline{\mathbf{F}}_n(X_{n-k,n})}{t\overline{\mathbf{F}}(t)} \frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} - \sqrt{k} \int_1^\infty \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)} dx.$$

So after adding and Subtract some terms we can decompose $\dfrac{\sqrt{k}(\widehat{\mu}_2 - \mu_2)}{t\overline{\mathbf{F}}(t)}$ into the sum of:

$$\mathbf{I}_1 := \sqrt{k} \frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} \frac{\overline{\mathbf{F}}_n(X_{n-k,n})}{\overline{\mathbf{F}}(t)} \frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} \left[\frac{X_{n-k,n}}{t} - 1 \right]$$

$$\mathbf{I}_2 := \sqrt{k} \frac{\overline{\mathbf{F}}_n(X_{n-k,n})}{\overline{\mathbf{F}}(t)} \frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} \left[\frac{\widehat{\gamma}_1}{1-\widehat{\gamma}_1} - \frac{\gamma_1}{1-\gamma_1} \right]$$

$$\mathbf{I}_3 := \sqrt{k} \frac{\gamma_1}{1-\gamma_1} \frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(t)} \left[\frac{\overline{\mathbf{F}}_n(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})} - 1 \right]$$

$$\mathbf{I}_4 := \sqrt{k} \frac{\gamma_1}{1-\gamma_1} \left[\frac{\overline{\mathbf{F}}(X_{n-k,n})}{\overline{\mathbf{F}}(t)} - \left(\frac{X_{n-k,n}}{t} \right)^{-\frac{1}{\gamma_1}} \right]$$

$$\mathbf{I}_5 := \sqrt{k} \frac{\gamma_1}{1-\gamma_1} \left[\left(\frac{X_{n-k,n}}{t} \right)^{-\frac{1}{\gamma_1}} - 1 \right]$$

$$\mathbf{I}_6 := \sqrt{k} \left[\frac{\gamma_1}{1-\gamma_1} - \int_1^\infty \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)} dx \right].$$

For $\mathbf{I}_1$, we have, $\widehat{\gamma}_1 \to \gamma_1$ and $X_{n-k,n}/t \to 1$. Since $\overline{\mathbf{F}}$ is regular variation we obtain $\overline{\mathbf{F}}(X_{n-k,n}) =$ $= (1 + o_{\mathbf{P}}(1))\overline{\mathbf{F}}(t)$. From remark 4.1 of [1], we have $\overline{\mathbf{F}}_n(X_{n-k,n})/\overline{\mathbf{F}}(X_{n-k,n}) \to 1$. So,

$$\sqrt{k}\mathbf{I}_1 = (1 + o_{\mathbf{P}}(1))\sqrt{k} \left(\frac{X_{n-k,n}}{t} - 1 \right).$$

From Theorem 2.1 of [1] we have

$$\sqrt{k} \left(\frac{X_{n-k,n}}{t} - 1 \right) - \gamma \mathbf{W}(1) = o_{\mathbf{P}}(1),$$

then

$$\sqrt{k}\mathbf{I}_1 = (1 + o_{\mathbf{P}}(1)) \frac{\gamma_1 \gamma}{1-\gamma_1} \mathbf{W}(1). \tag{18}$$

For $\mathbf{I}_2$, by using a similar way of $\mathbf{I}_1$, we prove that:

$$\sqrt{k}\mathbf{I}_2 = (1 + o_{\mathbf{P}}(1)) \frac{1}{(1-\gamma_1)^2} \sqrt{k}(\widehat{\gamma}_1 - \gamma_1). \tag{19}$$

From Theorem 3.1 of [2] we have

$$\sqrt{k}(\widehat{\gamma}_1 - \gamma_1) = \frac{\sqrt{k}\mathbf{A}_\circ(n/k)}{1-\tau_1} - \gamma \mathbf{W}(1) + \frac{\gamma}{\gamma_1+\gamma_2} \int_0^1 (\gamma_2 - \gamma_1 - \gamma \log s) s^{-\frac{\gamma}{\gamma_2}-1} \mathbf{W}(s) ds + o_{\mathbf{P}}(1).$$

For $\mathbf{I}_3$ we have

$$\sqrt{k}\mathbf{I}_3 = (1+o_{\mathbf{P}}(1))\frac{\gamma_1\gamma}{1-\gamma_1}\sqrt{k}\left(\frac{\overline{\mathbf{F}}_n(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})}-1\right).$$

From Theorem 4.1 of [1] we have

$$\sqrt{k}\left(\frac{\overline{\mathbf{F}}_n(X_{n-k,n})}{\overline{\mathbf{F}}(X_{n-k,n})}-1\right) = \frac{\gamma_2}{\gamma_1+\gamma_2}\mathbf{W}(1) + \frac{\gamma_1\gamma_2}{(\gamma_1+\gamma_2)^2}\int_0^1 s^{-\frac{\gamma}{\gamma_2}-1}\mathbf{W}(s)ds + o_{\mathbf{P}}(1).$$

So,

$$\begin{aligned}\sqrt{k}\mathbf{I}_3 =& (1+o_{\mathbf{P}}(1))\frac{\gamma_1\gamma_2}{(\gamma_1+\gamma_2)}\mathbf{W}(1)+\\ &+ (1+o_{\mathbf{P}}(1))\frac{\gamma_1\gamma_2^2}{(\gamma_1+\gamma_2)^2(1-\gamma_1)}\int_0^1 s^{-\frac{\gamma}{\gamma_2}-1}\mathbf{W}(s)ds + o_{\mathbf{P}}(1).\end{aligned} \tag{20}$$

For $\mathbf{I}_4$, after the second-order condition of regular variation

$$\sqrt{k}\mathbf{I}_4 = o_{\mathbf{P}}(1). \tag{21}$$

For $\mathbf{I}_5$, using the mean value theorem with $X_{n-k,n}/t \to 1$, we get

$$\sqrt{k}\mathbf{I}_5 = -(1+o_{\mathbf{P}}(1))\frac{1}{1-\gamma_1}\sqrt{k}\left(\frac{X_{n-k,n}}{t}-1\right). \tag{22}$$

From Theorem 2.1 of [1] we have

$$\sqrt{k}\left(\frac{X_{n-k,n}}{t}-1\right) - \gamma\mathbf{W}(1) = o_{\mathbf{P}}(1),$$

then

$$\sqrt{k}\mathbf{I}_5 = -(1+o_{\mathbf{P}}(1))\frac{\gamma}{1-\gamma_1}\mathbf{W}(1).$$

For $\mathbf{I}_6$, we have

$$\int_1^\infty x^{-1/\gamma_1}dx = \frac{\gamma_1}{1-\gamma_1},$$

then

$$\mathbf{I}_6 = \int_1^\infty x^{-1/\gamma_1}dx - \int_1^\infty \frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)}dx.$$

Then, by applying the uniform inequality of regularly varying functions (see, e.g., Theorem 2.3.9 in [9, page 48]) together with the regular variation of $|\mathbf{A}_\circ|$, we show that

$$\sqrt{k}\mathbf{I}_6 \sim \frac{\sqrt{k}\mathbf{A}_\circ(t)}{(\gamma_1+\tau_1-1)(1-\gamma_1)}. \tag{23}$$

Summing up above equations, we get

$$\begin{aligned}\frac{\sqrt{k}\left(\widehat{\mu}_2-\mu_2\right)}{t\overline{\mathbf{F}}(t)} =& \left(\frac{\gamma_1\gamma_2-2\gamma(\gamma_1+\gamma_2)}{(1-\gamma_1)(\gamma_1+\gamma_2)}\right)\mathbf{W}(1) - \frac{\gamma^2}{\gamma_1+\gamma_2}\int_0^1 s^{-\frac{\gamma}{\gamma_2}-1}W(s)\log s ds+\\ &+ \frac{\gamma_1^2\gamma_2(\gamma_2-\gamma_1)}{(\gamma_1+\gamma_2)^2(1-\gamma_1)}\int_0^1 s^{-\frac{\gamma}{\gamma_2}-1}\mathbf{W}(s)ds + \frac{\sqrt{k}\mathbf{A}_\circ(n/k)}{1-\tau_1}+\\ &+ \frac{\sqrt{k}\mathbf{A}_\circ(t)}{(\gamma_1+\tau_1-1)(1-\gamma_1)}.\end{aligned} \tag{24}$$

Finally, Summing up equations 17 and 24 achieves the proof.

4.2. Proof of Corollary 2.1

We set:

$$\frac{\sqrt{k}(\widehat{\mu}-\mu)}{\overline{\mathbf{F}}(X_{n-k,n})X_{n-k,n}} = \Delta + \frac{(\gamma_1+\tau_1-1)(1-\gamma_1)+(1-\tau_1)}{(1-\tau_1)(\gamma_1+\tau_1-1)(1-\gamma_1)}\sqrt{k}\mathbf{A}_\circ(n/k),$$

where $\Delta = c_1\Delta_1 + c_2\Delta_2 + c_3\Delta_3 + c_4\Delta_4 + c_5\Delta_5$ with

$$\Delta_1 = \mathbf{W}(1), \quad \Delta_2 = \int_0^1 s^{-\frac{2\gamma_1}{\gamma}}\mathbf{W}(s)ds, \quad \Delta_3 = \int_0^1 s^{-\gamma_1}\mathbf{W}(s)ds,$$
$$\Delta_4 = \int_0^1 s^{-\frac{\gamma}{\gamma_2}-1}\log(s)\mathbf{W}(s)ds, \quad \Delta_5 = \int_0^1 s^{-\frac{\gamma}{\gamma_2}-1\cdot}\mathbf{W}(s)ds.$$

After elementary but tedious computations, we find the following covariance as asymptotic variance: $\mathbf{\Gamma\Sigma\Gamma}^t$, where

$$\mathbf{\Gamma} = \left(\frac{p(1-p)}{1-\gamma_1}, -p\gamma_1, p(1-p), \gamma_1 p^2(1-p), p(1-p) + \frac{\gamma_1^2 p}{1-\gamma_1}\right)$$

and $\mathbf{\Gamma}^t$ is the transpose of $\mathbf{\Gamma}$, Σ is the variance-covariance matrix:

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{1,5} \\ \alpha_{1,2} & \alpha_2 & \alpha_{2,3} & \alpha_{2,4} & \alpha_{2,5} \\ \alpha_{1,3} & \alpha_{2,3} & \alpha_3 & \alpha_{3,4} & \alpha_{3,5} \\ \alpha_{1,4} & \alpha_{2,4} & \alpha_{3,4} & \alpha_4 & \alpha_{4,5} \\ \alpha_{1,5} & \alpha_{2,5} & \alpha_{3,5} & \alpha_{4,5} & \alpha_5 \end{bmatrix},$$

$$\mathbf{E}(\Delta_1^2) = 1, \quad \alpha_2 := \mathbf{E}(\Delta_2^2) = \frac{2p^2}{(-2+p)(-4+3p)},$$

$$\alpha_3 := \mathbf{E}(\Delta_3^2) = \frac{(1-2p)}{p^4(1-p)},$$

$$\alpha_4 := \ \mathbf{E}(\Delta_4^2) = \frac{1-2p}{p^4(1-p)^2} - \frac{2\gamma_1 p}{(1-p)^3} - \frac{2(1-p)^{-2}}{(-1-p)} + \frac{1}{(1-p)^2(2p-1)^2},$$

$$\alpha_5 := \mathbf{E}(\Delta_5^2) = \frac{4p-3}{-p(1-p)^2(2p-1)},$$

$$\alpha_{1,2} := \mathbf{E}(\Delta_1\Delta_2) = \frac{p}{-2(1-p)},$$

$$\alpha_{1,3} := \mathbf{E}(\Delta_1\Delta_3) = \frac{1}{-\gamma_1+2},$$

$$\alpha_{1,4} := \mathbf{E}(\Delta_1\Delta_4) = -\frac{1}{p^2},$$

$$\alpha_{1,5} := \mathbf{E}(\Delta_1\Delta_5) = \frac{1}{p},$$

$$\alpha_{2,3} := \mathbf{E}(\Delta_2\Delta_3) = \frac{3p^3}{2(-2+p)(p-1)(-2+\gamma_1 p+3p)} + \frac{p}{(-2+p)(-\gamma_1+2)},$$

$$\alpha_{2,4} := \ \mathbf{E}(\Delta_2\Delta_4) = \frac{3p^2}{2(p-1)}\left(\frac{p}{2} - \frac{1}{4-p}\right)^2 + \frac{3p-2}{6}\left(\frac{p}{1+p}\right)^2,$$

$$\alpha_{2,5} := \mathbf{E}(\Delta_2\Delta_5) = \frac{-p^3\gamma_1}{2(-1+p)(-2+p)(-1-p+\gamma_1(-2+p))} + \frac{1}{-2+p},$$

$$\alpha_{3,4} := \mathbf{E}(\Delta_3\Delta_4) = \frac{-1}{(\gamma_1+2)(-\gamma_1+p+1)^2} + \frac{1}{(-\gamma_1+1)}\left[\left(\frac{p}{-1+p^2}\right)^2 + \left(\frac{1}{1-p}\right)^2\right],$$

$$\alpha_{3,5} := \mathbf{E}(\Delta_3\Delta_5) = \frac{1}{(-\gamma_1+2)(-\gamma_1+p+1)} + + \frac{p^3\gamma_1^3}{(-\gamma_1+1)(-p\gamma_1-p\gamma_1^2-p^2\gamma_1^2-p+1)},$$

$$\alpha_{4,5} := \mathbf{E}(\Delta_4\Delta_5) = \frac{(1-p)^2}{p\gamma_1(-\gamma_1-1)(2p-1)} + \frac{1-p}{p^2}.$$

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Оценка среднего распределения с тяжелыми хвостами при случайном усечении

Бен Дахман Хансса
Бенатия Фатех
Брахими Брахим
Лаборатория прикладной математики
Университет Мохамеда Хидера
Бискра, Алжир

Аннотация. Вдохновленные работой Л. Пэна по оценке среднего значения распределения с тяжелыми хвостами в случае полных данных, мы предлагаем альтернативную оценку и изучаем ее асимптотическую нормальность, когда дело касается усеченной справа случайной величины. Имитационное исследование выполняется для анализа поведения конечной выборки на предлагаемой оценке.

Ключевые слова: случайное усечение, оценка Хилла, оценка Линдена-Белла, распределения с тяжелыми хвостами.

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A Note on the Conjugacy Between Two Critical Circle Maps

Utkir A. Safarov*
Turin Politechnic University in Tashkent
Tashkent, Uzbekistan
Tashkent State University of Economics
Tashkent, Uzbekistan

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Abstract. We study a conjugacy between two critical circle homeomorphisms with irrational rotation number. Let f_i, $i = 1, 2$ be a C^3 circle homeomorphisms with critical point $x_{cr}^{(i)}$ of the order $2m_i + 1$. We prove that if $2m_1 + 1 \neq 2m_2 + 1$, then conjugating between f_1 and f_2 is a singular function.

Keywords: circle homeomorphism, critical point, conjugating map, rotation number, singular function.

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1. Introduction and preliminaries

Denjoy's classical theorem [4] states, that if the C^2 circle diffeomorphism f and irrational rotation number $\rho = \rho_f$ then f is topologically conjugate to the linear rotation f_ρ, that is, there exists a circle homeomorphism φ with $f = \varphi^{-1} \circ f_\rho \circ \varphi$.

It is well known that a circle homeomorphisms f with irrational rotation number is strictly ergodic, i.e. it has a unique f-invariant probability measure ν_f. A remarkable fact is that the conjugacy φ can be defined by $\varphi(x) = \nu_f([0, x])$, which shows, that the regularity properties of conjugacy φ and the absolute continuity of invariant measure ν_f are closely related. The problem of smoothness of the conjugacy φ for diffeomorphisms is one of the important problems of circle dynamics. The fundamental results were obtained by V. I. Arnold [1] , J. Moser [15], M. Herman [9], J. Yoccoz [17], Ya. G. Sinai and K. Khanin [12], Y. Katsnelson and D. Ornstein [13]. Notice that for sufficiently smooth circle deffeomorphisms f with a typical irrational rotation number the conjugacy φ is C^1-diffeomorphism. Consequently, the invariant measure ν_f is absolutely continuous with respect to Lebesgue measure μ on S^1.

Since the works of Mostow, Margulis, Sullivan, and others, rigidity problems occupy a central place in the theory of holomorphic dynamical systems. This type of problems is classical in dynamics: a rigidity theorem postulates that in a certain class of dynamical systems equivalence (combinatorial, continuous, smooth, etc.) automatically has a higher regularity. The dynamical systems considered in this paper are critical circle maps, that is smooth homeomorphisms of the circle with a single critical point having an odd type. These maps have been a subject of intensive study since the early 1980's as one of the two main examples of universality in transition to chaos. Yoccoz in [17] generalized Denjoy's classical result, a critical circle homeomorphism with irrational rotation number is topologically conjugate to an irrational rotation.

*safarovua@mail.ru

Definition 1.1. *The point $x_{cr} \in S^1$ is called non-flat critical point of a homeomorphism f with order $(2m+1)$, $m \in N$, if for a some δ-neighborhood $U_\delta(x_{cr})$, the function f belongs to the class of $C^{2m+1}(U_\delta(x_{cr}))$ and*

$$f'(x_{cr}) = f''(x_{cr}) = \cdots = f^{(2m)}(x_{cr}) = 0, \quad f^{(2m+1)}(x_{cr}) \neq 0.$$

The order of the critical point x_{cr} is $2m+1$. By a *critical circle map* we define an orientation preserving circle homeomorphism with exactly one non-flat critical point of odd type.

An important one-parameter family of examples of critical circle maps are the Arnold's maps defined by

$$f_\theta(x) := x + \theta + \frac{1}{2\pi}\sin 2\pi x \ (mod 1), \quad x \in S^1.$$

For every $\theta \in R^1$ the map f_θ is a critical map with critical point 0 of cubic type.

Graczyk and Swiatek in [7] proved that if f is C^3 smooth circle homeomorphism with finitely many critical points of polynomial type and an irrational rotation number of bounded type, then the conjugating map φ is singular function on S^1 i.e. $\varphi'(x) = 0$ a.e. on S^1. Consequently, the invariant measure of critical circle homeomorphisms is singular w.r.t. Lebesque measure on S^1. Hence the problem of regularity of the conjugacy between two critical maps with identical irrational rotation number arises naturally. This is called the rigidity problem for critical circle homeomorphisms. For the critical circle maps the rigidity problem is developed by de Faria, de Melo, Yampolsky, Khanin and Teplinsky, Guarino among others.

The first result concerning on rigidity for critical maps was proven by de Melo and de Faria [6].

Theorem 1.1 (see [6])**.** *If f_1, f_2 are C^3 critical circle mappings with the same irrational rotation number of bounded type and the same power-law at the critical point, then there exists a $C^{1+\alpha}$ conjugacy h between f_1 and f_2 for some universal $\alpha > 0$.*

The following result of D. Khmelev and M. Yampolski [14] seemed to indicate that the analytic case could be different.

Theorem 1.2 ([14])**.** *There exists a universal constant $\alpha > 0$ such that the following holds. Let f_1 and f_2 be two analytic critical circle maps with the same irrational rotation number. Denote $h : S^1 \to S^1$ conjugacies between f_1 and f_2 fixing the critical points. Then h is $C^{1+\alpha}$ at the critical point.*

K. Khanin and A. Teplinskii [11] proved that any two f_1 and f_2 analytic critical circle maps with the same order of critical points and the same irrational rotation number are C^1-smoothly conjugate to each other. Later, A. Avila [2] showed, that there exist f_1 and f_2 analytic homeomorphisms with the same irrational rotation number such that h is not $C^{1+\alpha}$ for any $\alpha > 0$.

Next we formulate the result of P. Guarino, M. Martens, and W. de Melo [8].

Theorem 1.3 ([8])**.** *Let f_1 and f_2 be two analytic C^4-circle homeomorphisms with the same irrational rotation number and with a unique critical point of the same odd type. Then they are C^1-smoothly conjugate to each other. The conjugacy is $C^{1+\alpha}$ for Lebesgue almost every rotation number.*

The present work continuous and completes the above results. Namely we show that if the rotation numbers of two critical homeomorphisms coincide but the orders of critical points are different then the conjugacy h is a singular function. Now we formulate our main result.

Theorem 1.4. *Let f_1 and f_2 be C^3 critical circle maps with the same irrational rotation number. Suppose that the orders of critical points of f_1 and f_2 are different i.e. $2m_1+1 \neq 2m_2+1$. Then the conjugacy h between f_1 and f_2 is a singular function on S^1.*

2. Notations, terminology, background

Let f be a circle homeomorphism that preserves orientation, i.e. $f(x) = F(x)(mod\,1)$, $x \in S^1 \simeq [0,1)$, where F is continuous, strictly increasing on R^1 and $F(x+1) = F(x)+1$ for any $x \in R$. F is called *lift* of homeomorphism f. The important characteristic of the circle homeomorphism f is it's *rotation number* (see for instance [6]) ρ_f which defined by $\rho_f = \lim\limits_{n\to\infty} \frac{F^n(x)}{n}(mod\,1)$, here and later F^n denotes the n-th iteration of F. The rotation number ρ_f is rational if and only if f has periodic orbits.

2.1. Dynamical partition. Let f be an orientation preserving homeomorphism of the circle with lift F and irrational rotation number $\rho = \rho_f$. We denote by $\{a_n, n \in \mathbb{N}\}$ the sequence of entries in the continued fraction expansion of ρ, i.e. $\rho = [a_1, a_2, \ldots, a_n, \ldots]$. Denote by $p_n/q_n = [a_1, a_2, \ldots, a_n]$ the convergents of ρ. Their denominators q_n satisfy the recurrence relation, that is $q_{n+1} = a_{n+1}q_n + q_{n-1},\ n \geqslant 1,\ q_0 = 1,\ q_1 = a_1$.

For an arbitrary point $x_0 \in S^1$ we define $\Delta_0^{(n)}(x_0)$ the closed interval on S^1 with endpoints x_0 and $x_{q_n} = f^{q_n}(x_0)$. Note that for odd n the point x_{q_n} lies to the left of x_0 and for even n to the right. Denote by $\Delta_i^{(n)}(x_0)$ the iterates of the interval $\Delta_0^{(n)}(x_0)$ under f:$\Delta_i^{(n)}(x_0) := f^i(\Delta_0^{(n)}(x_0))$, $i \geqslant 1$.

Lemma 2.1 (see [12]). *Consider an arbitrary point $x_0 \in S^1$. A finite piece $\{x_i, 0 \leqslant i < q_n + q_{n-1}\}$ of the trajectory of this point divides the circle into the following disjoint (except for the endpoints) intervals: $\Delta_i^{(n-1)}(x_0),\ 0 \leqslant i < q_n$, $\Delta_j^{(n)}(x_0),\ 0 \leqslant j < q_{n-1}$.*

We denote the obtained partition by $\xi_n(x_0)$ and call it n-th *dynamical partition* of the circle. We now briefly describe the process of transition from $\xi_n(x_0)$ to $\xi_{n+1}(x_0)$. All intervals $\Delta_j^{(n)}(x_0),\ 0 \leqslant j < q_{n-1}$, are preserved, and each of the intervals $\Delta_i^{(n-1)}(x_0)$ is divided into $a_{n+1} + 1$ sub intervals:

$$\Delta_i^{(n-1)}(x_0) = \Delta_i^{(n+1)}(x_0) \cup \bigcup_{s=0}^{a_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^{(n)}(x_0).$$

Obviously one has $\xi_1(x_0) \leqslant \xi_2(x_0) \leqslant \ldots \leqslant \xi_n(x_0) \leqslant \ldots$.

Definition 2.1. *Let $K > 1$ be a constant. We call two intervals I_1 and I_2 of S^1 are K-comparable, if the inequalities $K^{-1}\mu(I_2) \leqslant \mu(I_1) \leqslant K\mu(I_2)$ hold.*

Next we formulate the lemma, that is proved in the similar way as in [16].

Let $x_{cr} \in S^1$ be a critical point of homeomorphism f. For any $x_0 \in S^1$, consider the dynamical partition $\xi_n(x_0)$. For definiteness we assume that n is odd. Then $x_{q_n} \prec x_0 \prec x_{q_{n-1}}$. The structure of the dynamical partition implies that $\overline{x}_{cr} = f^{-p}(x_{cr}) \in [x_{q_n}, x_{q_{n-1}}]$, for some p, $0 < p < q_n$. Let I_1 and I_2 be any elements of a dynamical partition $\xi_m(\overline{x}_{cr})$, $m \geqslant n$ having a common endpoints.

Lemma 2.2. *Let $f \in C^3(S^1)$ be a critical circle homeomorphism with irrational rotation number. Then there exists a constant $K > 1$ depending only on f such that the intervals I_1 and I_2 are K-comparable.*

It follows from the Lemma 2.2 that the trajectory of each point is dense in S^1. Hence it follows that there exists conjugation map φ between f and f_ρ, i.e. $\varphi(f(x)) = f_\rho(\varphi(x))$ for any $x \in S^1$.

We assume that $\Delta^{(m+k)}$ is element of partitioning $\xi_{m+k}(\overline{x}_{cr})$, while $\Delta^{(m)}$ is an element of partitioning $\xi_m(\overline{x}_{cr})$ that contains $\Delta^{(m+k)}$.

Lemma 2.3 (see [10])**.** *There exist constants $\lambda_1(f) < \lambda_2(f) < 1$ such that*

$$\ell(\Delta^{(m+k)}) \leqslant const\, \lambda_2^k(f)\ell(\Delta^{(m)}), \quad \ell(\Delta_0^{(m)}) \geqslant const\, \lambda_1^m(f).$$

2.2. Cross-ratio tools. In the proof of our main theorem the tool of cross-ratio plays a key role.

Definition 2.2. *The cross-ratio of four points (z_1, z_2, z_3, z_4), $z_1 < z_2 < z_3 < z_4$ is the number*

$$Cr(z_1, z_2, z_3, z_4) = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_1)(z_4 - z_2)}.$$

Definition 2.3. *Given four real numbers (z_1, z_2, z_3, z_4) with $z_1 < z_2 < z_3 < z_4$ and a strictly increasing function $F : \mathbb{R}^1 \to \mathbb{R}^1$. The distortion of their cross-ratio under F is given by*

$$Dist(z_1, z_2, z_3, z_4; F) = \frac{Cr(F(z_1), F(z_2), F(z_3), F(z_4))}{Cr(z_1, z_2, z_3, z_4)}.$$

For $m \geqslant 3$ and $z_i \in S^1$, $1 \leqslant i \leqslant m$, suppose that $z_1 \prec z_2 \prec \cdots \prec z_m \prec z_1$ (in the sense of the ordering on the circle). Then we set $\hat{z}_1 := z_1$ and

$$\hat{z}_i := \begin{cases} z_i & \text{if } z_1 < z_i < 1, \\ 1 + z_i & \text{if } 0 < z_i < z_1. \end{cases}$$

for $2 \leqslant i \leqslant m$.

Obviously, $\hat{z}_1 < \hat{z}_2 < \ldots < \hat{z}_m$. The vector $(\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_m)$ is called the lifted vector of $(z_1, z_2, \ldots, z_m) \in (S^1)^m$.

Let f be a circle homeomorphism with lift F. We define the cross-ratio distortion of (z_1, z_2, z_3, z_4), $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$ with respect to f by $Dist(z_1, z_2, z_3, z_4; f) = Dist(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4; F)$, where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of (z_1, z_2, z_3, z_4). We need the following lemma.

Lemma 2.4 ([5])**.** *Let $z_i \in S^1$, $i = 1, 2, 3, 4$, $z_1 \prec z_2 \prec z_3 \prec z_4$. Consider a circle homeomorphism f with $f \in C^{2+\varepsilon}([z_1, z_4])$, $\varepsilon > 0$, and $f'(x) \geqslant const > 0$ for $x \in [z_1, z_4]$. Then there is a positive constant $C_1 = C_1(f)$ such that*

$$|\, Dist(z_1, z_2, z_3, z_4; f) - 1 \,| \leqslant C_1 |\hat{z}_4 - \hat{z}_1|^{1+\varepsilon},$$

where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of (z_1, z_2, z_3, z_4).

We now consider the case when the interval $[z_1, z_4]$ contains a critical point x_{cr} of the homeomorphism f. More precisely, suppose that $z_2 = x_{cr}$. We define numbers α, β, γ, ξ and η as follows:

$$\alpha := \hat{z}_2 - \hat{z}_1, \quad \beta := \hat{z}_3 - \hat{z}_2, \quad \gamma := \hat{z}_4 - \hat{z}_3, \quad \xi := \frac{\beta}{\alpha}, \quad \eta := \frac{\beta}{\gamma},$$

where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of (z_1, z_2, z_3, z_4).

Thus we need the following lemma.

Lemma 2.5. *Suppose that a homeomorphism f with lift F has a critical point x_{cr} with order $2m+1$, $m \in N$. Then for any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon) > 0$, such that for all $z_i \in U_\delta(x_{cr})$, $i = \overline{1,n}$, $z_1 \prec z_2 = x_{cr} \prec z_3 \prec z_4$ one has*

$$\left|Dist(z_1, z_2, z_3, z_4; f) - \frac{1}{1-\xi+\xi^2-\cdots+\xi^{2m}} \times \frac{e_{2m}\eta^{2m} + e_{2m-1}\eta^{2m-1} + \cdots + e_1\eta + 1}{\eta^{2m} + C_{2m}^1\eta^{2m-1} + \cdots + C_{2m}^{2m-1}\eta + 1}\right| < R_0\varepsilon,$$

where the constants $e_{2m} = 2m+1$, $e_i = C_{2m}^i + C_{2m-1}^{i-1} + \cdots + C_{2m-i}^0$ and R_0 depends only on function f.

Proof. Fix a number ε. It is easy to check that for any $z_i \in S^1$, $i = \overline{1,n}$, $z_1 \prec z_2 \prec z_3 \prec z_4$ one has

$$F(z_1) = F(\hat{z}_2) - F'(\hat{z}_2)(\hat{z}_2 - \hat{z}_1) + \cdots + \frac{F^{(2m)}(\hat{z}_2)}{2m!}(\hat{z}_2 - \hat{z}_1)^{2m} - \frac{1}{2m!}\int_{\hat{z}_1}^{\hat{z}_2} F^{(2m+1)}(y)(y - \hat{z}_1)^{2m}dy,$$

$$\begin{aligned} F(\hat{z}_s) =& F(\hat{z}_2) + F'(\hat{z}_2)(\hat{z}_s - \hat{z}_2) + \cdots + \frac{F^{(2m)}(\hat{z}_2)}{2m!}(\hat{z}_s - \hat{z}_2)^{2m} + \\ &+ \frac{1}{2m!}\int_{\hat{z}_2}^{\hat{z}_s} F^{(2m+1)}(y)(\hat{z}_s - y)^{2m}dy, \quad s = 3, 4. \end{aligned} \tag{2.1}$$

By the assumption of the lemma, $z_2 = x_{cr}$, and using the (2.1) we write $Cr(f(z_1), f(z_2), f(z_3), f(z_4))$ as follows

$$\begin{aligned} Cr(f(z_1), f(z_2), f(z_3), f(z_4)) = \frac{(F(\hat{z}_2) - F(\hat{z}_1))(F(\hat{z}_4) - F(\hat{z}_3))}{(F(\hat{z}_3) - F(\hat{z}_1))(F(\hat{z}_4) - F(\hat{z}_2))} = \\ = \frac{\int_{\hat{z}_1}^{\hat{z}_2} F^{(2m+1)}(y)(y - \hat{z}_1)^{2m}dy}{\int_{\hat{z}_2}^{\hat{z}_3} F^{(2m+1)}(y)(\hat{z}_3 - y)^{2m}dy + \int_{\hat{z}_1}^{\hat{z}_2} F^{(2m+1)}(y)(y - \hat{z}_1)^{2m}dy} \times \\ \times \frac{\int_{\hat{z}_2}^{\hat{z}_4} F^{(2m+1)}(y)(\hat{z}_4 - y)^{2m}dy - \int_{\hat{z}_2}^{\hat{z}_3} F^{(2m+1)}(y)(\hat{z}_3 - y)^{2m}dy}{\int_{\hat{z}_2}^{\hat{z}_4} F^{(2m+1)}(y)(\hat{z}_4 - y)^{2m}dy}, \end{aligned} \tag{2.2}$$

where $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$ is the lifted vector of (z_1, z_2, z_3, z_4). Since $F^{(2l+1)} \in C(U_\omega(x_{cr}))$, there exist $\delta(\varepsilon) > 0$, such that for any $x, y \in (x_{cr} - \omega, x_{cr} + \omega)$ the inequality $|F^{(2m+1)}(x) - F^{(2m+1)}(y)| < \varepsilon$ is true.

Hence from (2.2) we have

$$\begin{aligned} Cr(f(z_1), f(z_2), f(z_3), f(z_4)) = \\ = \frac{\int_{\hat{z}_1}^{\hat{z}_2} F^{(2m+1)}(x_{cr})(y - \hat{z}_1)^{2m}dy(1 + O(\varepsilon))}{\left(\int_{\hat{z}_2}^{\hat{z}_3} F^{(2m+1)}(x_{cr})(\hat{z}_3 - y)^{2m}dy + \int_{\hat{z}_1}^{\hat{z}_2} F^{(2m+1)}(x_{cr})(y - \hat{z}_1)^{2m}dy\right)(1 + O(\varepsilon))} \times \\ \times \frac{\left(\int_{\hat{z}_2}^{\hat{z}_4} F^{(2m+1)}(x_{cr})(\hat{z}_4 - y)^{2m}dy - \int_{\hat{z}_2}^{\hat{z}_3} F^{(2m+1)}(x_{cr})(\hat{z}_3 - y)^{2m}dy\right)(1 + O(\varepsilon))}{\int_{\hat{z}_2}^{\hat{z}_4} F^{(2m+1)}(x_{cr})(\hat{z}_4 - y)^{2m}dy(1 + O(\varepsilon))} = \end{aligned}$$

$$= \frac{\alpha^{2m+1}}{\alpha^{2m+1} + \beta^{2m+1}} \cdot \frac{(\gamma+\beta)^{2m+1} - \beta^{2m+1}}{(\gamma+\beta)^{2m+1}} (1 + O(\varepsilon)).$$

From the last equality it follows that

$$Dist(z_1, z_2, z_3, z_4; f) = \frac{1}{1 - \xi + \xi^2 - \cdots + \xi^{2m}} \times$$

$$\times \frac{(1+\eta)^{2m} + (1+\eta)^{2m-1}\eta + \cdots + (1+\eta)\eta^{2m-1} + \eta^{2m}}{(1+\eta)^{2m}} (1 + O(\varepsilon)) =$$

$$= \frac{1}{1 - \xi + \xi^2 - \cdots + \xi^{2m}} \times \frac{e_{2m}\eta^{2m} + e_{2m-1}\eta^{2m-1} + \cdots + e_1\eta + 1}{\eta^{2m} + C^1_{2m}\eta^{2m-1} + \cdots + C^{2m-1}_{2m}\eta + 1} (1 + O(\varepsilon)).$$

Thus Lemma 2.5 is proved. □

Next suppose the interval $[z_1, z_4]$ is a subset of the interval $U_\omega(x_{cr})$ but does not contain a critical point x_{cr} of the homeomorphism f. Let $d = \min_{1 \leqslant s \leqslant 4} \ell([z_s, x_{cr}])$. We now state an assertion from [10].

Lemma 2.6 (see [10])**.** *Suppose that a homeomorphism f satisfies the conditions of Lemma 2.5. Then the following equality holds*

$$Dist(z_1, z_2, z_3, z_4; f) = 1 + O\left(\left(\frac{\alpha + \beta + \gamma}{d}\right)^2\right).$$

3. Proof of Theorem 1.4

In order to prove Theorem 1.4 we need several lemmas which we formulate next. Their proofs will be given later. We consider two copies of the unit circle S^1. The homeomorphism f_1 acts on the first circle and f_2 acts on the second one. Assume that f_i, $i = 1, 2$ satisfies the conditions of Theorem 1.4.

Let φ_1 and φ_2 be conjugations of f_1 and f_2 to linear rotation f_ρ, i.e. $\varphi_1 \circ f_1 = f_\rho \circ \varphi_1$ and $\varphi_2 \circ f_2 = f_\rho \circ \varphi_2$. It is easy to check that the homeomorphisms f_1 and f_2 are conjugated by $h = \varphi_2 \circ \varphi_1^{-1}$, i. e. $h \circ f_1(x) = f_2 \circ h(x), \forall x \in S^1$. Recall that every φ_i, $i = 1, 2$ is unique up to an additional constant. This gives us a possibility to choose h with initial condition $h(x^{(1)}_{cr}) = x^{(2)}_{cr}$.

Notice the conjugation $h(x)$ is continuous function on S^1. It suffices to show that $h'(x) = 0$ for almost all x with respect to the Lebesgue measure. The derivative $h'(x) = 0$ exists for almost all x with respect to the Lebesgue measure because the function h is monotonic. Let us show that $h'(x) = 0$ at all points where the derivative is defined.

Lemma 3.1 (see [5])**.** *Assume, that the conjugating homeomorphism $h(x)$ has a positive derivative $h'(x_0) = \omega_0$ at some point $x_0 \in S^1$, and that the following conditions hold for the points $z_i \in S^1$, $i = 1, \ldots, 4$, with $z_1 \prec z_2 \prec z_3 \prec z_4$, and some constant $R_1 > 1$:*

- *(a) the intervals $[z_1, z_2], [z_2, z_3], [z_3, z_4]$ are pairwise R_1-comparable;*
- *(b) $\max_{1 \leqslant i \leqslant 4} \ell([z_i, x_0]) \leqslant R_1 \ell([z_1, z_2])$.*

Then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|Dist(z_1, z_2, z_3, z_4; h) - 1| \leqslant C_2\varepsilon, \tag{3.1}$$

if $z_i \in (x_0 - \delta,\ x_0 + \delta)$ for all $i = 1, 2, 3, 4$, where the constant $C_2 > 0$ depends only on R_1, ω_0 and not on ε.

Suppose that $h'(x_0) = \omega_0$, where $x_0 \in S^1$. Let $\xi_n(x_0)$ be its n-th dynamical partition. Put $t_0 := h(x_0)$ and consider the dynamical partition $\tau_n(t_0)$ of t_0 on the second circle determined by the homeomorphism f_2, i.e.

$$\tau_n(t_0) = \{I_i^{(n-1)}(t_0),\quad 0 \leqslant i \leqslant q_n - 1\} \cup \{I_j^{(n)}(t_0),\quad 0 \leqslant j \leqslant q_{n-1} - 1\}$$

with $I_0^{(n)}(t_0)$ the closed interval with endpoints t_0 and $f_2^{q_n}(t_0)$. Choose an odd natural number $n_1 = n(f_1, f_2)$ such that the n_1-th renormalization neighborhoods $[x_{q_{n_1}}, x_{q_{n_1-1}}]$ and $[t_{q_{n_1}}, t_{q_{n_1-1}}]$ do not contain critical point of f_1 and f_2 respectively. Since the identical rotation number ρ of f_1 and f_2 is irrational, the order of the points on the orbit $\{f_1^k(x_0),\ k \in \mathbb{Z}\}$ on the first circle will be precisely the same as the one for the orbit $\{f_2^k(t_0),\ k \in \mathbb{Z}\}$ on the second one. This together with the relation $h(f_1(x)) = f_2(h(x))$ for $x \in S^1$ implies that

$$h(\Delta_i^{(n_1-1)}) = I_i^{(n_1-1)},\quad 0 \leqslant i \leqslant q_{n_1} - 1,\quad h(\Delta_j^{(n_1)}) = I_j^{(n_1)},\quad 0 \leqslant j \leqslant q_{n_1-1} - 1. \tag{3.2}$$

The structure of the dynamical partitions implies that $\overline{x}_{cr}^{(1)}(n_1) = f_1^{-l}(x_{cr}^{(1)}) \in [x_{q_{n_1}}, x_{q_{n_1-1}}]$, where $l \in (0, q_{n_1-1})$ if $\overline{x}_{cr}^{(1)}(n_1) \in [x_{q_{n_1}}, x_0]$, and $l \in (0, q_{n_1})$ if $\overline{x}_{cr}^{(1)}(n_1) \in [x_0, x_{q_{n_1-1}}]$. Since h conjugation between f_1 and f_2, we get

$$f_2^l(h(\overline{x}_{cr}^{(1)})) = f_2^{l-1}(f_2(h(\overline{x}_{cr}^{(1)}))) = f_2^{l-1}(h(f_1(\overline{x}_{cr}^{(1)}))) = \cdots = h(f_1^l(\overline{x}_{cr}^{(1)})) = h(x_{cr}^{(1)}) = x_{cr}^{(2)}.$$

Hence $\overline{x}_{cr}^{(2)}(n_1) = f_2^{-l}(x_{cr}^{(2)}) \in [t_{q_{n_1}}, t_{q_{n_1-1}}]$. The points $\overline{x}_{cr}^{(1)}(n_1)$ and $\overline{x}_{cr}^{(2)}(n_1)$ are called the q_{n_1}-*pre-images* of the critical points $x_{cr}^{(1)}$ and $x_{cr}^{(2)}$, respectively.

Next we introduce the concept of a "regular" cover of the critical point. Let $z_i \in S^1,\ i = \overline{1,4}$, $z_1 \prec z_2 \prec z_3 \prec z_4 \prec z_1$. Define for each j, $0 < j < q_n$

$$\xi_{f_1}(j) = \frac{\ell\big([f_1^j(z_2), f_1^j(z_3)]\big)}{\ell\big([f_1^j(z_1), f_1^j(z_2)]\big)},\quad \eta_{f_1}(j) = \frac{\ell\big([f_1^j(z_2), f_1^j(z_3)]\big)}{\ell\big([f_1^j(z_3), f_1^j(z_4)]\big)}.$$

Definition 3.1. *Let $M > 1$, $\zeta \in (0,1)$, $\delta > 0$ be constant numbers, n is a positive integer and $x_0 \in S^1$. We say that a triple of intervals $([z_1, z_2], [z_2, z_3], [z_3, z_4])$, $z_i \in S^1$, $i = 1,2,3,4$, covers the critical point of $x_{cr}^{(1)}$ "$(M, \zeta, \theta, \delta; x_0)$-regularly", if the following conditions hold:*

1) $[z_1, z_4] \subset (x_0 - \delta, x_0 + \delta)$, and the system of intervals $\{f_1^j([z_1, z_4]),\ 0 \leqslant j \leqslant q_n - 1\}$ cover critical point $x_{cr}^{(1)}$ only once;

2) $z_2 = f_1^{-l}(x_{cr}^{(1)})$ for some l, $0 < l < q_n$;

3) $\xi_{f_1}(l) < \zeta$ and $\eta_{f_1}(l) \geqslant M$.

Denote

$$L = \min\{2m_1 + 1,\ 2m_2 + 1,\ 2|m_1 - m_2|\}.$$

Lemma 3.2. *Suppose that the homeomorphisms f_i, $i = 1,2$ satisfy the conditions of Theorem 1.4. Then for any $x_0 \in S^1$ and $\delta > 0$ there exist constant $M_0 > 1$ and $\zeta_0 \in (0,1)$, such that for all triples of intervals $[z_s, z_{s+1}] \subset (x_0 - \delta,\ x_0 + \delta)$, $s = 1,2,3$, and $[h(z_s), h(z_{s+1})]$, $s = 1,2,3$, covering the critical points $x_{cr}^{(1)}$ and $x_{cr}^{(2)}$ regularly with constants M_0 and ζ_0 the following inequalities hold:*

$$\Big|\frac{1}{1 - \xi_{f_1}(l) + \cdots + \xi_{f_1}^{2m_1}(l)} \times \frac{e_{2m_1}\eta_{f_1}^{2m_1}(l) + e_{2m_1-1}\eta_{f_1}^{2m_1-1}(l) + \cdots + 1}{\eta_{f_1}^{2m_1}(l) + C_{2m_1}^1\eta_{f_1}^{2m_1-1}(l) + \cdots + 1} - (2m_1 + 1)\Big| < \frac{L}{16},$$

$$\Big|\frac{1}{1-\xi_{f_2}(l)+\cdots+\xi_{f_2}^{2m_2}(l)}\times\frac{e_{2m_2}\eta_{f_2}^{2m_2}(l)+e_{2m_2-1}\eta_{f_2}^{2m_2-1}(l)+\cdots+1}{\eta_{f_2}^{2m_2}(l)+C_{2m_2}^{1}\eta_{f_2}^{2m_2-1}(l)+\cdots+1}-(2m_2+1)\Big|<\frac{L}{16},$$

where m_1 and m_2 are orders of critical points $x_{cr}^{(1)}$ and $x_{cr}^{(2)}$ respectively.

Assume that the homeomorphism f_1 satisfies the conditions of Theorem 1.4. Let $\xi_n(x_{cr}^{(1)})$ be a dynamical partition of the circle by f_1. We take a natural number r, such that $\Delta_0^{(r)}(x_{cr}^{(1)})\cup\Delta_0^{(r-1)}(x_{cr}^{(1)})\subset U_{\omega_1}(x_{cr}^{(1)})$. Suppose that $h'(x_0)=p_0>0$ for some $x_0\in S^1$. Consider the dynamical partition $\xi_n(x_0)$ of the point x_0 under f_1. Suppose that $n>r$ an odd natural number. Let $\overline{x}_{cr}^{(1)}=f^{-l}(x_{cr}^{(1)})\in[x_{q_n},x_{q_{n-1}}]$.

Let $\{\xi_{n+k}(\overline{x}_{cr}^{(1)})\}_{k=0}^{\infty}$ be a sequence of dynamical partitions of the point $\overline{x}_{cr}$. We define the points z_i, $i=1,2,3,4$ as follows

$$z_1=f^{q_{n+k_0}}(\overline{x}_{cr}^{(1)}),\quad z_2=\overline{x}_{cr}^{(1)},\quad z_3=f^{q_{n+k_0+k_1}}(\overline{x}_{cr}^{(1)}),\quad z_4=f^{q_{n+k_0+k_1}+q_{n+k_2}}(\overline{x}_{cr}^{(1)}).$$

Lemma 3.3. *Suppose that the homeomorphisms f_1 and f_2 satisfies the conditions of Theorem 1.4. Let $h'(x_0)=p_0>0$ for some $x_0\in S^1$, $\delta\in(0,1)$ and $k_0\in N$. Then there exist natural numbers k_1, k_2 such that for sufficiently large n, the triple of intervals $[z_s,z_{s+1}]\subset(x_0-\delta,\ x_0+\delta)$, $s=1,2,3$ satisfies the following properties:*

(1) the intervals $\{[f_1^j(z_1),f_1^j(z_4)],\ 0\leqslant j\leqslant q_n\}$ cover each point at most once;

(2) the intervals $[z_s,z_{s+1}]$ and $[f_1^{q_n}(z_s),f_1^{q_n}(z_{s+1})]$, $s=1,2,3$ satisfy conditions (a) and (b) of Lemma 3.1 with some constant $R_1>1$ depending on k_0, k_1, k_2;

(3) the triples of intervals ($[z_s,z_{s+1}]$, $s=1,2,3$) and ($[h(z_s),h(z_{s+1})]$, $s=1,2,3$) cover the critical points $x_{cr}^{(1)}$, $x_{cr}^{(2)}$, "$(M_0,\zeta_0,\delta;x_0)$-regularly" and "$(M_0,\zeta_0,\delta;h(x_0))$-regularly", respectively.

Lemma 3.4. *Suppose the circle homeomorphisms f_1 and f_2 satisfy the conditions of Theorem 1.4. Then there exists natural number k_0 such that for intervals $[z_s,z_{s+1}]$, $s=1,2,3$ satisfying conditions (1)–(3) of Lemma 3.3, and for sufficiently large n the following inequality holds*

$$\Big|\frac{Dist(z_1,z_2,z_3,z_4;f_1^{q_n})}{Dist(h(z_1),h(z_2),h(z_3),h(z_4);f_2^{q_n})}-1\Big|\geqslant R_2>0,\tag{3.3}$$

where the constant R_2 depends only on f_1 and f_2.

Proof of Theorem 1.4. Let f_1 and f_2 be circle homeomorphisms satisfying the conditions of Theorem 1.4. The lift $H(x)$ of the conjugating map $h(x)$ is a continuous and monotone increasing function on R^1. Hence $H(x)$ has a finite derivative $H'(x)$ for almost all x with respect to Lebesgue measure. We claim that $h'(x)=0$ at all points x where the finite derivative exists. Suppose $h'(x_0)>0$ for some point $x_0\in S^1$. Fix $\varepsilon>0$. We take a triple of intervals $[z_s,z_{s+1}]\subset(x_0-\delta,\ x_0+\delta)$, $s=1,2,3$, satisfying the conditions of Lemma 3.4.

Using the assertion of Lemma 3.1 we obtain

$$\Big|Dist(z_1,z_2,z_3,z_4;h)-1\Big|\leqslant C_3\varepsilon,\tag{3.4}$$

$$\Big|Dist(f_1^{q_n}(z_1),f_1^{q_n}(z_2),f_1^{q_n}(z_3),f_1^{q_n}(z_4);h)-1\Big|\leqslant C_3\varepsilon.\tag{3.5}$$

Hence

$$\Big|\frac{Dist(z_1,z_2,z_3,z_4;h)}{Dist(f_1^{q_n}(z_1),f_1^{q_n}(z_2),f_1^{q_n}(z_3),f_1^{q_n}(z_4);h)}-1\Big|\leqslant C_4\varepsilon,\tag{3.6}$$

where the constant $C_4>0$ does not depend on ε and n.

Since h is conjugating f_1 and f_2 we can readily see that

$$Cr(h(f_1^{q_n}(z_1)), h(f_1^{q_n}(z_2)), h(f_1^{q_n}(z_3)), h(f_1^{q_n}(z_4))) =$$

$$= Cr(f_2^{q_n}(h(z_1)), f_2^{q_n}(h(z_2)), f_2^{q_n}(h(z_3)), f_2^{q_n}(h(z_4))).$$

Hence we obtain

$$\frac{Dist(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4); h)}{Dist(z_1, z_2, z_3, z_4; h)} =$$

$$= \frac{Cr(h(f_1^{q_n}(z_1)), h(f_1^{q_n}(z_2)), h(f_1^{q_n}(z_3)), h(f_1^{q_n}(z_4)))}{Cr(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4))} \times$$

$$\times \frac{Cr(z_1, z_2, z_3, z_4)}{Cr(h(z_1), h(z_2), h(z_3), h(z_4))} = \frac{Cr(f_2^{q_n}(h(z_1)), f_2^{q_n}(h(z_2)), f_2^{q_n}(h(z_3)), f_2^{q_n}(h(z_4)))}{Cr(h(z_1), h(z_2), h(z_3), h(z_4))} :$$

$$: \frac{Cr(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4))}{Cr(z_1, z_2, z_3, z_4)} = \frac{Dist(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{q_n})}{Dist(z_1, z_2, z_3, z_4; f_1^{q_n})}.$$

This, together with (3.6) obviously implies that

$$\Big| \frac{Dist(z_1, z_2, z_3, z_4; f_1^{q_n})}{Dist(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{q_n})} - 1 \Big| \leqslant C_5 \varepsilon,$$

where the constant $C_5 > 0$ does not depend on ε and n. This contradicts equation (3.3). Therefore Theorem 1.4 is completely proved. □

4. The proofs of Lemmas 3.2–3.4

Proof of Lemma 3.2. Denote

$$\psi_1(\xi_{f_1}(l)) = \frac{1}{1 - \xi_{f_1}(l) + \cdots + \xi_{f_1}^{2m_1}(l)},$$

and

$$\psi_2(\eta_{f_1}(l)) = \frac{e_{2m_1}\eta_{f_1}^{2m_1}(l) + e_{2m_1-1}\eta_{f_1}^{2m_1-1}(l) + \cdots + 1}{\eta_{f_1}^{2m_1}(l) + C_{2m_1}^1 \eta_{f_1}^{2m_1-1}(l) + \cdots + 1}.$$

It is easy to check that for $\eta_{f_1}(l) > 0$ the function $\psi_2(\eta_{f_1}(l))$ is monotone increasing and $1 < \psi_2(\eta_{f_1}(l)) < 2m_1 + 1$. Obviously

$$\lim_{\xi_{f_1}(l)\to 0} \psi_1(\xi_{f_1}(l)) = 1, \quad \lim_{\eta_{f_1}(l)\to\infty} \psi_2(\eta_{f_1}(l)) = 2m_1 + 1.$$

Taking these remarks into account and using the explicit form of the functions $\psi_1(\xi_{f_1}(l))$ and $\psi_2(\eta_{f_1}(l))$ we can now estimate $\mid \psi_1 \cdot \psi_2 - (2m_1 + 1) \mid$. Firstly, we estimate ψ_2 for large value of $\eta_{f_1}(l)$. Using the explicit form of the function $\psi_2(\eta_{f_1}(l))$, we see that the inequality

$$|\psi_2 - (2m_1 + 1)| = O\left(\frac{1}{\eta_{f_1}(l)}\right) \leqslant R_3 \left(\frac{1}{\eta_{f_1}(l)}\right), \tag{4.1}$$

where the constant $R_3 > 0$ depends only on f_1. If we choose $\eta_{f_1}(l)$ satisfying the inequality $R_2\left(\frac{1}{\eta_{f_1}(l)}\right) < \frac{L}{32}$, then

$$|\psi_2(\eta_{f_1}(l)) - (2m_1 + 1)| < \frac{L}{32},$$

for $\eta_{f_1}(l) > \dfrac{32R_3}{L}$.

We next estimate $|\psi_1 - 1|$ for small value of $\xi_{f_1}(l)$. Using the explicit form of the function $\psi_1(\xi_{f_1}(l))$, we see that $|\psi_1(\xi_{f_1}(l)) - 1| = O(\xi_{f_1}(l)) \leqslant R_4\xi_{f_1}(l)$. It follows from this together with (4.1) that $|\psi_1 \cdot \psi_2 - (2m_1+1)| \leqslant |\psi_2 - (2m_1+1)| + |\psi_2| \cdot |\psi_1 - 1| \leqslant \dfrac{L}{32} + (2m_1+1)R_4\xi_{f_1}(l)$. If we take

$$\zeta_1 := \min\Big\{\frac{L}{32(2m_1+1)R_5}, 1\Big\}, \quad M_1 := \max\Big\{\frac{32R_5}{L}, 1\Big\},$$

where $R_5 = \max\{R_3, R_4\}$, then for all $\xi_{f_1}(l) < \zeta_1$ and $\eta_{f_1}(l) > M_1$ the following inequality holds

$$|\psi_1 \cdot \psi_2 - (2m_1+1)| \leqslant \frac{L}{16}.$$

Similarly it can be shown that with

$$\zeta_2 := \min\Big\{\frac{L}{32(2m_2+1)R_6}, 1\Big\}, \quad M_2 := \max\Big\{\frac{32R_6}{L}, 1\Big\}, \tag{4.2}$$

and $\xi_{f_2}(l) < \zeta_2$ and $\eta_{f_2}(l) > M_2$, the second assertion of Lemma 3.2 holds. In (4.2) the constants $R_6 > 0$ depends only on f_2. Finally, if we set $\zeta_0 := \min\{\zeta_1, \zeta_2\}$ and $M_0 := \max\{M_1, M_2\}$, then Lemma 3.2 holds for $\xi_{f_1}(l), \xi_{f_2}(l) \in [0, \zeta_0)$ and $\eta_{f_1}(l), \eta_{f_2}(l) \geqslant M_0$. Lemma 3.2 is proved. □

Proof of Lemma 3.3. Firstly, we prove the third assertion of the lemma. By the construction of the points z_i, $i = 1,2,3,4$, it implies that the intervals $[z_s, z_{s+1}]$ and $[h(z_s), h(z_{s+1})]$, $s = 1,2,3$ satisfy the 1) and 2) conditions of definition of "regularly" covering. We consider dynamical partition $\xi_n(x_{cr}^{(1)})$. According to Lemma 2.2 the intervals $\Delta_0^{(n)}(x_{cr}^{(1)})$ and $\Delta_0^{(n-1)}(x_{cr}^{(1)})$ are K-comparable, i.e. there exist constant $K > 1$ such that $K^{-1}\ell(\Delta_0^{(n-1)}(x_{cr}^{(1)})) \leqslant \ell(\Delta_0^{(n)}(x_{cr}^{(1)})) \leqslant K\ell(\Delta_0^{(n-1)}(x_{cr}^{(1)}))$. Thus it follows that there exists $k_1^{(1)} \in N$ such that the following inequality holds

$$\frac{\ell\big([x_{cr}^{(1)}, f_1^{q_{n+k_0+k_1^{(1)}}}(x_{cr}^{(1)})]\big)}{\ell\big([f_1^{q_{n+k_0}}(x_{cr}^{(1)}), x_{cr}^{(1)}]\big)} < \zeta_0. \tag{4.3}$$

Indeed, it is clear that

$$\frac{\ell\big(\Delta_0^{(q_{n+k_0+3})}(x_{cr}^{(1)})\big)}{\ell\big(\Delta_0^{(q_{n+k_0+1})}(x_{cr}^{(1)})\big)} = \frac{1}{\frac{\ell\left(\Delta_0^{(q_{n+k_0+1})}(x_{cr}^{(1)})\right)}{\ell\left(\Delta_0^{(q_{n+k_0+3})}(x_{cr}^{(1)})\right)}} \leqslant \frac{1}{1+\frac{1}{K}} = \frac{K}{K+1}.$$

Hence $\ell\big(\Delta_0^{(q_{n+k_0+3})}(x_{cr}^{(1)})\big) \leqslant \frac{K}{K+1}\ell\big(\Delta_0^{(q_{n+k_0+1})}(x_{cr}^{(1)})\big)$. Using the last inequality we obtain that for any k

$$\ell\big(\Delta_0^{(q_{n+k_0+k})}(x_{cr}^{(1)})\big) \leqslant \Big(\frac{K}{K+1}\Big)^k \ell\big(\Delta_0^{(q_{n+k_0+1})}(x_{cr}^{(1)})\big).$$

Since $\Delta_0^{(q_{n+k_0+1})}(x_{cr}^{(1)})$ and $\Delta_0^{(q_{n+k_0})}(x_{cr}^{(1)})$ are K-comparable, there exists a $k_1^{(1)} \in N$ such that the inequality (4.3) is true. Similarly, we can show that there exists a $k_2^{(1)} \in N$ such that the following inequality holds

$$\frac{\ell\big([x_{cr}^{(1)}, f_1^{q_{n+k_0+k_1}}(x_{cr}^{(1)})]\big)}{\ell\Big([f_1^{q_{n+k_0+k_1}}(x_{cr}^{(1)}), f_1^{q_{n+k_0+k_1^{(1)}}+q_{n+k_2^{(1)}}}(x_{cr}^{(1)})]\Big)} > M_0.$$

Similarly, it can be shown that with natural numbers $k_1^{(2)}$ and $k_1^{(2)}$ the inequalities

$$\frac{\ell\big([x_{cr}^{(2)}, f_2^{q_{n+k_0+k_1^{(2)}}}(x_{cr}^{(2)})]\big)}{\ell\big([f_2^{q_{n+k_0}}(x_{cr}^{(2)}), x_{cr}^{(2)}]\big)} < \zeta_0, \quad \frac{\ell\big([x_{cr}^{(2)}, f_2^{q_{n+k_0+k_1^{(2)}}}(x_{cr}^{(2)})]\big)}{\ell\Big([f_2^{q_{n+k_0+k_1^{(2)}}}(x_{cr}^{(2)}), f_2^{q_{n+k_0+k_1}+q_{n+k_2^{(2)}}}(x_{cr}^{(2)})]\Big)} > M_0$$

hold. If we take $k_1 = \max\{k_1^{(1)}, k_1^{(2)}\}$ and $k_2 = \max\{k_2^{(1)}, k_2^{(2)}\}$ then the third assertion of Lemma 3.3 holds for k_1 and k_2. By the definition of the points z_i, $i = 1, 2, 3$ it implies the first assertion of the lemma.

Let $\xi_n(\overline{x}_{cr}^{(1)})$ be a dynamical partition of the point $\overline{x}_{cr}^{(1)}$. According to Lemma 2.2 the intervals $\Delta_0^{(n)}(\overline{x}_{cr}^{(1)})$ and $\Delta_0^{(n-1)}(\overline{x}_{cr}^{(1)})$ are K-comparable. Hence, it implies that the intervals $[z_s, z_{s+1}]$, $s = 1, 2, 3$ are pairwise $K^{k_1+k_2}$- comparable. It is easy to see that the intervals $[f_1^{q_n}(z_s), f_1^{q_n}(z_{s+1})]$, $s = 1, 2, 3$ are pairwise $K^{k_1+k_2}$-comparable. Obviously,

$$\frac{1}{K^{k_0+1}} \leqslant \frac{\ell\big(\Delta_0^{(n-1)}(\overline{x}_{cr}^{(1)})\big)}{\ell\big([z_1, z_2]\big)} \leqslant K^{k_0+1}, \quad \frac{1}{K^{k_0+1}} \leqslant \frac{\ell\big(\Delta_0^{(n-1)}(\overline{x}_{cr}^{(1)})\big)}{\ell\big([f_1^{q_n}(z_1), f_1^{q_n}(z_2)]\big)} \leqslant K^{k_0+1}.$$

Since the intervals $\Delta_0^{(n-1)}(\overline{x}_{cr}^{(1)})$ and $\Delta_0^{(n-1)}\big(f_1^{-q_{n-1}}(\overline{x}_{cr}^{(1)})\big)$ are K-comparable and $x_0 \in \Delta_0^{(n-1)}\big(f_1^{-q_{n-1}}(\overline{x}_{cr}^{(1)})\big) \cup \Delta_0^{(n-1)}\big(\overline{x}_{cr}^{(1)}\big)$ we get

$$\max_{1\leqslant i\leqslant 4} \{\ell\big([f^{q_n}(z_i), x_0]\big), \ell([z_i, x_0])\} \leqslant (K+1)K^{k_0+1}\ell\big([z_1, z_2]\big).$$

If we take $R_1 = (K+1)K^{k_0+k_1+k_2}$, then we obtain the proof of the second assertion of Lemma 3.3 with constant R_1. Lemma 3.3 is proved. □

Proof of Lemma 3.4. Suppose, the triples of intervals ($[z_s, z_{s+1}]$, $s=1, 2, 3$) and ($[h(z_s), h(z_{s+1})]$, $s = 1, 2, 3$) satisfy the conditions of Lemma 3.3. We want to compare the distortion $Dist(z_1, z_2, z_3, z_4; f_1^{q_n})$ and $Dist(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{q_n})$. We estimate only the first distortion, the second one can be estimated analogously. Obviously

$$Dist(z_1, z_2, z_3, z_4; f_1^{q_n}) = \prod_{i=0}^{q_n-1} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1).$$

We denote

$$J_r(x_{cr}^{(1)}) = \Delta_0^{(r)}(x_{cr}^{(1)}) \cup \Delta_0^{(r-1)}(x_{cr}^{(1)}), \quad A = \{i : (f_1^i(z_1), f_1^i(z_4)) \cap J_r(x_{cr}^{(1)}) = \emptyset\},$$
$$B = \{i : (f_1^i(z_1), f_1^i(z_4)) \cap J_r(x_{cr}^{(1)}) \neq \emptyset\}.$$

It is clear that $A \cup B = \{0, 1, \ldots, q_n\}$.

Next we rewrite $Dist(z_1, z_2, z_3, z_4; f_1^{q_n})$ in the form

$$Dist(z_1, z_2, z_3, z_4; f_1^{q_n}) = \prod_{i\in A} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1)\times$$
$$\times \prod_{i\in B} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1). \tag{4.4}$$

We estimate the first factor in (4.4). Using the Lemmas 2.4 we obtain

$$\Big|\prod_{i\in A} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1) - 1\Big| = \Big|\prod_{i\in A}\Big(1 + O\big(\ell([f_1^i(z_1), f_1^i(z_4)])\big)^{1+\nu}\Big) - 1\Big| =$$
$$= \max_i \big(\ell([f_1^i(z_1), f_1^i(z_4)])\big)^{\nu} O\Big(\sum_{i\in A} \ell\big([f_1^i(z_1), f_1^i(z_4)]\big)\Big) = O(\lambda_{f_1}^{n\nu}),$$

where $\nu > 0$ and $0 < \lambda_{f_1} < 1$. We fix $\varepsilon > 0$. There exists $N_0 = N_0(\varepsilon) \geqslant 1$ such that for any $n \geqslant N_0$ the estimate

$$\Big| \prod_{i \in A} Dist\big(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1\big) - 1 \Big| < C_6 \varepsilon \tag{4.5}$$

holds. We now estimate the second factor in (4.4). We rewrite the second factor in the following form

$$\prod_{i \in B} Dist\big(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1\big) =$$

$$= \prod_{i \in B, i \neq l} Dist\big(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1\big) \times Dist\big(f_1^l(z_1), f_1^l(z_2), f_1^l(z_3), f_1^l(z_4); f_1\big). \tag{4.6}$$

By applying Lemmas 2.5 and 3.2 we obtain

$$|Dist\big(f_1^l(z_1), f_1^l(z_2), f_1^l(z_3), f_1^l(z_4); f_1\big) - (2m_1 + 1)| < \frac{L}{8}. \tag{4.7}$$

Using Lemma 2.6 for the first factor in (4.6), we get

$$\Big| \prod_{i \in B, i \neq l} Dist\big(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1\big) - 1 \Big| = \Big| \prod_{i \in B, i \neq l} \bigg(1 + O\Big(\frac{\ell([f_1^i(z_1), f_1^i(z_4)])}{d_i}\Big)^2\bigg) - 1 \Big| =$$

$$= \Big| \exp\Big\{ \sum_{i \in B, i \neq l} \log\bigg(1 + O\Big(\frac{\ell([f_1^i(z_1), f_1^i(z_4)])}{d_i}\Big)^2\bigg)\Big\} - 1 \Big| \leqslant const \sum_{i \in B, i \neq l} \bigg(\frac{\ell([f_1^i(z_1), f_1^i(z_4)])}{d_i}\bigg)^2 =$$

$$= const \sum_{q=0}^{n-r} \sum_{i : [f_1^i(z_1), f_1^i(z_4)] \subset (J_{n-q}(x_{cr}^{(1)}) \setminus J_{n-q+1}(x_{cr}^{(1)})), i \neq l} \bigg(\frac{\ell([f_1^i(z_1), f_1^i(z_4)])}{d_i}\bigg)^2.$$

Obviously,

$$\sum_{i : [f_1^i(z_1), f_1^i(z_4)] \subset (J_{n-q}(x_{cr}^{(1)}) \setminus J_{n-q+1}(x_{cr}^{(1)})), i \neq l} \bigg(\frac{\ell([f_1^i(z_1), f_1^i(z_4)])}{d_i}\bigg) = const$$

and it follows from Lemma 2.3 that $\dfrac{\ell([f_1^i(z_1), f_1^i(z_4)])}{d_i} \leqslant const\, \lambda_{f_1}^{k_0+1+q}$. Consequently

$$\Big| \prod_{i \in B, i \neq l} Dist(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1) - 1 \Big| \leqslant C_7 \lambda_{f_1}^{k_0}, \tag{4.8}$$

where $C_7 > 0$ depends only on f_1.

Similarly one can show that for the triple of intervals $([h(z_s), h(z_{s+1})],\ s = 1, 2, 3)$ the following inequality

$$\Big| \prod_{i \in B, i \neq l} Dist(f_2^i(h(z_1)), f_2^i(h(z_2)), f_2^i(h(z_3)), f_2^i(h(z_4)); f_2) - 1 \Big| \leqslant C_8 \lambda_{f_2}^{k_0}, \tag{4.9}$$

where $C_8 > 0$ depends only on f_2 and $0 \leqslant \lambda_{f_2} \leqslant 1$ is defined in Lemma 2.3.

If we choose

$$k_0 = \max\Big\{ \Big[\log_{\lambda_{f_1}} \frac{L}{(16m_1 + 8 + L)C_7}\Big] + 1, \Big[\log_{\lambda_{f_2}} \frac{L}{(16m_2 + 8 + L)C_8}\Big] + 1 \Big\},$$

where constants $0 \leqslant \lambda_{f_1}, \lambda_{f_2} \leqslant 1$ are defined in Lemma 2.3, then from the relations (4.4)–(4.8) it implies that for sufficiently large n

$$|Dist(z_1, z_2, z_3, z_4; f_1^{q_n}) - (2m_1 + 1)| < \frac{L}{4}. \tag{4.10}$$

Similarly

$$|Dist(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{q_n}) - (2m_2 + 1)| < \frac{L}{4}. \tag{4.11}$$

The inequalities (4.10) and (4.11) implies

$$\frac{Dist(z_1, z_2, z_3, z_4; f_1^{q_n})}{Dist(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{q_n})} - 1 \geqslant \frac{8(m_1 - m_2) - 2L}{8m_2 + L + 4} > 0, \tag{4.12}$$

if $m_1 > m_2$, and

$$\frac{Dist(z_1, z_2, z_3, z_4; f_1^{q_n})}{Dist(h(z_1), h(z_2), h(z_3), h(z_4); f_2^{q_n})} - 1 \leqslant \frac{8(m_1 - m_2) + 2L}{8m_2 - L + 4} < 0, \tag{4.13}$$

if $m_1 < m_2$. If we set

$$R_2 := \min\left\{\frac{|8(m_1 - m_2) - 2L|}{8m_2 - L + 4}, \frac{|8(m_1 - m_2) + 2L|}{8m_2 + L + 4}\right\}, \tag{4.14}$$

then it follows from (4.12)–(4.14) that the assertion of the lemma holds.

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О сопряжение между двумя критическими отображениями окружности

Уткир А. Сафаров
Туринский политехнический университет
Ташкент, Узбекистан
Ташкентский государственный экономический университет
Ташкент, Узбекистан

Аннотация. В статье изучается сопряжение между двумя критическими гомеоморфизмами окружности с иррациональным числом вращения. Пусть f_i, $i = 1, 2$ являются C^3-гомеоморфизмы окружности с критической точкой $x_{cr}^{(i)}$ порядка $2m_i + 1$. Доказано, что если $2m_1 + 1 \neq 2m_2 + 1$, то сопряжение между f_1 и f_2 — сингулярная функция.

Ключевые слова: гомеоморфизм окружности, критическая точка, сопрягащий гомеоморфизм, число вращения, сингулярная функция.

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Limits of Risks Ratios of Shrinkage Estimators under the Balanced Loss Function

Mekki Terbeche*
University of Sciences and Technology, Mohamed Boudiaf, Oran
Laboratory of Analysis and Application of Radiation (LAAR), USTO-MB
Oran, Algeria

Abdelkader Benkhaled†
Mascara University, Mustapha Stambouli
Laboratory of Geomatics, Ecology and Environment (LGEO2E), Mascara University
Mascara, Algeria

Abdenour Hamdaoui‡
University of Sciences and Technology, Mohamed Boudiaf, Oran
Laboratory of Statistics and Random Modelisations (LSMA) of Tlemcen University
Oran, Algeria

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Abstract. In this paper we study the estimation of a multivariate normal mean under the balanced loss function. We present here a class of shrinkage estimators which generalizes the James-Stein estimator and we are interested to establish the asymptotic behaviour of risks ratios of these estimators to the maximum likelihood estimators (MLE). Thus, in the case where the dimension of the parameter space and the sample size are large, we determine the sufficient conditions for that the estimators cited previously are minimax.

Keywords: balanced Loss Function, James-Stein estimator, multivariate Gaussian random variable, non-central chi-square distribution, shrinkage estimators.

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Introduction

The multivariate analysis plays an essential role in statistical data analysis. Thus, the mean parameters estimation of the multivariate Gaussian distribution is of interest to many users. Stein [1] showed the inadmissibility of the usual estimator when the dimension of the parameter space is greater than or equal to three by considering an alternative estimator with uniformly smaller risk than the latter, the improvement being substantial for the mean close to the origin. A central focus is on the general technique, namely, shrinkage estimation. This is systematically applied to derive the MLE of the mean parameters. A large amount of research have been carried

*mekki.terbeche@gmail.com
†benkhaled08@yahoo.fr
‡abdenour.hamdaoui@yahoo.fr; abdenour.hamdaoui@univ-usto.dz

out to develop the properties of shrinkage estimators and to compare them with the MLE. For a selected review of the subject matter of shrinkage estimation, interested readers may refer to Stein [1], James and Stein [2] and Efron and Morris [3].

When the dimension of the parameter space and the sample size are large, Benmansour and Hamdaoui [4] have taken the model $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ where the parameter σ^2 is unknown and estimated by S^2 ($S^2 \sim \sigma^2\chi^2_n$). The authors established the analogous results obtained by Casella and Hwang [5]. Benkhaled and Hamdaoui [6], have considered the same model given by Benmansour and Hamdaoui [4], namely $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ where σ^2 is unknown. They studied two different forms of shrinkage estimators of θ: estimators of the form $\delta^{\psi} = (1-\psi(S^2, \|X\|^2)S^2/\|X\|^2)X$, and estimators of Lindley-Type given by $\delta^{\varphi} = (1-\varphi(S^2, T^2)S^2/T^2)(X-\overline{X})+\overline{X}$, that shrink the components of the MLE X to the random variable $\overline{X}$. The authors showed that if the shrinkage function ψ (respectively φ) satisfies the new conditions different from the known results in the literature, then the estimator δ^{ψ} (respectively δ^{φ}) is minimax. When the sample size and the dimension of parameters space tend to infinity, they studied the behaviour of risks ratio of these estimators to the MLE. Hamdaoui et al. [7], have treated the minimaxity and limits of risks ratios of shrinkage estimators of a multivariate normal mean in the Bayesian case. The authors have considered the model $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ where σ^2 is unknown and have taken the prior law $\theta \sim N_p\left(\upsilon, \tau^2 I_p\right)$. They constructed a modified Bayes estimator δ^*_B and an empirical modified Bayes estimator δ^*_{EB}. When n and p are finite, they showed that the estimators δ^*_B and δ^*_{EB} are minimax. The authors have also interested in studying the limits of risks ratios of these estimators, to the MLE X, when n and p tend to infinity. The majority of these authors have been considered the quadratic loss function for computing the risk.

Zellner [8] proposes a balanced loss function that takes error of estimation and goodness of fit into account. This balanced loss function consists of weighting the predictive loss function and the goodness of fit term. In addition for estimation under the balanced loss function we cite for example, Guikai et al. [9], Karamikabir et al. [10]. Sanjari Farsipour and Asgharzadeh [11] have considered the model: $X_1, \ldots, X_n$ to be a random sample from $N_p\left(\theta, \sigma^2\right)$ with σ^2 known and the aim is to estimate the parameter θ. They studied the admissibility of the estimator of the form $a\overline{X}+b$ under the balanced loss function. Selahattin and Issam [12] introduced and derived the optimal extended balanced loss function (EBLF) estimators and predictors and discussed their performances. Under the balanced loss function, Hamdaoui et al. [13] studied the behavior of risks ratios of James-Stein estimator and the positive-part of James-Stein estimator to the MLE, when the dimension of the parameter space tends to infinity and the sample size is fixes and when the dimension of the parameter space and the sample size tend simultaneously to the infinity. They showed that these risks ratios tend to values less than 1. Thus, the authors have assured the stability of minimaxity property of the James-Stein estimator and the positive-part of James-Stein estimator in the large values of the dimension of the parameter space p and the sample size n.

In this work, we deal with the model $X \sim N_p\left(\theta, \sigma^2 I_p\right)$, where the parameter σ^2 is unknown and estimated by S^2 ($S^2 \sim \sigma^2\chi^2_n$). Our aim is to estimate the unknown parameter θ by shrinkage estimators deduced from the MLE. The adopted criterion to compare two estimators is the risk associated to the balanced loss function. The paper is organized as follows. In Section 1, we recall some preliminaries that are useful for our main results. In Section 2, we present the main results. Under the balanced loss function, we consider the general class of shrinkage estimators $\delta_{\varphi} = (1-\varphi(S^2, \|X\|^2)S^2/\|X\|^2)X$ which containing the James-Stein estimator and we study the behavior of risks ratio of these estimators to the MLE. Thus we generalized some obtained results in the our published papers for the case where the risks functions calculated relatively to the quadratic loss function.

1. Preliminaries

We recall that if X is a multivariate Gaussian random $N_p\left(\theta,\sigma^2 I_p\right)$ in $\mathbb{R}^p$, then $\dfrac{\|X\|^2}{\sigma^2}\sim\chi_p^2\left(\lambda\right)$ where $\chi_p^2\left(\lambda\right)$ denotes the non-central chi-square distribution with p degrees of freedom and non-centrality parameter $\lambda=\dfrac{\|\theta\|^2}{2\sigma^2}$. We also recall the following definition given in formula (1.2) by Arnold [14]. It will be used to calculate the expectation of functions of a non-central chi-square law's variable.

Definition 1. *Let $U\sim\chi_p^2\left(\lambda\right)$ be non-central chi-square with p degrees of freedom and non-centrality parameter λ. The density function of U is given by*

$$f(x)=\sum_{k=0}^{+\infty}\frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^k}{k!}\frac{x^{(p/2)+k-1}e^{-x/2}}{\Gamma(\frac{p}{2}+k)2^{(p/2)+k}},\quad 0<x<+\infty.$$

The right hand side (RHS) of this equality is none other than the formula

$$\sum_{k=0}^{+\infty}\frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^k}{k!}\chi_{p+2k}^2,$$

where χ_{p+2k}^2 is the density of the central χ^2 distribution with $p+2k$ degrees of freedom.

To this definition we deduce that if $U\sim\chi_p^2\left(\lambda\right)$, then for any function $f:\mathrm{R}_+\longrightarrow\mathrm{R}$, $\chi_p^2\left(\lambda\right)$ integrable, we have

$$\begin{aligned} E\left[f(U)\right] &= E_{\chi_p^2(\lambda)}\left[f(U)\right]= \\ &= \int_{\mathrm{R}_+}f(x)\chi_p^2\left(\lambda\right)dx= \\ &= \sum_{k=0}^{+\infty}\left[\int_{\mathrm{R}_+}f(x)\chi_{p+2k}^2\left(0\right)dx\right]e^{-\frac{\lambda}{2}}\frac{\left(\frac{\lambda}{2}\right)^k}{k!}= \\ &= \sum_{k=0}^{+\infty}\left[\int_{\mathrm{R}_+}f(x)\chi_{p+2k}^2dx\right]P\left(\frac{\lambda}{2};dk\right), \end{aligned} \tag{1}$$

where $P\left(\frac{\lambda}{2};dk\right)$ being the Poisson distribution of parameter $\dfrac{\lambda}{2}$ and χ_{p+2k}^2 is the central chi-square distribution with $p+2k$ degrees of freedom.

Using the Definition 1 and the Lemma 1 in Benmansour and Hamdaoui [4], we deduce that if $X\sim N_p\left(\theta,\sigma^2 I_p\right)$, then

$$\frac{1}{\sigma^2\left(p-2+\frac{\|\theta\|^2}{\sigma^2}\right)}\leqslant E\left(\frac{1}{\|X\|^2}\right)=\frac{1}{\sigma^2}E\left(\frac{1}{p-2+2K}\right)\leqslant\frac{p}{\sigma^2(p-2)\left(p+\frac{\|\theta\|^2}{\sigma^2}\right)}. \tag{2}$$

We recall the following Lemma given by Stein [15], that we will use often in the next.

Lemma 1. *Let X be a $N\left(\upsilon,\sigma^2\right)$ real random variable and let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be an indefinite integral of the Lebesgue measurable function, f' essentially the derivative of f. Suppose also that $E\left|f'\left(X\right)\right|<+\infty$, then*

$$E\left[\left(\frac{X-\upsilon}{\sigma}\right)f\left(X\right)\right]=E\left(f'\left(X\right)\right).$$

For the next, assume that $X\sim N_p\left(\theta,\sigma^2 I_p\right)$ where σ^2 is unknown and estimated by S^2 ($S^2\sim\sigma^2\chi^2_n$). Our aim is to estimate the unknown parameter θ under the balanced loss function defined as, for any estimator δ of θ:

$$L_\omega(\delta,\theta)=\omega\|\delta-\delta_0\|^2+(1-\omega)\|\delta-\theta\|^2,$$

where $0\leqslant\omega<1$. We associate to this balanced loss function the risk function defined by

$$R_\omega(\delta,\theta)=E(L_\omega(\delta,\theta)).$$

In this model, it is clear that the MLE is $\delta_0=X$, its risk function is $(1-\omega)p\sigma^2$.

Indeed:

$$R_\omega(X,\theta)=\omega E(\|X-X\|^2)+(1-\omega)E(\|X-\theta\|^2).$$

As $X\sim N_p\left(\theta,\sigma^2 I_p\right)$, then $\dfrac{X-\theta}{\sigma}\sim N_p\left(0,I_p\right)$, thus $\dfrac{\|X-\theta\|^2}{\sigma^2}\sim\chi^2_p$.

Hence

$$E(\|X-\theta\|^2)=E(\sigma^2\chi^2_p)=\sigma^2 p.$$

It is well known that δ_0 is minimax and inadmissible for $p\geqslant 3$, thus any estimator dominates it is also minimax.

Now, we consider the shrinkage estimator

$$\delta_\varphi=\Big(1-\varphi(S^2,\|X\|^2)\frac{S^2}{\|X\|^2}\Big)X. \tag{3}$$

In the special case when $\varphi(S^2,\|X\|^2)=a$, (i.e. $\delta_a=\Big(1-a\dfrac{S^2}{\|X\|^2}\Big)X$) where a is a real constant may depend on n and p. It is easy to show that a sufficient condition for that δ_a dominating the MLE, thus it is minimax, is that

$$0\leqslant a\leqslant\frac{2(p-2)(1-\omega)}{n+2}.$$

For $a=\widehat{a}=\dfrac{(p-2)(1-\omega)}{n+2}$, we obtain the estimator that minimizes the risk function of the estimators δ_a, and its called the James-Sten estimator given by

$$\delta_{JS}=\delta_{\widehat{a}}=\left(1-\widehat{a}\frac{S^2}{\|X\|^2}\right)X=\left(1-\frac{(1-\omega)(p-2)}{n+2}\frac{S^2}{\|X\|^2}\right)X. \tag{4}$$

Using the Definition 1 and the Lemma 1, one can prove that the risk function of δ_{JS} is

$$R_\omega(\delta_{JS},\theta)=(1-\omega)p\sigma^2-(1-\omega)^2(p-2)^2\frac{n}{n+2}\sigma^2 E\left(\frac{1}{p-2+2K}\right), \tag{5}$$

where $K\sim P\left(\dfrac{\|\theta\|^2}{2\sigma^2}\right)$.

From the formula 5, it is trivial that the James-Stein estimator δ_{JS} dominate the MLE, thus it is minimax. Furthermore, the Theorem 4.1 given in Hamdaoui et al [13] show that

$$\lim_{n,p\to+\infty} \frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)} = \frac{\omega+c}{1+c}. \tag{6}$$

Then one can deduce that the James-Stein estimators dominates the MLE, for the large values of n and p.

2. Main results

In the next we need the following Lemma that shows a explicit formula of the risk function of the estimator δ_φ given in (3), which helps us to compute the limit of risks ratio.

Lemma 2. *Assume the estimator δ_φ given in (3). Then*

$$\begin{aligned}\Delta_{\varphi,JS} &:= R_\omega(\delta_\varphi,\theta) - R_\omega(\delta_{JS},\theta) = \\ &= E\left((d-\varphi(S^2,\|X\|^2))^2\frac{(S^2)^2}{\|X\|^2} - 2d(d-\varphi(S^2,\|X\|^2))\frac{(S^2)^2}{\|X\|^2}\right) + \\ &+ 2(1-\omega)\times E\left((d-\varphi(S^2,\|X\|^2))S^2 - \lambda(d-\varphi(\sigma^2\chi^2_n,\sigma^2\chi^2_{p+2}(\lambda)))\frac{\sigma^2\chi^2_n}{\chi^2_{p+2}(\lambda)}\right),\end{aligned}$$

where $\delta_{JS} = \left(1 - d\frac{S^2}{\|X\|^2}\right)X$, $d = \frac{(1-\omega)(p-2)}{n+2}$ *and* $\lambda = \frac{\|\theta\|^2}{\sigma^2}$.

Proof.

$$\begin{aligned}R_\omega(\delta_\varphi,\theta) &= \omega E(\|\delta_\varphi - X\|^2) + (1-\omega)E(\|\delta_\varphi-\theta\|^2) = \\ &= \omega E(\|\delta_\varphi - \delta_{JS} + \delta_{JS} - X\|^2) + (1-\omega)E(\|\delta_\varphi - \delta_{JS} + \delta_{JS} - \theta\|^2) = \\ &= \omega\{E\left(\|\delta_\varphi-\delta_{JS}\|^2 + \|\delta_{JS}-X\|^2 + 2\langle\delta_\varphi-\delta_{JS},\delta_{JS}-X\rangle\right)\} + \\ &+ (1-\omega)\{E\left(\|\delta_\varphi-\delta_{JS}\|^2 + \|\delta_{JS}-\theta\|^2 + 2\langle\delta_\varphi-\delta_{JS},\delta_{JS}-\theta\rangle\right)\} = \\ &= R_\omega(\delta_{JS},\theta) + E(\|\delta_\varphi-\delta_{JS}\|^2) + 2E(\langle\delta_\varphi-\delta_{JS},\delta_{JS}-X\rangle) + \\ &+ 2(1-\omega)E(\langle\delta_\varphi-\delta_{JS},X-\theta\rangle).\end{aligned} \tag{7}$$

As

$$E(\|\delta_\varphi-\delta_{JS}\|^2) = E\left((d-\varphi(S^2,\|X\|^2))^2\frac{(S^2)^2}{\|X\|^2}\right), \tag{8}$$

$$E(\langle\delta_\varphi-\delta_{JS},\delta_{JS}-X\rangle) = -E\left(d(d-\varphi(S^2,\|X\|^2))\frac{(S^2)^2}{\|X\|^2}\right) \tag{9}$$

and

$$\begin{aligned}E(\langle\delta_\varphi-\delta_{JS},X-\theta\rangle) &= E\left(\langle(d-\varphi(S^2,\|X\|^2))\frac{S^2}{\|X\|^2}X, X-\theta\rangle\right) = \\ &= E((d-\varphi(S^2,\|X\|^2))S^2) - E\left(\langle X,\theta\rangle(d-\varphi(S^2,\|X\|^2))\frac{S^2}{\|X\|^2}\right) = \\ &= E((d-\varphi(S^2,\|X\|^2))S^2) - \\ &- \lambda E\left((d-\varphi(\sigma^2\chi^2_n,\sigma^2\chi^2_{p+2}(\lambda)))\frac{\chi^2_n}{\chi^2_{p+2}(\lambda)}\right).\end{aligned} \tag{10}$$

The last equality comes from the conditional expectation and the formula (2.7) given in Benmansour and Mourid Benmansour and Mourid [16]. Using the formulas (7–9) and (10), we deduce the desired result. $\square$

Theorem 1. *Assume the estimator δ_φ given in (3), with φ satisfies the conditions*

(H1) $\varphi \geqslant \dfrac{\sqrt{(1-\omega)}(p-2)}{n+2}$,

(H2) $|d-\varphi| \leqslant g(S^2)$ *a.s., where* $E\left[(g^2(S^2))^{1+\gamma}\right] = O\left(\dfrac{1}{n^{2(1+\gamma)}}\right)$ *for some* $\gamma > 0$, *in the neighborhood of* $+\infty$.

If $\lim\limits_{p\to+\infty} \dfrac{\|\theta\|^2}{p\sigma^2} = c$, *then*

$$\lim_{n,p\to+\infty} \frac{R_\omega(\delta_\varphi,\theta)}{R_\omega(X,\theta)} = \frac{\omega+c}{1+c}.$$

Proof. From (H2) we have

$$\begin{aligned}
\Delta_{\varphi,JS} &\leqslant E\left((d-\varphi(S^2,\|X\|^2))^2\frac{(S^2)^2}{\|X\|^2} + 2d|d-\varphi(S^2,\|X\|^2)|\frac{(S^2)^2}{\|X\|^2}\right) + 2(1-\omega)\times \\
&\times E\left(|d-\varphi(S^2,\|X\|^2)|S^2 + \lambda|d-\varphi(\sigma^2\chi^2_n,\sigma^2\chi^2_{p+2}(\lambda))|\frac{\sigma^2\chi^2_n}{\chi^2_{p+2}(\lambda)}\right) \leqslant \\
&\leqslant E\left(g^2(S^2)\frac{(S^2)^2}{\|X\|^2}\right) + 2dE\left(g(S^2)\frac{(S^2)^2}{\|X\|^2}\right) + 2(1-\omega)E(g(S^2)S^2) + \\
&+ 2(1-\omega)\lambda E\left(g(S^2)\frac{S^2}{\chi^2_{p+2}(\lambda)}\right).
\end{aligned}$$

From the independence between $\|X\|^2$ and S^2 and the holder inequality, we get

$$\begin{aligned}
\Delta_{\varphi,JS} &\leqslant E^{\frac{1}{1+\gamma}}\left((g(S^2))^{2(1+\gamma)}\right) E^{\frac{\gamma}{1+\gamma}}\left((S^2)^{2(\frac{1+\gamma}{\gamma})}\right) E\left(\frac{1}{\|X\|^2}\right) + \\
&+ 2dE^{\frac{1}{2(1+\gamma)}}\left((g(S^2))^{2(1+\gamma)}\right) E^{\frac{1+2\gamma}{2(1+\gamma)}}\left((S^2)^{\frac{4(1+\gamma)}{1+2\gamma}}\right) E\left(\frac{1}{\|X\|^2}\right) + \\
&+ 2(1-\omega)E^{\frac{1}{2(1+\gamma)}}\left((g(S^2))^{2(1+\gamma)}\right) E^{\frac{1+2\gamma}{2(1+\gamma)}}\left((S^2)^{\frac{2(1+\gamma)}{1+2\gamma}}\right) + \\
&+ 2(1-\omega)\lambda E^{\frac{1}{2(1+\gamma)}}\left((g(S^2))^{2(1+\gamma)}\right) E^{\frac{1+2\gamma}{2(1+\gamma)}}\left((S^2)^{\frac{2(1+\gamma)}{1+2\gamma}}\right) E\left(\frac{1}{\chi^2_{p+2}(\lambda)}\right).
\end{aligned}$$

As

$$E\left(\frac{1}{\|X\|^2}\right) = \frac{1}{\sigma^2}E\left(\frac{1}{p-2+2K}\right) \leqslant \frac{1}{\sigma^2}\frac{1}{p-2},$$

$$E\left(\frac{1}{\chi^2_{p+2}(\lambda)}\right) = E\left(\frac{1}{p+2K}\right) \leqslant \frac{1}{p-2}$$

and

$$d = \frac{(1-\omega)(p-2)}{n+2},$$

we obtain

$$\frac{\Delta_{\varphi,JS}}{R_\omega(X,\theta)} \leqslant \frac{4}{(1-\omega)p(p-2)} E^{\frac{1}{1+\gamma}}\left((g(S^2))^{2(1+\gamma)}\right)\left(\frac{\Gamma(\frac{n}{2}+\frac{2(1+\gamma)}{\gamma})}{\Gamma(\frac{n}{2})}\right)^{\frac{\gamma}{1+\gamma}} +$$
$$+\frac{8}{p(n+2)} E^{\frac{1}{2(1+\gamma)}}\left((g(S^2))^{2(1+\gamma)}\right)\left(\frac{\Gamma(\frac{n}{2}+\frac{4(1+\gamma)}{1+2\gamma})}{\Gamma(\frac{n}{2})}\right)^{\frac{1+2\gamma}{2(1+\gamma)}} +$$
$$+\frac{4}{p} E^{\frac{1}{2(1+\gamma)}}\left((g(S^2))^{2(1+\gamma)}\right)\left(\frac{\Gamma(\frac{n}{2}+\frac{2(1+\gamma)}{1+2\gamma})}{\Gamma(\frac{n}{2})}\right)^{\frac{1+2\gamma}{2(1+\gamma)}} +$$
$$+\frac{4}{\sigma^2(p-2)}\frac{\|\theta\|^2}{p\sigma^2} E^{\frac{1}{2(1+\gamma)}}\left((g(S^2))^{2(1+\gamma)}\right)\left(\frac{\Gamma(\frac{n}{2}+\frac{2(1+\gamma)}{1+2\gamma})}{\Gamma(\frac{n}{2})}\right)^{\frac{1+2\gamma}{2(1+\gamma)}}.$$

Now, from stirling's formula which expresses that in the neighborhood of $+\infty$, we have

$$\Gamma(y+1) \simeq \sqrt{2\pi} y^{y+\frac{1}{2}} e^{-y}$$

and the fact that

$$\lim_{n\to+\infty}\left(1+\frac{\alpha}{n}\right)^n = e^\alpha,$$

we have

$$\left(\frac{\Gamma(\frac{n}{2}+\frac{2(1+\gamma)}{\gamma})}{\Gamma(\frac{n}{2})}\right)^{\frac{\gamma}{1+\gamma}} \simeq \left(\frac{n}{2}+\frac{2}{\gamma}+1\right)^2,$$

$$\left(\frac{\Gamma(\frac{n}{2}+\frac{4(1+\gamma)}{1+2\gamma})}{\Gamma(\frac{n}{2})}\right)^{\frac{1+2\gamma}{2(1+\gamma)}} \simeq \left(\frac{n}{2}+\frac{2}{1+2\gamma}+1\right)^2$$

and

$$\left(\frac{\Gamma(\frac{n}{2}+\frac{2(1+\gamma)}{1+2\gamma})}{\Gamma(\frac{n}{2})}\right)^{\frac{1+2\gamma}{2(1+\gamma)}} \simeq \frac{n}{2}+\frac{1}{1+2\gamma}.$$

Then, in the neighborhood of $+\infty$ we have

$$\frac{\Delta_{\varphi,JS}}{R_\omega(X,\theta)} \leqslant \frac{4}{(1-\omega)p(p-2)} E^{\frac{1}{1+\gamma}}\left((g(S^2))^{2(1+\gamma)}\right)\left(\frac{n}{2}+\frac{2}{\gamma}+1\right)^2 +$$
$$+\frac{8}{p(n+2)} E^{\frac{1}{2(1+\gamma)}}\left((g(S^2))^{2(1+\gamma)}\right)\left(\frac{n}{2}+\frac{2}{1+2\gamma}+1\right)^2 +$$
$$+\frac{4}{p} E^{\frac{1}{2(1+\gamma)}}\left((g(S^2))^{2(1+\gamma)}\right)\left(\frac{n}{2}+\frac{1}{1+2\gamma}\right) +$$
$$+\frac{4}{\sigma^2(p-2)}\frac{\|\theta\|^2}{p\sigma^2} E^{\frac{1}{2(1+\gamma)}}\left((g(S^2))^{2(1+\gamma)}\right)\left(\frac{n}{2}+\frac{1}{1+2\gamma}\right).$$

Using the condition $E((g^2(S^2))^{2(1+\gamma)}) = O\left(\frac{1}{n^{2(1+\gamma)}}\right)$, then it exists $M>0$ such that

$$\lim_{n,p\to+\infty}\frac{\Delta_{\varphi,JS}}{R_\omega(X,\theta)} \leqslant \lim_{n,p\to+\infty}\left\{\frac{4}{(1-\omega)p(p-2)} M^{\frac{1}{1+\gamma}}\frac{1}{n^2}\left(\frac{n}{2}+\frac{2}{\gamma}+1\right)^2 +\right.$$

$$+\frac{8}{p(n+2)}M^{\frac{1}{2(1+\gamma)}}\frac{1}{n}\left(\frac{n}{2}+\frac{2}{1+2\gamma}+1\right)^2+$$
$$+\frac{4}{p}M^{\frac{1}{2(1+\gamma)}}\frac{1}{n}\left(\frac{n}{2}+\frac{1}{1+2\gamma}\right)+$$
$$\left.+\frac{4}{\sigma^2(p-2)}\frac{\|\theta\|^2}{p\sigma^2}M^{\frac{1}{2(1+\gamma)}}\frac{1}{n}\left(\frac{n}{2}+\frac{1}{1+2\gamma}\right)\right\}+$$
$$=0,$$

from formula 6, we get

$$\lim_{n,p\to+\infty}\frac{R_\omega(\delta_\varphi,\theta)}{R_\omega(X,\theta)}\leqslant\lim_{n,p\to+\infty}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}=\frac{\omega+c}{1+c}. \tag{11}$$

In the other hand

$$R_\omega(\delta_\varphi,\theta)=\omega E\left(\varphi^2(S^2,\|X\|^2)\frac{(S^2)^2}{\|X\|^2}\right)+(1-\omega)E\left(\left\|\left(1-\varphi(S^2,\|X\|^2)\frac{S^2}{\|X\|^2}\right)X-\theta\right\|^2\right),$$

and

$$E\left\|\left(1-\varphi(S^2,\|X\|^2)\frac{S^2}{\|X\|^2}\right)X-\theta\right\|^2=E\left\{\sum_{i=1}^{p}\left[\left(1-\varphi(S^2,\|X\|^2)\frac{S^2}{\|X\|^2}\right)X_i-\theta_i\right]^2\right\}=$$
$$=E\left\{\sum_{i=1}^{p}\left(1-\varphi(S^2,\|X\|^2)\frac{S^2}{\|X\|^2}\right)^2X_i^2+\sum_{i=1}^{p}\theta_i^2-2\sum_{i=1}^{p}\left(1-\varphi(S^2,\|X\|^2)\frac{S^2}{\|X\|^2}\right)X_i\theta_i\right\}=$$
$$=E\left\{\left(1-\varphi(S^2,\|X\|^2)\frac{S^2}{\|X\|^2}\right)^2\|X\|^2+2\sigma^2K-2\left(1-\varphi(S^2,\|X\|^2)\frac{S^2}{\|X\|^2}\right)\sum_{i=1}^{p}X_i\theta_i\right\}.$$

Using (b) of Lemma 3.1 in Hamdaoui and Benmansour [17], we obtain

$$E\left\|\left(1-\varphi(S^2,\|X\|^2)\frac{S^2}{\|X\|^2}\right)X-\theta\right\|^2=\sigma^2E\left\{\left(1-\varphi(\sigma^2\chi_n^2,\sigma^2\chi_{p+2K}^2)\frac{\sigma^2\chi_n^2}{\sigma^2\chi_{p+2K}^2}\right)^2\chi_{p+2K}^2+\right.$$
$$\left.+2K-4\left(1-\varphi(\sigma^2\chi_n^2,\sigma^2\chi_{p+2K}^2)\frac{\sigma^2\chi_n^2}{\sigma^2\chi_{p+2K}^2}\right)\right\}=$$
$$=\sigma^2E\left\{\left(\varphi(\sigma^2\chi_n^2,\sigma^2\chi_{p+2K}^2)\frac{\chi_n^2}{\chi_{p+2K}^2}-1+\frac{2K}{\chi_{p+2K}^2}\right)^2\chi_{p+2K}^2\right\}+$$
$$+\sigma^2E\left\{p-\frac{(\chi_{p+2K}^2-2K)^2}{\chi_{p+2K}^2}\right\}.$$

Using the conditional expectation we get

$$E\left\{p-\frac{(\chi_{p+2K}^2-2K)^2}{\chi_{p+2K}^2}\right\}=E\left\{E\left[p-\chi_{p+2K}^2-\frac{4K^2}{\chi_{p+2K}^2}+4K|K\right]\right\}=$$
$$=E\left\{p-(p+2K)-\frac{4K^2}{p-2+2K}+4K\right\}=$$
$$=E\left\{p-2-(p-2+2K)-\frac{4K^2}{p-2+2K}+4K\right\}=$$

$$= E\left\{p - 2 - \frac{(p-2)^2}{p-2+2K}\right\}.$$

From the hypotheses (H1), we deduce that

$$R_\omega(\delta_\varphi,\theta) \geqslant \omega\left[\frac{(1-\omega)(p-2)^2 n\sigma^2}{n+2} E\left(\frac{1}{p-2+2K}\right)\right] + (1-\omega)\sigma^2 E\left\{p-2-\frac{(p-2)^2}{p-2+2K}\right\}.$$

Using the last formula and the formula (2), we have

$$\frac{R_\omega(\delta_\varphi,\theta)}{R_\omega(X,\theta)} \geqslant \left[\frac{\omega(p-2)^2 n}{p(n+2)} \frac{1}{p-2+\frac{\|\theta\|^2}{\sigma^2}})\right] + (1-\omega)\sigma^2 E\left\{1-\frac{2}{p}-(p-2)^2\frac{1}{(p-2)(p+\frac{\|\theta\|^2}{\sigma^2})}\right\}.$$

From the condition $\lim\limits_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2} = c$, we obtain

$$\lim_{n,p\to+\infty}\frac{R_\omega(\delta_\varphi,\theta)}{R_\omega(X,\theta)} \geqslant \frac{\omega+c}{1+c}. \tag{12}$$

The formulas 11 and 12 give the desired result. □

The following Proposition gives the same result as Theorem 1 for a particular shrinkage function φ. Indeed, we will choose g in L^2 and note in $L^{2(1+\gamma)}$ but with the constraint that the function g is monotone non-increasing.

Proposition 1. *Assume the estimator δ_φ given in (3), with φ satisfies the condition*

(H1) $\varphi \geqslant \dfrac{(1-\omega)^{1/2}(p-2)}{n+2}$,

(H2) $|d-\varphi| \leqslant g(S^2)$ *a.s., where g is monotone non-increasing and* $E\left((g^2(S^2))\right) = O\left(\dfrac{1}{n^2}\right)$ *in the neighborhood of* $+\infty$.

If $\lim\limits_{p\to+\infty}\dfrac{\|\theta\|^2}{p\sigma^2} = c$, *then*

$$\lim_{n,p\to+\infty}\frac{R_\omega(\delta_\varphi,\theta)}{R_\omega(X,\theta)} = \frac{\omega+c}{1+c}.$$

Proof. From (H2) we have

$$\begin{aligned}
\Delta_{\varphi,JS} &\leqslant E\left((d-\varphi(S^2,\|X\|^2))^2\frac{(S^2)^2}{\|X\|^2} + 2d|d-\varphi(S^2,\|X\|^2)|\frac{(S^2)^2}{\|X\|^2}\right) + 2(1-\omega)\times\\
&\times E\left(|d-\varphi(S^2,\|X\|^2)|S^2 + \lambda|d-\varphi(\sigma^2\chi^2_n,\sigma^2\chi^2_{p+2}(\lambda))|\frac{\sigma^2\chi^2_n}{\chi^2_{p+2}(\lambda)}\right) \leqslant\\
&\leqslant E\left((g(S^2))^2\frac{(S^2)^2}{\|X\|^2}\right) + 2dE\left(g(S^2)\frac{(S^2)^2}{\|X\|^2}\right) + 2(1-\omega)E(g(S^2)S^2) +\\
&+ 2(1-\omega)\lambda E\left(g(S^2)\frac{S^2}{\chi^2_{p+2}(\lambda)}\right).
\end{aligned}$$

As g is monotone non-increasing, the covariance of two functions, one increasing and the other decreasing is negative and the fact that $E\left(\dfrac{1}{\|X\|^2}\right) = \dfrac{1}{\sigma^2}E\left(\dfrac{1}{p-2+2K}\right)$, we obtain

$$\Delta_{\varphi,JS} \leqslant E((g(S^2))^2)\sigma^2 n(n+2)E\left(\frac{1}{p-2+2K}\right) + 2\frac{(1-\omega)(p-2)}{n+2}E(g(S^2))\sigma^2\times$$

$$\times\, n(n+2)E\left(\frac{1}{p-2+2K}\right)+2n(1-\omega)E(g(S^2))\left\{\sigma^2+\lambda E\left(\frac{1}{p+2K}\right)\right\}.$$

Then

$$\frac{\Delta_{\varphi,JS}}{R_\omega(X,\theta)}\leqslant\frac{n(n+2)}{p(1-\omega)}E((g(S^2))^2)E\left(\frac{1}{\chi^2_{p+2K}}\right)+\frac{2n(p-2)}{p}E(g(S^2))E\left(\frac{1}{\chi^2_{p+2K}}\right)+$$
$$+\frac{2n}{p}E(g(S^2))+\frac{2n\lambda}{p\sigma^2}E(g(S^2))E\left(\frac{1}{\chi^2_{p+2+2K}}\right).$$

From condition $E((g(S^2))^2)=O\left(\frac{1}{n^2}\right)$ and using the Schwarz inequality, when n is in the neighbourhood of $+\infty$, we obtain

$$E(g(S^2))\leqslant E^{1/2}((g(S^2))^2)\leqslant\sqrt{M}\frac{1}{n},$$

where M is a real strictly positive. Then, when n is in the neighbourhood of $+\infty$, we have

$$\frac{\Delta_{\varphi,JS}}{R_\omega(X,\theta)}\leqslant\frac{M}{p(1-\omega)}E\left(\frac{1}{p-2+2K}\right)+\frac{2(p-2)\sqrt{M}}{p}E\left(\frac{1}{p-2+2K}\right)+$$
$$+\frac{2\sqrt{M}}{p}+\frac{2\sqrt{M}}{\sigma^2}\frac{\|\theta\|^2}{p\sigma^2}E\left(\frac{1}{p+2K}\right)=$$
$$=\frac{M}{p(1-\omega)}\left(\frac{p}{p-2}\right)\left(\frac{1}{p+\frac{\|\theta\|^2}{\sigma^2}}\right)+\frac{2(p-2)\sqrt{M}}{p}\left(\frac{p}{p-2}\right)\left(\frac{1}{p+\frac{\|\theta\|^2}{\sigma^2}}\right)+$$
$$+\frac{2\sqrt{M}}{p}+\frac{2\sqrt{M}}{\sigma^2}\left(\frac{\|\theta\|^2}{p\sigma^2}\right)\left(\frac{p+2}{p}\right)\left(\frac{1}{p+2+\frac{\|\theta\|^2}{\sigma^2}}\right).$$

As $\lim\limits_{p\to+\infty}\dfrac{\|\theta\|^2}{p\sigma^2}=c$, then

$$\lim_{p\to+\infty}\frac{\Delta_{\varphi,JS}}{R_\omega(X,\theta)}\leqslant 0,$$

thus

$$\lim_{n,p\to+\infty}\frac{R_\omega(\delta_\varphi,\theta)}{R_\omega(X,\theta)}\leqslant\lim_{n,p\to+\infty}\frac{R_\omega(\delta_{JS},\theta)}{R_\omega(X,\theta)}=\frac{\omega+c}{1+c}.$$

The proof of

$$\lim_{n,p\to+\infty}\frac{R_\omega(\delta_\varphi,\theta)}{R_\omega(X,\theta)}\geqslant\frac{\omega+c}{1+c}.$$

is the same given in the Theorem 1. □

Conclusion

In this work, we studied the estimation of the multivariate normal mean distribution $X\sim N_p\left(\theta,\sigma^2I_p\right)$ under the balanced loss function. We considered the class of estimators defined by $\delta_\varphi=\left(1-\varphi(S^2,\|X\|^2)S^2/\|X\|^2\right)X$ which are not necessarily minimax, and containing the James-Stein estimator δ_{JS} and we interested to establish the sufficient conditions for that the estimators δ_φ dominates the MLE X in the case where the dimension of the parameter spaces p

and the sample size n are large. If the limit of the ratio $\|\theta\|^2/p\sigma^2$ is a constant $c > 0$ when p tends to infinity, we showed that the risks ratio $R_\omega(\delta_\varphi, \theta)/R_\omega(X, \theta)$ tends to $(\omega+c)/(1+c)$ $(0 \leqslant \omega < 1)$ when n and p tend simultaneously to infinity. Thus we ensured that the estimators δ_φ which are not necessarily minimax, dominate the MLE X, even if the dimension of the parameter spaces p and the sample size n tend simultaneously to infinity. An extension of this work is to obtain the similar results in the case where the model has a symmetrical spherical distribution.

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Пределы отношений рисков оценщиков усадки при сбалансированной функции потерь

Мекки Тербече
Университет наук и технологий, Мохамед Будиаф, Оран
Лаборатория анализа и применения излучения, USTO-MB
Оран, Алжир

Абделькадер Бенхалед
Университет Туши, Мустафа Стамбули
Лаборатория геоматики, экологии и окружающей среды, Университет Маскара
Тушь, Алжир

Абденур Хамдауи
Университет наук и технологий, Мохамед Будиаф, Оран
Лаборатория статистики и случайных моделей (LSMA) Университета Тлемсена
Оран, Алжир

Аннотация. В этой статье мы изучаем оценку многомерного нормального среднего при сбалансированной функции потерь. Мы представляем здесь класс оценок усадки, который обобщает оценку Джеймса-Стейна, и мы заинтересованы в установлении асимптотического поведения отношений рисков этих оценок к оценкам максимального правдоподобия (MLE). Таким образом, в случае, когда размерность пространства параметров и размер выборки велики, мы определяем достаточные условия для того, чтобы приведенные ранее оценки были минимаксными.

Ключевые слова: сбалансированная функция потерь, оценка Джеймса-Стейна, многомерная гауссова случайная величина, нецентральное распределение хи-квадрат, оценки усадки.

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Uniqueness and Stability Results for Caputo Fractional Volterra-Fredholm Integro-Differential Equations

Ahmed A. Hamoud*
Department of Mathematics
Taiz University
Taiz, Yemen

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Abstract. In this paper, we established some new results concerning the uniqueness and Ulam's stability results of the solutions of iterative nonlinear Volterra-Fredholm integro-differential equations subject to the boundary conditions. The fractional derivatives are considered in the Caputo sense. These new results are obtained by applying the Gronwall–Bellman's inequality and the Banach contraction fixed point theorem. An illustrative example is included to demonstrate the efficiency and reliability of our results.

Keywords: Volterra–Fredholm integro-differential equation, Caputo sense, Gronwall–Bellman's inequality, Banach contraction fixed point theorem.

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Introduction

In recent years, there has been a growing interest in the linear and nonlinear integro-differential equations which are a combination of differential and integral equations [3, 16, 18, 21]. The nonlinear integro-differential equations play an important role in many branches of nonlinear functional analysis and their applications in the theory of engineering, mechanics, physics, electrostatics, biology, chemistry and economics [13] and signal processing [25].

The challenging work is to find the solution while dealing with Volterra–Fredholm fractional integro-differential equations. Therefore, many researchers have tried their best to use different techniques to find the analytical and numerical solutions of these problems [1, 2, 4, 6–8, 10, 14, 22, 23, 29].

The study of iterative differential and integro-differential equations is linked to the wide applications of calculus in mathematical sciences. These equations are vital in the study of infection models. Many papers have dealt with the existence, uniqueness and other properties of solutions of special forms of the iterative differential equations and integro-differential equations [12, 15, 16, 19, 20].

Recently, Cheng et al. (2009), in [5,20] investigated analytic and exact solutions of an iterative functional differential equation

$$\begin{aligned} &u'(x) = f(x, u(h(x) + g(u(x)))),\\ &u(x_0) = x_0. \end{aligned}$$

*drahmedselwi985@hotmail.com https://orcid.org/0000-0002-8877-7337

Lauran (2011) [19], investigated the existence and uniqueness results for first order differential and iterative differential equations with deviating argument of the type

$$\begin{aligned} &u'(t) = f(t, u(t), u(u(t)), u(\lambda u(t))), \\ &u(t_0) = x_0. \end{aligned}$$

In [15], Ibrahim (2013) investigated the existence and uniqueness of solution for iterative differential equations of the type

$$\begin{aligned} &D^\alpha u(t) = f(t, u(u(t))), \\ &u(0) = u_0. \end{aligned}$$

Kendre et al. (2015), [16] investigated the existence of solution for iterative integro-differential equations of the type

$$\begin{aligned} &u'(t) = f(t, u(u(t)), \int_{t_0}^{t} k(t,s)u(u(s))ds), \\ &u(t_0) = x_0. \end{aligned}$$

Unhale and Kendre (2019), in [28] established the existence and uniqueness of solution for iterative integro-differential equations of the type

$$\begin{aligned} &D^\alpha u(t) = f(t) + \int_0^t h(t,s)u(\lambda u(s))ds, \\ &u(0) = u_0. \end{aligned}$$

Motivated by these problems, in this paper, we discuss new uniqueness and stability results for nonlinear fractional Volterra–Fredholm integro-differential equation with deviating argument of the type

$$D^\alpha u(x) = f(x) + \int_0^x h(x,s)u(u(s))ds + \int_0^T k(x,s)u(u(s))ds, \quad x, s \in J := [0, T], \tag{1}$$

with the boundary condition

$$au(0) + bu(T) = c, \quad a, b, c \in \mathbb{R}, \ a + b \neq 0, \tag{2}$$

where $D^\alpha(.)$, $0 < \alpha < 1$, is the Caputo fractional derivative, $f(t)$, $h(x,s)$ and $k(x,s)$ are given continuous functions, $u(x)$ is the unknown function to be determined.

The main objective of the present paper is to study the new uniqueness and stability results for iterative nonlinear fractional Volterra–Fredholm integro-differential equation with deviating argument.

The rest of the paper is organized as follows: In Section 1, some essential notations, definitions and Lemmas related to fractional calculus are recalled. In Section 2, the new uniqueness and stability results of the solution for nonlinear fractional Volterra-Fredholm integro-differential equation have been proved. In Section 3, we investigate the Ulam–Hyers stability and generalized Ulam–Hyers stability for the problem (1)–(2). In Section 4, focuses on an example to illustrate the theory. Finally, we will give a report on our paper and a brief conclusion.

1. Preliminaries

The mathematical definitions of fractional derivative and fractional integration are the subject of several different approaches. The most frequently used definitions of the fractional calculus involves the Riemann–Liouville fractional derivative, Caputo derivative, etc. The following observations are taken from [7, 9–11, 17, 18, 24, 26, 29].

Definition 1.1 ([16]). *The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function f is defined as*

$$\begin{aligned} J^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad x > 0, \quad \alpha \in \mathbb{R}^+, \\ J^0 f(x) &= f(x), \end{aligned} \tag{3}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 1.2 ([16]). *The Riemann–Liouville derivative of order α with the lower limit zero for a function $f : [0,1) \longrightarrow \mathbb{R}$ can be written as*

$${}^{L}D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^{\alpha}} dt, \quad x > 0, \quad 0 < \alpha < 1. \tag{4}$$

Definition 1.3 ([24]). *The Caputo derivative of order α for a function $f : [0,1) \longrightarrow \mathbb{R}$ can be written as*

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{f'(t)}{(x-t)^{\alpha}} dt, \quad x > 0, \quad 0 < \alpha < 1.$$

Definition 1.4 ([26]). *The fractional derivative of $f(x)$ in the Caputo sense is defined by*

$$\begin{aligned} {}^{c}D^{\alpha} f(x) &= J^{n-\alpha} D^n f(x) = \\ &= \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x (x-t)^{n-\alpha-1} \frac{d^n f(t)}{dt^n} dt, & n-1 < \alpha < n, \\ \dfrac{d^n f(x)}{dx^n}, & \alpha = n, \end{cases} \end{aligned} \tag{5}$$

where the parameter α is the order of the derivative, in general it is real or even complex.

Definition 1.5 ([26]). *The Riemann-Liouville fractional derivative of order $\alpha > 0$ is normally defined as*

$$D^{\alpha} f(x) = D^m J^{m-\alpha} f(x), \qquad m-1 < \alpha \leqslant m. \tag{6}$$

Lemma 1.1 ([24], Gronwall–Bellman's Inequality). *Let $u(x)$ and $f(x)$ be nonnegative continuous functions defined on $J = [\alpha, \alpha + h]$ and c be a nonnegative constant. If*

$$u(x) \leqslant c + \int_{\alpha}^{x} f(s)u(s)ds, \quad x \in J,$$

then

$$u(x) \leqslant c \exp\left(\int_{\alpha}^{x} f(s)ds \right), \quad x \in J.$$

Theorem 1.1 ([26], Banach contraction principle). *Let (X, d) be a complete metric space, then each contraction mapping $\mathcal{T} : X \longrightarrow X$ has a unique fixed point x of $\mathcal{T}$ in X i.e. $\mathcal{T}x = x$.*

2. Main results

In this section, we shall give an existence and uniqueness results of Eq. (1), with the boundary condition (2). Let $B = C(J, J)$ be the Banach space equipped with the norm

$\|u\| = \max_{x\in[0,T]} |u(x)|$. For convenience, we are listing the following hypotheses used in our further discussion: **(A1)** There exist two constants β_h and β_k such that

$$\beta_h = \sup\{|h(t,s)| : 0 \leqslant s \leqslant t \leqslant T\}.$$
$$\beta_k = \sup\{|k(t,s)| : 0 \leqslant s \leqslant t \leqslant T\}.$$

(A2) There exists a constant $M > 0$ such that

$$|u(t_1) - u(t_2)| \leqslant M|t_1 - t_2|^\alpha, \text{ for } u \in B, \ t_1, t_2 \in J, \ t_1 \leqslant t_2.$$

(A3) There exists a constant $L > 0$ such that $L = \sup\{|f(t)| : 0 \leqslant t \leqslant T\}$.

(A4) Let $\rho := \dfrac{T^\alpha(L + T^3(\beta_h + \beta_k))}{\Gamma(\alpha+1)}\left[1 + \dfrac{|b|}{|a+b|}\right] + \dfrac{|c|}{|a+b|} \leqslant T \leqslant M.$

Lemma 2.1. *If a function $u \in C[0,T]$ satisfies (1)–(2) in the closed interval $[0,T]$, then the problems (1)–(2) are equivalent to the problem of finding a continuous solution of the integral equation*

$$\begin{aligned} u(x) &= \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds\Big)dt - \\ &- \frac{1}{a+b}\Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds\Big)dt - c\Big]. \end{aligned}$$

Theorem 2.1. *Suppose that the hypotheses (A1)–(A4) are satisfied and*

$$\left[\frac{T^{\alpha+1}(\beta_h + \beta_k)(M+1)}{\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right)\right] < 1.$$

Then there is a unique solution to the problems (1)–(2).

Proof. Let $B(\rho) = \{u \in B : \ 0 \leqslant u \leqslant \rho, |u(t_1) - u(t_2)| \leqslant M|t_1 - t_2|^\alpha\}$.

To apply Banach contraction principle, we define an operator $\Psi : B(\rho) \longrightarrow B(\rho)$ by

$$\begin{aligned} (\Psi u)(x) = &\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds\Big)dt - \\ &- \frac{1}{a+b}\Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds\Big)dt - c\Big]. \end{aligned}$$

So, we have

$$
\begin{aligned}
0 \leqslant |\Psi u| = & \Big| \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds \Big) dt - \\
& - \frac{1}{a+b} \Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds \Big) dt - c \Big] \Big| \leqslant \\
& \leqslant \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(|f(t)| + \int_0^t |h(t,s)||u(u(s))|ds + \int_0^T |k(t,s)||u(u(s))|ds \Big) dt + \\
& + \frac{1}{|a+b|} \int_0^T \frac{|b|(T-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(|f(t)| + \int_0^t |h(t,s)||u(u(s))|ds + \int_0^T |k(t,s)||u(u(s))|ds \Big) dt + \\
& + \frac{|c|}{|a+b|} \leqslant \\
& \leqslant \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} (L + (\beta_h + \beta_k)T^3) ds + \frac{1}{|a+b|} \int_0^T \frac{|b|(T-t)^{\alpha-1}}{\Gamma(\alpha)} (L + (\beta_h + \beta_k)T^3) dt + \\
& + \frac{|c|}{|a+b|} \leqslant \\
& \leqslant \frac{T^\alpha (L + T^3(\beta_h + \beta_k))}{\Gamma(\alpha+1)} \Big[1 + \frac{|b|}{|a+b|} \Big] + \frac{|c|}{|a+b|} = \\
& = \rho.
\end{aligned}
$$

Also, for each $0 \leqslant x_1 \leqslant x_2 \leqslant T$, we have

$$
\begin{aligned}
& |\Psi u(x_2) - \Psi u(x_1)| \leqslant \\
& \leqslant \Big| \int_0^{x_1} \frac{(x_2-t)^{\alpha-1} - (x_1-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds \Big) dt \Big| + \\
& + \Big| \int_{x_1}^{x_2} \frac{(x_2-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds \Big) dt \Big| \leqslant \\
& \leqslant \frac{1}{\Gamma(\alpha)} \int_0^{x_1} \Big[(x_1-t)^{\alpha-1} - (x_2-t)^{\alpha-1} \Big] \Big(|f(t)| + \int_0^t |h(t,s)|\, |u(u(s))| ds + \\
& + \int_0^T |k(t,s)|\, |u(u(s))| ds \Big) dt + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2-t)^{\alpha-1} \Big(|f(t)| + \int_0^t |h(t,s)|\, |u(u(s))| ds + \\
& + \int_0^T |k(t,s)|\, |u(u(s))| ds \Big) dt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\Psi u(x_2) - \Psi u(x_1)| & \leqslant \frac{1}{\Gamma(\alpha)} \int_0^{x_1} \Big[(x_1-t)^{\alpha-1} - (x_2-t)^{\alpha-1} \Big] [L + T^3(\beta_h + \beta_k)] dt + \\
& + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2-t)^{\alpha-1} [L + T^3(\beta_h + \beta_k)] dt \leqslant \\
& \leqslant \frac{[L + T^3(\beta_h + \beta_k)]}{\Gamma(\alpha+1)} \Big[x_1^\alpha - x_2^\alpha + 2(x_2 - x_1)^\alpha \Big] \leqslant \\
& \leqslant \frac{2[L + T^3(\beta_h + \beta_k)]}{\Gamma(\alpha+1)} |x_2 - x_1|^\alpha.
\end{aligned}
$$

This shows that Ψ maps from $B(\rho) \longrightarrow B(\rho)$. Now, for all $u, v \in B(\rho)$, we have

$$
\begin{aligned}
&|\Psi u(x) - \Psi v(x)| \leqslant \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \Big(\int_0^t |h(t,s)|\, |u(u(s)) - v(v(s))| ds + \int_0^T |k(t,s)|\, |u(u(s)) - v(v(s))| ds \Big) dt + \\
&+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \Big(\int_0^t |h(t,s)|\, |u(u(s)) - v(v(s))| ds + \\
&+ \int_0^T |k(t,s)|\, |u(u(s)) - v(v(s))| ds \Big) dt \leqslant \\
&\leqslant \frac{(\beta_h + \beta_k)}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \Big(\int_0^t |u(u(s)) - u(v(s))| + |u(v(s)) - v(v(s))| ds + \\
&+ \int_0^T |u(u(s)) - u(v(s))| + |u(v(s)) - v(v(s))| ds \Big) dt + \\
&+ \frac{|b|(\beta_h + \beta_k)}{|a+b|\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \Big(\int_0^t |u(u(s)) - u(v(s))| + |u(v(s)) - v(v(s))| ds + \\
&+ \int_0^T |u(u(s)) - u(v(s))| + |u(v(s)) - v(v(s))| ds \Big) dt \leqslant \\
&\leqslant \frac{(\beta_h + \beta_k)}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \Big(\int_0^t (M\, |u(s) - v(s)| + |u(s) - v(s)|) ds \Big) dt + \\
&+ \frac{|b|(\beta_h + \beta_k)}{|a+b|\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \Big(\int_0^t (M\, |u(s) - v(s)| + |u(s) - v(s)|) ds \Big) dt \leqslant \\
&\leqslant \frac{(\beta_h + \beta_k)}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \Big(\int_0^t ((M+1)\, |u(s) - v(s)|) ds \Big) dt + \\
&+ \frac{|b|(\beta_h + \beta_k)}{|a+b|\Gamma(\alpha)} \int_0^T (T-t)^{\alpha-1} \Big(\int_0^t ((M+1)\, |u(s) - v(s)|) ds \Big) dt \leqslant \\
&\leqslant \frac{T(\beta_h + \beta_k)(M+1)}{\Gamma(\alpha)} \|u - v\| \int_0^x (x-t)^{\alpha-1} dt + \frac{|b| T(\beta_h + \beta_k)(M+1)}{|a+b|\Gamma(\alpha)} \|u - v\| \int_0^T (T-t)^{\alpha-1} dt \leqslant \\
&\leqslant \frac{T^{\alpha+1}(\beta_h + \beta_k)(M+1)}{\Gamma(\alpha+1)} \|u - v\| + \frac{|b| T T^{\alpha+1}(\beta_h + \beta_k)(M+1)}{|a+b|\Gamma(\alpha+1)} \|u - v\| \leqslant \\
&\leqslant \Big[\frac{T^{\alpha+1}(\beta_h + \beta_k)(M+1)}{\Gamma(\alpha+1)} \Big(1 + \frac{|b|}{|a+b|} \Big) \Big] \|u - v\|.
\end{aligned}
$$

Since

$$
\Big[\frac{T^{\alpha+1}(\beta_h + \beta_k)(M+1)}{\Gamma(\alpha+1)} \Big(1 + \frac{|b|}{|a+b|} \Big) \Big] < 1,
$$

by the Banach contraction principle, Ψ has a unique fixed point. This means that the problems (1)–(2) has unique solution. □

The above theorem shows that there exists a unique solution to the problems (1)–(2). However, it does not tell us how to find this solution. To find the solution of the problems (1)–(2), we will define the following sequence

$$
\begin{aligned}
u_{n+1}(x) &= \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s) u_n(u_n(s)) ds + \int_0^T k(t,s) u_n(u_n(s)) ds \Big) dt - \\
&- \frac{1}{a+b} \Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s) u_n(u_n(s)) ds + \int_0^T k(t,s) u_n(u_n(s)) ds \Big) dt - c \Big],
\end{aligned}
$$

where $n = 0, 1, 2, \ldots$ and $u_0(x)$ is fixed functions of the class C^1 mapping $[0,T] \longrightarrow [0,T]$ such that $|u_0(x)| \leqslant T$. For this, we have the following theorem.

Theorem 2.2. *If the assumptions of the Theorem 2.1 are satisfied then the sequences defined in* (7) *converges uniformly to the unique solution of the problems* (1)–(2).

Proof. Let $U_k = \max_{x\in J} |u_k(x) - u_{k-1}(x)|$. Then

$$U_1 = \max_{x\in J} |u_1(x) - u_0(x)| =$$
$$= \max_{x\in J} \Bigg| \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u_0(u_0(s))ds + \int_0^T k(t,s)u_0(u_0(s))ds \Big) dt -$$
$$- \frac{1}{a+b} \Bigg[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)} \Bigg(f(t) + \int_0^t h(t,s)u_0(u_0(s))ds + \int_0^T k(t,s)u_0(u_0(s))ds \Bigg) dt - c \Bigg] - u_0(x) \Bigg| \leqslant$$
$$\leqslant \frac{T^\alpha (L + T^3(\beta_h + \beta_k))}{\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|} \right) + \frac{|c|}{|a+b|} \leqslant$$
$$\leqslant T.$$

Since $u_0 : [0,T] \longrightarrow [0,T]$, we have $U_1 \leqslant T$.

$$U_2 = \max_{x\in J} |u_2(x) - u_1(x)| =$$
$$= \max_{x\in J} \Bigg| \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u_1(u_1(s))ds + \int_0^T k(t,s)u_1(u_1(s))ds \Big) dt -$$
$$- \frac{1}{a+b} \Bigg[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u_1(u_1(s))ds + \int_0^T k(t,s)u_1(u_1(s))ds \Big) dt - c \Bigg] -$$
$$- \Bigg\{ \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u_0(u_0(s))ds + \int_0^T k(t,s)u_0(u_0(s))ds \Big) dt -$$
$$- \frac{1}{a+b} \Bigg[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u_0(u_0(s))ds + \int_0^T k(t,s)u_0(u_0(s))ds \Big) dt - c \Bigg] dt \Bigg\} \Bigg| \leqslant$$
$$\leqslant \max_{x\in J} \Bigg\{ \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(|f(t)| + \int_0^t |h(t,s)|\, |u_1(u_1(s)) - u_0(u_0(s))| ds +$$
$$+ \int_0^T |k(t,s)|\, |u_1(u_1(s)) - u_0(u_0(s))| ds \Big) dt - \frac{1}{|a+b|} \int_0^T \frac{|b|(T-t)^{\alpha-1}}{\Gamma(\alpha)} \times$$
$$\times \Big(|f(t)| + \int_0^t |h(t,s)|\, |u_1(u_1(s)) - u_0(u_0(s))| ds + \int_0^T |k(t,s)|\, |u_1(u_1(s)) - u_0(u_0(s))| ds \Big) dt \Bigg\} \leqslant$$
$$\leqslant TU_1 \leqslant T^2.$$

Assume that result is true for n i.e. $U_n \leqslant TU_{n-1} \leqslant T^n$. Now, we show that result holds for $n+1$

$$U_{n+1} = \max_{x\in J} |u_{n+1}(x) - u_n(x)| =$$
$$= \max_{x\in J} \Bigg| \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u_1(u_n(s))ds + \int_0^T k(t,s)u_n(u_n(s))ds \Big) dt -$$
$$- \frac{1}{a+b} \Bigg[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)} \Big(f(t) + \int_0^t h(t,s)u_n(u_n(s))ds + \int_0^T k(t,s)u_n(u_n(s))ds \Big) dt - c \Bigg] -$$

$$
\begin{aligned}
&-\Big\{\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_{n-1}(u_{n-1}(s))ds+\int_0^T k(t,s)u_{n-1}(u_{n-1}(s))ds\Big)dt-\\
&-\frac{1}{a+b}\Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t)+\int_0^t h(t,s)u_{n-1}(u_{n-1}(s))ds+\\
&+\int_0^T k(t,s)u_{n-1}(u_{n-1}(s))ds\Big)dt-c\Big]dt\Big\}\Big|\leqslant\\
&\leqslant \max_{x\in J}\Big\{\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(|f(t)|+\int_0^t |h(t,s)|\,|u_n(u_n(s))-u_{n-1}(u_{n-1}(s))|ds+\\
&+\int_0^T |k(t,s)|\,|u_n(u_n(s))-u_{n-1}(u_{n-1}(s))|ds\Big)dt-\frac{1}{|a+b|}\int_0^T \frac{|b|(T-t)^{\alpha-1}}{\Gamma(\alpha)}\times\\
&\times\Big(|f(t)|+\int_0^t |h(t,s)|\,|u_n(u_n(s))-u_{n-1}(u_{n-1}(s))|ds+\\
&+\int_0^T |k(t,s)|\,|u_n(u_n(s))-u_{n-1}(u_{n-1}(s))|ds\Big)dt\Big\}\leqslant\\
&\leqslant TU_n\leqslant T^{n+1}.
\end{aligned}
$$

Thus by induction, we have $U_k \leqslant T^k$. Since

$$\frac{T^\alpha(L+T^3(\beta_h+\beta_k))}{\Gamma(\alpha+1)}\left[1+\frac{|b|}{|a+b|}\right]+\frac{|c|}{|a+b|}\leqslant T<1.$$

Hence U_k tends to zero as k tends to infinity. Since the family $\{U_k\}$ is the Arzelà–Ascoli family thus for every subsequence $\{u_{kj}\}$ of $\{U_k\}$ there exists a subsequence $\{u_{kj}\}$ uniformly convergent and the limit needs to be a solution of the problem (1)–(2). Thus, the sequence $\{U_k\}$ tends uniformly to the unique solution of the problem (1)–(2). □

3. Stability results

In this section, we investigate the Ulam–Hyers stability and generalized Ulam–Hyers stability for the problem (1)–(2).

Definition 3.1 ([27]). *The Eq. (1) is Ulam–Hyers stable if there exists a real number $\Omega > 0$ such that for each $\epsilon > 0$ and for each solution $v \in C^1(J, J)$ of the inequality*

$$\Big|D^\alpha v(x)-f(x)-\int_0^x h(x,s)v(v(s))ds-\int_0^T k(x,s)v(v(s))ds\Big|\leqslant\epsilon,\quad x\in J, \tag{7}$$

there exists a solution $u \in C^1(J, J)$ of Eq. (1) with

$$|v(x)-u(x)|\leqslant\Omega\epsilon. \tag{8}$$

Definition 3.2 ([27]). *The Eq. (1) is generalized Ulam–Hyers stable if there exists $\Theta \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\Theta(0) = 0$ such that for each $\epsilon > 0$ and for each solution $v \in C^1(J, J)$ of the inequality*

$$\Big|D^\alpha v(x)-f(x)-\int_0^x h(x,s)v(v(s))ds-\int_0^T k(x,s)v(v(s))ds\Big|\leqslant\epsilon,\quad x\in J, \tag{9}$$

there exists a solution $u \in C^1(J, J)$ of Eq. (1) with

$$|v(x)-u(x)|\leqslant\Theta(\epsilon). \tag{10}$$

Theorem 3.1. *If the assumptions of the Theorem 2.1 are satisfied, then the problem (1)–(2) is Ulam–Hyers stable.*

Proof. Let $\epsilon > 0$ and let function $v \in C^1(J,J)$ which satisfies the inequality

$$D^\alpha v(x) - f(x) - \int_0^x h(x,s)v(v(s))ds - \int_0^T k(x,s)v(v(s))ds \leqslant \epsilon, \tag{11}$$

and let $u \in C(J,J)$ be the unique solution of the following problem

$$\begin{aligned} &D^\alpha u(x) = f(x) + \int_0^x h(x,s)u(u(s))ds + \int_0^T k(x,s)u(u(s))ds,\\ &u(0) = v(0), \quad u(T) = v(T). \end{aligned}$$

from Lemma 2.1, we obtain

$$\begin{aligned} u(x) &= \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds\Big)dt - \\ &- \frac{1}{a+b}\Big[\int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds\Big)dt - c\Big] = \\ &= \Delta_u + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds\Big)dt, \end{aligned}$$

where

$$\Delta_u = \frac{1}{a+b}\Big[c - \int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds\Big)dt\Big].$$

Let

$$\Delta_v = \frac{1}{a+b}\Big[c - \int_0^T \frac{b(T-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)v(v(s))ds + \int_0^T k(t,s)v(v(s))ds\Big)dt\Big].$$

On the other hand, if $u(0) = v(0)$, $u(T) = v(T)$, then $\Delta_u = \Delta_v$ and

$$u(x) = \Delta_v + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)u(u(s))ds + \int_0^T k(t,s)u(u(s))ds\Big)dt.$$

From inequality (11) we have

$$-\epsilon \leqslant D^\alpha v(x) - f(x) - \int_0^x h(x,s)v(v(s))ds - \int_0^T k(x,s)v(v(s))ds \leqslant \epsilon. \tag{12}$$

If we integrate each term of the above inequality and appling the boundary conditions, then we have

$$\Big|v(x) - \Delta_v - \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)v(v(s))ds + \int_0^T k(t,s)v(v(s))ds\Big)dt\Big| \leqslant \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}.$$

For any $x \in J$, we have

$$\begin{aligned} &|v(x) - u(x)| \leqslant \\ \leqslant\ &\Big|v(x) - \Delta_v - \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(f(t) + \int_0^t h(t,s)v(v(s))ds + \int_0^T k(t,s)v(v(s))ds\Big)dt\Big| + \end{aligned}$$

$$
\begin{aligned}
&+\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Big(\int_0^t h(t,s)|v(v(s))-u(u(s))|ds+\int_0^T k(t,s)|v(v(s))-u(u(s))|ds\Big)dt \leqslant\\
&\leqslant \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}+\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\Bigg(\beta_h\int_0^t \big[|v(v(s))-v(u(s))|+|v(u(s))-u(u(s))|\big]ds+\\
&+\beta_k\int_0^T [|v(v(s))-v(u(s))|+|v(u(s))-u(u(s))|]ds\Bigg)dt \leqslant\\
&\leqslant \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}+\frac{\beta_h}{\Gamma(\alpha)}\int_0^x\int_0^t (x-t)^{\alpha-1}(M+1)|v(s)-u(s)|dsdt+\\
&+\frac{\beta_k}{\Gamma(\alpha)}\int_0^x\int_0^T (x-t)^{\alpha-1}(M+1)|v(s)-u(s)|dsdt \leqslant\\
&\leqslant \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}+\frac{T^\alpha(\beta_h+\beta_k)(M+1)}{\Gamma(\alpha+1)}\int_0^x |v(s)-u(s)|ds.
\end{aligned}
$$

Using Gronwall's inequality, we get

$$
\Big|v(x)-u(x)\Big| \leqslant \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}\left[1+\frac{\xi T^\alpha(\beta_h+\beta_k)(M+1)}{\Gamma(\alpha+1)}\right] := \Omega\epsilon,
$$

where $\xi=\xi(\alpha)$ a constant, which completes the proof.

Moreover, if we set $\Theta(\epsilon)=\Omega\epsilon$, $\Theta(0)=0$, then boundary value problem (1)–(2) is generalized Ulam–Hyers stable. □

4. An example

We consider the nonlinear iterative fractional integro-differential equation (1)–(2) with

$$
\alpha=0.5,\quad T=0.5,\quad L=0.2,\quad M=0.4,\quad \beta_h=\beta_k=0.5,\quad a=b=1,\quad \text{and}\quad c=0.
$$

New, we have

$$
\begin{aligned}
\frac{T^\alpha(L+T^3(\beta_h+\beta_k))}{\Gamma(\alpha+1)}\Big(1+\frac{|b|}{|a+b|}\Big)+\frac{|c|}{|a+b|} &= \frac{0.5^{0.5}(0.2+0.5^3(0.5+0.5))}{\Gamma(0.5+1)}\Big(1+\frac{1}{2}\Big)+0\\
&= \frac{0.2298098}{\Gamma(1.5)}(1.5)\\
&= \frac{0.3447145}{0.886227}\\
&= 0.38897\\
&< 0.5=T.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\frac{T^{\alpha+1}(M+1)(\beta_h+\beta_k)}{\Gamma(\alpha+1)}\Bigg(1+\frac{|b|}{|a+b|}\Bigg) &= \frac{0.5^{0.5+1}(0.4+1)(0.5+0.5)}{\Gamma(0.5+1)}\Big(1+\frac{1}{2}\Big)\\
&= \frac{0.494975}{0.886227}(1.5)\\
&= 0.8378\\
&< 1.
\end{aligned}
$$

Since all the hypotheses of Theorem 2.1 are fulfilled, then there exists a unique solution of the given equation.

Conclusion

The main purpose of this paper was to present new existence and uniqueness results as well as the Ulam–Hyers stability and generalized Ulam–Hyers stability results of the solution for Caputo fractional iterative Volterra–Fredholm integro-differential. The techniques used to prove our results are a variety of tools such as the Gronwall–Bellman's inequality, some properties of fractional calculus and the Banach contraction fixed point theorem. Moreover, the results of references [15, 16, 28] appear as a special case of our results.

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Результаты единственности и устойчивости для Капуто дробных интегро-дифференциальных уравнений Вольтерра-Фредгольма

Ахмед А. Хамуд
Кафедра математики
Университет Таиз
Таиз, Йемен

Аннотация. В этой статье мы установили некоторые новые результаты, касающиеся единственности и устойчивости Улама решений итерационных нелинейных интегро-дифференциальных уравнений Вольтерра–Фредгольма с граничными условиями. Дробные производные рассматриваются в смысле Капуто. Эти новые результаты получены путем применения неравенства Гронуолла–Беллмана и теоремы Банаха о сжатии неподвижной точки. Включен наглядный пример, чтобы продемонстрировать эффективность и надежность результатов.

Ключевые слова: интегро-дифференциальное уравнение Вольтерра–Фредгольма, смысл Капуто, неравенство Гронуолла–Беллмана, теорема Банаха о сжатии неподвижной точки.

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On Transcendental Systems of Equations

Alexander M. Kytmanov*
Olga V. Khodos†
Siberian Federal University
Krasnoyarsk, Russian Federation

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Abstract. Several types of transcendental systems of equations are considered: the simplest ones, special, and general. Since the number of roots of such systems, as a rule, is infinite, it is necessary to study power sums of the roots of negative degree. Formulas for finding residue integrals, their relation to power sums of a negative degree of roots and their relation to residue integrals (multidimensional analogs of Waring's formulas) are obtained. Various examples of transcendental systems of equations and calculation of multidimensional numerical series are given.

Keywords: transcendental systems of equations, power sums of roots, residue integral.

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Introduction

Based on the multidimensional logarithmic residue, for systems of non-linear algebraic equations in $\mathbb{C}^n$ formulas for finding power sums of the roots of a system without calculating the roots themselves were earlier obtained (see [1–3]). For different types of systems such formulas have different forms. Based on this, a new method for the study of systems of algebraic equations in $\mathbb{C}^n$ have been constructed. It arose in the work of L. A. Aizenberg [1], its development was continued in monographs [2–4]. The main idea is to find power sums of roots of systems (for positive powers) and then, to use one-dimensional or multidimensional recurrent Newton formulas (see [5]). Unlike the classical method of elimination, it is less labor-intensive and does not increase the multiplicity of roots. It is based on the formula (see [1]) for a sum of the values of an arbitrary polynomial in the roots of a given systems of algebraic equations without finding the roots themselves.

For systems of transcendental equations, formulas for the sum of the values of the roots of the system, as a rule, cannot be obtained, since the number of roots of a system can be infinite and a series of coordinates of such roots can be diverging. Nevertheless, such transcendental systems of equations may very well arise, for example, in the problems of chemical kinetics [6, 7]. Thus, this is an important task to consider such systems.

In the works [8–21] power sums of roots are considered for a negative power for different systems of non-algebraic (transcendental) equations. To compute these power sums, a residue integral is used, the integration is carried out over skeletons of polycircles centered at the origin. Note that this residue integral is not, generally speaking, a multidimensional logarithmic residue or a Grothendieck residue. For various types of lower homogeneous systems of functions included in the system, formulas are given for finding residue integrals, their relationship with power sums of the roots of the system to a negative degree are established.

*AKytmanov@sfu-kras.ru https://orcid.org/0000-0002-7394-1480
†khodos_o@mail.ru

The paper [12] investigated more complex systems in which the lower homogeneous parts are decomposed into linear factors and integration cycles in residue integrals are constructed from these factors. In [11], a system is studied that arises in the Zel'dovich–Semenov model (see [6,7]) in chemical kinetics.

The object of this study is transcendental systems of equations in which the lower homogeneous parts of the functions included in the system form a non-degenerate system of algebraic equations: formulas are found for calculating the residue integrals, power sums of roots for a negative power, their relationship with the residue integrals are established. See [21].

1. The simplest transcendental systems of equations

Consider a system of functions of the form

$$f_1(z),\ f_2(z), \dots, f_n(z),$$

holomorphic in a neighborhood of the point $0 \in \mathbb{C}^n$, $z = (z_1, z_2, \dots, z_n)$ and having the following form:

$$f_j(z) = z^{\beta^j} + Q_j(z), \quad j = 1, 2, \dots, n, \tag{1}$$

where $\beta^j = (\beta_1^j, \beta_2^j, \dots, \beta_n^j)$ is a multi-index with integer non-negative coordinates, $z^{\beta^j} = z_1^{\beta_1^j} \cdot z_2^{\beta_2^j} \cdots z_n^{\beta_n^j}$ and $\|\beta^j\| = \beta_1^j + \beta_2^j + \ldots + \beta_n^j = k_j$, $j = 1, 2, \dots, n$. The functions Q_j can be expanded in absolutely and uniformly converging Taylor series in a neighborhood of the origin of the form

$$Q_j(z) = \sum_{\|\alpha\| > k_j} a_\alpha^j z^\alpha, \tag{2}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_j \geqslant 0$, $\alpha_j \in \mathbb{Z}$, a $z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_n^{\alpha_n}$.

Consider the cycles $\gamma(r) = \gamma(r_1, r_2, \dots, r_n)$, which are skeletons of polydisks:

$$\gamma(r) = \{z \in \mathbb{C}^n : |z_s| = r_s,\ s = 1, 2, \dots, n\}, \quad r_1 > 0, \dots, r_n > 0.$$

For sufficiently small r_j, the cycles $\gamma(r)$ lie in the domain of holomorphy of functions f_j, therefore the series

$$\sum_{\|\alpha\| > k_j} |a_\alpha^j| r_1^{\alpha_1} \cdots r_n^{\alpha_n}, \quad j = 1, \dots n,$$

converge. Then on the cycle $\gamma(tr) = \gamma(tr_1, tr_2, \dots, tr_n)$ for sufficiently small $t > 0$ we have

$$|z|^{\beta^j} = t^{k_j} \cdot r_1^{\beta_1^j} \cdot r_2^{\beta_2^j} \cdots r_n^{\beta_n^j} = t^{k_j} \cdot r^{\beta^j},$$

and

$$|Q_j(z)| = \left| \sum_{\|\alpha\| > k_j} a_\alpha^j z^\alpha \right| \leqslant$$

$$\leqslant \sum_{\|\alpha\| > k_j} t^{\|\alpha\|} |a_\alpha^j| r^\alpha \leqslant t^{k_j + 1} \sum_{\|\alpha\| \geqslant 0} |a_\alpha^j| r^\alpha, \quad j = 1, \dots, n.$$

Therefore, for such t on the cycle $\gamma(tr)$ the inequalities hold

$$|z|^{\beta^j} > |Q_j(z)|, \quad j = 1, 2, \dots, n. \tag{3}$$

Thus

$$f_j(z) \neq 0 \quad \text{on} \quad \gamma(tr), \quad j = 1, 2, \dots, n.$$

In what follows, we will assume that $t = 1$. Consider a system of equations of the form

$$\begin{cases} f_1(z) = 0, \\ f_2(z) = 0, \\ \dots\dots\dots \\ f_n(z) = 0. \end{cases} \tag{4}$$

From (3) it follows that for sufficiently small r_j the following integrals are defined

$$\int\limits_{\gamma(r)} \frac{1}{z^{\beta+I}} \cdot \frac{df}{f} = \int\limits_{\gamma(r_1,r_2,\dots,r_n)} \frac{1}{z_1^{\beta_1+1} \cdot z_2^{\beta_2+1} \cdots z_n^{\beta_n+1}} \cdot \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \dots \wedge \frac{df_n}{f_n},$$

where $\beta_1 \geqslant 0,\ \beta_2 \geqslant 0, \dots, \beta_n \geqslant 0,\ \beta_j \in \mathbb{Z},\ I = (1, 1, \dots, 1)$. We call them *residue integrals* ([22]).

The logarithmic residue theorem does not apply to these integrals, and they are not standard Grothendieck residues.

Since condition (3) is satisfied on the cycles $\gamma(r)$, by the Cauchy–Poincaré theorem, these integrals are independent of $(r_1, \dots, r_n)$. Let us denote

$$J_\beta = \frac{1}{(2\pi i)^n} \int\limits_{\gamma(r)} \frac{1}{z^{\beta+I}} \cdot \frac{df}{f}.$$

Theorem 1. *Under the assumptions made, for a function f_j of the form (1), (2) the next formulas are valid*

$$J_\beta = \sum_{\|\alpha\| \leqslant \|\beta\| + \min(n, k_1 + \ldots + k_n)} \frac{(-1)^{\|\alpha\|}}{(\beta + (\alpha_1+1)\beta^1 + \ldots + (\alpha_n+1)\beta^n)!} \times$$

$$\times \frac{\partial^k (\Delta \cdot Q^\alpha)}{\partial z^{\beta + (\alpha_1+1)\beta^1 + \ldots + (\alpha_n+1)\beta^n}} \Bigg|_{z=0} =$$

$$= \sum_{\|\alpha\| \leqslant \|\beta\| + \min(n, k_1 + \ldots + k_n)} (-1)^{\|\alpha\|} \mathfrak{M} \left[\frac{\Delta \cdot Q^\alpha}{z^{\beta + (\alpha_1+1)\beta^1 + \ldots + (\alpha_n+1)\beta^n}} \right],$$

where $k = \|\beta + (\alpha_1+1)\beta^1 + \ldots + (\alpha_n+1)\beta^n\|$, $\beta! = \beta_1! \cdot \beta_2! \cdots \beta_n!$, $Q^\alpha = Q_1^{\alpha_1} \cdot Q_2^{\alpha_2} \cdots Q_n^{\alpha_n}$, $\dfrac{\partial^{\|\beta\|}}{\partial z^\beta} = \dfrac{\partial^{\|\beta\|}}{\partial z_1^{\beta_1} \partial z_2^{\beta_2} \cdots \partial z_n^{\beta_n}}$, Δ is the Jacobian of the system of functions (1) and, finally, $\mathfrak{M}$ is a linear functional assigning to the Laurent series (under the sign of the functional $\mathfrak{M}$) its free term.

Corollary 1. *If all $\beta^j = (0, 0, \dots, 0)$, $j = 1, \dots, n$, then the integral*

$$J_\beta = \sum_{\|\alpha\| \leqslant \|\beta\|} (-1)^{\|\alpha\|} \mathfrak{M} \left[\frac{\Delta \cdot Q^\alpha}{z^\beta} \right] = \sum_{\|\alpha\| \leqslant \|\beta\|} \frac{(-1)^{\|\alpha\|}}{\beta!} \frac{\partial^{\|\beta\|}}{\partial z^\beta} (\Delta \cdot Q^\alpha) \Bigg|_{z=0}.$$

Our further goal is to relate the considered integrals to power sums of roots of the system (4). To do this, we will narrow the function class f_j. First, we take as functions Q_j $(j = 1, 2, \dots, n)$ polynomials of the form

$$Q_j(z) = \sum_{\alpha \in M_j} a_\alpha^j z^\alpha, \tag{5}$$

where M_j is a finite set of multi-indices such that for $\alpha \in M_j$ the coordinates $\alpha_k \leqslant \beta^j_k$, $k = 1, 2, \dots, n,\ k \neq j$. (But it is still assumed that $\|\alpha\| > k_j$ for all $\alpha \in M_j$).

Denote

$$\sigma_{\beta+I} = \sigma_{(\beta_1+1,\beta_2+1,\dots,\beta_n+1)} = \sum_{k=1}^{M} \frac{1}{z_{1(k)}^{\beta_1+1} \cdot z_{2(k)}^{\beta_2+1} \cdots z_{n(k)}^{\beta_n+1}},$$

where $\beta = (\beta_1, \dots, \beta_n)$ is some multi-index. This expression is a power sum of roots that do not lie on the coordinate planes of the system (4), but in negative power (or a power sum of the reciprocal of the roots).

Theorem 2. *For the system (4) with functions f_j of the form (1) and polynomials Q_j of the form (5) the next formulas are valid*

$$J_\beta = (-1)^n \sigma_{\beta+I},$$

i.e.

$$\sigma_{\beta+I} = \sum_{\|\alpha\| \leqslant \|\beta\| + \min(n, k_1+\ldots+k_n)} (-1)^{\|\alpha\|+n} \mathfrak{M}\left[\frac{\Delta \cdot Q^\alpha}{z^{\beta+(\alpha_1+1)\beta^1+\ldots+(\alpha_n+1)\beta^n}}\right].$$

Consider a system of equations in three complex variables

$$\begin{cases} f_1(z_1, z_2, z_3) = 1 + a_1 z_1 = 0, \\ f_2(z_1, z_2, z_3) = 1 + b_1 z_1 + b_2 z_2 = 0, \\ f_3(z_1, z_2, z_3) = 1 + c_1 z_1 + c_2 z_2 + c_3 z_3 = 0. \end{cases} \tag{6}$$

Here the functions do not satisfy the conditions of Theorem 2, but they satisfy the conditions of Theorem 1. We find the integral

$$J_{(\beta,0,0)} = \frac{1}{(2\pi i)^3} \int_{\gamma(r)} \frac{1}{z_1^{\beta_1+1} z_2 z_3} \cdot \frac{df_1 \wedge df_2 \wedge df_3}{f_1 \cdot f_2 \cdot f_3} =$$

$$= \frac{1}{(2\pi i)^3} \int_{\gamma(r)} \frac{1}{z_1^{\beta_1+1} z_2 z_3} \cdot \frac{a_1 b_2 c_3 dz_1 \wedge dz_2 \wedge dz_3}{(1 + a_1 z_1)(1 + b_1 z_1 + b_2 z_2)(1 + c_1 z_1 + c_2 z_2 + c_3 z_3)} =$$

$$= \frac{a_1 b_2 c_3}{\beta!} \cdot \frac{\partial^\beta}{\partial z_1^\beta} \cdot \left[\frac{1}{(1 + a_1 z_1)(1 + b_1 z_1)(1 + c_1 z_1)}\right]\Bigg|_{z_1=0}.$$

To calculate the last derivative, we transform the expression

$$\frac{1}{(1 + a_1 z_1)(1 + b_1 z_1)(1 + c_1 z_1)} = \frac{A}{1 + a_1 z_1} + \frac{B}{1 + b_1 z_1} + \frac{C}{1 + c_1 z_1},$$

$$\begin{cases} A = \dfrac{a_1^2}{(a_1 - b_1)(a_1 - c_1)}, \\ B = -\dfrac{b_1^2}{(a_1 - b_1)(b_1 - c_1)}, \\ C = \dfrac{c_1^2}{(a_1 - c_1)(b_1 - c_1)}, \end{cases} \tag{7}$$

assuming that $a_1 \neq b_1,\ a_1 \neq c_1,\ b_1 \neq c_1$, then

$$J_{(\beta,0,0)} = (-1)^\beta a_1 b_2 c_3 \times$$

$$\times\left[\frac{a_1^{\beta+2}}{(a_1-b_1)(a_1-c_1)}-\frac{b_1^{\beta+2}}{(a_1-b_1)(b_1-c_1)}+\frac{c_1^{\beta+2}}{(a_1-c_1)(b_1-c_1)}\right].$$

The roots of the system (6)

$$z_1=-\frac{1}{a_1},\quad z_2=\frac{b_1-a_1}{a_1b_2},\quad z_3=\frac{b_2c_1-b_1c_2+a_1c_2-a_1b_2}{a_1b_2c_3}.$$

If the numerator in the formula for z_3 is 0, then this root lies on a coordinate plane, and we should not take it into consideration.

Therefore, the power sum

$$\sigma_{(\beta_1+1,1)}=\frac{(-1)^{\beta+1}a_1^{\beta+3}b_2^2c_3}{(b_1-a_1)(b_2c_1-b_1c_2+a_1c_2-a_1b_2)},$$

i.e.

$$J_{(\beta,0,0)}=-\sigma_{(\beta_1+1,1)}-\frac{(-1)^{\beta}a_1^2b_2^2c_3b_1^{\beta+1}}{(b_1-a_1)(b_2c_1-b_1c_2+a_1c_2-a_1b_2)}+$$
$$+\frac{(-1)^{\beta+1}a_1b_2c_2c_3}{(b_2c_1-b_1c_2+a_1c_2-a_1b_2)}\times$$
$$\times\left[-a_1c_2\cdot\frac{a_1^{\beta+1}-c_1^{\beta+1}}{a_1-c_1}+(b_1c_2-b_2c_1)\cdot\frac{b_1^{\beta+1}-c_1^{\beta+1}}{b_1-c_1}\right]. \tag{8}$$

We recall the well-known expansions of the sine into an infinite product and the power series:

$$\frac{\sin\sqrt{z}}{\sqrt{z}}=\prod_{k=1}^{\infty}\left(1-\frac{z}{k^2\pi^2}\right)=\sum_{k=0}^{\infty}\frac{(-1)^kz^k}{(2k+1)!},$$

which converge uniformly and absolutely on the complex plane.

Consider the system of equations

$$\begin{cases} f_1(z_1,z_2,z_3)=\dfrac{\sin\sqrt{z_1-a^2}}{\sqrt{z_1-a^2}}=\prod\limits_{k=1}^{\infty}\left(1-\dfrac{z_1-a^2}{k^2\pi^2}\right)=0,\\ f_2(z_1,z_2,z_3)=\dfrac{\sin\sqrt{z_2-z_1-a^2}}{\sqrt{z_2-z_1-a^2}}=\prod\limits_{m=1}^{\infty}\left(1-\dfrac{z_2-z_1-a^2}{m^2\pi^2}\right)=0,\\ f_3(z_1,z_2,z_3)=\dfrac{\sin\sqrt{z_3-z_2-a^2}}{\sqrt{z_3-z_2-a^2}}=\prod\limits_{s=1}^{\infty}\left(1-\dfrac{z_3-z_2-a^2}{s^2\pi^2}\right)=0. \end{cases} \tag{9}$$

Each of the functions of this system can be expanded into an infinite product of functions from system (6).

The roots of the system (9) are the points $(\pi^2k^2+a^2,\pi^2(k^2+m^2)+2a^2,\pi^2(k^2+m^2+s^2)+3a^2)$, $k,m,s\in\mathbb{N}$. Therefore, the power sum $\sigma_{(\beta+1,1,1)}$ is equal to the sum of the series

$$\sigma_{(\beta+1,1,1)}=\sum_{k,m,s=1}^{\infty}\frac{1}{(\pi^2k^2+a^2)^{(\beta+1)}(\pi^2(k^2+m^2)+2a^2)(\pi^2(k^2+m^2+s^2)+3a^2)},$$

which converges as $a\neq\pi ki$.

For the system (9)

$$f_1=\sum_{k=0}^{\infty}\frac{(-1)^k(z_1-a^2)^k}{(2k+1)!},$$

$$f_2=\sum_{k=0}^{\infty}\frac{(-1)^k(z_2-z_1-a^2)^k}{(2k+1)!},$$

$$f_3=\sum_{k=0}^{\infty}\frac{(-1)^k(z_3-z_2-a^2)^k}{(2k+1)!},$$

therefore

$$f_1(0,0,0)=f_2(0,0,0)=f_3(0,0,0)=\sum_{k=0}^{\infty}\frac{a^2k}{(2k+1)!}=\frac{\text{sha}}{a}.$$

Therefore, to apply the formula from Theorem 1, we need to divide the functions f_1, f_2, f_3 by these constants (normalize).

Consider the integral $J_{(\eta,0,0)}$ for the system (9). Using the form of the roots of the system (9), we obtain that

$$a_1=-\frac{1}{\pi^2k^2+a^2},\quad b_1=\frac{1}{\pi^2m^2+a^2},\quad b_2=-\frac{1}{\pi^2m^2+a^2},\quad c_2=\frac{1}{\pi^2s^2+a^2},\quad c_3=-\frac{1}{\pi^2s^2+a^2}.$$

$$J_{(\beta,0,0)}=-\sigma_{(\beta+1,1,1)}+(-1)^{\beta+1}\times$$

$$\times\sum_{k,m,s=1}^{\infty}\frac{1}{(m^2\pi^2+a^2)^{\beta+1}(\pi^2(k^2+m^2)+2a^2)(\pi^2(k^2+m^2+s^2)+3a^2)}+$$

$$+(-1)^{\beta+1}\sum_{k,m,s=1}^{\infty}\frac{1}{(\pi^2s^2+a^2)(\pi^2(k^2+m^2+s^2)+3a^2)}\times\left[\frac{1}{(m^2\pi^2+a^2)^{\beta+1}}+\frac{(-1)^{\beta}}{(k^2\pi^2+a^2)^{\beta+1}}\right],$$

$$J_{(\beta,0,0)}=\sum_{k,m,s=1}^{\infty}\frac{1}{(\pi^2(k^2+m^2)+2a^2)(\pi^2(k^2+m^2+s^2)+3a^2)}\times$$

$$\times\left[\frac{-1}{(k^2\pi^2+a^2)^{\beta+1}}+\frac{(-1)^{\beta+1}}{(m^2\pi^2+a^2)^{\beta+1}}\right]+$$

$$+(-1)^{\beta+1}\sum_{k,m,s=1}^{\infty}\frac{1}{(\pi^2s^2+a^2)(\pi^2(k^2+m^2+s^2)+3a^2)}\times\left[\frac{1}{(m^2\pi^2+a^2)^{\beta+1}}+\frac{(-1)^{\beta}}{(k^2\pi^2+a^2)^{\beta+1}}\right].$$

$$\times$$

For odd β the integral $J_{(\beta,0,0)}=0$, and for even $\beta=2n$ we obtain the following formula for finding the sum of the series

$$J_{(2n,0,0)}=\sum_{\|\alpha\|\leqslant 2n}\mathfrak{M}\left[\frac{\Delta\cdot Q^{\alpha}}{z_1^{2n}}\right]=-2\sigma_{(2n+1,1,1)}-$$

$$-2\sum_{k,m,s=1}^{\infty}\frac{1}{(\pi^2k^2+a^2)^{(2n+1)}\cdot(\pi^2s^2+a^2)\cdot(\pi^2(k^2+m^2+s^2)+3a^2)}.$$

Let us calculate, for example:

$$J_{(0,0,0)}=\mathfrak{M}[\Delta]=\mathfrak{M}\left[\frac{\partial f_1}{\partial z_1}\cdot\frac{\partial f_2}{\partial z_2}\cdot\frac{\partial f_3}{\partial z_3}\right]=\left(\frac{1}{2a^2}-\frac{1}{2a}\text{ctha}\right)^3.$$

Applying the identity

$$\frac{1}{(\pi^2k^2+a^2)(\pi^2(k^2+m^2)+2a^2)}+\frac{1}{(\pi^2m^2+a^2)(\pi^2(k^2+m^2)+2a^2)}=$$

$$= \frac{1}{(\pi^2k^2 + a^2)(\pi^2m^2 + a^2)},$$

We get that

$$2\sigma_{(1,1,1)} = \sum_{k,m,s=1}^{\infty} \frac{1}{(\pi^2k^2 + a^2)\cdot(\pi^2s^2 + a^2)\cdot(\pi^2(k^2 + m^2 + s^2) + 3a^2)}.$$

Thus, we get

$$\sum_{k,m,s=1}^{\infty} \frac{1}{(\pi^2k^2 + a^2)\cdot(\pi^2(k^2 + m^2) + 2a^2)\cdot(\pi^2(k^2 + m^2 + s^2) + 3a^2)}) =$$

$$= \frac{(a\mathrm{ctha} - 1)^3}{48a^6}.$$

2. Special systems of equations

Consider a system of functions $f_1(z),\ f_2(z), \ldots, f_n(z)$ of the form

$$\begin{cases} f_1(z) = (1 - a_{11}z_1)^{m_{11}} \cdots (1 - a_{1n}z_n)^{m_{1n}} + Q_1(z), \\ f_2(z) = (1 - a_{21}z_1)^{m_{21}} \cdots (1 - a_{2n}z_n)^{m_{2n}} + Q_2(z), \\ \ldots \\ f_n(z) = (1 - a_{n1}z_1)^{m_{n1}} \cdots (1 - a_{nn}z_n)^{m_{nn}} + Q_n(z), \end{cases} \tag{10}$$

where m_{ij} are natural numbers, a_{ij} are complex numbers that are different for fixed j, $Q_i(z)$ are entire functions, $i = 1, \ldots, n$. Let $J = (j_1, \ldots, j_n)$ be a multi-index, where $(j_1 \ldots j_n)$ is a permutation of $(1, \ldots, n)$. Let us define $a_J = (a_{1j_1}, \ldots, a_{nj_n})$ for a multi-index J. We denote

$$q_i(z_1, \ldots, z_n) = (1 - a_{i1}z_1)^{m_{i1}} \cdots (1 - a_{in}z_n)^{m_{in}}, \quad i = 1, \ldots, n, \tag{11}$$

then the system (10) can be rewritten as

$$f_i(z_1, \ldots, z_n) = q_i(z_1, \ldots, z_n) + Q_i(z_1, \ldots, z_n), \quad i = 1, \ldots, n. \tag{12}$$

For each m we define the function

$$h_m(z) = \begin{cases} q_m(z) & \text{if } a_{mj} \neq 0 \ \text{ for all } j; \\ q_m(z) \cdot \dfrac{1}{z_{j_1}} \cdot \ldots \cdot \dfrac{1}{z_{j_k}} & \text{if } a_{mj_1} = \ldots = a_{ij_k} = 0. \end{cases} \tag{13}$$

A system

$$h_m(z) = 0, \quad i = 1, \ldots, n, \tag{14}$$

has $n!$ isolated roots in $\overline{\mathbb{C}}^n$, where $\overline{\mathbb{C}}^n = \overline{\mathbb{C}} \times \cdots \times \overline{\mathbb{C}}$. Recall that $\overline{\mathbb{C}}$ is a compactification of the complex plane $\mathbb{C}$ (the Riemann sphere). Then $\overline{\mathbb{C}}^n$ is one of the well-known compactifications of $\mathbb{C}^n$ (the function theory space). The roots of the system (14) are equal

$$\tilde{a}_J = \begin{cases} \left(1/a_{1j_1}, \ldots, 1/a_{nj_n}\right) & \text{if } a_{kj_k} \neq 0 \text{ for } \ k = 1, \ldots, n, \\ \left(1/a_{1j_1}, \ldots, \infty_{[i_1]}, \ldots, \infty_{[i_k]}, \ldots, 1/a_{nj_n}\right) & \text{if } a_{i_1j_{i_1}} = \ldots = a_{i_kj_{i_k}} = 0, \end{cases}$$

where $k, j = 1, \ldots, n$. If $a_{j_1,i_1} = 0$, then in $\tilde{a}_J$ we write ∞, this is the point at infinity in $\overline{\mathbb{C}}$.

By Γ_h we denote the (global) cycle:

$$\Gamma_h = \{z \in \mathbb{C}^n \colon |h_m| = r_m,\ r_i > 0,\ m = 1, \dots, n\}. \tag{15}$$

In the case when all $a_{k,j_k} \neq 0$, we define the (local) cycle $\Gamma_{h,\tilde{a}_J}$ as follows

$$\begin{cases} |1 - a_{1j_1} z_1| = r_1, \\ |1 - a_{2j_2} z_2| = r_2, \\ \dots \\ |1 - a_{nj_n} z_n| = r_n. \end{cases} \tag{16}$$

If $a_{i_1 j_{i_1}} = \ldots = a_{i_k j_{i_k}} = 0$ for some $i_1, \dots, i_k$, then $\Gamma_{h,\tilde{a}_J}$ is defined as

$$\begin{cases} |1 - a_{1j_1} z_1| = r_1, \\ \dots \\ |1/z_{i_1}| = r_{i_1}, \\ \dots \\ |1/z_{i_k}| = r_{i_k}, \\ \dots \\ |1 - a_{nj_n} z_n| = r_n. \end{cases} \tag{17}$$

Lemma 1. *For sufficiently small r_m, the global cycle Γ_h has connected components (local cycles) in the neighborhood of the roots a_J. Moreover, Γ_h is homologous to the sum of local cycles $\Gamma_{h,\tilde{a}_J}$.*

Consider the system of equations

$$F_m(z,t) = q_m(z) + t \cdot Q_m(z) = 0, \quad m = 1, \dots, n, \tag{18}$$

depending on the real parameter $t \geqslant 0$. Let $r_1 > 0, \dots, r_n > 0$ be fixed real numbers. Then, for sufficiently small $t > 0$, the inequalities

$$\big|q_m(z)\big| > \big|t \cdot Q_m(z)\big|, \quad m = 1, \dots, n$$

on cycles

$$\Gamma_h = \{z \in \mathbb{C}^n \colon |h_m| = r_m, \quad m = 1, \dots, n\}$$

because Γ_h is compact.

We denote by $J_\gamma(t)$ the residue integral

$$J_\gamma(t) = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_h} \frac{1}{z^{\gamma+I}} \cdot \frac{dF}{F} = \tag{19}$$

$$= \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_h} \frac{1}{z_1^{\gamma_1+1} \cdot z_2^{\gamma_2+1} \cdots z_n^{\gamma_n+1}} \cdot \frac{dF_1}{F_1} \wedge \frac{dF_2}{F_2} \wedge \ldots \wedge \frac{dF_n}{F_n},$$

where $\gamma = (\gamma_1, \dots, \gamma_n)$ is a multi-index, and $I = (1, 1, \dots, 1)$.

We denote by $\Delta = \Delta(t)$ the Jacobian of the system of functions $F_1(z,t), \dots, F_n(z,t)$ in the variables $z_1, \dots, z_n$.

Theorem 3. *Under the assumptions made on the functions F_i defined by formulas (18), the following expressions for $J_\gamma(t)$ are absolutely convergent (for sufficiently small t) series:*

$$J_\gamma(t) = {\sum_J}' \sum_\alpha \frac{(-1)^{s(J)}(-t)^{||\alpha||+\|\beta(\alpha,J)|+n}}{\beta(\alpha,J)!\cdot a_J^{\beta+I}}\times$$

$$\times \frac{\partial^{||\beta(\alpha(J)||}}{\partial z^{\beta(\alpha,J)}}\left[\frac{\Delta(t)}{z_1^{\gamma_1+1}\cdot\ldots\cdot z_n^{\gamma_n+1}}\cdot\frac{Q^\alpha}{q^{\alpha+I}(J)}\right]\Bigg|_{z=\tilde a_J},$$

where $(-1)^{s(J)}$ is the parity of the permutation J, $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a multi-index of length n, $q^{\alpha+I}(J) = q_1^{\alpha_1+1}[j_1]\cdots q_n^{\alpha_n+1}[j_n]$, and $q_s[j_s]$ is the product of all $(1-a_{j1}z_1)^{m_{j1}}\cdots(1-a_{jn}z_n)^{m_{jn}}$, except $(1-a_{sj_s}z_s)^{m_{sj_s}}$,

$$\beta(\alpha,J) = \big(m_{1j_1}(\alpha_{j_1}+1)-1,\ldots,m_{nj_n}(\alpha_{j_n}+1)-1\big),$$

$$\beta(\alpha,J)! = \prod_p \big(m_{pj_p}(\alpha_{j_p}+1)-1\big)!,$$

$$a_j^{\beta+I} = a_{1j_1}^{m_{1j_1}(\alpha_{j_1}+1)}\cdots a_{nj_n}^{m_{nj_n}(\alpha_{j_n}+1)},$$

$$\frac{\partial^{||\beta(\alpha(J)||}}{\partial z^{\beta(\alpha,J)}} = \frac{\partial^{m_{1j_1}(\alpha_{j_1}+1)-1+\ldots+m_{nj_n}(\alpha_{j_n}+1)-1}}{\partial z_1^{m_{1j_1}(\alpha_{j_1}+1)-1}\ldots\partial z_n^{m_{nj_n}(\alpha_{j_n}+1)-1}}.$$

The dash at the summation sign means that the summation is performed over all multi-indices J for which there are no zero coordinates in a_J.

Suppose $Q_s(z)$ are polynomials:

$$Q_s(z) = z_1\cdots z_n \sum_{|\alpha\|\geqslant 0} C_\alpha^s z^\alpha \quad s=1,\ldots,n, \tag{20}$$

where α is a multi-index, $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$, and $\deg_{z_j} Q_s \leqslant m_{sj}$, $s,j=1,\ldots,n$ for all nonzero a_{sj}. If $a_{sj}=0$, then there are no restrictions on the degree $\deg_{z_j} Q_s$.

Assuming that all $w_j \neq 0$, we make the change $z_j = \dfrac{1}{w_j}$, $j=1,\ldots,n$ in the functions

$$F_s(z,t) = \big(q_s(z) + t\cdot Q_s(z)\big), \quad s=1,\ldots,n.$$

Hence, for $s=1,\ldots,n$ we have

$$F_s\left(\frac{1}{w_1},\ldots,\frac{1}{w_n},t\right) = q_s\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right) + t\cdot Q_s\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right) =$$
$$= \left(1-a_{s1}\frac{1}{w_1}\right)^{m_{s1}}\cdots\left(1-a_{sn}\frac{1}{w_n}\right)^{m_{sn}} + t\cdot Q_s\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right) =$$
$$= \left(\frac{1}{w_1}\right)^{m_{s1}}\cdots\left(\frac{1}{w_n}\right)^{m_{sn}}\cdot(w_1-a_{s1})^{m_{s1}}\cdots(w_n-a_{sn})^{m_{sn}} + t\cdot Q_s\left(\frac{1}{w_1},\ldots,\frac{1}{w_n}\right).$$

Then we arrive at the formula

$$F_s\left(\frac{1}{w_1},\ldots,\frac{1}{w_n},t\right) = \left(\frac{1}{w_1}\right)^{m_{s1}}\cdots\left(\frac{1}{w_n}\right)^{m_{sn}}\cdot\Big(\widetilde{q}_s(w) + t\cdot\widetilde{Q}_s(w)\Big), \tag{21}$$

where $\widetilde{q}_s$ are functions

$$\widetilde{q}_s = (w_1-a_{s1})^{m_{s1}}\cdots(w_n-a_{sn})^{m_{sn}},$$

and $\widetilde{Q}_i$ are polynomials

$$\widetilde{Q}_s = w_1^{m_{s1}} \cdots w_n^{m_{sn}} \cdot Q_s\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}\right).$$

From the formula (20) we obtain

$$\deg_{w_j} \widetilde{Q}_s < m_{sj}, \quad s, j = 1, \dots, n.$$

We denote

$$\widetilde{F_s} = \widetilde{F_s}(w, t) = \widetilde{q}_s(w) + t \cdot \widetilde{Q}_s(w), \quad s = 1, \dots, n. \tag{22}$$

If $0 \leqslant t \leqslant 1$, then the system (22) has a finite number of roots in $\mathbb{C}^n$ that depend on t. Moreover, (22) has no infinite roots in $\overline{\mathbb{C}}^n$.

Consider the cycle

$$\widetilde{\Gamma}_h = \left\{ w \in \mathbb{C}^n : \left| h_s\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}\right)\right| = \varepsilon_s, \quad s = 1, \dots, n \right\},$$

for t close enough to zero. The compactness of the cycle $\widetilde{\Gamma}_h$ implies

$$\left| q_s\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}\right)\right| > \left| t \cdot Q_s\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}\right)\right|, \quad s = 1, \dots, n.$$

Therefore, $\widetilde{\Gamma}_h$ is homologous to the sum of cycles $\widetilde{\Gamma}_{h,\tilde{a}_J}$:

$$\begin{cases} \left|1 - a_{1j_1}\dfrac{1}{w_1}\right| = \varepsilon_1, \\ \left|1 - a_{2j_2}\dfrac{1}{w_2}\right| = \varepsilon_2, \\ \dots\dots\dots\dots \\ \left|1 - a_{nj_n}\dfrac{1}{w_n}\right| = \varepsilon_n, \end{cases} \tag{23}$$

obtained from the cycles $\Gamma_{h,\tilde{a}_J}$ by replacing $z_j = \dfrac{1}{w_j}$.

The equation

$$\left|1 - a_{js_j}\frac{1}{w_j}\right| = \varepsilon$$

defines a circle. Indeed, we rewrite it as

$$|w_j - a_{js_j}| = \varepsilon |w_j| \quad \text{or} \quad |w_j - a_{js_j}|^2 = \varepsilon^2 |w_j|^2,$$

then

$$(1 - \varepsilon^2)\left|w_j - \frac{a_{js_j}}{1 - \varepsilon^2}\right|^2 = \frac{\varepsilon^2 \cdot |a_{js_j}|^2}{(1 - \varepsilon^2)}$$

or

$$\left|w_j - \frac{a_{js_j}}{1 - \varepsilon^2}\right|^2 = \frac{\varepsilon^2 \cdot |a_{js_j}|^2}{(1 - \varepsilon^2)^2}, \quad j = 1, \dots, n,$$

for sufficiently small ε the point a_{js_j} lies outside the circle and, therefore, $\widetilde{\Gamma}_{h,\tilde{a}_J}$ is homologous to the cycle $\widetilde{\Gamma}_{h,a_J}$:

$$\begin{cases} |w_1 - a_{1j_1}| = \varepsilon_1, \\ |w_2 - a_{2j_2}| = \varepsilon_2, \\ \dots\dots\dots\dots \\ |w_n - a_{nj_n}| = \varepsilon_n. \end{cases}$$

Here some a_{ij} can be zero.

Lemma 2. *The residue integral (19) is*

$$J_\gamma(t)=\frac{(-1)^n}{(2\pi i)^n}\int\limits_{\widetilde{\Gamma}_h} w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\cdot\frac{d\widetilde{F_1}}{\widetilde{F_1}}\wedge\frac{d\widetilde{F_2}}{\widetilde{F_2}}\wedge\ldots\wedge\frac{d\widetilde{F}_n}{\widetilde{F}_n}. \tag{24}$$

Theorem 4. *The following equalities are valid*

$$\sum_{j=1}^{p}\frac{1}{z_{j1}(t)^{\gamma_1+1}\cdot z_{j2}(t)^{\gamma_2+1}\cdots z_{jn}(t)^{\gamma_n+1}}=$$

$$=\sum_{K\in\Re}(-t)^{||K||+n}\sum_{J}\frac{(-1)^{s(J)}}{\beta(K,J)!}\cdot\frac{\partial^{||\beta(K,J)||}}{\partial w^{\beta}(K,J)}\times\left[\widetilde{\Delta}(t)\cdot w_1^{\gamma_1+1}\cdots w_n^{\gamma_n+1}\cdot\frac{\widetilde{Q}^K}{\widetilde{q}^{K+I}(J)}\right]\Bigg|_{w=a_J}.$$

Since zeros of (22) are polynomials in t, the equality (4) also holds for $t=1$. We denote

$$\sigma_{\gamma+I}=\sum_{j=1}^{p}\frac{1}{z_{j1}^{\gamma_1+1}\cdot z_{j2}^{\gamma_2+1}\cdots z_{jn}^{\gamma_n+1}},$$

where $z^{(j)}=(z_{j1},\ldots,z_{jn})=(z_{j1}(1),\ldots,z_{jn}(1))$, $j=1,\ldots,n$.

Theorem 5. *For the system (10) with functions f_j defined in (12) and Q_i defined in (20), the following formulas hold:*

$$\sigma_{\gamma+I}=\sum_{j=1}^{p}\frac{1}{z_{j1}^{\gamma_1+1}\cdot z_{j2}^{\gamma_2+1}\cdots z_{jn}^{\gamma_n+1}}=$$

$$=\frac{1}{(2\pi i)^n}\sum_{\|K\|\geqslant 0}(-1)^{\|K\|+n}\sum_{J}(-1)^{s(J)}\times$$

$$\times\int\limits_{\widetilde{\Gamma}_{h,a_J}}\widetilde{\Delta}\cdot w_1^{\gamma_1+1}\ldots w_n^{\gamma_n+1}\cdot\frac{\widetilde{Q}_1^{k_1}\cdot\ldots\cdot\widetilde{Q}_n^{k_n}}{\widetilde{q}_1^{k_1+1}\cdot\ldots\cdot\widetilde{q}_n^{k_n+1}}dw=$$

$$=\sum_{K\in\Re}(-1)^{||K||+n}\sum_{J}\frac{(-1)^{s(J)}}{\beta(K,J)!}\cdot\frac{\partial^{||\beta(K,J)||}}{\partial w^{\beta}(K,J)}\left[\widetilde{\Delta}\cdot w_1^{\gamma_1+1}\cdots w_n^{\gamma_n+1}\cdot\frac{\widetilde{Q}^K}{\widetilde{q}^{K+I}(J)}\right]\Bigg|_{w=a_J}.$$

Consider the following system of equations in two complex variables:

$$\begin{cases} f_1(z_1,z_2)=(1-a_2z_2)^2+a_3z_1z_2^2=0,\\ f_2(z_1,z_2)=(1-b_1z_1)^2(1-b_2z_2)+b_3z_1^2z_2=0.\end{cases} \tag{25}$$

Then Q_m, $m=1,2$ have the form (20). The system (25) has, as is easy to verify, 5 roots (z_{j1},z_{j2}), $j=1,2,3,4,5$. If $a_2\neq b_2$, then these roots do not lie on the coordinate planes.

Let us change the variables $z_1=\dfrac{1}{w_1}$, $z_2=\dfrac{1}{w_2}$. Our system will take the form

$$\begin{cases} \widetilde{f}_1=w_1(w_2-a_2)^2+a_3=0,\\ \widetilde{f}_2=(w_1-b_1)^2(w_2-b_2)+b_3=0.\end{cases}. \tag{26}$$

Jacobian of the system (26)

$$\widetilde{\Delta}=\begin{vmatrix}(w_2-a_2)^2 & 2w_1(w_2-a_2)\\ 2(w_1-b_1)(w_2-b_2) & (w_1-b_1)^2\end{vmatrix}=$$

$$= (w_1 - b_1)^2 (w_2 - a_2)^2 - 4w_1(w_1 - b_1)(w_2 - a_2)(w_2 - b_2).$$

Then, by Theorem 5, we obtain

$$\sigma_\gamma = \sum_{j=1}^{5} \frac{1}{z_{j1}^{\gamma_1+1}} \cdot \frac{1}{z_{j2}^{\gamma_2+1}} = \sum_J \sum_{K \in \Re} \frac{(-1)^{\|K\|+s(j)}}{(2\pi i)^2} \times$$

$$\times \int\limits_{\widetilde{\Gamma}_{h,a_J}} \frac{w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdot a_3^{k_1} \cdot b_3^{k_2} \cdot \widetilde{\Delta} \cdot dw_1 \wedge dw_2}{w_1^{k_1+1}(w_2 - a_2)^{2(k_1+1)} \cdot (w_1 - b_1)^{2(k_2+1)}(w_2 - b_2)^{k_2+1}}. \tag{27}$$

Here the multi-indices $\Re = \{K = (k_1, k_2)| \quad \exists m : \gamma_m + 2 > k_1 + k_2, \quad m = 1, 2\}$. The cycles $\widetilde{\Gamma}_{h,a_J}$ are cycles of the form $\{|w_1| = r_{11}, |w_2 - b_2| = r_{22}\}$, taken with positive orientation, and $\{|w_2 - a_2| = r_{12}, |w_1 - b_1| = r_{21}\}$ are with negative orientation.

In particular, calculating $J_{(0,0)}$, after some transformations we obtain

$$\sigma_{(1,1)} = 4a_2 b_1 - \frac{a_3 b_2}{(b_2 - a_2)^2} \tag{28}$$

without finding the roots.

Consider a system of equations in three complex variables:

$$\begin{cases} f_1(z_1, z_2, z_3) = 1 - a_1 z_1 - a_2 z_2 - a_3 z_3 + a_1 a_2 z_1 z_2 + a_1 a_3 z_1 z_3 + a_2 a_3 z_2 z_3 = \\ = (1 - a_1 z_1)(1 - a_2 z_2)(1 - a_3 z_3) + a_1 a_2 a_3 z_1 z_2 z_3 = 0, \\ f_2(z_1, z_2, z_3) = 1 - b_1 z_1 - b_2 z_2 - b_3 z_3 + b_1 b_2 z_1 z_2 + b_1 b_3 z_1 z_3 + b_2 b_3 z_2 z_3 = \\ = (1 - b_1 z_1)(1 - b_2 z_2)(1 - b_3 z_3) + b_1 b_2 b_3 z_1 z_2 z_3 = 0, \\ f_3(z_1, z_2, z_3) = 1 - c_1 z_1 - c_2 z_2 - c_3 z_3 + c_1 c_2 z_1 z_2 + c_1 c_3 z_1 z_3 + c_2 c_3 z_2 z_3 = \\ = (1 - c_1 z_1)(1 - c_2 z_2)(1 - c_3 z_3) + c_1 c_2 c_3 z_1 z_2 z_3 = 0. \end{cases} \tag{29}$$

The roots of the system (29) are (z_{j1}, z_{j2}, z_{j3}), $j = 1, \ldots, 12$.

Change the variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$ and $z_3 = \dfrac{1}{w_3}$. Our system will take the form

$$\begin{cases} \widetilde{f}_1 = w_1 w_2 w_3 - a_1 w_2 w_3 - a_2 w_1 w_3 - a_3 w_1 w_2 + a_1 a_2 w_3 + a_1 a_3 w_2 + a_2 a_3 w_1 = \\ = (w_1 - a_1)(w_2 - a_2)(w_3 - a_3) + a_1 a_2 a_3 = 0, \\ \widetilde{f}_2 = w_1 w_2 w_3 - b_1 w_2 w_3 - b_2 w_1 w_3 - b_3 w_1 w_2 + b_1 b_2 w_3 + b_1 b_3 w_2 + b_2 b_3 w_1 = \\ = (w_1 - b_1)(w_2 - b_2)(w_3 - b_3) + b_1 b_2 b_3 = 0, \\ \widetilde{f}_3 = w_1 w_2 w_3 - c_1 w_2 w_3 - c_2 w_1 w_3 - c_3 w_1 w_2 + c_1 c_2 w_3 + c_1 c_3 w_2 + c_2 c_3 w_1 = \\ = (w_1 - c_1)(w_2 - c_2)(w_3 - c_3) + c_1 c_2 c_3 = 0. \end{cases} \tag{30}$$

The Jacobian of the system (30)

$$\widetilde{\Delta} = (w_2 - a_2)(w_3 - a_3)[(w_1 - b_1)(w_3 - b_3)(w_1 - c_1)(w_2 - c_2) - (w_1 - b_1)(w_2 - b_2)(w_1 - c_1)(w_3 - c_3)] -$$

$$-(w_1 - a_1)(w_3 - a_3)[(w_2 - b_2)(w_3 - b_3)(w_1 - c_1)(w_2 - c_2) - (w_1 - b_1)(w_2 - b_2)(w_2 - c_2)(w_3 - c_3)] +$$

$$+(w_1 - a_1)(w_2 - a_2)[(w_2 - b_2)(w_3 - b_3)(w_1 - c_1)(w_3 - c_3) - (w_1 - b_1)(w_3 - b_3)(w_2 - c_2)(w_3 - c_3)].$$

Then, by Theorem 5, we obtain $J_{(0,0,0)} = \sum_J (-1)^s (J)$

$$\sum_{\|k\|<2} \frac{(-1)^{\|k\|}}{(2\pi i)^2} \int\limits_{\widetilde{\Gamma}_{q,a_J}} \frac{w_1 w_2 w_3 \cdot (a_1 a_2 a_3)^{k_1} (b_1 b_2 b_3)^{k_2} (c_1 c_2 c_3)^{k_3} \cdot \widetilde{\Delta}}{(w_1 - a_1)^{k_1+1} (w_2 - a_2)^{k_1+1} (w_3 - a_3)^{k_1+1}} \times$$

$$\times \frac{dw_1 \wedge dw_2 \wedge dw_3}{(w_1-b_1)^{k_2+1}(w_2-b_2)^{k_2+1}(w_3-b_3)^{k_2+1} \cdot (w_1-c_1)^{k_3+1}(w_2-c_2)^{k_3+1}(w_3-c_3)^{k_3+1}}, \tag{31}$$

where $\widetilde{\Gamma}_{q,a_J}$ are cycles of the form $\{|w_1-a_1|=r_{11}, |w_2-b_2|=r_{22}, |w_3-c_3|=r_{33}\}$; $\{|w_3-a_3|=r_{13}, |w_1-b_1|=r_{21}, |w_2-c_2|=r_{32}\}$; $\{|w_2-a_2|=r_{12}, |w_3-b_3|=r_{23}, |w_1-c_1|=r_{31}\}$, taken with a positive orientation, and $\{|w_1-a_1|=r_{11}, |w_3-b_3|=r_{23}, |w_2-c_2|=r_{32}\}$; $\{|w_2-a_2|=r_{12}, |w_1-b_1|=r_{21}, |w_3-c_3|=r_{33}\}$; $\{|w_3-a_3|=r_{13}, |w_2-b_2|=r_{22}, |w_1-c_1|=r_{31}\}$ with negative orientation.

Calculating these integrals, we get

$$-\sigma_{(1,1,1)} = J_{(0,0,0)} = a_1b_2c_3 + a_1b_3c_2 + a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 + a_3b_2c_1+ \tag{32}$$
$$+\frac{a_3c_1c_2c_3}{a_3-c_3}\cdot\left[\frac{b_1}{b_1-c_1}+\frac{b_2}{b_2-c_2}\right]+\frac{a_1b_1b_2b_3}{a_1-b_1}\cdot\left[\frac{c_3}{c_3-b_3}+\frac{c_2}{c_2-b_2}\right]+$$
$$+\frac{a_2b_1b_2b_3}{a_2-b_2}\cdot\left[\frac{c_3}{c_3-b_3}+\frac{c_1}{c_1-b_1}\right]+\frac{a_3b_1b_2b_3}{a_3-b_3}\cdot\left[\frac{c_2}{c_2-b_2}+\frac{c_1}{c_1-b_1}\right]+$$
$$+\frac{a_1c_1}{a_1-c_1}\cdot\left[\frac{b_2c_2c_3}{b_2-c_2}+\frac{b_3c_2c_3}{b_3-c_3}+\frac{a_2a_3b_2}{a_2-b_2}+\frac{a_2a_3b_3}{a_3-b_3}\right]+$$
$$+\frac{a_2c_2}{a_2-c_2}\cdot\left[\frac{b_1c_1c_3}{b_1-c_1}+\frac{b_3c_1c_3}{b_3-c_3}+\frac{a_1a_3b_3}{a_3-b_3}+\frac{a_1a_3b_1}{a_1-b_1}\right].$$

So, we found the sums of the roots $\sigma_{(1,1,1)}$ without calculating the roots of the system themselves.

3. General systems of transcendental equations

Let $f_1(z), \dots, f_n(z)$ be a system of functions holomorphic in a neighborhood of the origin in the multidimensional complex space $\mathbb{C}^n$, $z = (z_1, \dots, z_n)$.

We expand the functions $f_1(z), \dots, f_n(z)$ in Taylor series in the vicinity of the origin and consider a system of equations of the form

$$f_j(z) = P_j(z) + Q_j(z) = 0, \quad i = 1, \dots, n, \tag{33}$$

where P_j is the lowest homogeneous part of the Taylor expansion of the function $f_j(z)$. The degree of all monomials (with respect to the totality of variables) included in P_j, is equal to m_j, $j = 1, \dots, n$. In the functions Q_jl, the degrees of all monomials are strictly greater than m_j.

The expansion of the functions Q_j, P_j, $j = 1, \dots, n$ in a neighborhood of zero in Taylor series converging absolutely and uniformly in this neighborhood has the form

$$Q_j(z) = \sum_{\|\alpha\|>m_j} a^j_\alpha z^\alpha, \tag{34}$$

$$P_j(z) = \sum_{\|\beta\|=m_j} b^j_\beta z^\beta, \tag{35}$$
$$j = 1, \dots, n,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ are multi-indexes, i.e. α_j and β_j are non-negative integers, $j = 1, \dots, n$, $\|\alpha\| = \alpha_1 + \ldots + \alpha_n$, $\|\beta\| = \beta_1 + \ldots + \beta_n$, and monomials $z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_n^{\alpha_n}$, $z^\beta = z_1^{\beta_1} \cdot z_2^{\beta_2} \cdots z_n^{\beta_n}$.

In what follows, we will assume that the system of polynomials $P_1(z), \dots, P_n(z)$ is *nondegenerate*, that is, its common zero is only point 0, the origin.

Consider an open set (a special analytic polyhedron) of the form

$$D_P(r_1,\dots,r_n)=\{z:\ |P_j(z)|<r_j,\ i=j,\dots,n\},$$

where $r_1,\dots,r_n$ are positive numbers. Its *skeleton* has the form

$$\Gamma_P(r_1,\dots,r_n)=\Gamma_P(r)=\{z:\ |P_j(z)|=r_j,\ j=1,\dots,n\}.$$

Let us start with a statement.

Lemma 3. *The next equality is true*

$$J_\gamma=\frac{1}{(2\pi i)^n}\int\limits_{\Gamma_P}\frac{1}{z_1^{\gamma_1+1}\cdot z_2^{\gamma_2+1}\cdots z_n^{\gamma_n+1}}\cdot\frac{df_1}{f_1}\wedge\frac{df_2}{f_2}\wedge\dots\wedge\frac{df_n}{f_n}=$$

$$=\frac{(-1)^n}{(2\pi i)^n}\int\limits_{\Gamma_{\tilde P}}w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\cdot\frac{d\tilde f_1}{\tilde f_1}\wedge\frac{d\tilde f_2}{\tilde f_2}\wedge\dots\wedge\frac{d\tilde f_n}{\tilde f_n}=(-1)^n\tilde J_\gamma.$$

For what follows, we need a generalized formula for transforming the Grothendieck residue.

Theorem 6. *Let $h(w)$ be a holomorphic function, and the polynomials $f_k(w)$ and $g_j(w)$, $j,k=1,\dots,n$, are related by the relations*

$$g_j=\sum_{k=1}^n a_{jk}f_k,\quad j=1,2,\dots,n,$$

the matrix $A=\|a_{jk}\|_{j,k=1}^n$ consists of polynomials. Consider the cycles

$$\Gamma_f=\{w:\ |f_j(w)|=r_j,\ j=1,\dots,n\},$$

$$\Gamma_g=\{w:\ |g_j(z)|=r_j,\ j=1,\dots,n\},$$

where all $r_j>0$. Then the equality

$$\int\limits_{\Gamma_f}h(w)\frac{dw}{f^\alpha}=\sum_{K,\sum\limits_{s=1}^n k_{sj}=\beta_s}\frac{\beta!}{\prod\limits_{s,j=1}^n(k_{sj})!}\int\limits_{\Gamma_g}h(w)\frac{\det A\prod\limits_{s,j=1}^n a_{sj}^{k_{sj}}\,dw}{g^\beta},\tag{36}$$

holds. Here $\beta!=\beta_1!\beta_2!\dots\beta_n$, $\beta=(\beta_1,\beta_2,\dots,\beta_n)$, the summation in the formula is over all non-negative integer matrices $K=\|k_{sj}\|_{s,j=1}^n$ with the conditions that the sum $\sum\limits_{s=1}^n k_{sj}=\alpha_j$, then $\beta_j=\sum\limits_{j=1}^n k_{js}$. Here$f^\alpha=f_1^{\alpha_1}\cdots f_n^{\alpha_n}$, $g^\beta=g_1^{\beta_1}\cdots g_n^{\beta_n}$.

Theorem 7. *The next formulas are valid*

$$\sum_{j=1}^p\frac{1}{z_{j1}^{\gamma_1+1}\cdot z_{j2}^{\gamma_2+1}\cdots z_{jn}^{\gamma_n+1}}=$$

$$=(2\pi i)^n)\int\limits_{\Gamma_{\tilde P}}w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\cdot\frac{d\tilde f_1}{\tilde f_1}\wedge\frac{d\tilde f_2}{\tilde f_2}\wedge\dots\wedge\frac{d\tilde f_n}{\tilde f_n}=$$

$$= \sum_{\|\alpha\|\leqslant\|\gamma\|+n} \frac{(-1)^{n+\|\alpha\|}}{(2\pi i)^n} \int\limits_{\Gamma_{\tilde{P}}} w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\times$$

$$\times\frac{\tilde{\Delta}\cdot\tilde{Q}_1^{\alpha_1}\cdot\tilde{Q}_2^{\alpha_2}\cdots\tilde{Q}_n^{\alpha_n}dw_1\wedge dw_2\wedge\ldots\wedge dw_n}{\tilde{P}_1^{\alpha_1+1}\cdot\tilde{P}_2^{\alpha_2+1}\cdots\tilde{P}_n^{\alpha_n+1}} =$$

$$= \sum_{\|K\|\leqslant\|\gamma\|+n} \frac{(-1)^{\|K\|+n}\prod\limits_{s=1}^{n}\left(\sum\limits_{j=1}^{n}k_{sj}\right)!}{\prod\limits_{s,j=1}^{n}(k_{sj})!}\mathfrak{M}\left[\frac{w^{\gamma+I}\cdot\tilde{\Delta}\cdot\det A\cdot Q^{\alpha}\prod\limits_{s,j=1}^{n}a_{sj}^{k_{sj}}}{\prod\limits_{j=1}^{n}w_j^{\beta_jN_j+\beta_j+N_j}}\right],$$

where $\|K\| = \sum\limits_{s,j=1}^{n} k_{sj}$*, and the functional* $\mathfrak{M}$ *assigns its free term to the Laurent polynomial.*

In fact, in Theorem 7, analogs of the classical Waring formulas for finding power sums of roots of a system of algebraic equations are obtained.

Consider a system of equations in two complex variables

$$\begin{cases} f_1(z_1,z_2) = a_1z_1 - a_2z_2 + z_1^2 = 0,\\ f_2(z_1,z_2) = b_1z_1 + b_2z_2 + z_2^2 = 0.\end{cases} \tag{37}$$

It satisfies the conditions on $Q_j(z)$ We will assume that $a_1b_2 + a_2b_1 \neq 0$, i.e. the system of lower homogeneous polynomials is nondegenerate.

Let us change variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will take the form

$$\begin{cases} \tilde{f}_1 = -a_2w_1^2 + a_1w_1w_2 + w_2 = 0,\\ \tilde{f}_2 = b_2w_1w_2 + b_1w_2^2 + w_1 = 0.\end{cases} . \tag{38}$$

This system has 4 roots, on the coordinate planes there is one root — (0,0).

The Jacobian of the system (38)

$$\tilde{\Delta} = \begin{vmatrix} -2a_2w_1 + a_1w_2 & a_1w_1 + 1\\ b_2w_2 + 1 & 2b_1w_2 + b_2w_1\end{vmatrix} =$$

$$= -2a_2b_2w_1^2 - 4a_2b_1w_1w_2 + 2a_1b_1w_2^2 - a_1w_1 - b_2w_2 - 1.$$

Notice that

$$\tilde{Q}_1 = w_2, \quad \tilde{Q}_2 = w_1. \tag{39}$$

$$\tilde{P}_1 = -a_2w_1^2 + a_1w_1w_2, \quad \tilde{P}_2 = b_2w_1w_2 + b_1w_2^2. \tag{40}$$

To find the matrix A we use Example 8.3 from [4].

We introduce the matrix

$$\mathrm{Res} = \begin{pmatrix} -a_2 & a_1 & 0 & 0\\ 0 & -a_2 & a_1 & 0\\ 0 & b_2 & b_1 & 0\\ 0 & 0 & b_2 & b_1\end{pmatrix}.$$

The determinant Δ of the matrix Res is $\Delta = a_2b_1(a_2b_1 + a_1b_2)$. Let us calculate some minors according to example 8.3 from [4]:

$$\tilde{\Delta}_1 = \begin{vmatrix} -a_2 & a_1 & 0 \\ b_2 & b_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = -a_2b_1^2 - a_1b_1b_2, \quad \tilde{\Delta}_2 = -\begin{vmatrix} a_1 & 0 & 0 \\ b_2 & b_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = -a_1b_1^2,$$

$$\widetilde{\Delta}_3 = \begin{vmatrix} a_1 & 0 & 0 \\ -a_2 & a_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = a_1^2b_1, \quad \tilde{\Delta}_4 = -\begin{vmatrix} a_1 & 0 & 0 \\ -a_2 & a_1 & 0 \\ b_2 & b_1 & 0 \end{vmatrix} = 0.$$

$$\Delta_1 = -\begin{vmatrix} 0 & -a_2 & a_1 \\ 0 & b_2 & b_1 \\ 0 & 0 & b_2 \end{vmatrix} = 0, \quad \Delta_2 = \begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & b_2 & b_1 \\ 0 & 0 & b_2 \end{vmatrix} = -a_2b_2^2,$$

$$\Delta_3 = -\begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & -a_2 & a_1 \\ 0 & 0 & b_2 \end{vmatrix} = -a_2^2b_2, \quad \Delta_4 = \begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & -a_2 & a_1 \\ 0 & b_2 & b_1 \end{vmatrix} = a_2^2b_1 + a_1a_2b_2.$$

Therefore, the elements a_{ij} of the matrix A are

$$a_{11} = \frac{1}{\Delta}\left(\tilde{\Delta}_1w_1 + \tilde{\Delta}_2w_2\right) = \frac{1}{\Delta}\left((-a_2b_1^2 - a_1b_1b_2)w_1 - a_1b_1^2w_2\right),$$

$$a_{12} = \frac{1}{\Delta}\left(\widetilde{\Delta}_3w_1 + \widetilde{\Delta}_4w_2\right) = \frac{a_1^2b_1w_1}{\Delta}, \qquad a_{21} = \frac{1}{\Delta}\left(\Delta_1w_1 + \Delta_2w_2\right) = \frac{-a_2b_2^2w_2}{\Delta},$$

$$a_{22} = \frac{1}{\Delta}\left(\Delta_3w_1 + \Delta_4w_2\right) = \frac{1}{\Delta}\left(-a_2^2b_2w_1 + (a_2^2b_1 + a_1a_2b_2)w_2\right).$$

Then it is easy to check that

$$w_1^3 = a_{11}\tilde{P}_1 + a_{12}\tilde{P}_2, \quad w_2^3 = a_{21}\tilde{P}_1 + a_{22}\tilde{P}_2.$$

We calculate $\det A$:

$$\det A = \frac{1}{\Delta}\left(a_2b_2w_1^2 - a_2b_1w_1w_2 - a_1b_1w_2^2\right).$$

By Theorem 7

$$J_{(0,0)} = \sum_{\|K\|\leqslant 2} \frac{(-1)^{\|K\|}\cdot(k_{11}+k_{12})!\cdot(k_{21}+k_{22})!}{k_{11}!\cdot k_{12}!\cdot k_{21}!\cdot k_{22}!}\times$$

$$\times\mathfrak{M}\left[\frac{\tilde{\Delta}\cdot\det A\cdot\tilde{Q}_1^{k_{11}+k_{21}}\cdot\tilde{Q}_2^{k_{12}+k_{22}}\cdot a_{11}^{k_{11}}\cdot a_{12}^{k_{12}}\cdot a_{21}^{k_{21}}\cdot a_{22}^{k_{22}}}{w_1^{3(k_{11}+k_{12})+1}\cdot w_2^{3(k_{21}+k_{22})+1}}\right].$$

We denote $\bar{\Delta} = a_2b_1 + a_1b_2$. Cumbersome but simple calculations (using the definition of the functional $\mathfrak{M}$) give that

$$J_{(0,0)} = \frac{1}{\bar{\Delta}} - \frac{2a_1b_2}{a_2b_1\bar{\Delta}} + \frac{6a_1^2b_2^2}{a_2b_1\bar{\Delta}^2} - \frac{b_2^3}{b_1\bar{\Delta}^2} + \frac{a_1^3}{a_2\bar{\Delta}^2} + \frac{8a_1b_2}{\bar{\Delta}^2} - \frac{4}{a_2b_1} =$$

$$= \frac{a_1^3}{a_2\bar{\Delta}^2} - \frac{a_1b_2}{\bar{\Delta}^2} - \frac{3a_2b_1}{\bar{\Delta}^2} - \frac{b_2^3}{b_1\bar{\Delta}^2}.$$

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О трансцендентных системах уравнений

Александр М. Кытманов
Ольга В. Ходос
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. Рассмотрены различные типы систем трансцендентных уравнений: простейшие, специальные и общие. Поскольку число корней таких систем, как правило, бесконечно, то необходимо изучить степенные суммы корней в отрицательной степени. Получены формулы для нахождения вычетных интегралов, их связь со степенными суммами корней в отрицательной степени, многомерные аналоги формул Варинга. Приведены различные примеры трансцендентных систем уравнений и вычислены суммы многомерных числовых рядов.

Ключевые слова: трансцендентные системы уравнений, степенные суммы корней, вычетные интегралы.

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Sharply 3-transitive Groups with Finite Element

Evgeny B. Durakov*
Siberian Federal University
Krasnoyarsk, Russian Federation

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Abstract. In this paper we study sharply 3-transitive groups. The local finiteness of sharply triply transitive permutation groups of characteristic $p > 3$ containing a finite element of order p is proved.

Keywords: group, sharply k-transitive group, sharply 3-transitive group, locally finite group, near-domain, near-field.

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Introduction

We recall that the group G of permutations of the set F ($|F| \geqslant k$) is called *exactly k-transitive* on F if for any two ordered sets $(\alpha_1, \dots, \alpha_k)$ and $(\beta_1, \dots, \beta_k)$ elements from F such that $\alpha_i \neq \alpha_j$ and $\beta_i \neq \beta_j$ for $i \neq j$, there is exactly one element of the group G taking α_i to β_i $(i = 1, \dots, k)$.

In 1872, K. Jordan described the class of finite sharply k-transitive groups for $k \geqslant 4$ ([1, page 215]).

In infinite groups J. Tits and M. Hall established that for $k \geqslant 4$ infinite sharply k-transitive groups do not exist ([1, page 215], [2, page 86–87]).

Unlike the cases $k \geqslant 4$, the sets of finite exactly 2- and 3-transitive groups are countable, and the locally finite sets are continuous.

Sharply 2- and 3-transitive groups are closely related algebraic structures such as near-fields, near-domains, KT-fields (Kerby-Tits fields), etc. (see [1, Ch. V], [2, chap. 20]).

Finite exactly 2- and 3-transitive groups and near-fields were classified by G. Zassenhaus [1, ch. IV and Theorem V.5.2]. Complete description of locally finite sharply 3-transitive groups in 1967 got O. Kegel [3].

The study of the class of infinite exactly 2- and 3-transitive groups is actively continued at the present time. In 2000 V. D. Mazurov in [4] fully described exactly 3 - transitive groups with abelian stabilizers of two points. In 2011, T. Grundhöfer and E. Jabara proved the local finiteness of the binary finite sharply doubly transitive groups [5]. In 2013, in the paper [6], A. I. Sozutov established a similar fact for the periodic groups of Shunkov.

In the paper [7], in the class of sharply triply transitive groups, the local finiteness of permutation groups with a periodic stabilizer of two points was proved and, as a consequence, the local finiteness of the periodic sharply 3-transitive groups.

In the papers [8, 9], examples of sharply doubly transitive groups of characteristic 2 that do not contain regular abelian normal subgroups are constructed, and in [10], there are similar examples of sharply 3-transitive groups. These examples show that there are near-domains of characteristic 2 that are not near-fields and KT-fields, (F, σ), in which near-domains $(F, +, \cdot)$ are not near-fields. This provides a basis for studying these structures with additional restrictions.

*durakov@mail.ru

Recall that a nonidentity element k of a group G is called *finite in* G if for any $g \in G$ the subgroup $\langle k, k^g \rangle$ is finite.

Let G be sharply 3-transitive on X, J the set of involutions in G, $J^2 = \{kv | k, v \in J\}$. The characteristic G ($Char(G)$) is defined as follows [1]:

1. $Char(G) = 2$, if elements from J do not fix points from X;
2. $Char(G) = 0$ if each $g \in J^2 \setminus \{1\}$ is of infinite order;
3. $Char(G) = p$, where p is odd prime, if the order of each $g \in J^2 \setminus \{1\}$ is p.

In continuation of the research started in [7] and [11], in this work a special case of Theorem 6 announced in [12] is proved:

Theorem 1. *A sharply triple transitive permutation group of characteristic $p > 3$, containing a finite element of order p, is locally finite.*

Proof of the theorem

Let G be an infinite sharply triply transitive permutation group of the set $X = F \cup \{\infty\}$. By B we denote the stabilizer G_α of the point $\alpha \in X$ and through H — stabilizer $G_{\alpha\,\beta} = G_\alpha \cap G_\beta$ of two points $\alpha = \infty \in X$, $\beta \in F$. Let also J be the set of involutions of the group G, and J_m be the set involutions stabilizing exactly m points, $m = 0, 1, 2$. Let us also formulate the well-known properties of involutions from groups $G = T_3(F, v)$ and $B = T_2(F)$(see, for example, [1, Ch. V]) with comments.

Lemma 1. *The following statements are true:*

1. *The group $B = G_\infty$ is regular on the set F an elementary abelian p-subgroup of U and $B = U \leftthreetimes H$ — Frobenius group.*
2. *U — Sylow p -subgroup of the group G, $B = N_G(U)$, $U^\# = a^H$, $C_G(u) = U$ for any element $u \in U^\#$ and $U \cap U^x = 1$ for any element $x \in G \setminus B$.*
3. *$H = G_\infty \cap G_\alpha$, H contains the only involution z, $z \in J_2$, $C_G(z) = N_G(H)$.*
4. *Each subgroup of order qr in H, where q, r not necessarily different primes, cyclic, and $H \cap H^x = 1$ for any element $x \in G \setminus N_G(H)$.*
5. *$N = N_G(H) = H \leftthreetimes \langle v \rangle$, where v is an involution from J_2, $C_H(v) = \langle z \rangle$.*
6. *If $N \cap N^x \neq 1$ for $x \in G \setminus N$, then $N \cap N^x = \langle t \rangle$, where $t = t(x)$ is an involution.*
7. *$G = B \cup BvU$ and $B \cap B^x = H^b$ for any $x \in G$ setminus$B$ and a suitable $b = b(x) \in B$.*

Proof. 1. The statement follows from [6, Theorem 2].

2. The statement easily follows from the exact 3-transitivity of G (see also [7, Lemma 1], [13], item 1 of the lemma and finiteness of elements from U. Non-trivial element from $U \cap U^x$ must stabilize two points, which is impossible in view of item 1.

3. The statement is well known [1, 6, 14].

4. The statement follows from Burnside's theorem [15, Theorem 1.2], 3-transitivity of G and equality $B \cap B^x = G_\infty \cap G_{\infty^x}$.

5. This statement and statement 6 are obvious.

7. Follows from 2- (and even 3-) transitivity and items $1, 5$ of the lemma.

The lemma is proved. □

The groups H and $N = C_G(z)$ will also be denoted by H_z and N_z, and for $k = z^g$ by H_k and N_k we will denote subgroups H^g and N^g.

Lemma 2. *The following statements are true:*

1. *Either $J = J_2$, or $J = J_0 \cup J_2$, while $J_2 = v^G$.*
2. *For each involution j the set $vN \cap j^G$ is infinite.*
3. *For each involution $j \in J$ the set $J_2 \cap C_G(j)$ is infinite.*
4. *Every Sylow 2-subgroup in H is (locally) cyclic, or (locally) quaternionic; are they conjugate, isomorphic, we do not know yet.*
5. *Every Sylow 2-subgroup of T from N whose order is greater than 4, is a Sylow 2-subgroup of G.*
6. *If a Sylow 2-subgroup T of N is a proper subgroup of a Sylow 2-subgroup R of G, then R is a (locally) dihedral group.*
7. *G contains no elementary abelian subgroups of order 8, containing an involution from J_2. The rank of Sylow 2-subgroups in N is 2. The rank of any Sylow 2-subgroup of G containing an involution from J_2, is equal to 2.*

Proof. 1. The inequalities $0 \leqslant m \leqslant 2$ follow from the sharply 3-transitivity of the group G. Lemma 1 implies that the partitions $J = J_1 \cup J_2$ and $J = J_0 \cup J_1 \cup J_2$ are impossible, and it is obvious that the sets J_1 and J_2 are conjugacy classes. Since $\operatorname{Char} G = p > 2$, then either $J = J_2$ or $J = J_0 \cup J_2$.

2. In each such class j^G there is an involution k permuting the points α and β. Further, we apply Ditzmann's lemma [16, Lemma 2.3].

3. The involution j is contained in the subgroup $N_{\gamma\delta}$, if the permutation j contains a cycle $(\gamma\,\delta)$.

4. Follows from Shunkov's theorem [16, Theorem 2.15].

5. The subgroup $\langle z\rangle$ is characteristic in T and $x \in N_G(T)$ implies $x \in N = C_G(z)$.

6. Follows from the fact that $C_R(z) = T$. In particular, potentially R can be an infinite locally dihedral group.

7. If $E_8 \leqslant N$, then $H \cap E_8 = E_4$, which contradicts the uniqueness of the involution z. Further we use item 6 of the lemma. The lemma is proved. □

Lemma 3. *The set of all 2-elements of the group H invertible involution v, is a (locally) cyclic 2-subgroup of S. If $x \in H \setminus S$ and $x^2 \in S$, then the order of the element $x^{-1}vxv$ is infinite.*

Proof. The assertions of the lemma are proved in [13, Lemmas 5, 6]

By the conditions of the theorem, all subgroups $L_x = \langle a, a^x\rangle$ in G are finite, and for $x \in J$, the subgroups K_x are also finite. Let's find out their structure. Let's start with the subgroups $L = \langle a, a^v\rangle$, $K = \langle a, v\rangle$.

Lemma 4. *The subgroup $L = \langle a, a^v\rangle$ is isomorphic to the group $L_2(p^n)$ for some n.*

Proof. It is clear that $|K : L| \leqslant 2$. According to Lemma 1, $P = L \cap U$ and $P_2 = L \cap U^x$ — elementary Abelian Sylow p-subgroups of L, with Silov p-subgroups of L are pairwise coprime, in particular, L is not an abelian group.

It is clear that $B_1 = N_L(P) = L \cap B$. If $B_1 = P$, then $P \cap P^x = 1$ for any $x \in L \setminus P$, and by the Frobenius theorem $L = M \leftthreetimes P$ is the Frobenius group with nilpotent kernel M [15, Thompson's Theorem 1.5] and the cyclic complement $P = \langle a \rangle$ [15, Burnside's Theorem 1.2]. By Lemma 2, the 2-rank of the group K (and the group L) does not exceed 2, and if $2 \in \pi(M)$, then the order of the center of a Sylow 2-subgroup from the Frobenius kernel M is 4. By the conditions $p > 3$ and, therefore, $2 \notin \pi(M)$.

Obviously, $|B \cap K| = 2p$ and by Frattini's argument and Lemma 1 $N_K(P) = \langle a \rangle \leftthreetimes \langle k \rangle = D$ — dihedral group, where $k \in v^K$. Hence, by virtue of the same Burnside theorem [15, Theorem 1.2] $C_Z(k) \neq 1$ for the center Z of each Sylow q-subgroups of M. Obviously, $C_Z(k) < H^x$ for some x, and in view of item 4 of Lemma 1, $|\Omega_1(Z)| = q$. Hence, the dihedral group $B \cap P$ is contained in the group of automorphisms of a cyclic group of order q, a contradiction, therefore, $B \cap P \neq P$.

Note that by Frattini's argument and Lemma 1 the group K contains the group anyway dihedral $D = \langle a \rangle \leftthreetimes \langle k \rangle$, where $k \in v^K$. Let M be the minimal normal subgroup in K from L. Consider the case when M — elementary abelian q-group. As proved above, $q \neq 2$. Since P is strongly isolated in $L = \langle P, P^v \rangle$ as above, we have $q \neq p$, $M \leftthreetimes P$ is a Frobenius group, $P = \langle a \rangle$, $C_M(k) \neq 1$, $|M| = q$ and $D \leqslant \operatorname{Aut} M$, a contradiction. Hence, M is a direct product of non-abelian simple groups, and since the 2-rank of the group M does not exceed 2, then M is a simple group of 2-rank 2.

If $P \nleqslant M$, then by Frattini's lemma $P \cap N_L(S) \neq 1$ for some Sylow 2-subgroup S of M and each element from $P^{\#} \cap N_L(S)$ acts on S regularly, which is impossible, since the 2-rank of G is at most 2 and $p > 3$. Therefore, $P \leqslant M$ and $|L : M| \leqslant 2$, and therefore $M = \langle P, P^v \rangle = L$.

If a Sylow 2-subgroup S in L is dihedral (Lemma 2), then by the Gorenstein-Walter theorem [17, p. 27] $L \simeq L_2(q)$, q is odd, or $L \simeq A_7$.

Let's exclude the group $L \simeq A_7$. For $p = 7$, by Kerby's theorem, H contains a unique subgroup of order 3, and in A_7 is an elementary abelian subgroup E_9, which contradicts Lemma 1. Hence, $p = 5$. The involution k inverting a cyclic subgroup of order 5 is obviously contained in J_2. It is easy to check (see, for example, cite [Proposition 14] LSS), that $C_L(k)$ contains the only subgroup $\langle b \rangle \leqslant E_9$ of order 3, which is contained in H_k. But $E_9 \leqslant C_L(b) \nleqslant H_k$, which contradicts Lemma 1. Therefore, L cannot be isomorphic to A_7.

Let $L \simeq L_2(q)$. If $q \neq p^n$ then $P = \langle a \rangle$ and p divides either $q-1$ or $q+1$. Since $C_G(P)$ — $2'$ is a group, then either $q - 1 = 2p$ or $q + 1 = 2p$. Note that then $t \in L \cap J_2$, $C_L(t) \leqslant N_L(P)$, in this case either $|C_L(t)| = q + 1$, or $|C_L(t)| = q - 1$. However, this is not possible. Therefore, $L \simeq L_2(p^n)$. If $v \notin L$, using Lemmas 1–3 and information from [19, p. 8–10], apparently it can be shown that $K \simeq PGL_2(p^n)$.

Let a Sylow 2-subgroup S in L be not dihedral. Since $v \in J_2$, in view of item 6 of Lemma 2, this means that $J \cap L \subset J_2$. As Alperin, Brower and Gorenstein proved [20] finite simple groups of 2-rank 2, up to isomorphism, are the following groups: $L_2(q)$, A_7, $L_3(s)$, $U_3(r)$, M_{11}, $U_3(4)$, where q, s, r are odd and $q > 3$.

First, let's exclude the groups $U_3(4)$ and M_{11} from this list. In $U_3(4)$ all involutions are conjugate and the Sylow 2-subgroup S is of order 64, all its involutions lie in the center of Z, $|Z| = 4$ (see, for example, [18, Proposition 13]). If $v \in L$, then $Z^{\#} \subset J_2$, which contradicts Lemmas 1, 2. If $Z^{\#} \subset J_0$, then $v \notin L$, which contradicts Lemma 2. In M_{11} all involutions are conjugate, the Sylow 2-subgroup S is a semidihedral group of order 16 and the centralizer of the involution is isomorphic to $GL_2(3)$ (see, for example [18, clause 14]). As noted above, $J \cap S \subset J_2$. Therefore, $S < N_k$, where k is the central involution from S.

The group S contains a cyclic subgroup of index 2, suitable for the role intersection of $S \cap H_k$, but each involution from $S \cap H$ centralizes an element of order 4 in $S \cap H_k$, which is impossible by Lemmas 1, 2. Hence, L cannot be isomorphic group M_{11}.

Assume that L is isomorphic to $L_3(s)$, or $U_3(r)$. Then, by [18, Proposition 11], all involutions and quadruple groups in L are conjugate, L contains an element of order 8 and a Sylow 2-subgroup S in L is isomorphic to either a semidihedral group

$$SD_m = \langle s, k \mid s^{2^{m+1}} = k^2 = 1,\ s^k = s^{-1+2^m} \rangle,\ m \geqslant 2, \quad \text{or woven group} \tag{1}$$

$$WR_m = \langle s_1, s_2, k \mid s_1^{2^m} = s_2^{2^m} = k^2 = 1,\ s_1 s_2 = s_2 s_1,\ s_1^k = s_2,\ s_2^k = s_1 \rangle,\ m \geqslant 3. \tag{2}$$

Recall that in the case under consideration $S \cap J \subset J_2$ and, therefore, $S \leqslant N_j$ for the involution $j \in Z(S)$. In the group $S = WR_m$ from (2), each subgroup of index 2 contains the subgroup E_4, which is impossible by Lemma 1. And in the cyclic subgroup of order 8 from the group $S = SD_m$ is a subgroup of order 4 commuting with all involutions from S, which again contradicts Lemma 1. Therefore, in all cases $L \simeq L_2(q)$. As proved above, $q = p^n$, and the lemma is proved. □

Lemma 5. *For any element $c \in U^v$ the subgroup $L = \langle a, c \rangle$ is isomorphic to the group $L_2(p^n)$ for some $n = n(a, c)$.*

Proof. By virtue of the finiteness condition for the element a and items 1–2 of Lemma 1 the subgroup L is finite. Further, as in the proof of Lemma 4, $P = L \cap U$ and $P_2 = L \cap U^x$ — elementary Abelian Sylow p-subgroups in L, Sylow p-subgroups in L are pairwise coprime and L is not an abelian group. To continue to follow the logic of the proof of Lemma 4, we prove that the 2-rank of the group L does not exceed 2. If $L \cap J_2$ is nonempty, then the desired follows from Lemma 2. Let $L \cap J_2 = \emptyset$. Note that by claim 3 of Lemma 1 the involution $z \in H$, and by claim 1 of the same lemma, z inverts the elements a and c: $a^z = a^{-1}$, $c^z = c^{-1}$. Therefore, $z \in N_G(L)$, the subgroup $K = \langle a, c, z \rangle$ is finite, $|K : L| \leqslant 2$, $K \cap J_2 \neq 2$ and for K the boundedness of the 2-rank follows from Lemma 2. Hence, the 2-rank of the group L does not exceed 2, and $D = \langle a, z \rangle$ — dihedral group, $D \leqslant K$. Moreover, in the case $L \cap J = \emptyset$, by Lemma 2 the Sylow 2-subgroups in K (and in L) are dihedral. Taking into account these remarks, part of the proof of Lemma 4, on the structure of L groups with dihedral Sylow 2-subgroup, carries over literally to the case under consideration. The lemma is proved. □

Lemma 6. *For any non-permutable elements $x, s \in a^G$ the subgroup $L = \langle s, x \rangle$ is finite and isomorphic to the group $L_2(p^n)$ for suitable $n = n(s, x)$.*

Proof. Due to the arbitrary initial choice of the element a from the class of conjugate elements of a^G it follows that statement of Lemma 5 is true for any $s \in U^{\#}$ and $x \in U^v \cap a^G = U^{v\#}$. Since G is 3-transitive on the set U^G, we conclude that that the lemma is true. □

Proof of the theorem. According to [19, p. 9] the group $L = \langle a, a^v \rangle$, isomorphic $L_2(q)$ by Lemma 4, has $\dfrac{q(q+1)}{2}$ cyclic subgroups of order $\dfrac{(q-1)}{2}$ (Cartan subgroups), of these, $(B \cap L) \cup (B^v \cap L)$ contains $2q-1$ such subgroups. Since $\dfrac{q(q+1)}{2} > 2q-1$ for $q > 3$, then there is a pair of dots $\gamma, \delta \in X \setminus \{\alpha, \beta\}$ for which the intersection $L \cap G_{\alpha\beta}$ is cyclic subgroup conjugate to the Cartan subgroup $L \cap H$ of order $\dfrac{(q-1)}{2}$. The group G acts on the set J_2 twice transitively, since it is twice transitive on the set H^G, and each the subgroup H^g is defined by its unique central involution z^g from J_2 (Lemma 1). Hence we deduce that any pair of involutions from $H \cap J_2$ is contained in an appropriate subgroup conjugate to the subgroup L. This means that the involution v is finite in the group N, and by [16, Corollary 2.30] the subgroup N is locally finite. By Theorem 2 in [21], the group G is locally finite. The theorem is proved. □

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Точно трижды транзитивные группы с конечным элементом

Евгений Б. Дураков
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. В настоящей работе исследуются точно трижды транзитивные группы. Доказана локальная конечность точно трижды транзитивных групп подстановок характеристики $p > 3$, содержащих конечный элемент порядка p.

Ключевые слова: группа, точно k-транзитивная группа, точно трижды транзитивная группа, локально конечная группа, почти-область, почти-поле.

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Effective Acoustic Equations for a Layered Material Described by the Fractional Kelvin-Voigt Model

Alexey S. Shamaev*
Vladlena V. Shumilova†
Ishlinsky Institute for Problems in Mechanics RAS
Moscow, Russian Federation

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Abstract. The paper is devoted to the construction of effective acoustic equations for a two-phase layered viscoelastic material described by the Kelvin–Voigt model with fractional time derivatives. For this purpose, the theory of two-scale convergence and the Laplace transform with respect to time are used. It is shown that the effective equations are partial integro-differential equations with fractional time derivatives and fractional exponential convolution kernels. In order to find the coefficients and the convolution kernels of these equations, several auxiliary cell problems are formulated and solved.

Keywords: homogenization, acoustic equations, viscoelasticity, fractional Kelvin–Voigt model.

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The study of macroscopic acoustic behavior of heterogeneous viscoelastic materials with periodic microstructure is one of the most significant problems in acoustical engineering when dealing with polymer based composites. The most rigorous and widely accepted mathematical tool for the theoretical part of this study is the theory of homogenization. Using techniques of homogenization, the actual highly inhomogeneous periodic viscoelastic composite can be replaced by the corresponding effective (homogenized) material with the similar acoustic properties.

It is well known that short memory effects in microheterogeneous viscoelastic Kelvin–Voigt materials lead to the appearance of long memory effects in the corresponding effective media (see [1–3]). In other words the acoustic equations for these materials, which are partial differential equations, become partial integro-differential equations after homogenization. The same result was observed for two-phase materials, in which the first phase is an elastic material whilst the second one is a viscoelastic Kelvin–Voigt material [4, 5].

In recent years there has been an increasing number of papers devoted to the development of fractional models in viscoelasticity (see, for instance, [6–8] and the reference therein). Such models consist of differential or integro-differential equations with fractional derivatives. The growing popularity of fractional models is explained by their ability of describing the complex behaviour of viscoelastic materials using a small number of parameters.

In this paper, we consider a mathematical model describing small displacements of a two-phase layered viscoelastic material whose behavior is described by the fractional Kelvin–Voigt model. This model consists of a system of partial differential equations with fractional time

*https://orcid.org/0000-0003-2766-6382
†v.v.shumilova@mail.ru https://orcid.org/0000-0003-3830-7924

derivatives and rapidly oscillating piecewise constant coefficients, conditions of ideal contact between layers, and homogeneous initial and outer boundary conditions. Using the two-scale convergence method [9, 10] and applying the Laplace transform, we show that the corresponding effective model involves a system of partial integro-differential equations with fractional time derivatives and constant coefficients. By solving a number of auxiliary cell problems, we calculate these coefficients and find that the integral parts of the effective equations are of convolution type and their kernels are fractional exponential Rabotnov's functions. Thus, we rigorously establish that long memory effects mentioned above also appear in the effective material that corresponds to the fractional Kelvin–Voigt material.

1. Original acoustic equations

Consider a bounded domain $\Omega = (0, L)^3$ occupied by two-phase viscoelastic material with a periodic microstructure. Let $\varepsilon \ll L$ be a small positive parameter characterizing the heterogeneity period of the viscoelastic material. We suppose that every phase is isotropic and consists of the union of layers that are parallel to the Ox_2x_3 plane. More precisely, denote

$$D_{2\varepsilon} = (0, L) \cap \left(\bigcup_{k=0}^{\infty} (\varepsilon(h_1 + k), \varepsilon(h_2 + k)) \right), \quad D_{1\varepsilon} = (0, L) \setminus \overline{D}_{2\varepsilon},$$

$$h_1 = \frac{1-h}{2}, \quad h_2 = \frac{1+h}{2}, \quad 0 < h < 1$$

and assume that the sets $\Omega_{1\varepsilon} = D_{1\varepsilon} \times (0, L)^2$ and $\Omega_{2\varepsilon} = D_{2\varepsilon} \times (0, L)^2$ are occupied by the first and the second phase, respectively.

Note that the periodicity cell Y_ε of the above layered material may be extracted in different ways. For our convenience, we will assume that $Y_\varepsilon = \varepsilon Y$, where $Y = (0, 1)^3$ is a unite cube. The cube Y can be decomposed into two parts Y_1 and Y_2 with a common boundary S as follows:

$$Y_1 = ((0, h_1) \cup (h_2, 1)) \times (0, 1)^2, \quad Y_2 = (h_1, h_2) \times (0, 1)^2,$$

$$S = (\{h_1\} \cup \{h_2\}) \times (0, 1)^2.$$

It is obvious that $Y_\varepsilon = \varepsilon Y_1 \cup \varepsilon Y_2 \cup \varepsilon S$. The part εY_1 represents the first phase and consists of two layers with the same thickness $\varepsilon(1 - h)/2$ while the part εY_2 represents the second phase of the layered material and consists of one layer with the thickness εh (see Fig. 1).

The viscoelastic material we propose to study is described by the fractional Kelvin–Voigt model. Its constitutive equations between the components of the stress and strain tensors have the form

$$\sigma^\varepsilon_{ij} = a^\varepsilon_{ijkh}(x) e_{kh}(u^\varepsilon) + b^\varepsilon_{ijkh}(x) e_{kh}\left(D^\alpha_t u^\varepsilon\right), \quad 0 < \alpha < 1, \tag{1}$$

where $u^\varepsilon(x, t)$ is the displacement vector, $a^\varepsilon(x) = a(\varepsilon^{-1}x)$ and $b^\varepsilon(x) = b(\varepsilon^{-1}x)$ are Y_ε-periodic tensors describing the elastic and viscous properties of the material, σ^ε and $e(u^\varepsilon)$ are the stress and strain tensors, and D^α_t is the Caputo fractional time derivative of order α,

$$e_{kh}(u^\varepsilon) = \frac{1}{2}\left(\frac{\partial u^\varepsilon_k}{\partial x_h} + \frac{\partial u^\varepsilon_h}{\partial x_k}\right), \quad D^\alpha_t u^\varepsilon = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial u^\varepsilon}{\partial \tau}\, d\tau,$$

$$a_{ijkh}(y) = \lambda_s \delta_{ij}\delta_{kh} + \mu_s(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \quad y \in Y_s,$$

$$b_{ijkh}(y) = \zeta_s \delta_{ij}\delta_{kh} + \eta_s(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}), \quad y \in Y_s,$$

$$y = \varepsilon^{-1}x, \quad s = 1, 2, \quad 1 \leqslant i, j, k, h \leqslant 3.$$

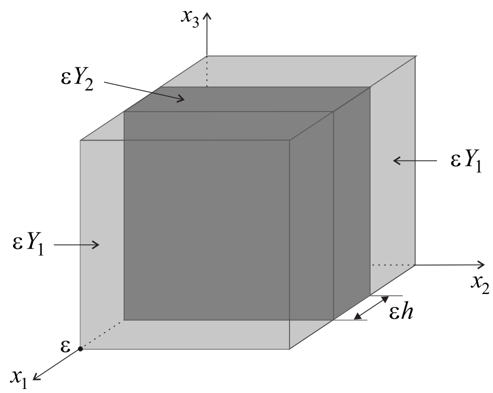

Fig. 1. The first and the second phases of the layered material

Here λ_s and μ_s are the Lamé parameters of $\Omega_{s\varepsilon}$, ζ_s and η_s are parameters describing the viscous behavior of $\Omega_{s\varepsilon}$, $\Gamma(\alpha)$ is Euler's gamma function, and δ_{ij} is Kroneker's delta. Note that in (1) and everywhere below we assume summation with respect to repeated indices.

The motion of the viscoelastic material in the phase $\Omega_{s\varepsilon}$ is described by the system of partial differential equations with fractional time derivative

$$\rho_s \frac{\partial^2 u_i^\varepsilon}{\partial t^2} = \frac{\partial \sigma_{ij}^\varepsilon}{\partial x_j} + f_i(x,t) \ \text{ in } \ \Omega_{s\varepsilon} \times (0,T), \quad s = 1, 2, \tag{2}$$

where $\rho_s = \text{const} > 0$ is the density of the material in $\Omega_{s\varepsilon}$ and $f_i(x,t)$ are the components of the volume external force vector.

On the boundaries between the layers we assume the condition of ideal contact. It means the continuity of displacements and normal stresses at each layer interface and is written as

$$[u^\varepsilon]|_{S_\varepsilon} = 0, \quad [\sigma_{i1}^\varepsilon]|_{S_\varepsilon} = 0, \tag{3}$$

where the square brackets $[\cdot]|_{S_\varepsilon}$ denote the jump in the enclosed quantity across the boundary $S_\varepsilon = \partial\Omega_{1\varepsilon} \cap \partial\Omega_{2\varepsilon}$.

Finally, we accept that the boundary conditions on $\partial\Omega$ for displacements as well as the initial conditions for displacements and velocities are homogeneous, i.e.

$$u^\varepsilon|_{\partial\Omega} = 0, \quad u^\varepsilon|_{t=0} = 0, \quad \frac{\partial u^\varepsilon}{\partial t}\Big|_{t=0} = 0. \tag{4}$$

Problem (2)–(4) is a mathematical model describing the general motion of the two-phase viscoelastic material. Our aim now is to deduce the corresponding effective (homogenized) model that describes the limit dynamic behavior of the original two-phase viscoelastic material as $\varepsilon \to 0$.

2. Effective acoustic equations

To construct the homogenized problem, we will use the method proposed in [5, 11, 12]. This method was developed for the homogenization of acoustics equations in two-phase dissipative media with periodic microstructure. Its main tools are the Laplace transform and the concept of two-scale convergence introduced by G. Nguetseng [9].

First, applying the Laplace transform $u^\varepsilon(x,t) \to u^\varepsilon_\lambda(x)$ and $f(x,t) \to f_\lambda(x)$, we convert the evolutionary problem (2)–(4) into the stationary one. As a result, we obtain the following boundary value problem for Laplace transforms:

$$\begin{gathered}\rho_s \lambda^2 u^\varepsilon_{\lambda i} = \frac{\partial \sigma^{\lambda\varepsilon}_{ij}}{\partial x_j} + f_{\lambda i}(x) \text{ in } \Omega_{s\varepsilon}, \quad s = 1,2, \\ u^\varepsilon_\lambda|_{\partial\Omega} = 0, \quad [u^\varepsilon_\lambda]|_{S_\varepsilon} = 0, \quad [\sigma^{\lambda\varepsilon}_{i1}]|_{S_\varepsilon} = 0,\end{gathered} \tag{5}$$

where

$$\sigma^{\lambda\varepsilon}_{ij} = \left(a^\varepsilon_{ijkh}(x) + \lambda^\alpha b^\varepsilon_{ijkh}(x)\right) e_{kh}(u^\varepsilon_\lambda).$$

Next, using the basic properties of two-scale convergence and repeating the same arguments as in [5, 11, 12], we can show that the homogenized problem that corresponds to problem (5) and which is constructed for $\varepsilon \to 0$ has the form

$$\rho_0 \lambda^2 u_{\lambda i} = \frac{\partial \sigma^\lambda_{ij}}{\partial x_j} + f_{\lambda i}(x) \text{ in } \Omega, \quad u_\lambda|_{\partial\Omega} = 0, \tag{6}$$

where

$$\begin{gathered}\rho_0 = \rho_1(1-h) + \rho_2 h, \quad \sigma^\lambda_{ij} = d^\lambda_{ijkh} e_{kh}(u_\lambda), \\ d^\lambda_{ijkh} = \int_Y \left(c^\lambda_{ijkh}(y) + c^\lambda_{ijlm}(y) e^y_{lm}(Q^{kh}_\lambda)\right) dy, \\ c^\lambda_{ijkh}(y) = a_{ijkh}(y) + \lambda^\alpha b_{ijkh}(y), \quad e^y_{lm}(Q^{kh}_\lambda) = \frac{1}{2}\left(\frac{\partial Q^{kh}_{\lambda l}}{\partial y_m} + \frac{\partial Q^{kh}_{\lambda m}}{\partial y_l}\right).\end{gathered} \tag{7}$$

Here the vector-valued functions $Q^{kh}_\lambda(y)$ are Y-periodic solutions to the following cell problems:

$$\begin{gathered}\frac{\partial}{\partial y_j}\left(c^\lambda_{ijkh}(y) + c^\lambda_{ijlm}(y) e^y_{lm}(Q^{kh}_\lambda)\right) = 0 \text{ in } Y, \quad \int_Y Q^{kh}_\lambda dy = 0, \\ [Q^{kh}_\lambda]\big|_{y_1 = h_s} = 0, \quad \left[c^\lambda_{ijkh}(y) + c^\lambda_{ijlm}(y) e^y_{lm}(Q^{kh}_\lambda)\right]\big|_{y_1 = h_s} = 0, \quad s = 1,2.\end{gathered} \tag{8}$$

Now we apply the inverse Laplace transform to the homogenized stationary problem (6). We have

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i(x,t) \text{ in } \Omega \times (0,T), \tag{9}$$

$$u|_{\partial\Omega} = 0, \quad u|_{t=0} = 0, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 0$$

with

$$\sigma_{ij} = d_{ijkh}(t) * e_{kh}(u), \tag{10}$$

where the symbol $*$ denotes the operation of convolution with respect to time t.

3. Solutions of auxiliary cell problems

Passing to the inverse Laplace transforms in (8) we see that $Q^{kh}(y,t)$ depends on the Dirac function $\delta(t)$ and cannot be expressed in explicit form without some additional explanations. In order to do this and at the same time derive direct formula for calculation of components of the tensor $d(t)$, we will proceed in the following way. Let us represent the solutions $Q_\lambda^{kh}(y)$ to problems (8) in the form

$$Q_\lambda^{kh}(y) = Z^{kh}(y) + \frac{V^{kh}(y)}{\lambda^\alpha - M^{kh}}, \quad M^{kh} = \text{const}, \tag{11}$$

where vector-valued functions $Z^{kh}(y)$, $V^{kh}(y)$ and parameters M^{kh} are to be specified.

In a first step, let us define the vector-valued functions $Z^{kh}(y)$ as Y-periodic solutions to the cell problems

$$\begin{gathered} \frac{\partial}{\partial y_j}\big(b_{ijkh}(y) + b_{ijlm}(y)e^y_{lm}(Z^{kh})\big) = 0 \ \text{ in } \ Y, \quad \int_Y Z^{kh}dy = 0, \\ [Z^{kh}]\big|_{y_1=h_s} = 0, \quad \big[b_{ijkh}(y) + b_{ijlm}(y)e^y_{lm}(Z^{kh})\big]\big|_{y_1=h_s} = 0, \quad s = 1,2. \end{gathered} \tag{12}$$

In a second step, using the solutions $Z^{kh}(y)$ to problems (12), we define the vector-valued functions $V^{kh}(y)$ as Y-periodic solutions to the cell problems

$$\begin{gathered} \frac{\partial}{\partial y_j}\big(a_{ijkh}(y) + a_{ijlm}(y)e^y_{lm}(Z^{kh}) + b_{ijlm}(y)e^y_{lm}(V^{kh})\big) = 0 \ \text{ in } \ Y, \\ \big[a_{ijkh}(y) + a_{ijlm}(y)e^y_{lm}(Z^{kh}) + b_{ijlm}(y)e^y_{lm}(V^{kh})\big]\big|_{y_1=h_s} = 0, \quad s = 1,2, \\ \int_Y V^{kh}dy = 0, \quad [V^{kh}]\big|_{y_1=h_s} = 0. \end{gathered} \tag{13}$$

To write out solutions to problems (12) and (13), we introduce 1-periodic piecewise linear function $z(y_1)$ defined by

$$z(y_1) = \begin{cases} \dfrac{y_1 h}{1-h}, & y_1 \in (0,h_1), \\ -y_1 + \dfrac{1}{2}, & y_1 \in (h_1,h_2), \\ \dfrac{(y_1-1)h}{1-h}, & y_1 \in (h_2,1). \end{cases}$$

It is easy to check that $Z^{kh}(y) = Z^{hk}(y)$ and $V^{kh}(y) = V^{hk}(y)$, so that we need only to find $Z^{kh}(y)$ and $V^{kh}(y)$ for $k \leqslant h$. Solving problems (12) for $k \leqslant h$, we obtain

$$Z^{11}(y) = (c_1 z(y_1), 0, 0), \quad Z^{22}(y) = Z^{33}(y) = (c_2 z(y_1), 0, 0),$$

$$Z^{12}(y) = (0, c_3 z(y_1), 0), \quad Z^{13}(y) = (0, 0, c_3 z(y_1)), \quad Z^{23}(y) = (0,0,0),$$

where

$$c_1 = \frac{1}{b_{12}}(1-h)(b_2 - b_1), \quad c_2 = \frac{1}{b_{12}}(1-h)(\zeta_2 - \zeta_1), \quad c_3 = \frac{1}{\eta_{12}}(1-h)(\eta_2 - \eta_1),$$

$$b_{12} = b_1 h + b_2(1-h), \quad \eta_{12} = \eta_1 h + \eta_2(1-h), \quad b_s = \zeta_s + 2\eta_s.$$

Substituting $Z^{kh}(y)$ into problems (13) and solving them for $k \leqslant h$, we derive

$$V^{11}(y) = (c_4 z(y_1), 0, 0), \quad V^{22}(y) = V^{33}(y) = (c_5 z(y_1), 0, 0),$$

$$V^{12}(y) = (0, c_6 z(y_1), 0), \quad V^{13}(y) = (0, 0, c_6 z(y_1)), \quad V^{23}(y) = (0,0,0),$$

where

$$c_4 = \frac{1}{b_{12}^2}(1-h)(b_1 a_2 - b_2 a_1), \quad c_5 = \frac{1-h}{b_{12}^2}\left((\lambda_2-\lambda_1)b_{12} - (\zeta_2-\zeta_1)a_{12}\right),$$
$$c_6 = \frac{1}{\eta_{12}^2}(1-h)(\eta_1\mu_2 - \eta_2\mu_1), \quad a_{12} = a_1 h + a_2(1-h), \quad a_s = \lambda_s + 2\mu_s.$$

Now, after defining $Z^{kh}(y)$ and $V^{kh}(y)$ in (11), we can find parameters M^{kh}. It follows from (8), (12), and (13) that M^{kh} satisfies the system

$$\frac{\partial}{\partial y_j}\left(a_{ijlm}(y)e_{lm}(V^{kh}) + M^{kh}b_{ijlm}(y)e^y_{lm}(V^{kh})\right) = 0 \text{ in } Y, \tag{14}$$

$$\left[a_{ijlm}(y)e_{lm}(V^{kh}) + M^{kh}b_{ijlm}(y)e^y_{lm}(V^{kh})\right]\Big|_{y_1=h_s} = 0, \quad s=1,2. \tag{15}$$

Substitute $V^{kh}(y)$ found above into (14) and (15). It is easy to check that equations (14) are always fulfilled for any parameters M^{kh}. Further, from the boundary conditions (15) we calculate the required values of M^{kh}:

$$M^{11} = M^{22} = M^{33} = -\frac{a_{12}}{b_{12}},$$
$$M^{12} = M^{21} = M^{13} = M^{31} = -\frac{\mu_{12}}{\eta_{12}}.$$

Applying the inverse Laplace transform to (11), we get

$$Q^{kh}(y,t) = \delta(t)Z^{kh}(y) + R_{\alpha-1}(M^{kh},t)V^{kh}(y),$$

where $R_\nu(\beta,t)$ denotes fractional exponential Rabotnov's function [13]:

$$R_\nu(\beta,t) = t^\nu \sum_{n=0}^{\infty} \frac{\beta^n t^{n(1+\nu)}}{\Gamma[(1+n)(1+\nu)]}.$$

Next we substitute the decomposition (11) into (7) to obtain

$$d^\lambda_{ijkh} = A_{ijkh} + \lambda^\alpha B_{ijkh} + G_{ijkh}(\lambda),$$

where the components of the tensors A, B, and $G(\lambda)$ are given by the formulas

$$A_{ijkh} = \int_Y \left(a_{ijkh}(y) + a_{ijlm}(y)e^y_{lm}(Z^{kh}) + b_{ijlm}(y)e^y_{lm}(V^{kh})\right) dy, \tag{16}$$

$$B_{ijkh} = \int_Y \left(b_{ijkh}(y) + b_{ijlm}(y)e^y_{lm}(Z^{kh})\right) dy, \tag{17}$$

$$G_{ijkh}(\lambda) = \frac{1}{\lambda^\alpha - M^{kh}} \int_Y \left(a_{ijlm}(y)e^y_{lm}(V^{kh}) + M^{kh}b_{ijlm}(y)e^y_{lm}(V^{kh})\right) dy.$$

Therefore, the constitutive equations (10) take the form

$$\sigma_{ij} = A_{ijkh}e_{kh}(u) + B_{ijkh}e_{kh}(D^\alpha_t u) + G_{ijkh}(t) * e_{kh}(u), \tag{18}$$

where $G_{ijkh}(t)$ are the inverse Laplace transforms of $G_{ijkh}(\lambda)$:

$$G_{ijkh}(t) = R_{\alpha-1}\left(M^{kh},t\right) \int_Y \left(a_{ijlm}(y)e_{lm}(V^{kh}) + M^{kh}b_{ijlm}(y)e^y_{lm}(V^{kh})\right) dy. \tag{19}$$

From (18) we see that the effective acoustic equations (9) are partial integro-differential equations with fractional time derivative and constant coefficients. It is interesting to note that their kernels are expressed via two different Rabotnov's functions $R_{\alpha-1}(-a_{12}/b_{12},t)$ and $R_{\alpha-1}(-\mu_{12}/\eta_{12},t)$.

4. Components of the tensors A, B, and $G(t)$

Before proceeding to the calculation of the tensors A, B, and $G(t)$, let us note that

$$A_{ijkh} = A_{jikh} = A_{khij}, \quad A_{ijkh} = 0 \text{ whenever } \delta_{ij}\delta_{kh} + \delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk} = 0,$$

$$A_{2222} = A_{3333}, \quad A_{1122} = A_{1133}, \quad A_{1212} = A_{1313}, \quad A_{2222} - A_{2233} = 2A_{2323}$$

and similarly for the tensors B and $G(t)$. Moreover, it is easy to see that

$$A_{2323} = \mu_1(1-h) + \mu_2 h, \quad B_{2323} = \eta_1(1-h) + \eta_2 h, \quad G_{2323}(t) = 0.$$

Therefore, it is sufficient to find the components of A, B, and $G(t)$ with indexes $\{1111\}$, $\{2222\}$, $\{1122\}$, and $\{1212\}$. To do this, we first substitute the found solutions to problems (12) and (13) into formulas (16) and (17). This yeilds

$$A_{1111} = a_1(1-h) + a_2 h + c_1 h(a_1 - a_2) + c_4 h(b_1 - b_2),$$

$$A_{2222} = a_1(1-h) + a_2 h + c_2 h(\lambda_1 - \lambda_2) + c_5 h(\zeta_1 - \zeta_2),$$

$$A_{1122} = \lambda_1(1-h) + \lambda_2 h + c_2 h(a_1 - a_2) + c_5 h(b_1 - b_2),$$

$$A_{1212} = \mu_1(1-h) + \mu_2 h + c_3 h(\mu_1 - \mu_2) + c_6 h(\eta_1 - \eta_2),$$

$$B_{1111} = b_1(1-h) + b_2 h + c_1 h(b_1 - b_2), \quad B_{2222} = b_1(1-h) + b_2 h + c_2 h(\zeta_1 - \zeta_2),$$

$$B_{1212} = \eta_1(1-h) + \eta_2 h + c_3 h(\eta_1 - \eta_2), \quad B_{1122} = \zeta_1(1-h) + \zeta_2 h + c_2 h(b_1 - b_2).$$

Taking into account the above values of constants c_i and using trivial transformations, we obtain

$$A_{1111} = \frac{1}{b_{12}^2}\left(a_2 b_1^2 h + a_1 b_2^2(1-h)\right), \quad A_{1212} = \frac{1}{\eta_{12}^2}\left(\mu_2 \eta_1^2 h + \mu_1 \eta_2^2 (1-h)\right),$$

$$A_{2222} = a_1(1-h) + a_2 h + \frac{h(1-h)(\zeta_1 - \zeta_2)}{b_{12}^2}\left(a_{12}(\zeta_1 - \zeta_2) - 2b_{12}(\lambda_1 - \lambda_2)\right),$$

$$A_{1122} = \frac{1}{b_{12}}\left(b_1\lambda_2 h + b_2\lambda_1(1-h)\right) + \frac{h}{b_{12}^2}(1-h)(\zeta_2 - \zeta_1)(a_1 b_2 - a_2 b_1),$$

$$B_{1111} = \frac{b_1 b_2}{b_{12}}, \quad B_{2222} = b_1(1-h) + b_2 h - \frac{h}{b_{12}}(1-h)(\zeta_1 - \zeta_2)^2,$$

$$B_{1122} = \frac{1}{b_{12}}\left(b_1\zeta_2 h + b_2\zeta_1(1-h)\right), \quad B_{1212} = \frac{\eta_1\eta_2}{\eta_{12}}.$$

In order to find the components of $G(t)$, we substitute the solutions $V^{kh}(y)$ to problems (13) and the parameters M^{kh} into formulas (19). As a result, we get

$$G_{1111}(t) = -\frac{h(1-h)}{b_{12}^3}(a_1 b_2 - a_2 b_1)^2 R_{\alpha-1}\left(-\frac{a_{12}}{b_{12}}, t\right),$$

$$G_{2222}(t) = -\frac{h(1-h)}{b_{12}^3}\left((\lambda_1 - \lambda_2)b_{12} - (\zeta_1 - \zeta_2)a_{12}\right)^2 R_{\alpha-1}\left(-\frac{a_{12}}{b_{12}}, t\right),$$

$$G_{1122}(t) = -\frac{h(1-h)}{b_{12}^3}(a_1 b_2 - a_2 b_1)\left((\lambda_1 - \lambda_2)b_{12} - (\zeta_1 - \zeta_2)a_{12}\right) R_{\alpha-1}\left(-\frac{a_{12}}{b_{12}}, t\right),$$

$$G_{1212}(t) = -\frac{h(1-h)}{\eta_{12}^3}(\mu_1\eta_2 - \mu_2\eta_1)^2 R_{\alpha-1}\left(-\frac{\mu_{12}}{\eta_{12}}, t\right).$$

To conclude, we note that our results can be considered as a generalization of those obtained in the case of two-phase layered viscoelastic material described by a standard Kelvin-Voigt model ($\alpha = 1$). Indeed, the effective acoustic equations for the last material also have form (9) with the constitutive equations (18), where A_{ijkh} and B_{ijkh} are defined by the same formulas as above. Moreover, the components of $G(t)$ are found by using the formulas presented here, in which we should put $\alpha = 1$ and take into account that

$$R_0\left(-\frac{a_{12}}{b_{12}}, t\right) = \exp\left(-\frac{a_{12}t}{b_{12}}\right), \quad R_0\left(-\frac{\mu_{12}}{\eta_{12}}, t\right) = \exp\left(-\frac{\mu_{12}t}{\eta_{12}}\right).$$

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Эффективные уравнения акустики для слоистого материала, описываемого дробной моделью Кельвина-Фойгта

Алексей С. Шамаев
Владлена В. Шумилова
Институт проблем механики им. А. Ю. Ишлинского РАН
Москва, Российская Федерация

Аннотация. Статья посвящена построению эффективных уравнений акустики для двухфазного слоистого вязкоупругого материала, описываемого моделью Кельвина–Фойгта с дробными производными по времени. Для этой цели используется теория двухмасштабной сходимости и преобразование Лапласа по времени. Показано, что эффективные уравнения являются интегро-дифференциальными уравнениями в частных производных с дробными производными по времени и дробно-экспоненциальными ядрами свертки. Для того чтобы найти коэффициенты и ядра сверток этих уравнений, сформулированы и решены несколько вспомогательных задач.

Ключевые слова: усреднение, уравнения акустики, вязкоупругость, дробная модель Кельвина–Фойгта.

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Analytic Continuation of Diagonals of Laurent Series for Rational Functions

Dmitry Yu. Pochekutov*
Siberian Federal University
Krasnoyarsk, Russian Federation

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Abstract. We describe branch points of complete $\boldsymbol{q}$-diagonals of Laurent series for rational functions in several complex variables in terms of the logarithmic Gauss mapping. The sufficient condition of non-algebraicity of such a diagonal is proven.

Keywords: diagonals of Laurent series, hyperplane amoeba, logarithmic Gauss mapping, zero pinch, monodromy.

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1. Introduction and preliminaries

We use the notation $\mathbb{C}^\times$ for the one-dimensional complex torus $\mathbb{C}\setminus\{0\}$. For vectors $\boldsymbol{z} = (z_1,\dots,z_n)$ in $\mathbb{C}^n$ or $(\mathbb{C}^\times)^n$ and $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n)$ in $\mathbb{Z}^n$, denote by $\boldsymbol{z}^{\boldsymbol{\alpha}}$ the monomial $z_1^{\alpha_1}\dots z_n^{\alpha_n}$.

Consider a Laurent series for a rational function $F(\boldsymbol{z}) = \dfrac{P(\boldsymbol{z})}{Q(\boldsymbol{z})}$ of n complex variables centered at the origin:

$$F(\boldsymbol{z}) = \sum_{\boldsymbol{\alpha}\in\mathbb{Z}^n} C_{\boldsymbol{\alpha}}\boldsymbol{z}^{\boldsymbol{\alpha}}. \tag{1}$$

Let $L\subset\mathbb{Z}^n$ be a sublattice of the n-dimensional integer lattice. Then the generating function for the subsequence $\{C_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha}\in L}$ of the coefficients indexed by L is called the *complete diagonal* of the Laurent series (1). Throughout the paper, we consider the sublattice of rank 1 generated by the irreducible vector $\boldsymbol{q} = (q_1,\dots,q_n)$ from $\mathbb{Z}^n\setminus\{0\}$. We will call the corresponding diagonal

$$d_{\boldsymbol{q}}(t) = \sum_{k=-\infty}^{\infty} C_{\boldsymbol{q}\cdot k}t^k$$

a *complete* $\boldsymbol{q}$*-diagonal* of the Laurent series (1). Such a diagonal can be written naturally as a sum of two subseries $d^+_{\boldsymbol{q}}(t)$ and $d^-_{\boldsymbol{q}}(t)$ with only non-negative and negative powers of t, correspondingly. We call them *one-sided* $\boldsymbol{q}$*-diagonals*. Clearly, we have the equality $d_{\boldsymbol{q}}(t) = d^+_{\boldsymbol{q}}(t)$ in the case of Taylor series. For the unit vector $\boldsymbol{I} = (1,\dots,1)$, we denote $d_{\boldsymbol{I}}(t)$ by $d(t)$, and refer to $\boldsymbol{I}$-diagonal simply as a diagonal.

Further, we consider irreducible polynomials $P(\boldsymbol{z})$ and $Q(\boldsymbol{z})$. It is well-known that domains of absolute convergence of power series are logarithmically convex. In the case of the Laurent

*dpotchekutov@sfu-kras.ru https://orcid.org/0000-0002-4545-2129

series (1), it is convenient to use the notion of an amoeba of the denominator $Q(\boldsymbol{z})$ of the rational function $F(\boldsymbol{z})$ in the description of such domains. Recall [1, Section 6.1] that the *amoeba* of a polynomial Q is the image of a hypersurface $Z^\times(Q)$ under the logarithmic mapping $\Lambda : (\mathbb{C}^\times)^n \to \mathbb{R}^n$ defined by

$$\Lambda(\boldsymbol{z}) = (\log|z_1|, \ldots, \log|z_n|),$$

where $Z^\times(Q)$ is defined in the complex torus $(\mathbb{C}^\times)^n$ by zeroes of the polynomial Q.

The complement $\mathbb{R}^n \backslash \mathcal{A}_Q$ consists of a finite number of connected components E that are open and convex. The preimages $\Lambda^{-1}(E)$ of these components are domains of absolute convergence for Laurent expansions (1) (centered at the origin) for the rational function $F(\boldsymbol{z})$ (see Section 2).

Amoebas are closely related to the notion of the *logarithmic Gauss mapping*

$$\gamma_Q : \operatorname{reg} Z^\times(Q) \to \mathbb{CP}^{n-1}$$

defined as

$$\gamma_Q(\boldsymbol{z}) = \left(z_1 \frac{\partial Q}{\partial z_1}(\boldsymbol{z}) : \ldots : z_n \frac{\partial Q}{\partial z_n}(\boldsymbol{z}) \right) \tag{2}$$

in regular points $\boldsymbol{z}$ of the hypersurface $Z^\times(Q)$. In fact, the set of critical points of the logarithmic projection $\Lambda : Z^\times(Q) \to \mathbb{R}^n$ contains the boundary $\partial\mathcal{A}_Q$ and coincides with $\gamma_Q^{-1}(\mathbb{RP}^{n-1})$.

The complete $\boldsymbol{q}$-diagonal $d_{\boldsymbol{q}}(t)$ of the Laurent series (1) that converges in the domain $\Lambda^{-1}(E)$ for a rational function F can be represented as the integral (see Section 2)

$$d_{\boldsymbol{q}}(t) = \frac{1}{(2\pi\imath)^n} \int_\Gamma \frac{P(\boldsymbol{z})}{Q(\boldsymbol{z})} \frac{\boldsymbol{z}^{\boldsymbol{q}}}{\boldsymbol{z}^{\boldsymbol{q}} - t} \frac{dz_1 \wedge \ldots \wedge dz_n}{z_1 \ldots z_n}$$

over the n-dimensional cycle $\Gamma = \Lambda^{-1}(\boldsymbol{y}_2) - \Lambda^{-1}(\boldsymbol{y}_1)$ in $(\mathbb{C}^\times)^n \setminus \{Z^\times(Q \cdot (\boldsymbol{z}^{\boldsymbol{q}} - t))\}$. The parameter t in the integral representation is chosen so that the amoeba of the polynomial $\boldsymbol{z}^{\boldsymbol{q}} - t$ (that is the hyperplane $\langle \boldsymbol{q}, \boldsymbol{u} \rangle = \log|t|$ with the normal vector $\boldsymbol{q}$) divides the component E into two parts, and points $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ are chosen from different parts of this partition. The ramification of the complete $\boldsymbol{q}$-diagonal happens when a value of the parameter t is such that the rank of the n-dimensional homology group $(\mathbb{C}^\times)^n \setminus \{Z^\times(Q \cdot (\boldsymbol{z}^{\boldsymbol{q}} - t))\}$ drops.

Since E is convex, the restriction of a linear function $\langle \boldsymbol{q}, \boldsymbol{u} \rangle$ to the closure of E in $\mathbb{R}^n$ attains extreme values on the boundary ∂E. Let $\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{q})$ be one of the points of the boundary ∂E such that the function specified above attains an extreme value. Then the branch points of $d_{\boldsymbol{q}}(t)$ should be among points of the form $\boldsymbol{p}^{\boldsymbol{q}}$, where $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q})$ is a point of the hypersurface $Z^\times(Q)$ such that $\Lambda(\boldsymbol{p}) = \boldsymbol{u}_0$.

The main result of the present paper is the theorem that characterises branch points of diagonals.

Theorem 1. *Let the Laurent series (1) for a rational function of n variables converge in the domain $\Lambda^{-1}(E)$, and let $d_{\boldsymbol{q}}(t)$ be its complete $\boldsymbol{q}$-diagonal. If $\boldsymbol{q} = \gamma_Q(\boldsymbol{p})$, where the point $\boldsymbol{p}$ is regular for the logarithmic Gauss mapping and $\Lambda(\boldsymbol{p}) \in \partial E$, then*

1. *In the case $n = 2k$ the point $t_0 = \boldsymbol{p}^{\boldsymbol{q}}$ is a branch point of finite order 2 of $d_{\boldsymbol{q}}(t)$.*
2. *In the case $n = 2k + 1$ the point $t_0 = \boldsymbol{p}^{\boldsymbol{q}}$ is a branch point of infinite order (logarithmic branch point) of $d_{\boldsymbol{q}}(t)$.*

In the context of enumerative combinatorics (see. [2, Section 6.1]), there is the following hierarchy of generating functions

$$\{\text{rational}\} \subset \{\text{algebraic}\} \subset \{D - \text{finite}\}.$$

It was proven in [3] that complete $\boldsymbol{q}$-diagonals of Laurent series for rational functions of two complex variables are algebraic. In expositions that deals with diagonals (see, for instance, [4, Section 2] or [2, Section 6.3]), treatment of the case of more than two variables is limited by pointing at the example of non-algebraic diagonal of the Taylor series for the rational function of three variables.

Since algebraic functions cannot have branch points of infinite order, Theorem 1 gives the sufficient condition of non-algebraicity of a diagonal in the case when the dimension n is odd.

Corollary 1. *Let the Laurent series (1) for a rational function of $2k+1$ variables converge in the domain $\Lambda^{-1}(E)$, and let $d_{\boldsymbol{q}}(t)$ be its complete $\boldsymbol{q}$-diagonal. If $\boldsymbol{q} = \gamma_Q(\boldsymbol{p})$, where the point $\boldsymbol{p}$ is regular for the logarithmic Gauss mapping and $\Lambda(\boldsymbol{p}) \in \partial E$, then $d_{\boldsymbol{q}}(t)$ is a non-algebraic function.*

2. Amoebas and integral representation for diagonal

From the moment of diagonals appeared on the mathematical scene (see [5, p. 280]), the important role in their study was played by integral representations. George Pólya showed the algebraicity of a diagonal of a bivariate rational Taylor series from a particular class in [6]. His proof was based on a representation of the diagonal by an integral over a contour in the complex plane. Exploiting a similar idea it was shown in [4, 7] that the diagonal of an analytic power series F in a bidisk $\{|z_1| < A, |z_2| < B\}$ can be represented as

$$d(t) = \frac{1}{2\pi\imath} \int_{|\zeta|=\varepsilon} F\left(\zeta, \frac{t}{\zeta}\right) \frac{d\zeta}{\zeta},$$

where $\varepsilon = \left(A + \dfrac{|t|}{B}\right)/2$. If, in addition, F converges to a rational function, then evaluating the integral by residues gives that the diagonal is algebraic, see [4, Section 2] and [2, Section 6.3].

Further, in [8] it was proved that the $\boldsymbol{q}$-diagonal of the Taylor series for a rational function $F(\boldsymbol{z}) = \dfrac{P(\boldsymbol{z})}{Q(\boldsymbol{z})}$ of n complex variables holomorphic at the origin has the integral representation

$$d_{\boldsymbol{q}}(t) = \frac{1}{(2\pi\imath)^n} \int_{\Gamma_{\boldsymbol{\rho}}} \frac{P(\boldsymbol{z})}{Q(\boldsymbol{z})} \frac{\boldsymbol{z}^{\boldsymbol{q}-\boldsymbol{I}}}{\boldsymbol{z}^{\boldsymbol{q}} - t} d\boldsymbol{z},$$

where the cycle $\Gamma_{\boldsymbol{\rho}} = \{\boldsymbol{z} \in \mathbb{C}^n : |z_1| = \rho_1, \ldots, |z_n| = \rho_n\}$ is chosen so that the closed polydisk $\{|z_1| \leqslant \rho_1, \ldots, |z_n| \leqslant \rho_n\}$ contains no poles of the function $F(\boldsymbol{z})$, and $\boldsymbol{\rho}^{\boldsymbol{q}} > |t|$. It will be convenient for us to use the following notation

$$\omega = \frac{1}{(2\pi\imath)^n} P(\boldsymbol{z}) \boldsymbol{z}^{\boldsymbol{q}-\boldsymbol{I}} d\boldsymbol{z}.$$

In order to describe the integral representation for a complete $\boldsymbol{q}$-diagonal of the Laurent series (1), we list necessary facts about amoebas of polynomials.

Recall that the *Newton polytope* Δ_Q of a polynomial Q is the convex hull in $\mathbb{R}^n$ of the set of exponents of the monomials occuring with non-zero coefficients in Q. According to Propositions 2.4–2.6 in [9], on the set $\{E\}$ of connected components of $\mathbb{R}^n \setminus \mathcal{A}_Q$ there exists an injective order mapping

$$\nu : \{E\} \mapsto \Delta_Q \cap \mathbb{Z}^n$$

such that the dual cone to Δ_Q at the point $\nu(E)$ coincides with the recession cone of the component E. Then it follows from this fact that the number of connected components of the amoeba complement is at most equal to the number of integer points in Δ_Q (see [9, Theorem 2.8]). Note that the proof of the injectivity of ν also establishes that components E are convex in $\mathbb{R}^n$.

Corollary 1.6 in [1] says that all centered at the origin Laurent expansions (1) of a rational function $F(\boldsymbol{z}) = \dfrac{P(\boldsymbol{z})}{Q(\boldsymbol{z})}$ are in a bijective correspondence with the connected components $\{E\}$. The sets $\Lambda^{-1}(E)$ are the convergence domains for the corresponding Laurent expansions. If the rational function $F(\boldsymbol{z})$ is holomorphic at the origin, then its Taylor expansion converges in the domain $\Lambda^{-1}(E)$, where $\nu(E) = (0, \ldots, 0)$, and the point $(0, \ldots, 0)$ is a vertice of the Newton polytope Δ_Q.

The following proposition from [3] generalizes the integral representation for diagonals of Taylor series that have been mentioned above.

Proposition 1. *Let the Laurent series (1) for a rational function of n variables converge in the domain $\Lambda^{-1}(E)$, where E is a connected component of the complement $\mathbb{R}^n \setminus \mathcal{A}_Q$, and let $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ are points in E such that the inequality $\langle \boldsymbol{q}, \boldsymbol{y}_1\rangle < \langle \boldsymbol{q}, \boldsymbol{y}_2\rangle$ holds for a non-zero $\boldsymbol{q} \in \mathbb{Z}^n$. Then the complete $\boldsymbol{q}$-diagonal $d_{\boldsymbol{q}}(t)$ of the Laurent series (1) has the integral representation*

$$d_{\boldsymbol{q}}(t) = \int_\Gamma \frac{\omega}{Q(\boldsymbol{z})(\boldsymbol{z}^{\boldsymbol{q}} - t)}, \tag{3}$$

where $\langle \boldsymbol{q}, \boldsymbol{y}_1\rangle < \log|t| < \langle \boldsymbol{q}, \boldsymbol{y}_2\rangle$, *and* $\Gamma = \Lambda^{-1}(\boldsymbol{y}_2) - \Lambda^{-1}(\boldsymbol{y}_1)$.

3. Proof of Theorem 1

Note that the differential form ω is regular in $(\mathbb{C}^\times)^n$, while the differential form in the integral representation (3) is meromorphic in $(\mathbb{C}^\times)^n$ with polar singularities on hypersurfaces

$$S_1 = Z^\times(Q), \quad S_2 = Z^\times(\boldsymbol{z}^{\boldsymbol{q}} - t).$$

Let $\boldsymbol{y}_1$, $\boldsymbol{y}_2$ be points in E chosen as specified in Proposition 1. The fibers $\Lambda^{-1}(\boldsymbol{y}_1)$, $\Lambda^{-1}(\boldsymbol{y}_2)$ of the logarithmic projection over these points are n-dimensional real tori $(\mathbb{C}^\times)^n$ that define classes in the reduced homology group $H_n((\mathbb{C}^\times)^n \setminus S_1 \cup S_2)$ with compact supports.

We want to show that the family $\{S_1, S_2\}$ has a so-called *quadratic zero-pinch* (see [10, Section IV.1]) at the point $\boldsymbol{p}$ for $t = t_0$, where $t_0 = \boldsymbol{p}^{\boldsymbol{q}}$. For this purpose, we introduce new coordinates $\boldsymbol{w} = (w_1, \ldots, w_n)$ in the n-dimensional torus $(\mathbb{C}^\times)^n$.

We first note that since vector $\boldsymbol{q}$ is irreducible, according to the Invariant Factor Theorem (see [11, Theorem 16.6]), there exists an unimodular transformation $A : \mathbb{Z}^n \to \mathbb{Z}^n$ that takes vector $\boldsymbol{q}$ to vector $\boldsymbol{e}_1 = (1, 0, \ldots, 0)$. This transformation induces the diffeomorphism $(\mathbb{C}^\times)^n \to (\mathbb{C}^\times)^n$ defined as

$$w_1 = \boldsymbol{z}^{\boldsymbol{a}_1}, \ldots, w_n = \boldsymbol{z}^{\boldsymbol{a}_n},$$

where $\boldsymbol{a}_j$'s are columns of the matrix for the transformation A, and $\boldsymbol{a}_1 = \boldsymbol{q}$. In new coordinates, the hypersurfaces S_1, S_2 are defined by equations

$$\tilde{Q}(\boldsymbol{w}) = 0, \ w_1 - t = 0,$$

correspondingly.

Next, assume, without loss of generality, that $\tilde{Q}_{w_1}(\tilde{p}) \neq 0$, where the point $\tilde{\boldsymbol{p}} = (\boldsymbol{p}^{a_1}, \ldots, \boldsymbol{p}^{a_n})$. Then, by the Implicit Function Theorem, there exists a sufficiently small neighbourhood U of the point $\tilde{\boldsymbol{p}}$ such that S_1 is defined in U as a graph of some analytic function,

$$S_1 \cap U = \{\boldsymbol{w} \in U : w_1 = f(w_2, \ldots, w_n)\}.$$

Therefore, the intersection $S_1 \cap S_2$ is defined in U as the zero set of the system

$$\begin{cases} w_1 - f(w_2, \ldots, w_n) = 0, \\ w_1 - t = 0. \end{cases}$$

From the definition of the logarithmic Gauss mapping (2), it follows that

$$\gamma_{\tilde{Q}}(\boldsymbol{w}) = (1 : -w_2 f_{w_2}(w_2, \ldots, w_n) : \ldots : -w_n f_{w_n}(w_2, \ldots, w_n))$$

for $\boldsymbol{w} \in U$. In particular, $\gamma_{\tilde{Q}}(\tilde{\boldsymbol{p}}) = (1 : 0 \ldots : 0)$. Since the (i,j)-component of the Jacobian matrix of the logarithmic Gauss mapping $\gamma_{\tilde{Q}}$ at the point $\tilde{\boldsymbol{p}} \in U$ has the form

$$\left(-\tilde{p}_i f_{w_i w_j}(\tilde{p}_2, \ldots, \tilde{p}_n)\right)_{i,j}, \quad i, j = 2, \ldots, n,$$

the Jacobian determinant of $\gamma_{\tilde{Q}}$ at $\tilde{\boldsymbol{p}}$ and the Hessian determinant of the function $f(w_2, \ldots, w_n)$ at the point $(\tilde{p}_2, \ldots, \tilde{p}_n)$ vanish simultaneously. If $\boldsymbol{p}$ is a regular point of γ_Q then $\tilde{\boldsymbol{p}}$ is a regular point of $\gamma_{\tilde{Q}}$. So the point $(\tilde{p}_2, \ldots, \tilde{p}_n)$ is a Morse critical point for the function $f(w_2, \ldots, w_n)$, and by the Morse lemma, there exist local coordinates $(\tilde{w}_2, \ldots, \tilde{w}_n)$ in a neighbourhood of this point such that $f = \tilde{w}_2^2 + \ldots + \tilde{w}_n^2 + \boldsymbol{p}^{\boldsymbol{q}}$. So the intersection $S_1 \cap S_2$ is given locally by the equation

$$\tilde{w}_2^2 + \ldots + \tilde{w}_n^2 + \boldsymbol{p}^{\boldsymbol{q}} - t = 0.$$

Therefore, the family of the hypersurfaces S_1, S_2 has the quadratic zero-pinch at the point $\boldsymbol{p}$ for $t = \boldsymbol{p}^{\boldsymbol{q}}$.

Thus, for the discriminant value $t_0 = \boldsymbol{p}^{\boldsymbol{q}}$ of the parameter t, we have the standart degeneration of type $P_i = P_2$ (in terms of the notation of [12, Section I.8]). The monodromy operator

$$\Phi : H_n((\mathbb{C}^\times)^n \setminus S_1 \cup S_2) \to H_n((\mathbb{C}^\times)^n \setminus S_1 \cup S_2),$$

defined by a small loop going around t_0 was calculated in [10, Part IV]. This operator reduces to the standart Picard–Lefschetz formula for the Morse singularity in $\mathbb{C}^{n-i+1} = \mathbb{C}^{n-1}$.

So, by Theorem 2.4 in [10, Part IV], we have that

$$\Phi([\Gamma]) = [\Gamma] + \iota[\Sigma]$$

where ι is a non-zero integer, and the class $[\Sigma]$ is defined as follows. According to the Thom Isotopy theorem, the monodromy acts identically outside a sufficiently small neighbourhood W of the point $\boldsymbol{p}$. Let σ be the vanishing sphere of the dimension $n-2$ in the intersection of $S_1 \cap S_2$ and W. Then $[\Sigma] = i_* \delta^2[\sigma]$, the homomorphism i_* is induced by the inclusion of W into $(\mathbb{C}^\times)^n$, and $\delta^2 : H_{n-2}(S_1 \cap S_2 \cap W) \to H_n(W \setminus (S_1 \cup S_2))$ is 2-iterated coboundary operator of Leray defined in Theorem 2 of [10, Part II].

Note that the Picard-Lefschetz formula also gives us

$$\Phi([\Sigma]) = (-1)^{n-1}[\Sigma].$$

Knowing the transformation of $[\Gamma]$ and $[\Sigma]$ by Φ allows us to continue the integral

$$d_{\boldsymbol{q}}(t) = \int_\Gamma \frac{\omega}{Q(\boldsymbol{z})(\boldsymbol{z}^{\boldsymbol{q}} - t)}$$

analytically along a small loop around the point t_0. Let

$$q(t) = \int_\Sigma \frac{\omega}{Q(\boldsymbol{z})(\boldsymbol{z}^{\boldsymbol{q}} - t)}.$$

Then during one traversal of the mentioned loop the integral for $d_{\boldsymbol{q}}(t)$ goes to

$$d_{\boldsymbol{q}}(t) + \iota q(t).$$

If the dimension $n = 2k$, the two traversals of the loop give

$$d_{\boldsymbol{q}}(t) + \iota q(t) + (-1)^{2k-1}\iota q(t) = d_{\boldsymbol{q}}(t).$$

So, the point t_0 is a branch point of order 2 for the diagonal $d_{\boldsymbol{q}}(t)$. If the dimension $n = 2k+1$, the two traversals of the loop give

$$d_{\boldsymbol{q}}(t) + \iota q(t) + (-1)^{2k}\iota q(t) = d_{\boldsymbol{q}}(t) + 2\iota q(t).$$

In this case, t_0 is a branch point of infinite order for the diagonal $d_{\boldsymbol{q}}(t)$. The theorem is proved.

4. The diagonal of the multivariate geometric series

The purpose of this section is to illustrate Theorem 1.

Consider the polynomial $L(z) = 1 - z_1 - \ldots - z_n$. The multivariate geometric series

$$\frac{1}{L(z)} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} \frac{(\alpha_1 + \ldots + \alpha_n)!}{\alpha_1! \ldots \alpha_n!} \boldsymbol{z}^{\boldsymbol{\alpha}}$$

converges in the domain $\Lambda^{-1}(E_0)$, where E_0 is the compoment of the complement $\mathbb{R}^n \setminus \mathcal{A}_L$ that corresponds to the constant term of L.

For convenience, we denote the diagonal of this Taylor series by

$$\mathfrak{d}_n(t) = \sum_{k=0}^{\infty} \frac{nk!}{(k!)^n} t^k. \tag{4}$$

The logarithmic Gauss mapping $\gamma_L : Z^\times(L) \to \mathbb{CP}^{n-1}$ is a birational isomorphism with the inverse given by

$$z_j = \frac{q_j}{q_1 + \ldots + q_n}, \quad j = 1, \ldots, n,$$

where $\boldsymbol{q} = (q_1 : \ldots : q_n) \in \mathbb{CP}^{n-1}$. Also, the point $\boldsymbol{p} = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ is projected by the logarithmic mapping Λ to the point of the boundary ∂E_0, so that, by Theorem 1, the point $t_0 = \boldsymbol{p}^{\boldsymbol{I}} = 1/n^n$ is a branch point for the diagonal $\mathfrak{d}_n(t)$, and the type of this branch point depends on the parity of n.

We note that

$$\mathfrak{d}_2(t) = \frac{1}{\sqrt{1-4t}},$$

by means of the generalized binomial expansion. Thus, the diagonal $\mathfrak{d}_2(t)$ is an algebraic function that has a branch point of the order 2 at $t_0 = \frac{1}{4}$.

In the case $n = 3$, the diagonal (4) is represented by the Gaussian hypergeometric function

$$\mathfrak{d}_3(t) = {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}, 1; 27t\right),$$

so that $t_0 = \frac{1}{27}$ is a branch point for the diagonal. Note that the parameters of this hypergeometric function are not in Schwarz's list of the cases when the Gaussian hypergeometric function is algebraic.

Proposition 1. *The diagonal $\mathfrak{d}_3(t)$ has the form*

$$\mathfrak{d}_3(t) = a_3(t)\log(1-27t) + b_3(t),$$

in a neighbourhood of the point $t_0 = \frac{1}{27}$, where the functions $a_3(t)$ and $b_3(t)$ are holomorphic and non-vanishing at the point $t_0 = \frac{1}{27}$.

Proof. According to [13, Section 16], we can write the hypergeometric function ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}, 1; 27t\right)$ as the integral

$$-\frac{1}{2\pi\imath}\frac{1}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)} \int\limits_{-\frac{1}{2}+\imath\mathbb{R}} \Gamma^2(-\zeta)\Gamma\left(\tfrac{1}{3}+\zeta\right)\Gamma\left(\tfrac{2}{2}+\zeta\right)(1-27t)^{\zeta}d\zeta$$

with the meromorphic integrand that has three groups of poles

$$\xi_k = k, \quad \zeta_k = -\frac{1}{3} - k, \quad \eta_k = -\frac{2}{3} - k, \quad k \in \mathbb{N} \cup \{0\}.$$

The poles ξ_k lie on the complex plane to the right of the integration contour, while the poles ζ_k, η_k lie to the left of it.

Evaluating the integral as the sum of residues in poles ξ_k of the first group gives us the desired representation. □

Further, it is clear from the representation

$$\mathfrak{d}_4(t) = {}_3F_2\left(\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}; 1, 1; 256t\right)$$

in the form of the generalized hypergeometric function that the diagonal $\mathfrak{d}_4(t)$ has a branch point at $t_0 = \frac{1}{256}$.

By a happy coincidence, the generalized hypergeometric function ${}_3F_2$ that corresponds to this specific set of parameters can be written in the form

$$\mathfrak{d}_4(t) = \left(F\left(\tfrac{1}{8}, \tfrac{3}{8}; 1; 256t\right)\right)^2 \tag{5}$$

with a help of Clausen's formula [14]. It allows us to describe a type of the branch point $t_0 = \frac{1}{256}$ in a way that is similar to the proof of Proposition 1.

Proposition 2. *The diagonal $\mathfrak{d}_4(t)$ has the form*

$$\mathfrak{d}_4(t) = a_4(t)(1-256t)^{\frac{1}{2}} + b_4(t),$$

in a neighbourhood of the point $t_0 = \frac{1}{256}$, where functions $a_4(t)$ and $b_4(t)$ are holomorphic and non-vanishing at the point $t_0 = \frac{1}{256}$.

Proof. According to [13, Section 16], we can write the hypergeometric function ${}_2F_1\left(\frac{1}{8},\frac{3}{8},1;256t\right)$ as the integral

$$-\frac{1}{2\pi\imath}\frac{1}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}\int\limits_{-\frac{1}{16}+\imath\mathbb{R}}\Gamma(-\zeta)\Gamma\left(\tfrac{1}{2}-\zeta\right)\Gamma\left(\tfrac{1}{8}+\zeta\right)\Gamma\left(\tfrac{3}{8}+\zeta\right)(1-256t)^{\zeta}d\zeta,$$

where the integration contour separates poles of the function $\Gamma(-\zeta)\Gamma\left(\frac{1}{2}-\zeta\right)$ of the form

$$\xi_k = k, \quad \zeta_k = \tfrac{1}{2}+k,$$

from the poles of $\Gamma\left(\frac{1}{8}+\zeta\right)\Gamma\left(\frac{3}{8}+\zeta\right)$ of the form

$$\eta_k = -\frac{1}{8}-k, \quad \varkappa_k = -\frac{3}{8}-k.$$

The parameter k ranges over the set $\mathbb{N}\cup\{0\}$.

We let $b(t)$ denote the sum of residues of the integrand at the point ξ_k. It occurs that $b(t)$ is holomorphic at $t_0=\dfrac{1}{256}$ and does not vanish at this point. At the same time, the sum of residues of the integrand at the points ζ_k has the form $a(t)(1-256t)^{1/2}$, where the function $a(t)$ is holomorphic at $t_0=\dfrac{1}{256}$ and is non-vanishing at this point.

Thus, the function ${}_2F_1\left(\frac{1}{8},\frac{3}{8},1;256t\right)$ has the representation

$$a(t)(1-256t)^{\frac{1}{2}}+b(t)$$

in some neighbourhood of the point $t_0=\dfrac{1}{256}$. Then the Proposition follows directly from the Clausen formula (5). □

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Аналитическое продолжение диагоналей рядов Лорана рациональных функций

Дмитрий Ю. Почекутов
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. Мы описываем точки ветвления полных $\boldsymbol{q}$-диагоналей рядов Лорана рациональных функций нескольких комплексных переменных в терминах логарифмического отображения Гаусса. Доказано достаточное условие неалгебраичности такой диагонали.

Ключевые слова: диагонали рядов Лорана, логарифмическое отображение Гаусса, амеба гиперповерхности, нулевой пинч, монодромия.

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Removable Singularities of Separately Harmonic Functions

Sevdiyor A. Imomkulov*
Khorezm Regional Branch of the V. I. Romanovsky Mathematical Institute
Academy of Sciences of the Republic of Uzbekistan
Urgench, Uzbekistan

Sultanbay M. Abdikadirov†
Karakalpak State University
Nukus, Uzbekistan

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Abstract. Removable singularities of separately harmonic functions are considered. More precisely, we prove harmonic continuation property of a separately harmonic function $u(x,y)$ in $D \setminus S$ to the domain D, when $D \subset \mathbb{R}^n(x) \times \mathbb{R}^m(y)$, $n, m > 1$ and S is a closed subset of the domain D with nowhere dense projections $S_1 = \{x \in \mathbb{R}^n : (x,y) \in S\}$ and $S_2 = \{y \in \mathbb{R}^m : (x,y) \in S\}$.

Keywords: separately harmonic function, pseudoconvex domain, Poisson integral, $\mathcal{P}$-measure.

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The theorem on removal of compact singularities (see [1, 2]) is one of the most important results in the theory of functions in several complex variables: *if a function f is holomorphic everywhere in the domain $\Omega \subset \mathbb{C}^n$ $(n > 1)$ except a set $K \Subset \Omega$, which does not divide the domain (i.e. such that $\Omega \setminus K$ is connected), then f can be extended holomorphically to whole domain Ω.* In the work [3], an analogue of this theorem was proved for separately harmonic functions, i.e. for functions which are harmonic in each variable separately: *let D be a domain in $\mathbb{R}^n(x) \times \mathbb{R}^m(y)$, $n, m > 1$, $K \Subset D$ a compact set such that $D \setminus K$ is connected. If the function $u(x,y)$ is separately harmonic in $D \setminus K$, then it harmonically continues to D.*

1. Separately harmonic functions

Definition 1. *If a function $u(x,y)$ is defined in the domain $D \subset \mathbb{R}^n(x) \times \mathbb{R}^m(y)$ and satisfies the following properties:*

1) *for any fixed $x^0 : \{x = x^0\} \cap D \neq \emptyset$, a function $u(x^0, y)$ is harmonic in y on $\{x = x^0\} \cap D$;*

2) *for any fixed $y^0 : \{y = y^0\} \cap D \neq \emptyset$, a function $u(x, y^0)$ is harmonic in x on $\{y = y^0\} \cap D$, then it is called a separately harmonic function in the domain D.*

One of the main methods of studying extension of harmonic functions is the transition to holomorphic functions, and then using the principles of holomorphic extensions.

*sevdiyor_i@mail.ru
†abdikadirov1983@inbox.ru

Lemma 1 ([5]). *For any domain $D \subset \mathbb{R}^n(x) \subset \mathbb{C}^n$ there is a domain of holomorphy $\widehat{D} \subset \mathbb{C}^n(z)$ such that $D \subset \widehat{D}$ and any harmonic function $u(x)$ in D holomorphically extends into the domain $\widehat{D}$, i.e. there is a holomorphic function $f_u(z)$ in $\widehat{D}$ such that $f_u|_D = u$.*

The existence of the domain $\widehat{D}$ follows easily from the representation of harmonic functions by the Poisson integral. Indeed, let $B = B(x^0, R) \Subset D$ be an arbitrary ball in D, and $u(x)$ be a harmonic function in D. Then the following formula holds

$$u(x) = \frac{1}{\sigma_n} \int\limits_{\partial B} \frac{R^2 - |x - x^0|^2}{R|x - y|^n} u(y) d\sigma(y),$$

where σ_n is the surface area of the unit sphere. It is clear that the Poisson kernel

$$P(x, y) = \frac{1}{\sigma_n} \frac{R^2 - |x - x^0|^2}{R|x - y|^n}$$

for any fixed $y \in \partial B$ holomorphically extends to some domain $\widehat{B} \in \mathbb{C}^n$, $\widehat{B} \supset B$. Eventually, $\widehat{B}$ is a Lie ball centered at $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)$ with the radius R (see [11])

$$\widehat{B} = \left\{ z \in \mathbb{C}^n : \sqrt{|z - x^0|^2 + \sqrt{|z - x^0|^4 - \left| \sum_{j=1}^{n} \left(z_j - x_j^0 \right)^2 \right|^2}} < R \right\}.$$

Consequently, every harmonic in B function holomorphically extends to $\widehat{B}$, which implies the existence a domain $\widehat{D}$, $D \subset \widehat{D} \subset \mathbb{C}^n$ satisfying the above properties.

It can be seen from the construction that for each fixed $z^0 \in \widehat{D}$ there is a constant M_{z^0} such that

$$|f_u(z^0)| \leqslant M_{z^0} \|u\|_D, \tag{1}$$

nevertheless, M_{z^0} is bounded on compact subsets of $\widehat{D}$ and

$$\lim_{z \to x \in D} M_z = 1.$$

2. Separately analytic functions

Let two domains $\mathbb{D} \subset \mathbb{C}^n$, $\mathbb{G} \subset \mathbb{C}^m$ and two subsets, $E \subset \mathbb{D}$, $F \subset \mathbb{G}$ be given. Assume that a function $f(z, w)$, determined firstly on the set $E \times F$, has the following properties:

a) for any fixed $w^0 \in F$, a function $f(z, w^0)$ holomorphically extends to the domain $\mathbb{D}$;

b) for any fixed $z^0 \in E$, a function $f(z^0, w)$ holomorphically extends to the domain $\mathbb{G}$.

In this case $f(z, w)$ defines some function on the set $X = (\mathbb{D} \times F) \cup (E \times \mathbb{G})$ and it is called *a separately-analytic* function on X.

We will use the following theorem on analytic continuation of separately-analytic functions (V. Zakharyuta [8], J. Sichak [9], and see also [7]): *let two domains $\mathbb{D} \subset \mathbb{C}^n$, $\mathbb{G} \subset \mathbb{C}^m$ be strongly pseudoconvex and two subsets $E \subset \mathbb{D}$, $F \subset \mathbb{G}$ be non-pluripolar Borel sets. If $f(z, w)$ is a separately analytic function on the set $X = (\mathbb{D} \times F) \cup (E \times \mathbb{G})$, then it extends holomorphically to the domain*

$$\widehat{X} = \{(z, w) \in \mathbb{D} \times \mathbb{G} : \omega^*(z, E, \mathbb{D}) + \omega^*(w, F, \mathbb{D}) < 1\}.$$

Here $\omega^*(z, E, \mathbb{D})$ is the $\mathcal{P}$-measure of the set E with respect to the domain $\mathbb{D}$ (see [7,8,10]). It is defined as an extremal plurisubharmonic function

$$\omega^*(z, E, \mathbb{D}) = \overline{\lim_{\zeta \to z}}\, \omega(\zeta, E, \mathbb{D}),$$

where

$$\omega(z, E, \mathbb{D}) = \sup\{u(z) : u \in psh(\mathbb{D}), u|_{\mathbb{D}} \leqslant 1, u|_E \leqslant 0\}.$$

3. On Lelong's theorem

P. Lelong [4] proved the following analogue of the fundamental theorem of Hartogs (see [1], Ch. 1): *if $u(x, y)$ is separately harmonic in the domain $D \subset \mathbb{R}^n(x) \times \mathbb{R}^m(y)$, then it is harmonic in D in both variables.*

The proof of Lelong's theorem can be obtained easily if we use the above theorem of V. Zakharyuta and J. Sichak: if $u(x, y)$ is separately harmonic in the domain $D \subset \mathbb{R}^n \times \mathbb{R}^m$ and $B_1 \subset \mathbb{R}^n$, $B_2 \subset \mathbb{R}^m$ are arbitrary balls such that $B_1 \times B_2 \subset D$, then by Lemma 1 it extends to the set $X = (\widehat{B}_1 \times B_2) \cup (B_1 \times \widehat{B}_2)$ as a separately analytic function. Therefore, $u(x, y)$ extends holomorphically to the domain

$$\widehat{X} = \left\{(z, w) \in \widehat{B}_1 \times \widehat{B}_2 : \omega^*(z, B_1, \widehat{B}_1) + \omega^*(w, B_2, \widehat{B}_2) < 1\right\}.$$

Since $B_1 \times B_2 \subset \widehat{X}$, the function $u(x, y)$ is infinitely differentiable in $B_1 \times B_2$ and therefore, harmonic in both variables. Since the balls are arbitrary, it follows that $u(x, y)$ is harmonic in both variables in the domain D.

4. The main results

Now we are ready to prove the main results of this paper.

Theorem 1. *Let S be a closed subset of the domain $D \subset \mathbb{R}^n(x) \times \mathbb{R}^m(y)$, $n, m > 1$, and its orthogonal projections $S_1 = \{x \in \mathbb{R}^n : (x, y) \in S\}$ and $S_2 = \{y \in \mathbb{R}^m : (x, y) \in S\}$ are nowhere dense. Then any function $u(x, y)$ which is separately harmonic in the domain $D \setminus S$ extends harmonically to the domain D.*

Proof. Let $u(x, y)$ be a separately harmonic function in the domain $D \setminus S$ and the projections of the closed set S:

$$S_1 = \{x \in \mathbb{R}^n : (x, y) \in S\}, \; S_2 = \{y \in \mathbb{R}^m : (x, y) \in S\},$$

are nowhere dense. We denote by $\tilde{S} \subset S$ the set of non-removable singularities for the function $u(x, y)$. Suppose that $\tilde{S} \neq \varnothing$. We take arbitrary balls $B_1 \subset \mathbb{R}^n(x)$ and $B_2 \subset \mathbb{R}^m(y)$ such that $B_1 \times B_2 \subset D$ and $(B_1 \times B_2) \cap \tilde{S} \neq \varnothing$. We denote by

$$\tilde{S}_1 = \{x \in B_1 : (x, y) \in (B_1 \times B_2) \cap \tilde{S}\}, \; \tilde{S}_2 = \{y \in B_2 : (x, y) \in (B_1 \times B_2) \cap \tilde{S}\}.$$

Since $(B_1 \times B_2) \cap \tilde{S} \subset \tilde{S}_1 \times \tilde{S}_2$, we have

$$(B_1 \times B_2) \setminus (\tilde{S}_1 \times \tilde{S}_2) = \Big(B_1 \times (B_2 \setminus \tilde{S}_2)\Big) \cup \Big((B_1 \setminus \tilde{S}_1) \times B_2\Big) \subset (B_1 \times B_2) \setminus \tilde{S}.$$

Hence, by Lemma 1, the function $u(x, y)$ can be extended analytically to the set $X = \left(\widehat{B}_1 \times (B_2 \setminus \tilde{S}_2)\right) \cup \left((B_1 \setminus \tilde{S}_1) \times \widehat{B}_2\right)$ as a separately analytic function. Consequently, $u(x, y)$ extends holomorphically to the domain

$$\widehat{X} = \left\{(z, w) \in \widehat{B}_1 \times \widehat{B}_2 : \omega^*(z, B_1 \setminus \tilde{S}_1, \widehat{B}_1) + \omega^*(w, B_2 \setminus \tilde{S}_2, \widehat{B}_2) < 1\right\}.$$

Since the sets $B_1 \setminus \tilde{S}_1$, $B_2 \setminus \tilde{S}_2$ are locally pluri-regular, we get

$$X \subset \widehat{X}, \text{ i.e. } (B_1 \times B_2) \setminus (\tilde{S}_1 \times \tilde{S}_2) \subset \widehat{X}.$$

(About pluri-regular sets and their properties, see [6,12]). Now we take an arbitrary point $a \in \tilde{S}_1$ and $x^0 \in U(a, \varepsilon) \setminus \tilde{S}_1$, where $U(a, \varepsilon) = \{x : |x - a| < \varepsilon\}$, $0 < \varepsilon < \frac{1}{2} dist(a, \partial B_1)$. For the point x^0 there is a point $a^0 \in \tilde{S}_1$ such that

$$d = |x^0 - a^0| = \inf\left\{|x^0 - x| : x \in \tilde{S}_1\right\}.$$

It is clear that the intersection $B_1 \cap \{x : |x^0 - x| < d\} \subset B_1 \setminus \tilde{S}_1$ contains the interval (x^0, a^0), which is not pluri-thin at the point $a^0 \in \tilde{S}_1$ (see [6], Proposition 4.1). Hence, it follows that

$$\omega^*(a^0, B_1 \setminus \tilde{S}_1, \widehat{B}_1) = 0.$$

On the other hand, there is a point $b^0 \in \tilde{S}_2$ such that $(a^0, b^0) \in \tilde{S}$ and by the definition of $\mathcal{P}$-measure there is also some number $\delta_2 : \omega^*(b^0, B_2 \setminus \tilde{S}_2, \widehat{B}_2) < \delta_2 < 1$. Now we take some number $\delta_1 > 0$ so that $\delta_1 + \delta_2 < 1$. Hence, an open neighborhood of the point

$$(a^0, b^0) \in \tilde{S} : \left\{z : \omega^*(z, B_1 \setminus \tilde{S}_1, \widehat{B}_1) < \delta_1\right\} \times \left\{w : \omega^*(w, B_2 \setminus \tilde{S}_2, \widehat{B}_2) < \delta_2\right\},$$

is contained in $\widehat{X}$, i.e. the point $(a^0, b^0) \in \tilde{S}$ is a removable singularity and this contradicts our assumption concerning $\tilde{S}$. Thus $\tilde{S} = \varnothing$. The theorem is proved. □

Using methods of V. Zahariuta on analytic extension of separately analytic functions we get the following result which generalizes Hamano's theorems [3].

Theorem 2. *Let two domains $D \subset \mathbb{R}^n$, $G \subset \mathbb{R}^m$ and two sets $E \subset D$, $F \subset G$ be given. If $E \Subset D$ is compact and F is a closed subset of G with nonempty complement $G \setminus F \neq \emptyset$, then any separately harmonic function $u(x, y)$ in $(D \times G) \setminus (E \times F)$ harmonically extends to the domain $D \times G$.*

Proof. According to Lemma 1 there is a pseudoconvex domain $\widehat{G} \subset \mathbb{C}^m$ such that $G \subset \widehat{G}$ and for each fixed $x \in D \setminus E$ a function $u(x, \cdot)$ holomorphically extends to $\widehat{G}$. Moreover, there is a sequence of strongly pseudoconvex domains $\widehat{G}_j, j = 1, 2, \dots$ such that $\widehat{G}_j \Subset \widehat{G}_{j+1} \Subset \widehat{G}$, $\widehat{G} = \bigcup_{j=1}^{\infty} \widehat{G}_j$ and $(G \cap \widehat{G}_1) \setminus F \neq \emptyset$. According to (1) for the set

$$K_\varepsilon = \{z \in D : dist(z, E) \leqslant \varepsilon\} \Subset D,$$

where $\varepsilon > 0$ is a small enough number, there is a sequence of positive real numbers M_j such that

$$|u(x, w)| \leqslant M_j \ \forall (x, w) \in \partial K_\varepsilon \times \widehat{G}_{j+1}.$$

Consequently, for any $l \in N$ there is a sequence of positive numbers $N_j^{(l)}$ such that the inequality

$$\sum_{|\alpha| \leqslant l} \left(\int_{\widehat{G}_j} \left| \frac{\partial^{|\alpha|} u(x,w)}{\partial w^\alpha} \right|^2 dV \right)^{\frac{1}{2}} \leqslant N_j^{(l)} \ \forall x \in \partial K_\varepsilon \tag{2}$$

holds.

Now we take a closed ball $\overline{B} \Subset (G \cap \widehat{G}_1) \setminus F$ and for a fixed j and a sequence of sets $\overline{B} \Subset \widehat{G}_j$ we consider a Hilbert space $H_0 \subset H_1$. For H_0 we take the closure of the space

$$\mathcal{O}(\widehat{G}) \cap h(G) \cap W_2^l(\widehat{G}_j), \ l > m.$$

(Here $\mathcal{O}(\widehat{G})$ is the space of holomorphic functions on $\widehat{G}$, $h(G)$ is the space of harmonic functions on G and $W_2^l(\widehat{G}_j)$ is the Sobolev space.) For H_1 we take the closure of the space $h(G) \cap L_2(\overline{B}, \sigma)$, where

$$L_2(\overline{B}, \mu) = \left\{ f : \left(\int_{\overline{B}} |f(w)|^2 d\sigma \right)^{\frac{1}{2}} \leqslant \infty \right\}$$

and $d\sigma = \left(dd^c \omega^*(w, \overline{B}, \widehat{G}_j) \right)^m$ (see [7, 8, 10]). Let $\{e_k(w)\}_{k=1}^\infty$ be the common orthogonal basis for spaces $H_0 \subset H_1$ such that $\|e_k\|_{H_0} = \mu_k$, $\|e_k\|_{H_1} = 1$, $\dfrac{1}{M} k^{\frac{1}{m}} \leqslant ln \mu_k \leqslant M k^{\frac{1}{m}}$, and M is a constant, $k = 1, 2, \ldots$ (see [8, 13]).

From the continuous embedding of $H_0 \subset C(\overline{\widehat{G}}_j) \cap \mathcal{O}(\widehat{G}_j)$ it follows that

$$|e_k(w)| \leqslant C \|e_k\|_{H_0} = C \mu_k, \ \ w \in \widehat{G}_j, \tag{3}$$

where C is a constant.

We consider the set $A_k = \{z \in \overline{B} : |e_k(y)| > k\}$. By Chebyshev's inequality we have

$$\sigma(A_k) \leqslant \frac{1}{k^2} \int_{\overline{B}} |e_k(y)|^2 d\sigma(y) = \frac{1}{k^2} \|e_k\|_{H_1} = \frac{1}{k^2}, \ \ k = 1, 2, \ldots .$$

Consequently, $\sum\limits_{k=1}^{\infty} \sigma(A_k) < \infty$ and $\lim\limits_{s \to \infty} \sigma \left(\bigcup\limits_{k=s}^{\infty} A_k \right) = 0$. We let $U_s = \overline{B} \setminus \bigcup\limits_{k=s}^{\infty} A_k$, $U = \bigcup\limits_{s=1}^{\infty} U_s$. Then $\sigma(\overline{B} \setminus U) = 0$. Therefore, $\omega^*(w, \overline{B}, \widehat{G}_j) = \omega^*(w, U, \widehat{G}_j)$, i.e. $\omega^*(w, U_s, \widehat{G}_j) \downarrow \omega^*(w, \overline{B}, \widehat{G}_j)$, $w \in \widehat{G}_j$ (see [7, 10]). Since $|e_k(y)| \leqslant k$, $w \in E_s$, $k \geqslant s$, taking into account (3), by two constants theorem we obtain the following estimation

$$|e_k(w)| \leqslant c(s) k \mu_k^{\omega^*(w, U_s, \widehat{G}_j)}, \ \ k \geqslant s, \ \ w \in \widehat{G}_j, \tag{4}$$

where $c(s)$ is a constant independent of k.

Now we compare the formal Fourier–Hartogs series to the function $u(x, w), (x, w) \subset D \times \widehat{G}_j$,

$$u(x, w) \sim \sum_{k=1}^{\infty} a_k(x) e_k(w), \tag{5}$$

where the coefficients are defined by the usual formulas of the space H_1:

$$a_k(x) = \int_{\overline{B}} u(x, w) \overline{e_k(w)} d\sigma, \ \ k = 1, 2, \ldots .$$

We show that the series (5) converges locally uniformly in the set $K_\varepsilon \times \widehat{G}_j$.

Since the function $u(x,y)$ is continuous and separately harmonic on the set $D \times \overline{B}$, it follows that $a_k(x)$ is harmonic on D. Moreover, for any fixed $x \in \partial K_\epsilon$ the function $u(x,w) \in H_0$, then $|a_k(x)| = (u(x,\cdot), e_k)_{H_1} = \mu_k^{-2}(u(x,\cdot), e_k)_{H_0}$. Consequently,

$$|a_k(x)| \leqslant \frac{1}{\mu_k^2}\|u(x,\cdot)\|_{H_0}\|e_k\|_{H_0} \leqslant \frac{\|u(x,\cdot)\|_{H_0}}{\mu_k}, \quad x \in \partial K_\varepsilon.$$

Hence, by the estimation (2) and the maximum principle we get the following estimation

$$|a_k(x)| \leqslant \frac{N_j^l}{\mu_k}, \; k = 1, 2, \ldots, \quad x \in K_\varepsilon. \tag{6}$$

Comparing the estimates (4) and (6), we obtain

$$|a_k(x)e_k(w)| \leqslant c(s)N_j k \mu_k^{\omega^*(w,U_s,\widehat{G}_j)-1} \leqslant c(s)N_j k e^{Mk^{\frac{1}{m}(\omega^*(w,U_s,\widehat{G}_j)-1)}},$$

$k \geqslant s$, $(x,w) \in K_\varepsilon \times \widehat{G}_j$, where $U_s \subset \overline{B}$, $\sigma(U_s) > 0$. The last estimation shows that the series (5) converges locally uniformly on the set $K_\epsilon \times \widehat{G}_j$ and its sum $\widetilde{u}(x,w)$ coincides with $u(x,w)$ on the set $\partial K_\varepsilon \times \widehat{G}_j$, i.e. $\widetilde{u}(x,w)$ is an analytic continuation of $u(x,w)$. Finally, letting j tend to infinity we obtain an analytic continuation of the function $u(x,w)$ on the set $K_\varepsilon \times \widehat{G}$ which contains the set $E \times F$, that is the function $u(x,y)$ can be separately harmonically extended to $D \times G$. The proof of Theorem 2 is completed. □

Comparing the ideas of proof of theorems above, one can easily prove the following theorem:

Theorem 3. *Let two domains $D \subset \mathbb{R}^n$, $G \subset \mathbb{R}^m$ and two sets $E \subset D$, $F \subset G$ be given. If E is a nowhere dense closed subset of the domain D and F is a closed subset of the domain G with a non-empty complement $G \setminus F \neq \emptyset$, then any separately harmonic function $u(x,y)$ on the domain $(D \times G) \setminus (E \times F)$ can be extended harmonically to the domain $D \times G$.*

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Стираемые особенности сепаратно-гармонических функций

Севдиёр А. Имомкулов
Хорезмский областной филиал Математического института им. В. И. Романовского
Академия наук Республики Узбекистан
Ургенч, Узбекистан
Султанбай М. Абдикадиров
Каракалпакский государственный университет
Нукус, Узбекистан

Аннотация. В работе рассматриваются устранимые особенности сепаратно-гармонических функций. Точнее, доказана теорема о гармоническом продолжении сеператно-гармонической в $D \setminus S$ функции $u(x, y)$ в область D, где $D \subset \mathbb{R}^n(x) \times \mathbb{R}^m(y)$, $n, m > 1$ и S — замкнутое подмножество области D, а ее проекции $S_1 = \{x \in \mathbb{R}^n : (x, y) \in S\}$ и $S_2 = \{y \in \mathbb{R}^m : (x, y) \in S\}$ нигде не плотны.

Ключевые слова: сепаратно-гармоническая функция, псевдовыпуклая область, интеграл Пуассона, $\mathcal{P}$-мера.

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The Time-fractional Airy Equation on the Metric Graph

Kamoladdin Rakhimov*
National University of Uzbekistan
Tashkent, Uzbekistan
Zarifboy Sobirov†
University of Geological Sciences
Tashkent, Uzbekistan
Nasridin Jabborov‡
National University of Uzbekistan
Tashkent, Uzbekistan

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Abstract. Initial boundary value problem for the time-fractional Airy equation on a graph with finite bonds is considered in the paper. Properties of potentials for this equation are studied. Using these properties the solutions of the considered problem were found. The uniqueness theorem is proved using the analogue of Grönwall-Bellman inequality and a-priory estimate.

Keywords: time-fractional Airy equation, IBVP, PDE on metric graphs, fundamental solutions, integral representation.

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Introduction

In recent years noticeable interest has been shown in the study of initial and initial-boundary value problems for equations of fractional order. This is due to the fact that fractional-integral calculus have applications in the study of diffusion and dispersion processes in various fields of science (see [1–5]).

The Schrodinger equation on metric graphs was studied (see [6, 7] and references therein). Such graphs sometimes called quantum graphs. The Schrodinger equation on the metric graph was also studied with Fokas unified transformation method [8].

The Airy equation on an interval was studied with Fokas unified transform method [9] and [10]. The potential theory for solutions of this equation was developed [11] and [12]. The linearised Airy equation on metric graphs was considered in [13–16] and [17]. M. Cavalcante considered non linearised KdV equation [18].

A. Pskhu studied properties of the Airy equation with time-fractional derivative. Fundamental solution of the equation was found and properties of potentials were studied (see [19]). Later, second fundamental solution was found and the properties of the some additional potentials were

*kamoliddin_ru@inbox.ru
†sobirovzar@gmail.com
‡jabborov61@mail.ru

studied [20, 21]. Using this results solutions of initial and some IBVPs over infinite and finite intervals were found.

In this paper we consider the initial boundary value problem (IBVP) on a closed star graph with finite bonds. The solutions are found with the use of the potential method developed in [19–21].

1. Basic concepts

The operator

$$ {}_C D^{\alpha}_{\eta,t} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_{\eta}^{t} \frac{g'(\xi)}{|t-\xi|^{\alpha}} d\xi, \quad 0 < \alpha < 1, \tag{1} $$

is called *fractional derivative (Caputo derivative)* (see [22]), where $\Gamma(x)$ is the Gamma function. Inverse of this operator is called operator of *fractional integration*

$$ J^{\alpha}_{\eta,t} g(t) = \frac{1}{\Gamma(\alpha)} \int_{\eta}^{t} \frac{g(\xi)}{|t-\xi|^{1-\alpha}} d\xi. \tag{2} $$

It is easy to show that

$$ {}_C D^{\alpha}_{\eta,t} g(t) = {}_C D^{\alpha}_{0,t-\eta} g(t). \tag{3} $$

Function

$$ \phi(\lambda, \mu; z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda n + \mu)}, \quad \lambda > -1, \mu \in \mathrm{C} \tag{4} $$

is called *Wright function* (see [23]). Wright function can be represented as

$$ \phi(\lambda, \mu; z) = \frac{1}{2\pi i} \int_{Ha} e^{\sigma + z\sigma^{-\lambda}} \frac{d\sigma}{\sigma^{\mu}}, $$

where the integral is taken along the Hankel contour (see [22]). We have following estimate (see [19])

$$ |\phi(-\lambda, \mu; z)| \leqslant C \exp\left(-\nu |z|^{\frac{1}{1-\lambda}}\right), \quad C = C(\lambda, \mu, \nu), \tag{5} $$

where $\nu < (1-\lambda)\lambda^{\frac{\lambda}{1-\lambda}} \cos\dfrac{\pi - |\arg z|}{1-\lambda}$, $\dfrac{1+\lambda}{2}\pi < |\arg z| \leqslant \pi$. The value of integral of this function is (see [19])

$$ \int_{0}^{+\infty} \phi(-\lambda, \mu; az) dz = -\frac{1}{a\Gamma(\mu+\lambda)}. \tag{6} $$

2. Formulation of the problem

The Cauchy problem for time-fractional Airy equation on a metric graph with infinite bonds was considered ([21]). Now we consider a graph with k incoming and m outgoing bonds. In the incoming bonds coordinates are set from L_j ($L_j < 0$, $j = \overline{1,k}$) to 0, and on the outgoing bonds the coordinates are set from 0 to L_i ($L_i > 0, i = \overline{k+1, k+m}$). The bonds of the graph are denoted by b_j, $j = \overline{1, k+m}$ (Fig. 1).

On each bond b_j ($j = \overline{1, k+m}$) of the graph, we consider the Airy equation with a fractional time derivative

$$ {}_C D^{\alpha}_{0,t} u_j(x,t) - \frac{\partial^3}{\partial x^3} u_j(x,t) = f_j(x,t), \quad 0 < t \leqslant T. \tag{7} $$

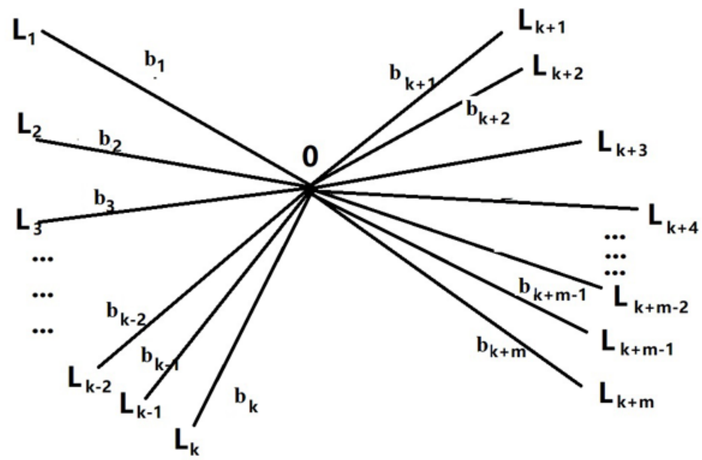

Fig. 1. Star-shaped graph

Let $0 \leqslant t \leqslant T$, and $x \in \overline{b_j}$, $j = \overline{1, k+m}$. We need to impose the following initial conditions

$$u(x, 0) = u_0(x), \tag{8}$$

vertex conditions

$$Au(0, t) = 0, \tag{9}$$

$$\frac{\partial}{\partial x} u^+(0, t) = B \frac{\partial}{\partial x} u^-(0, t), \tag{10}$$

where $u^- = (u_1, u_2, \dots, u_k)^T$, $u^+ = (u_{k+1}, u_{k+2}, \dots, u_{k+m})^T$, $u = \begin{pmatrix} u^+ \\ u^- \end{pmatrix}$,

$$A = \begin{pmatrix} 1 & -a_2 & 0 & \dots & 0 \\ 1 & 0 & a_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & -a_{k+m} \end{pmatrix}$$

and B is the constant m-by-k matrix.

We need impose the following conditions which are sometimes called the Kirchhoff conditions or the condition of conservation of flow rate at the vertex of the graph

$$C^- \frac{\partial^2 u^-(x, t)}{\partial x^2}\Big|_{x=0} = C^+ \frac{\partial^2 u^+(x, t)}{\partial x^2}\Big|_{x=0}, \tag{11}$$

where $C^- = \left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_k}\right)$, $C^+ = \left(\frac{1}{a_{k+1}}, \dots, \frac{1}{a_{k+m}}\right)$, $a_1 = 1$ and $a_j \neq 0$ for $j = \overline{2, k+m}$. Boundary conditions are

$$u(L, t) = \varphi(t), \quad \frac{\partial u^-(x, t)}{\partial x}\Big|_{x=L^-} = \phi(t), \tag{12}$$

where $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_{k+m})^T$ and $\phi = (\phi_1, \phi_2, \dots, \phi_k)^T$.

A regular solution of equation (7) is constructed on the graph defined above that satisfies conditions (8)–(12).

2.1. Uniqueness of solution

Theorem 1. *Let $B^TB - I_k$ be negative defined matrix. Then problem (7), (8)–(12) has at most one solution.*

Proof. Let us consider the following inequality [24]

$$\int_a^b v_C D_{0,t}^\alpha v dx \geqslant \frac{1}{2} {}_C D_{0,t}^\alpha \int_a^b v^2 dx$$

. Using the Cauchy inequality and conditions (8)–(9), we have

$${}_C D_{0,t}^\alpha ||u||_0^2 \leqslant (u^-)^T (B^TB - I_k)(u^-) + 2||u||_0||f||_0 \leqslant 2||u||_0||f||_0 \leqslant ||u||_0^2 + ||f||_0^2,$$

where

$$\|u\|_0^2 = \sum_{j=1}^{k+m} \int_{B_j} u_j^2 dx,$$

$u = (u_1, u_2, \ldots, u_{k+m})$.

Using the analogue of Grönwall's inequality [24], we obtain from the last inequality the following a priori estimate

$$||u||_0^2 \leqslant ||u_0||^2 E_\alpha(2t^\alpha) + \Gamma(\alpha) E_{\alpha,\alpha}(2t^\alpha) {}_C D_{0,t}^\alpha ||f||_0^2. \tag{13}$$

The proof of the theorem follows from (13). □

2.2. Fundamental solutions

We construct the solution of the problem with the use of the potential method. To begin with, we need to obtain a special solution of equation (7) that is called fundamental solution. A fundamental solution of the equation was found in the following form [19]

$$G_\alpha^{2\alpha/3}(x,t) = \frac{1}{3t^{1-2\alpha/3}} \begin{cases} \phi\Big(-\alpha/3, 2\alpha/3; \dfrac{x}{t^{\alpha/3}}\Big), & x < 0, \\ -2\mathrm{Re}\Big[e^{2\pi i/3}\phi\Big(-\alpha/3, 2\alpha/3; e^{2\pi i/3}\dfrac{x}{t^{\alpha/3}}\Big)\Big], & x > 0. \end{cases} \tag{14}$$

Using results from [21], second fundamental solution can be written in the following form

$$V_\alpha^{2\alpha/3}(x,t) = \frac{1}{3t^{1-2\alpha/3}} \mathrm{Im}\Big[e^{2\pi i/3}\phi\Big(-\alpha/3, 2\alpha/3; e^{2\pi i/3}\frac{x}{t^{\alpha/3}}\Big)\Big], \quad x > 0. \tag{15}$$

These functions have the following properties (see [19])

$${}_C D_{0,t}^\nu G_\sigma^\mu(x,t) = G_\sigma^{\mu-\nu}(x,t), \quad \frac{\partial^3}{\partial x^3} G_\sigma^\mu(x,t) = G_\sigma^{\mu-\sigma}(x,t) \tag{16}$$

with estimate

$$|{}_C D_{0,t}^\nu G_\sigma^\mu(x,t)| \leqslant C x^{-\theta} t^{\mu+\theta\sigma/3-1}, \tag{17}$$

where

$$\theta \geqslant \begin{cases} 0, & (-\mu) \notin \mathbf{N_0}, \\ 1, & (-\mu) \in \mathbf{N}_0. \end{cases}$$

Using these functions we define functions that are called potentials

$$w_1(x,t)=\int_0^t G_\alpha^{2\alpha/3}(x-a,t-\eta)\tau_1(\eta)d\eta, \quad w_2(x,t)=\int_0^t V_\alpha^{2\alpha/3}(x-a,t-\eta)\tau_2(\eta)d\eta,$$

$$w_3(x,t)=\int_0^t \frac{\partial^2}{\partial x^2}G_{\alpha^2\alpha/3}(x-a,t-\eta)\tau_3(\eta)d\eta, \quad w_4(x,t)=\int_0^t \frac{\partial^2}{\partial x^2}V_\alpha^{2\alpha/3}(x-a,t-\eta)\tau_4(\eta)d\eta,$$

$$w_5(x,t)=\int_a^b G_\alpha^{2\alpha/3}(x-\xi,t)\tau_5(\xi)d\xi \quad \text{and} \quad w_6(x,t)=\int_0^t\int_a^b G_\alpha^{2\alpha/3}(x-\xi,t-\eta)f(\xi,\eta)d\xi d\eta.$$

Let us show some properties of these functions in the following lemmas.

Lemma 1. *Let functions $\tau_k(t),\ k=1,2$ are continuous and bounded on $(0;+\infty)$. Then*

1. Functions $w_1(x,t)$ and $w_2(x,t)$ are solutions of the equation

$${}_C D_{0,t}^\alpha u_j(x,t)-\frac{\partial^3 u_j(x,t)}{\partial x^3}=0;$$

2. Functions $w_1(x,t)$ and $w_2(x,t)$ satisfy conditions

$$\lim_{t\to 0} w_k(x,t)=0, k=1,2.$$

Lemma 2. *Let $\tau_3(\eta),\tau_4(\eta)\in CVL(0,h)$. Then*

$$\lim_{x\to a-0} w_3(x,t)=\frac{1}{3}\tau_3(t), \quad \lim_{x\to a+0} w_3(x,t)=-\frac{2}{3}\tau_3(t), \quad \lim_{x\to a+0} w_4(x,t)=0.$$

The proofs of these lemmas can be found in [21].

Lemma 3. *Let $\tau_5(x)\in C[a,b]$. Then function $w_5(x,t)$ is the fundamental solution of equation (7) and*

$$\lim_{t\to 0} {}_C D_{0,t}^{\alpha-1} w_5(x,t)=\tau_5(x).$$

Proof. Let us show that function $w_5(x,t)$ is the fundamental solution of equation (7). Using relations (16), we obtain

$${}_C D_{0,t}^\alpha w_5(x,t)=\int_a^b {}_C D_{0,t}^\alpha G_\alpha^{2\alpha/3}(x-\xi,t)\tau_5(\xi)d\xi=\int_a^b G_\alpha^{-\alpha/3}(x-\xi,t)\tau_5(\xi)d\xi$$

and

$$\frac{\partial^3}{\partial x^3}w_5(x,t)=\int_a^b \frac{\partial^3}{\partial x^3}G_\alpha^{2\alpha/3}(x-\xi,t)\tau_5(\xi)d\xi=\int_a^b G_\alpha^{-\alpha/3}(x-\xi,t)\tau_5(\xi)d\xi.$$

Comparing these equalities, we obtain that function $w_5(x,t)$ is the fundamental solution of equation (7).

Let us find

$${}_C D_{0,t}^{\alpha-1} w_5(x,t)=\int_a^b {}_C D_{0,t}^{\alpha-1} G_\alpha^{2\alpha/3}(x-\xi,t)\tau_5(\xi)d\xi=\int_a^b G_\alpha^{1-\alpha/3}(x-\xi,t)\tau_5(\xi)d\xi.$$

Using inequality (17), we have the following estimate

$$|{}_C D_{0,t}^{\alpha-1} w_5(x,t)|=\left|\int_a^b G_\alpha^{1-\alpha/3}(x-\xi,t)\tau_5(\xi)d\xi\right|\leqslant\left|\max_{a\leqslant x\leqslant b}\tau_5(x)\int_a^b C|x-\xi|^{-\theta}t^{(1-\theta)\frac{\alpha}{3}}d\xi\right|,$$

where $1 > \theta \geqslant 0$. It shows that the integral form converges. Replacing $\dfrac{x-\xi}{t^{\alpha/3}}$ with y and taking into account that

$$\int_{-\infty}^{\infty} g_\alpha(y)dy = \int_{-\infty}^{\infty} t^{\alpha/3} G_\alpha^{1-\alpha/3}(yt^{\alpha/3}, t)dy =$$

$$= t^{\alpha/3} \int_{-\infty}^{0} \frac{1}{3t^{\alpha/3}} \phi(-\frac{\alpha}{3}, 1-\frac{\alpha}{3}; y)dy - 2t^{\alpha/3}\mathrm{Re}\left[e^{2\pi i/3} \int_0^\infty \frac{1}{3t^{\alpha/3}} \phi(-\frac{\alpha}{3}, 1-\frac{\alpha}{3}; e^{2\pi i/3}y)dy\right] =$$

$$= \frac{1}{3}\int_{-\infty}^{0} \phi\left(-\frac{\alpha}{3}, 1-\frac{\alpha}{3}; y\right) dy - 2\mathrm{Re}\left[e^{2\pi i/3}\frac{1}{3}\int_0^\infty \phi\left(-\frac{\alpha}{3}, 1-\frac{\alpha}{3}; e^{2\pi i/3}y\right) dy\right] =$$

$$= \frac{1}{3}\left(\frac{1}{\Gamma(1-\alpha/3+\alpha/3)} - 2\mathrm{Re}\left[-e^{2\pi i/3}\frac{1}{e^{2\pi i/3}\Gamma(1-\alpha/3+\alpha/3)}\right]\right) = 1,$$

we obtain

$$\lim_{t\to 0} {}_C D_{0,t}^{\alpha-1} w_5(x,t) = \lim_{t\to 0} \int_a^b G_\alpha^{1-\alpha/3}(x-\xi, t)\tau_5(\xi)d\xi =$$

$$= \lim_{t\to 0} \int_{\frac{x-b}{t^{\alpha/3}}}^{\frac{x-a}{t^{\alpha/3}}} t^{\alpha/3} G_\alpha^{1-\alpha/3}(yt^{\alpha/3}, t)\tau_5(x - t^{\alpha/3}y)dy = \frac{\tau_5(x)}{3} \int_{-\infty}^{+\infty} g_\alpha(y)dy = \tau_5(x).$$

The lemma is proved. □

Lemma 4. *The equation* ${}_C D_{0,t}^{\alpha} u(x,t) - \dfrac{\partial^3}{\partial x^3} u(x,t) = f(x,t)$ *with initial condition*

$${}_C D_{0,t}^{\alpha-1} u(x,t)|_{t=0} = 0$$

has a solution in the form

$$w_6(x,t) = \int_0^t d\eta \int_a^b G_\alpha^{2\alpha/3}(x-\xi, t-\eta) f(\xi,\eta)d\xi.$$

Proof. Using the results given in [19], it is easy to show that solution of the Cauchy problem for the homogeneous equation ${}_C D_{0,t}^{\alpha} v(x,t) - \dfrac{\partial^3}{\partial x^3} v(x,t) = 0$ with initial condition $v(x,0) = v_0(x)$ is

$$v(x,t) = {}_C D_{0,t}^{\alpha-1} \int_a^b G_\alpha^{2\alpha/3}(x-\xi, t) v_0(\xi)d\xi.$$

Let us determine the derivatives of function $w_6(x,t)$

$${}_C D_{0,t}^{\alpha} w_6(x,t) = \frac{d}{dt}\int_0^t d\eta \int_a^b {}_C D_{\eta,t}^{\alpha-1} G_\alpha^{2\alpha/3}(x-\xi, t-\eta) f(\xi,\eta)d\xi =$$

$$= \lim_{\eta\to t} \int_a^b {}_C D_{\eta,t}^{\alpha-1} G_\alpha^{2\alpha/3}(x-\xi, t-\eta) f(\xi,\eta)d\xi + \int_0^t d\eta \int_a^b \frac{d}{dt} {}_C D_{\eta,t}^{\alpha-1} G_\alpha^{2\alpha/3}(x-\xi, t-\eta) f(\xi,\eta)d\xi.$$

Taking into account (3) and relation (16), we obtain

$$\int_a^b {}_C D_{\eta,t}^{\alpha-1} G_\alpha^{2\alpha/3}(x-\xi, t-\eta) f(\xi,\eta)d\xi = \int_a^b {}_C D_{0,t-\eta}^{\alpha-1} G_\alpha^{2\alpha/3}(x-\xi, t-\eta) f(\xi,\eta)d\xi =$$

$$= \int_a^b G_\alpha^{1-\alpha/3}(x-\xi, t-\eta) f(\xi,\eta)d\xi.$$

It follows from relation (5) that integral I_1 converges uniformly. Substituting $\frac{x-\xi}{(t-\eta)^{\alpha/3}}$ for y in this integral and taking into account (6), we obtain

$$I_1 = \lim_{\eta\to t}\int_a^b {}_C D_{\eta,t}^{\alpha-1} G_\alpha^{2\alpha/3}(x-\xi,t-\eta)f(\xi,\eta)d\xi = \lim_{\eta\to t}\int_a^b G_\alpha^{1-\alpha/3}(x-\xi,t-\eta)f(\xi,\eta)d\xi =$$

$$= \lim_{\eta\to t}\int\limits_{\frac{x-b}{(t-\eta)^{\alpha/3}}}^{\frac{x-a}{(t-\eta)^{\alpha/3}}} G_\alpha^{1-\alpha/3}((t-\eta)^{\alpha/3}y,t-\eta)f(x-(t-\eta)^{\alpha/3}y,\eta)(t-\eta)^{\alpha/3}dy =$$

$$= \lim_{\eta\to t}\int\limits_0^{\frac{x-a}{(t-\eta)^{\alpha/3}}} \frac{-2f(x-(t-\eta)^{\alpha/3}y,\eta)(t-\eta)^{\alpha/3}}{3(t-\eta)^{\alpha/3}}\mathrm{Re}\left(e^{2\pi i/3}\phi(-\frac{\alpha}{3},1-\frac{\alpha}{3};e^{2\pi i/3}y)\right)dy+$$

$$+\lim_{\eta\to t}\int\limits_{\frac{x-b}{(t-\eta)^{\alpha/3}}}^0 \frac{1}{3(t-\eta)^{\alpha/3}}\phi\left(-\frac{\alpha}{3},1-\frac{\alpha}{3};y\right)f(x-(t-\eta)^{\alpha/3}y,\eta)(t-\eta)^{\alpha/3}dy =$$

$$= -\frac{2}{3}\lim_{\eta\to t}\int\limits_0^{\frac{x-a}{(t-\eta)^{\alpha/3}}} \mathrm{Re}\left(e^{2\pi i/3}\phi\left(-\frac{\alpha}{3},1-\frac{\alpha}{3};e^{2\pi i/3}y\right)\right)f(x-(t-\eta)^{\alpha/3}y,\eta)dy+$$

$$+\frac{1}{3}\lim_{\eta\to t}\int\limits_{\frac{x-b}{(t-\eta)^{\alpha/3}}}^0 \phi\left(-\frac{\alpha}{3},1-\frac{\alpha}{3};y\right)f(x-(t-\eta)^{\alpha/3}y,\eta)dy =$$

$$= -\frac{2}{3}\mathrm{Re}\left(\int_0^{+\infty} e^{2\pi i/3}\phi\left(-\frac{\alpha}{3},1-\frac{\alpha}{3};e^{2\pi i/3}y\right)f(x,t)dy\right)+\frac{1}{3}\int_{-\infty}^0\phi\left(-\frac{\alpha}{3},1-\frac{\alpha}{3};y\right)f(x,t)dy =$$

$$= -\frac{2}{3}\mathrm{Re}\left(-e^{2\pi i/3}\frac{1}{e^{2\pi i/3}\Gamma(1)}\right)f(x,t)+\frac{1}{3}\left(-\frac{1}{-\Gamma(1)}\right)f(x,t) = f(x,t).$$

Now we have $I_1 = f(x,t)$. Furthermore we show that $I_2 = \frac{\partial^3}{\partial x^3}u(x,t)$. We begin with

$$\begin{aligned} I_2 &= \int_0^t d\eta\int_a^b \frac{d}{dt}{}_C D_{\eta,t}^{\alpha-1}G_\alpha^{2\alpha/3}(x-\xi,t-\eta)f(\xi,\eta)d\xi = \\ &= \int_0^t d\eta\int_a^b \frac{d}{dt}{}_C D_{0,t-\eta}^{\alpha-1}G_\alpha^{2\alpha/3}(x-\xi,t-\eta)f(\xi,\eta)d\xi = \\ &= \int_0^t d\eta\int_a^b \frac{d}{dt}G_\alpha^{1-\alpha/3}(x-\xi,t-\eta)f(\xi,\eta)d\xi. \end{aligned} \tag{18}$$

To determine $\frac{\partial^3}{\partial x^3}u(x,t)$ we use relation (16). So, we have

$$\begin{aligned} \frac{\partial^3}{\partial x^3}u(x,t) &= \frac{\partial^3}{\partial x^3}\int_0^t d\eta\int_a^b G_\alpha^{2\alpha/3}(x-\xi,t-\eta)f(\xi,\eta)d\xi = \\ = \int_0^t d\eta\int_a^b \frac{\partial^3}{\partial x^3}G_\alpha^{2\alpha/3}(x-\xi,t-\eta)f(\xi,\eta)d\xi &= \int_0^t d\eta\int_a^b G_\alpha^{2\alpha/3-\alpha}(x-\xi,t-\eta)f(\xi,\eta)d\xi = \\ &= \int_0^t d\eta\int_a^b \frac{\partial^3}{\partial x^3}G_\alpha^{-\alpha/3}(x-\xi,t-\eta)f(\xi,\eta)d\xi. \end{aligned} \tag{19}$$

Comparing (18) and (19), we obtain $I_2 = \dfrac{\partial^3}{\partial x^3}u(x,t)$. The Lemma is proved. □

2.3. Existence of solutions

Let

$$\begin{aligned} F^- &= (F_1, \dots, F_k)^T, \quad & F^+ &= (F_{k+1}, \dots, F_{k+m})^T, \\ \alpha^- &= (\alpha_1, \dots, \alpha_k)^T, \quad & \alpha^+ &= (\alpha_{k+1}, \dots, \alpha_{k+m})^T, \\ \beta^- &= (\beta_1, \dots, \beta_k)^T, \quad & \beta^+ &= (\beta_{k+1}, \dots, \beta_{k+m})^T, \\ \gamma^- &= (\gamma_1, \dots, \gamma_k)^T, \quad & \gamma^+ &= (\gamma_{k+1}, \dots, \gamma_{k+m})^T, \\ \rho^- &= (\rho_1, \rho_2, \dots, \rho_k)^T, \quad & \rho^+ &= (\rho_{k+1}, \dots, \rho_{k+m})^T, \end{aligned}$$

$$\alpha = \begin{pmatrix} \alpha^- \\ \alpha^+ \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta^- \\ \beta^+ \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma^- \\ \gamma^+ \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho^- \\ \rho^- \end{pmatrix}, \quad F = \begin{pmatrix} F^- \\ F^+ \end{pmatrix}$$

and $b_j = \begin{cases} (L_j; 0), & j = \overline{1,k} \\ (0; L_j), & j = \overline{k+1, k+m} \end{cases}$.

Let us find solutions in the form

$$u_j(x,t) = \int_0^t G_\alpha^{2\alpha/3}(x - L_j, t-\tau)\,\alpha_j(\tau)d\tau + \int_0^t V_\alpha^{2\alpha/3}(x - L_j, t-\tau)\,\beta_j(\tau)d\tau +$$

$$+\int_0^t G_\alpha^{2\alpha/3}(x - 0, t-\tau)\,\gamma_j(\tau)d\tau + \int_0^t V_\alpha^{2\alpha/3}(x - 0, t-\tau)\,\rho_j(\tau)d\tau + F_j(x,t), \;\; j = \overline{1, k+m},$$

where functions α_j, γ_j $(j = \overline{1,k+m})$, β_j $(j = \overline{1,k})$, ρ_j $(j = \overline{k+1,k+m})$ are unknown functions, $\rho_j(t) = 0$, $(j = \overline{1,k})$; $\beta_i(t) = 0$, $i = \overline{k+1,k+m}$ and

$$F_j(x,t) = \int_{b_j} u_{0,j}(\xi)\,{}_C D_{0,t}^{\alpha-1} G_\alpha^{2\alpha/3}(x-\xi, t-0)d\xi + \int_0^t \int_{b_j} G_\alpha^{2\alpha/3}(x-\xi, t-0) f_j(\xi,\tau)d\xi d\tau.$$

It follows from Lemma 4 and the results given in [19] that these functions are the solutions of equation (7) and they satisfy initial conditions (8).

Taking into account condition (9), we have

$$a_j \int_0^t G_\alpha^{2\alpha/3}(-L_j, t-\tau)\alpha_j(\tau)d\tau + a_j \int_0^t V_\alpha^{2\alpha/3}(-L_j, t-\tau)\beta_j(\tau)d\tau +$$

$$+ a_j \int_0^t G_\alpha^{2\alpha/3}(0, t-\tau)\,\gamma_j(\tau)d\tau + a_j \int_0^t V_\alpha^{2\alpha/3}(0, t-\tau)\,\rho_j(\tau)d\tau + a_j F_j(0,t) =$$

$$= \int_0^t G_\alpha^{2\alpha/3}(-L_1, t-\tau)\,\alpha_1(\tau)d\tau + \int_0^t V_\alpha^{2\alpha/3}(-L_1, t-\tau)\,\beta_1(\tau)d\tau +$$

$$+ \int_0^t G_\alpha^{2\alpha/3}(0, t-\tau)\,\gamma_1(\tau)d\tau + F_1(0,t), \quad j = \overline{2, k+m}.$$

Furthermore

$$\int_0^t \Big(G_\alpha^{2\alpha/3}(-L_1, t-\tau)\,\alpha_1(\tau) + V_\alpha^{2\alpha/3}(-L_1, t-\tau)\,\beta_1(\tau) \Big) d\tau +$$

$$+\int_0^t \frac{\phi(-\frac{\alpha}{3},\frac{2\alpha}{3};0)}{3(t-\tau)^{1-2\alpha/3}}\gamma_1(\tau)d\tau + F_1(0,t) =$$

$$= a_j \int_0^t G_\alpha^{2\alpha/3}(-L_j, t-\tau)\,\alpha_j(\tau)d\tau + a_j \int_0^t V_\alpha^{2\alpha/3}(-L_j, t-\tau)\,\beta_j(\tau)d\tau +$$

$$+ a_j \int_0^t \frac{\phi(-\frac{\alpha}{3},\frac{2\alpha}{3};0)}{3(t-\tau)^{1-2\alpha/3}}\gamma_j(\tau)d\tau + \mathrm{Im}\left[a_j \int_0^t \frac{e^{2\pi i/3}\phi(-\frac{\alpha}{3},\frac{2\alpha}{3};0)}{3(t-\tau)^{1-2\alpha/3}}\rho_j(\tau)d\tau\right] + a_j F_j(0,t).$$

So, we have

$$a_j F_j(0,t) - F_1(0,t) = \int_0^t \frac{\frac{\sqrt{3}a_j}{2}\rho_j(\tau) - a_j\gamma_j(\tau) + \gamma_1(\tau)}{3\Gamma\left(\frac{2\alpha}{3}\right)(t-\tau)^{1-2\alpha/3}}d\tau -$$

$$-a_j \int_0^t G_\alpha^{2\alpha/3}(-L_j, t-\tau)\,\alpha_j(\tau)d\tau - a_j \int_0^t V_\alpha^{2\alpha/3}(-L_j, t-\tau)\,\beta_j(\tau)d\tau +$$

$$+\int_0^t G_\alpha^{2\alpha/3}(-L_1, t-\tau)\,\alpha_1(\tau)d\tau + \int_0^t V_\alpha^{2\alpha/3}(-L_1, t-\tau)\,\beta_1(\tau)d\tau.$$

and

$$\gamma_1(\tau) - a_j\gamma_j(\tau) + \frac{\sqrt{3}a_j}{2}\rho_j(\tau) = 3\,{}_C D_{0,t}^{2\alpha/3}\left(a_j F_j(0,t) - F_1(0,t)\right) -$$

$$-3a_j\,{}_C D_{0,t}^{2\alpha/3}\left(\int_0^t G_\alpha^{2\alpha/3}(-L_j, t-\tau)\,\alpha_j(\tau)d\tau + \int_0^t V_\alpha^{2\alpha/3}(-L_j, t-\tau)\,\beta_j(\tau)d\tau\right) -$$

$$-3\,{}_C D_{0,t}^{2\alpha/3}\left(\int_0^t G_\alpha^{2\alpha/3}(-L_1, t-\tau)\,\alpha_1(\tau)d\tau + \int_0^t V_\alpha^{2\alpha/3}(-L_1, t-\tau)\,\beta_1(\tau)d\tau\right).$$

From above relation we obtain

$$\gamma_1(\tau) - a_j\gamma_j(\tau) + \frac{\sqrt{3}a_j}{2}\rho_j(\tau) = 3\,{}_C D_{0,t}^{2\alpha/3}\left(a_j F_j(0,t) - F_1(0,t)\right) -$$

$$-3\left(\int_0^t G_\alpha^{0}(-L_1, t-\tau)\,\alpha_1(\tau)d\tau + \int_0^t V_\alpha^{0}(-L_1, t-\tau)\,\beta_1(\tau)d\tau\right) +$$

$$+3a_j\left(\int_0^t G_\alpha^{0}(-L_j, t-\tau)\,\alpha_j(\tau)d\tau + \int_0^t V_\alpha^{0}(-L_j, t-\tau)\,\beta_j(\tau)d\tau\right), \quad j = \overline{2, k+m}. \tag{20}$$

In a similar manner, we obtain from condition (10) that

$$B\gamma^-(t) - \gamma^+(t) + \frac{\sqrt{3}}{2}\rho^+(t) = 3\,{}_C D_{0,t}^{\alpha/3}\left(F_x^+(0,t) - BF_x^-(0,t)\right) -$$
$$-3\int_0^t \left(BG_\alpha^{\alpha/3}\left(L^-, t-\tau\right)\alpha^-(\tau) + BV_\alpha^{\alpha/3}\left(-L^-, t-\tau\right)\beta^-(\tau)\right)d\tau + \tag{21}$$
$$+3\int_0^t \left(G_\alpha^{\alpha/3}\left(-L^+, t-\tau\right)\alpha^+(\tau)\right)d\tau.$$

Taking into account condition (11) and using Lemmas given above, we have

$$C^-\gamma^-(t) + 2C^+\gamma^+(t) = 3C\lim_{x\to 0}\frac{\partial^2}{\partial x^2}\int_0^t G_\alpha^{2\alpha/3}(x-L, t-\tau)\alpha(\tau)d\tau +$$
$$+3C\lim_{x\to 0}\frac{\partial^2}{\partial x^2}\int_0^t V_\alpha^{2\alpha/3}(x-L, t-\tau)\beta(\tau)d\tau + 3CF_{xx}(0,t), \tag{22}$$

where $C = (-C^-, C^+)$.

Using conditions (12), we have

$$J_{0,t}^{2\alpha/3}\left(\alpha_j(t) + \frac{\sqrt{3}}{2}\beta_j(t)\right) + \int_0^t G_\alpha^{2\alpha/3}\left(L_j, t-\tau\right)\gamma_j(\tau)d\tau +$$
$$+\int_0^t V_\alpha^{2\alpha/3}\left(L_j, t-\tau\right)\rho_j(\tau)d\tau + F_j\left(L_j, t\right) = \varphi_j\left(t\right), \quad j = \overline{1, k+m}.$$

Applying the properties of fractional operators, we obtain

$$\alpha_j(t) + \frac{\sqrt{3}}{2}\beta_j(t) = {}_C D_{0,t}^{2\alpha/3}\left(\varphi_j\left(t\right) - F_j\left(L_j, t\right)\right) -$$
$$-{}_C D_{0,t}^{2\alpha/3}\left(\int_0^t G_\alpha^{2\alpha/3}\left(L_j, t-\tau\right)\gamma_j(\tau)d\tau + \int_0^t V_\alpha^{2\alpha/3}\left(L_j, t-\tau\right)\rho_j(\tau)d\tau\right), \quad j = \overline{1, k+m}.$$

Equations given above can be written in the following form

$$\begin{aligned}\alpha(t) + \frac{\sqrt{3}}{2}\beta(t) = -\int_0^t G_\alpha^0(L, t-\tau)\gamma(\tau)d\tau - \int_0^t V_\alpha^0(L, t-\tau)\rho(\tau)d\tau + \\ +{}_C D_{0,t}^{2\alpha/3}\left(\varphi(t) - F(L,t)\right).\end{aligned} \tag{23}$$

In a similar manner, we have from condition (12) that

$$\begin{aligned}\alpha^-(t) - \frac{\sqrt{3}}{2}\beta^-(t) = \int_0^t G_\alpha^0(L^-, t-\tau)\gamma^-(\tau)d\tau + \int_0^t V_\alpha^0(L^-, t-\tau)\rho^-(\tau)d\tau + \\ +{}_C D_{0,t}^{\alpha/3}\left(\phi(t) - F_x^-(L^-, t)\right).\end{aligned} \tag{24}$$

We obtain the following system of integral equations (20)–(24) with respect to unknowns $\Lambda(t)$

$$Q\Lambda(t) + \int_0^t K(t-\tau)\Lambda(\tau)d\tau = H, \tag{25}$$

where Λ is the unknown functions, Q is the $(3k+3m)$-by-$(3k+3m)$ matrix, K is the matrix of potentials. Using above system, the matrices can be written in the form

$$H = \begin{pmatrix} -3A_C D_{0,t}^{2\alpha/3}F(0,t) \\ 3_C D_{0,t}^{\alpha/3}\left(F_x^+(0,t) - BF_x^-(0,t)\right) \\ 3CF_{xx}(0,t) \\ {}_C D_{0,t}^{2\alpha/3}\left(\varphi(t) - F(L,t)\right) \\ {}_C D_{0,t}^{\alpha/3}\left(\phi(t) - F_x^-(L^-,t)\right) \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \rho \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & M \\ Q_1 & 0 \end{pmatrix},$$

where M is the matrix on the form

$$M = \begin{pmatrix} 1_{(k-1)\times 1} & -diag(a_2, \ldots, a_k) & 0 \\ 1_{(m)\times 1} & -diag(a_{k+1}, \ldots, a_{k+m}) & \frac{\sqrt{3}}{2}diag(a_{k+1}, \ldots, a_{k+m}) \\ B & -I_m & \frac{\sqrt{3}}{2}I_m \\ \frac{1}{a_1}\cdots\frac{1}{a_k} & \frac{2}{a_{k+1}}\cdots\frac{2}{a_{k+m}} & 0 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} I_k & 0 & \frac{\sqrt{3}}{2} I_k \\ 0 & I_m & 0 \\ I_k & 0 & -\frac{\sqrt{3}}{2} I_k \end{pmatrix} \text{ and } K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \text{ where}$$

$$K_1 = 3 \begin{pmatrix} -AG_\alpha^0(-L) & -AV_\alpha^0(-L) \\ -BG_\alpha^0(L^-) \,|\, G_\alpha^0(-L^+) & V_\alpha^0(-L^-) \\ C \lim_{x\to 0} \frac{\partial^2}{\partial x^2} G_\alpha^{2\alpha/3}(x-L) & -C^- \lim_{x\to 0} \frac{\partial^2}{\partial x^2} V_\alpha^{2\alpha/3}(x-L) \end{pmatrix},$$

$$K_2 = \begin{pmatrix} -G_\alpha^0(L) & -V_\alpha^0(L) \\ -G_\alpha^0(L^-) \,|\, 0 & V_\alpha^0(L^+) \end{pmatrix}.$$

It is obvious that $det(Q) \neq 0$ and elements of matrix $K(t,\tau)$ are bounded and continuous functions on $(0,T)$. It was proved that $det M \neq 0$ [21]. So, matrix integral equation (25) has unique solution in $(C[0,t])^{2k+m}$.

So, we arrive at the following theorem.

Theorem 2. *Let* $B^T B - I_k$ be negative defined matrix, functions $u_{j,0}(x) \in C(\bar{b}_j)$, $f_j(x,t) \in C^{0,1}(\bar{b}_j \times [0,T])$ *for* $j = \overline{1, k+m}$, $\varphi(t)$ *and* $\phi(t)$ *are differentiable functions on* $[0,T]$. *Then problem (7)–(12) has unique solution on* $0 \leqslant t \leqslant T$.

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Уравнение Эйри с дробной производной по времени на метрическом графе

Камоладдин Рахимов
Национальный университет Узбекистана
Ташкент, Узбекистан
Зарифбой Собиров
Университет геологических наук
Ташкент, Узбекистан
Насридин Жабборов
Национальный университет Узбекистана
Ташкент, Узбекистан

Аннотация. Мы рассматриваем задачу Коши и начально-краевую задачу для уравнении Эйри с дробной производной по времени на метрическом графе с ограниченными и с неограниченными ветвями. Мы изучали свойства потенциалов для этого уравнения и, используя эти свойства, нашли решения рассматриваемой задачи. Теорема единственности была доказана с помощью аналога неравенства Гронуолла–Беллмана и априорной оценки.

Ключевые слова: уравнение Эйри с дробной производной по времени, начально-краевая задача, уравнения в частных производных на метрическом графе, фундаментальные решения, интегральное представление.

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Delta-extremal Functions in $\mathbb{C}^n$

Nurbek Kh. Narzillaev*
National University of Uzbekistan
Tashkent, Uzbekistan

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Abstract. The article is devoted to properties of a weighted Green function. We study the (δ, ψ)-extremal Green function $V^*_\delta(z, K, \psi)$ defined by the class $\mathcal{L}_\delta = \{u(z) \in psh(\mathbb{C}^n) : \ u(z) \leqslant C_u + \delta \ln^+ |z|, \ z \in \mathbb{C}^n\}, \ \delta > 0$. We see that the notion of regularity of points with respect to different numbers δ differ from each other. Nevertheless, we prove that if a compact set $K \subset \mathbb{C}^n$ is regular, then δ-extremal function is continuous in the whole space $\mathbb{C}^n$.

Keywords: plurisubharmonic function, Green function, weighted Green function, δ-extremal function.

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1. Introduction and preliminaries

The Green function in the multidimensional complex space $\mathbb{C}^n$ is one of the main objects for the study of analytic and plurisubharmonic (psh) functions. The Green function was introduced and applied in the works of P. Lelong, J. Sichak, V. Zaharyuta, A. Zeriahi, A. Sadullaev and others (see [1–7]). Recall that a function $u(z) \in psh(\mathbb{C}^n)$ is said to be of logarithmic growth if there is a constant C_u such that

$$u(z) \leqslant C_u + \ln^+ |z|, \quad z \in \mathbb{C}^n,$$

where $\ln^+ |z| = \max\{\ln |z|, 0\}$. The family of all such functions is called the Lelong class and denoted by $\mathcal{L}$. We also introduce a class $\mathcal{L}^+$ as follows:

$$\mathcal{L}^+ := \{u(z) \in psh(\mathbb{C}^n), \quad c_u + \ln^+ |z| \leqslant u(z) \leqslant C_u + \ln^+ |z|\}.$$

For a fixed compact set $K \subset \mathbb{C}^n$ we put

$$V(z, K) = \sup\{u(z) : u(z) \in \mathcal{L}, u(z)|_K \leqslant 0\}.$$

Then the regularization of

$$V^*(z, K) = \overline{\lim_{w \to z}} V(w, K)$$

is called *the Green function* of the compact set K. For a non-pluripolar compact set K, the function $V^*(z, K)$ exists ($V^*(z, K) \not\equiv +\infty$) and belongs to the class $\mathcal{L}^+$. The Green function $V^*(z, K) \equiv +\infty$ if and only if K is pluripolar.

*n.narzillaev@nuu.uz https://orcid.org/0000-0002-3175-5516

Definition 1. *A compact set $K \subset \mathbb{C}^n$ is called globally pluri-regular at a point z_0 if $V^*(z^0, K) = 0$. It is called locally pluri-regular at a point z_0 if $V^*(z^0, K \cap B(z^0, r)) = 0$ for any ball $B(z^0, r))$, $r > 0$. A compact set K is globally pluri-regular if it is globally pluri-regular at every point of itself. A compact set K is locally pluri-regular if it is locally pluri-regular at every point of itself.*

Theorem 1.1 (see for example, J. Siciak [4], V. Zakharyuta [3])**.** *If a compact set K is globally pluriregular, then the function $V^*(z, K)$ is continuous in $\mathbb{C}^n$, and $V^*(z, K) = V(z, K)$.*

2. Weighted Green functions in $\mathbb{C}^n$

Let $\psi(z)$ be a bounded function on a compact set $K \subset \mathbb{C}^n$. Consider the class of functions

$$\mathcal{L}(K, \psi) := \{u(z) \in \mathcal{L},\ u(z)|_K \leqslant \psi(z)\}$$

and

$$V(z, K, \psi) := \sup\{u(z) : u(z) \in \mathcal{L}(K, \psi)\},\ z \in \mathbb{C}^n.$$

Then $V^*(z, K, \psi) = \overline{\lim_{w \to z}} V(w, K, \psi)$ is said to be a weighted Green function of K with respect to $\psi(z)$. Note that in the case $\psi(z) \equiv 0$ the function $V^*(z, K, \psi)$ coincides with the Green function $V^*(z, K)$, i.e., $V^*(z, K, 0) \equiv V^*(z, K, \psi)$. Extremal weighted Green functions are the subject of study by many authors (see [7, 10–13]). They are successfully applied in multidimensional complex analysis, in the approximation theory of functions, in multidimensional complex dynamical systems etc.

It is clear that for any compact set $K \subset \mathbb{C}^n$ we have the inequality

$$V^*(z, K) + \min_K \psi(z) \leqslant V^*(z, K, \psi) \leqslant V^*(z, K) + \max_K \psi(z). \tag{1}$$

If a function $\psi(z)$ extends to the space $\mathbb{C}^n$ as a function from the class $\mathcal{L}$, i.e. if there is a function

$$\Psi \in \mathcal{L} :\ \Psi|_K \equiv \psi, \tag{2}$$

then it is obvious $V(z, K, \psi) \geqslant \Psi(z)$ and

$$V(z, K, \psi) = \psi(z)\ \ \forall z \in K. \tag{3}$$

However, if the condition (2) is not met, then generally speaking, the equality (3) is not true.

Example 1. Let $K = \{|z| \leqslant 1\} \subset \mathbb{C}$ and $\psi(z) = 1 - |z|^2$. Then by the maximum principle

$$V(z, K, \psi) = V(z, K) = V(z, K) = \ln^+ |z|.$$

Therefore, $V(z, K, \psi) = 0 < \psi(z)\ \ \forall\ |\ z\ |< 1$.

According to this example, in order to introduce the concept of regularity, below we assume that the Green function satisfies the condition (3).

Definition 2. *We say that a compact set K is globally ψ-regular at z^0 if $V^*(z^0, K, \psi) = \psi(z^0)$. We say that a compact set K is locally ψ-regular at z^0 if $V^*(z^0, K \cap B(z^0, r), \psi) = \psi(z^0)$ for every ball $B(z^0, r)$, $r > 0$.*

A. Sadullaev [7] proved the following theorem.

Theorem 2.1. *Let K be a compact set, and $\psi(z)$ is a weight on K such that there exists a strictly plurisubharmonic function*

$$\Psi \in \mathcal{L} \cap C^2(\mathbb{C}^n): \quad dd^c\Psi > 0, \ \Psi|_K = \psi. \tag{4}$$

Then K is locally ψ-regular at $z^0 \in K$ if and only if K is globally ψ-regular at z^0.

Note that Theorem 2.1, generally speaking, is not true if Ψ is not a strictly plurisubharmonic function. For the weight function $\psi(z) \equiv 0$ and for the compact set $K = \{|z| = 1\} \cup \{z = 0\} \subset \mathbb{C}$ the point $z = 0$ is globally regular, but it is not locally regular. In this example K is not polynomially convex $\hat{K} \neq K$. In the work [5] A. Sadullaev constructed the following interesting example.

Example 2. The compact set $K = K_1 \cup K_2 \subset \mathbb{C}^2(z_1, z_2)$, where $K_1 = \{|z_1| < 1, z_2 = 0\}$, $K_2 = \{z_1 = e^{i\varphi}, Rez_2 = 0, \ 0 \leqslant Imz_2 \leqslant e^{\frac{1}{\cos\varphi - 1}}, -\pi \leqslant \varphi \leqslant \pi\}$, has the following properties:

a) K is polynomially convex, i.e., $\hat{K} = K$;

b) K is globally pluri-regular, i.e., $V^*(z, K) = 0, \forall z \in K$;

c) K is not locally pluri-regular at the points $z \in K_1$.

In connection with this example and with Theorem 2.1, the following problem arises (see [7]).

Problem 1. *Let K be a compact set in $\mathbb{C}^n$. Under a weaker condition that the weight function $\psi(z)$ continues only to a neighbourhood $U \supset K$ as a strictly plurisubharmonic function, prove that K is locally ψ-regular at $z_0 \in K$ if and only if K is globally ψ-regular at $z_0 \in K$.*

The following theorem relates to local regularity for different weight functions.

Theorem 2.2. *Let K be a compact set, and $\psi(z)$ is a weight on $K : \psi(z) \in C(K)$. Then K is locally ψ-regular at $z^0 \in K$ if and only if K is locally regular (case $\psi \equiv 0$) at z^0.*

Proof. Indeed, we use the inequality (1). If the point $z^0 \in K$ is not locally pluri-regular, i.e., if $V^*(z^0, K \cap \overline{B}) = \sigma > 0$ for some neighborhood $B: \quad z^0 \in B \subset \mathbb{C}^n$, then $V^*(z^0, K \cap \overline{B}_1) \geqslant \sigma$ for any $z^0 \in B_1 \subset B$. Therefore, by (1)

$$V^*(z^0, K \cap B_1, \psi) \geqslant V^*(z^0, K \cap B_1) + \min_{K \cap B_1} \psi(z) \geqslant \sigma + \min_{K \cap B_1} \psi(z). \tag{5}$$

Since $\psi(z)$ is continuous, choosing the neighborhood B_1 small enough we can make the right part of (5) to be greater than $\psi(z^0)$ i.e., $V^*(z, K \cap B_1, \psi) > \psi(z^0)$ and the point z^0 is not locally ψ-regular.

Reversing the roles of $V^*(z, K \cap B_1, \psi)$ and $V^*(z, K \cap B_1)$ from (1) we can prove the second part of the theorem: if the point $z^0 \in K$ is not locally ψ-regular, then it is not locally pluri-regular. □

It should be noted here that the conditions of continuity of the function $\psi(z)$ in Theorem 2.2 is essential. An example is given in [15], when the function $\psi(z)$ is discontinuous, Theorem 2.2 is false, i.e., some point $z^0 \in K \subset \mathbb{C}$ is a ψ-regular point, but it is not pluri-regular.

3. δ-extremal functions

Let $K \subset \mathbb{C}^n$ be a compact set and $\psi(z)$ be some bounded function on K. Consider the following generalization of the Lelong class

$$\mathcal{L}_\delta := \left\{u(z) \in psh(\mathbb{C}^n) \colon u(z) \leqslant C_u + \delta \ln^+ |z|,\ z \in \mathbb{C}^n\right\},\ \delta > 0.$$

It is clear that if $v(z) \in \mathcal{L}$, then $c \cdot v(z) \in \mathcal{L}_\delta$, where $0 < c \leqslant \delta$. Put

$$\mathcal{L}_\delta(K, \psi) := \{u(z) \in \mathcal{L}_\delta,\ u(z)|_K \leqslant \psi(z)\}.$$

Definition 3. *The function* $V_\delta^*(z, K, \psi) = \overline{\lim_{w\to z}} V_\delta(w, K, \psi)$ *is called a* δ*-extremal function of* K *with respect to* $\psi(z)$*, where*

$$V_\delta(z, K, \psi) := \sup\{u(z) : u(z) \in \mathcal{L}_\delta(K, \psi)\}, \quad z \in \mathbb{C}^n.$$

We list simple properties of δ-extremal functions:

1°. If $\delta_1 \leqslant \delta_2$, then $V_{\delta_1}(z, K, \psi) \leqslant V_{\delta_2}(z, K, \psi)$.

2°. If $\psi_1 \leqslant \psi_2, \forall z \in K$, then $V_\delta(z, K, \psi_1) \leqslant V_\delta(z, K, \psi_2)$.

3°. $V_\delta(z, K, \psi) = \delta V\left(z, K, \frac{\psi}{\delta}\right)$, in particular $V_\delta(z, K) = \delta V(z, K)$.

4°. $V_\delta(z, K, \psi + c) = c + V_\delta(z, K, \psi)$, $\forall c \in \mathbb{R}$.

If a function $\psi(z)$ extends to the space $\mathbb{C}^n$ as a function from the class $\mathcal{L}_\delta$, i.e. if there is a function

$$\Psi \in \mathcal{L}_\delta :\ \Psi|_K \equiv \psi, \tag{6}$$

then it is obvious $V_\delta(z, K, \psi) \geqslant \Psi(z)$ and

$$V_\delta(z, K, \psi) = \psi(z) \quad \forall z \in K. \tag{7}$$

However, if the condition (6) is not met, then generally speaking, the equality (7) is not true. In this section, as above we assume that the Green function $V_\delta(z, K, \psi)$ satisfies the condition (7). For such a function ψ we can introduce the concept of (δ, ψ)-regularity.

Definition 4. *We say that a compact set* K *is globally* (δ, ψ)*-regular at* z^0 *if* $V_\delta^*(z^0, K, \psi) = \psi(z^0)$. *We say that a compact set* K *is locally* (δ, ψ)*-regular at* z^0 *if* $V_\delta^*(z^0, K \cap B(z^0, r), \psi) = \psi(z^0)$ *for any ball* $B(z^0, r), r > 0$.

The following theorem is proved similarly to the proof of Theorem 2.2 and we omit it.

Theorem 3.1. *Let* K *be a compact set and* $\psi(z)$ *is a weight on* $K : \psi(z) \in C(K)$, $V_\delta(z, K, \psi) = \psi(z)$ $\forall z \in K$. *Then* K *is locally* (δ, ψ)*-regular at* $z^0 \in K$ *if and only if* K *is locally* $(\delta, 0)$*-regular at* z^0.

Similarly to Theorem 1.1 the continuity of the δ-extremal function takes place.

Theorem 3.2. *Let* $\psi(z)$ *be continuous on* K. *If* K *is globally* (δ, ψ)*-regular i.e. if* K *is globally* (δ, ψ)*-regular at a point* $z^0 \in K$, *then* $V_\delta^*(z, K, \psi) = V_\delta(z, K, \psi)$ *and* $V_\delta^*(z, K, \psi)$ *is continuous in* $\mathbb{C}^n$.

Proof. Let $\psi(z)$ be a function defined and continuous on K. It is well known that $\psi(z)$ can be extended continuously to K, i.e., there is a function $\Psi(z) \in C(\mathbb{C}^n)$ such that $\Psi(z)|_K = \psi(z)$ (see Whitney H. [8]). We use the standard approximation $u_j \downarrow V_\delta^*(z, K, \psi)$, where $u_j \in \mathcal{L}_\delta \cap C^\infty(\mathbb{C}^n)$. Since $V_\delta^*(z, K, \psi) \equiv \Psi(z),\ z \in K$, for any $\varepsilon > 0$ there is an open set $\{z \in \mathbb{C}^n,\ V_\delta^*(z, K, \psi) < \Psi(z) + \varepsilon\}$ contained K. Therefore, by the Hartogs lemma, there exists $j_0 \in \mathbb{N}$ such that $u_j(z) < \Psi(z) + 2\varepsilon = \psi(z) + 2\varepsilon,\ \forall z \in K,\ j > j_0$. From here, $u_j - 2\varepsilon \in \mathcal{L}_\delta(\psi, K)$ and

$$u_j - 2\varepsilon \leqslant V_\delta(z, K, \psi) \leqslant V_\delta^*(z, K, \psi) \leqslant u_j, \quad j > j_0, \quad z \in \mathbb{C}^n.$$

This means that the sequence u_j converges to $V_\delta^*(z, K, \psi)$ uniformly and $V_\delta^*(z, K, \psi) = V_\delta(z, K, \psi) \in C(\mathbb{C}^n)$. □

In the case when $\delta = 1$ and $\psi(z)$ continues throughout $\mathbb{C}^n$ as a continuous function of the class $\mathcal{L}$, Theorem 3.2 was proved by A. Sadullaev.

4. δ-extremal functions for different δ

Note that in the general case $V_\delta(z, K, \psi)$ and the weight function ψ do not have to be equal on K for all δ. In other words, the condition (7) may not be satisfied.

Example 3 (see Alan [10])**.** Let $K = \overline{B(0,1)}$ and $\psi(z) = |z|^2$. Then one can prove that

$$V_\delta(z, K, \psi) = \begin{cases} |z|^2, & |z| \leqslant \sqrt{\dfrac{\delta}{2}}, \\ \delta \ln |z| + \dfrac{\delta}{2} - \dfrac{\delta}{2} \ln \left|\dfrac{\delta}{2}\right|, & |z| > \sqrt{\dfrac{\delta}{2}}. \end{cases}$$

We see $V_\delta(z, K, \psi) = |z|^2,\ \forall z \in \left\{|z| \leqslant \sqrt{\dfrac{\delta}{2}}\right\}$ and $V_\delta(z, K, \psi) < |z|^2,\ \forall z \in \left\{\sqrt{\dfrac{\delta}{2}} < |z| \leqslant 1\right\}$.

We denote by $\Lambda = \Lambda(K, \psi)$ the set of numbers δ for which the equality of type (7) holds, i.e.

$$\Lambda = \Lambda(K, \psi) = \{\delta > 0 : V_\delta(z, K, \psi)|_K \equiv \psi(z)\}.$$

For Alan's example, $\Lambda = [2, +\infty)$. In fact,

$$V_2(z, K, \psi) = \begin{cases} |z|^2, & |z| \leqslant 1, \\ 2\ln|z| + 1, & |z| > 1. \end{cases}$$

So, $V_2(z, K, \psi)|_K \equiv \psi(z)$ and by property 1° from Section 3 $V_\delta(z, K, \psi) \geqslant V_2(z, K, \psi)$ for all $\delta \in [2, +\infty)$. If $\delta \in (0, 2)$ then there is a point $z^0 \in K$ such that $V_\delta(z^0, K, \psi) < \psi(z^0)$, that is $(0, 2) \cap \Lambda = \emptyset$.

The sets Λ may be empty. For example, for $K = \{|z| \leqslant 1\} \subset \mathbb{C}$ and $\psi(z) = 1 - |z|^2$, by property 3° we have

$$V_\delta(z, K, \psi) = V_\delta(z, K) = \delta V(z, K) = \delta \ln^+ |z|.$$

Therefore, for any $\delta > 0$, $V_\delta(z, K, \psi) < \psi(z)$, $\forall |z| < 1$. That is, in this case $\Lambda = \emptyset$.

If $\psi(z) \equiv c$, where c is a constant, then $V_\delta(z, K, c) = c + V_\delta(z, K) = c + \delta V(z, K)$. Since the Green function $V(z, K) \geqslant 0$,, for any $\delta > 0$ and $z \in K$ the equality $V_\delta(z, K, c) = c$ holds. This means that $\Lambda = (0, +\infty)$.

Let $\Lambda \neq \emptyset$. If $\delta \in \Lambda$, then from property 1° we easily get $\delta_1 \in \Lambda$ for $\delta_1 > \delta$. On the other hand

Proposition 1. *If* $\delta_j \in \Lambda,\ \forall j \in \mathbb{N}$ *and* $\delta_j \downarrow \delta_0 \neq 0$ *as* $j \to \infty$ *then* $\delta_0 \in \Lambda$.

Proof. Indeed, by the hypothesis we have $V_{\delta_j}(z, K, \psi) = \psi(z), z \in K$. Using properties 2° and 3°, we get

$$V_{\delta_j}(z, K, \psi) = \delta_j V\big(z, K, \tfrac{\psi}{\delta_j}\big) \leqslant \delta_j V\big(z, K, \tfrac{\psi}{\delta_0}\big).$$

Consequently, $\forall j \in \mathbb{N}$ we have $\psi(z) = V_{\delta_j}(z, K, \psi) \leqslant \delta_j V\big(z, K, \frac{\psi}{\delta_0}\big),\ z \in K$. As j tends to infinity, we get

$$\psi(z) \leqslant \delta_0 V\big(z, K, \tfrac{\psi}{\delta_0}\big) = V_{\delta_0}(z, K, \psi), \quad z \in K,$$

i.e. $\psi(z) = \delta_0 V\big(z, K, \frac{\psi}{\delta_0}\big) = V_{\delta_0}(z, K, \psi),\ z \in K$ and $\delta_0 \in \Lambda$. □

Proposition 1 follows, if $\Lambda \neq \emptyset$ then $\Lambda = (0, \infty)$ or $\Lambda = [\delta_0, +\infty), \delta_0 > 0$. Note that if $\delta \in \Lambda(K, \psi)$, then $V_\delta(z, K, \psi) = \psi(z),\ z \in K$. Therefore, by monotonicity $V_\delta(z, K \cap \bar{B}, \psi) = \psi(z),\ z \in K \cap \bar{B}$, for any ball $B \cap K \neq \emptyset$. It follows that if $\delta \in \Lambda(K, \psi)$, then $\delta \in \Lambda(K \cap B, \psi)$.

Definition 5. *Let* $\delta \in \Lambda(K)$. *A compact set* K *is called globally* (δ, ψ)*-regular at a point* $z^0 \in K$ *if* $V_\delta^*(z^0, K, \psi) = \psi(z^0)$. *It is called locally* (δ, ψ)*-regular at a point* $z^0 \in K$ *if for every nonempty ball* $B(z^0, r)$: $V_\delta^*(z^0, K \cap \bar{B}(z^0, r), \psi) = \psi(z^0)$. *A compact set* K *is globally* (δ, ψ)*-regular if it is globally* (δ, ψ)*-regular at every point of itself. A compact* K *is locally* (δ, ψ)*-regular if it is locally* (δ, ψ)*-regular at every point of itself.*

Note that global or local (δ, ψ)-regularity can only be defined for $\delta \in \Lambda$. It is easy to see that any locally (δ, ψ)-regular point is globally (δ, ψ)-regular. We denote by $\Lambda_{reg} = \Lambda_{reg}(K, \psi)$ the set of numbers $\delta \subset \Lambda$, for which K is globally regular, we denote by $\Lambda_{reg}^{loc} = \Lambda_{reg}^{loc}(K, \psi)$ the set of numbers $\delta \subset \Lambda$, for which K is locally regular. We see, $\Lambda_{reg}^{loc} \subset \Lambda_{reg} \subset \Lambda$.

Proposition 2. *Let* $\delta_1, \delta_2 \in \Lambda$ *and* $\delta_1 \leqslant \delta_2$. *If a point* z^0 *is* (δ_2, ψ)*-regular, then it is* (δ_1, ψ)*-regular.*

The proof follows from property 1° of Section 3. For a continuous function ψ there holds

Theorem 4.1. *Let* $\delta \in \Lambda$, *and a function* $\psi(z)$ *be continuous on* K. *Then a fixed point* $z^0 \in K \subset \mathbb{C}^n$ *is locally* (δ, ψ)*-regular if and only if it is locally pluri-regular.*

Proof. We show that for any compact set $K \subset \mathbb{C}^n$ the following is true:

$$\delta V^*(z, K) + \min_K \psi(z) \leqslant V_\delta^*(z, K, \psi) \leqslant \delta V^*(z, K) + \max_K \psi(z). \tag{8}$$

In fact, if $u \in \mathcal{L}_\delta(K, \psi)$, i.e., $u \in \mathcal{L}_\delta,\ \ u|_K \leqslant \psi$, then

$$u(z) - \max_K \psi(z) \in \mathcal{L}_\delta(K).$$

Therefore

$$u(z) - \max_K \psi(z) \leqslant V_\delta^*(z, K)$$

and

$$V_\delta^*(z, K, \psi) - \max_K \psi(z) \leqslant V_\delta^*(z, K) = \delta V^*(z, K), \quad \forall z \in \mathbb{C}^n.$$

Conversely, if $u \in \mathcal{L}_\delta(K)$, then $u(z) + \min_K \psi(z) \in \mathcal{L}_\delta(K, \psi)$. Therefore,

$$V_\delta^*(z, K) + \min_K \psi(z) = \delta V^*(z, K) + \min_K \psi(z) \leqslant V_\delta^*(z, K, \psi),$$

so that (8) holds.

Using (8) we can now prove the theorem. If a fixed point $z^0 \in K$ is not locally pluri-regular, i.e., if $V^*(z^0, K \cap \overline{B}) = \sigma > 0$ for some neighborhood $B: \ z^0 \in B \subset \mathbb{C}^n$, then $V^*(z^0, K \cap \overline{B}_1) \geqslant \sigma$ for any $z^0 \in B_1 \subset B$. Therefore, by (8)

$$V_\delta^*(z^0, K \cap B_1, \psi) \geqslant \delta V^*(z^0, K \cap B_1) + \min_{K \cap B_1} \psi(z) \geqslant \delta\sigma + \min_{K \cap B_1} \psi(z). \tag{9}$$

Since $\psi(z)$ is continuous, choosing a neighborhood B_1 small enough we can make the right part of (9) to be greater than $\psi(z^0)$ i.e., $V_\delta^*(z, K \cap B_1, \psi) > \psi(z^0)$. This means that the point z^0 is not locally (δ, ψ)-regular.

Reversing the roles of $V_\delta^*(z, K \cap B_1, \psi)$ and $V^*(z, K \cap B_1)$ from (8) we can prove the second part of the theorem: if a point $z^0 \in K$ is not locally (δ, ψ)-regular, then it is not locally pluri-regular. □

Corollary 1. *Let $\delta_1, \delta_2 \in \Lambda$ and a function $\psi(z)$ be continuous on K. Then a fixed point $z^0 \in K \subset \mathbb{C}^n$ is locally (δ_1, ψ)-regular if and only if it is locally (δ_2, ψ)-regular.*

Proposition 3. *If $\delta_j \in \Lambda_{reg},\ \forall j \in \mathbb{N}$ and $\delta_j \uparrow \delta$ as $j \to \infty$, then $\delta \in \Lambda_{reg}$.*

Proof. In fact, since $\psi(z) = V_{\delta_j}^*(z, K, \psi), z \in K$, we get

$$\psi(z) = V_{\delta_j}^*(z, K, \psi) = \delta_j V^*\big(z, K, \tfrac{\psi}{\delta_j}\big) \geqslant \delta_j V^*\big(z, K, \tfrac{\psi}{\delta}\big).$$

Therefore, $\forall j \in \mathbb{N}$ we have $\psi(z) \geqslant \delta_j V^*\big(z, K, \frac{\psi}{\delta}\big),\ z \in K$. As j tends to infinity, we get

$$\psi(z) \geqslant \delta V^*\big(z, K, \tfrac{\psi}{\delta}\big) = V_\delta^*(z, K, \psi),\ z \in K.$$

This means that $\delta \in \Lambda_{reg}$. □

Corollary 2. *If $\Lambda = [\delta_0, \infty)$, then* $\Lambda_{reg} = \begin{cases} or & [\delta_0, \delta_1] \\ or & [\delta_0, \infty). \end{cases}$

Corollary 3. *If $\Lambda = (0, \infty)$, then* $\Lambda_{reg} = \begin{cases} or & (0, \delta_1] \\ or & (0, \infty). \end{cases}$

In the paper [10] M. Alan studied the concepts of (δ, ψ)-regularity and posed the following problem

Problem 2 ([10]). *Let K be a compact set in $\mathbb{C}^n$, $\psi(z)$ extends to $\mathcal{L}_{\delta_1}^+$ (see (6)) and $0 < \delta_1 < \delta_2$. If K is (δ_1, ψ)-regular at $z_0 \in K$, then K is (δ_2, ψ)-regular at z_0.*

5. The property of (δ, ψ)-regularity

Further properties of δ-extremal function are associated with pluri-thin sets.

Definition 6. *Let $E \subset \mathbb{C}^n$ and let E' be its limit point set. Then E is said to be pluri-thin at z^0 if either $z^0 \notin E'$ or $z^0 \in E'$ but there exists a neighbourhood U of z^0 and a function $u(z) \in psh(U)$ such that*

$$\overline{\lim_{\substack{z \to z^0 \\ z \in E \setminus \{z^0\}}}}\, u(z) < u(z^0).$$

So, if the set E is not thin at the point z^0, then for any plurisubharmonic function $u(z)$ in the neighborhood of z^0

$$\overline{\lim_{\substack{z\to z^0\\ z\in E\setminus\{z^0\}}}} u(z) = \overline{\lim_{\substack{z\to z^0\\ z\in E}}} u(z) = u(z^0).$$

Proposition 4 ([16])**.** *If $E \subset \mathbb{C}^n$ is pluri-thin at a limit point z^0 of E, then there exists a plurisubharmonic function $u \in \mathcal{L}^+$ such that*

$$\overline{\lim_{\substack{z\to z^0\\ z\in E\setminus\{z^0\}}}} u(z) = -\infty < u(z^0).$$

Theorem 5.1. *If z^0 is a pluri-thin point of K, then z^0 is locally (δ,ψ)-irregular point of K. Here the function $\psi \in L^\infty(K)$ and $\delta \in \Lambda$.*

Proof. Let K be pluri-thin at the point $z^0 \in K$. Then, according to Proposition 4, there exists a function $u(z) \in \mathcal{L}_\delta$ such that

$$\lim_{\substack{z\to z^0\\ z\in E\setminus\{z^0\}}} u(z) = -\infty < u(z^0).$$

Without loss of generality, we can assume $u(z^0) > 0$ and find a ball $B(z^0, r)$ such that

$$\begin{cases} u(z) \leqslant \inf\limits_{z\in K} \psi(z) - \psi(z^0) \text{ for } z \in K \cap B \setminus \{z^0\}, \\ u(z^0) > 0. \end{cases}$$

Put $w(z) = u(z) + \psi(z^0)$. It is easy to see that $w(z) \in L_\delta(\psi, K \cap B \setminus \{z^0\})$, because for $z \in K \cap B \setminus \{z^0\}$

$$w(z) = u(z) + \psi(z^0) \leqslant \inf_{z\in K} \psi(z) - \psi(z^0) + \psi(z^0) = \inf_{z\in K} \psi(z) \leqslant \psi(z).$$

Consequently,

$$w(z) \leqslant V_\delta^*(z, K \cap B \setminus \{z^0\}, \psi) = V_\delta^*(z, K \cap B, \psi),\ \forall z \in \mathbb{C}^n.$$

From here

$$w(z^0) \leqslant V_\delta^*(z^0, K \cap B, \psi).$$

On the other hand

$$w(z^0) = u(z^0) + \psi(z^0) > \psi(z^0).$$

Therefore

$$\psi(z^0) < w(z^0) \leqslant V_\delta^*(z^0, K \cap B, \psi).$$

Hence, the point z^0 is a locally (δ,ψ) irregular point of the compact set K. □

Note that if $n > 1$, the necessary condition of Theorem 5.1, generally speaking, is not true.

Example 4. Let $(\delta,\psi) = (1,0)$ and $K = \{(z_1,z_2) \in \mathbb{C}^2 : |z| \leqslant 1\} \cup \{(z_1,z_2) \in \mathbb{C}^2 : z_2 = 0, |z_1| \leqslant 2\}$.

The compact set K is a union of the unit ball in $\mathbb{C}^2$ and a pluripolar set. We have

$$V(z,K) = \begin{cases} \ln^+ |z| & \text{for } z_2 \neq 2 \\ \ln^+ \left|\dfrac{z_1}{2}\right| & \text{for } z_2 = 0 \end{cases}$$

and

$$V^*(z,K) = \ln^+ |z|.$$

A point $(2,0) \in K$ is an irregular point, but it is not pluri-thin.

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Дельта-экстремальная функция в пространстве $\mathbb{C}^n$

Нурбек Х. Нарзиллаев
Национальный университет Узбекистана
Ташкент, Узбекистан

Аннотация. В этой статье мы изучаем (δ, ψ)-экстремальную функцию Грина $V_\delta^*(z, K, \psi)$, которая определяется при помощи класса $\mathcal{L}_\delta = \{u(z) \in psh(\mathbb{C}^n) : \ u(z) \leqslant C_u + \delta \ln^+ |z|, \ z \in \mathbb{C}^n\}, \ \delta > 0$. Покажем, что понятие регулярности точек для разных δ не совпадают. Тем не менее мы доказываем, что если компакт $K \subset \mathbb{C}^n$ регулярен, то δ-экстремальная функция Грина непрерывна во всем пространстве $\mathbb{C}^n$.

Ключевые слова: плюрисубгармонические функции, экстремальная функция Грина, функция Грина с весом, δ-экстремальная функция.