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СОДЕРЖАНИЕ

This issue of the journal is dedicated to the memory of Professor Yuri Belov.

Yuri Ya. Belov, a founder of the Krasnoyarsk School of inverse problems of mathematical physics, died December 9, 2019

Yuri Belov graduated the faculty of mechanics and mathematics at Novosibirsk state university. In 1970 he was assigned to work at Krasnoyarsk state university upon graduation of doctor's course where he held the position of assistant professor at the department of applied mathematics. In 1971 Yuri Belov defended Cand. Sc. (Phd) thesis on the topic "Certain mathematics aspects of the gas dynamics with viscosity". In 1982 he successfully defended the doctor's thesis "The approximation and correctness of boundary value problems for the systems of differential equations" and gotten the Doctor's degree (Doctor of Science). In 1986 he was given the academical title of professor.

In 1972 Yu. Ya. Belov took up his post as a head of the department of mathematical analysis and differential equations and was at the head of it to the last days. The department was found by professor L. A. Aizenberg as the department of mathematical analysis at Krasnoyarsk branch of Novosibirsk state university in 1965 and renamed in 1972. All next work activities of Yuri Yakovlevich were concerned with the science, the university and this department.

Yu. Ya. Belov is the recognized specialist on the boundary value problems for the partial differential equations and the inverse problems of mathematical physics. He published two monographs and more then 120 research papers and methodical works. His main works are on the unique solvability and approximation of certain boundary value problems for the systems of equations describing the ocean flows (1979); the decomposition of degenerate quasi-linear parabolic equation (1989); inverse problems for parabolic equations (1993).

At different times Yuri Yakovlevich held the position of dean at the faculty of mathematics and pro-rector at Krasnoyarsk state university. Hi showed exceptional talent as a teacher, administrator and scientist. Professor Yu. Ya. Belov directed his efforts towards creating good traditions at the faculty of mathematics and then the school of mathematics and computer science. These traditions took all the best of that Yu. Ya. Belov brought as the pupil of the schools of thought of Novosibirsk Academgorodok and his outstanding teacher academician N. N. Yanenko.

Yuri Belov gave impetus to many young researchers which became Candidates of Sciences and work on the doctor's thesis. At present time his pupils I. V. Frolenkov, O. N. Cherepanova, T. N. Shipina, A. Sh. Lyubanova, R. V. Sorokin, S. V. Polintseva work at the department of mathematical analysis and differential equations and continue his research. The memory of professor Yu. Ya. Belov lives in his collegues and pupils.

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Solution of a Two-Layer Flow Problem with Inhomogeneous Evaporation at the Thermocapillary Interface

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Abstract. The Ostroumov–Birikh type exact solution of thermodiffusion convection equations is constructed in the frame of mathematical model considering evaporation through the liquid–gas interface and the influence of direct and inverse thermodiffusion effects. It is interpreted as a solution describing steady flow of evaporating liquid driven by co-current gas-vapor flux on a working section of a plane horizontal channel. Functional form of required functions is presented. An algorithm for finding all the constants and parameters contained in the solution is outlined, and their explicit expressions are written. The solution is derived for the case of vapor absorption on the upper wall of the channel which is set with the help of the first kind boundary condition for the function of vapor concentration. Applicability field of the solution is briefly discussed.

Keywords: mathematical model, boundary value problem, exact solution, evaporative convection

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Introduction

The widespread use of two-phase systems in different technologies motivates the intensive development of the experimental and theoretical methods for studying the features of convective flows accompanied by evaporation in the frame of various approaches [1]. Examples of such technologies are the fluidic cooling, thermal coating or drying processes etc. Full-scale experimental elaboration and testing of the real technological systems can be very expensive and sometimes impossible (for example, if it is expected that these systems will be used in microgravity). Thus, preliminary theoretical investigation based on the mathematical modeling is the necessary requirement and an indispensable part when solving the optimization problems of fluid technologies and in the search for innovative technical solutions.

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Currently, mathematical models built on the basis of the Navier–Stokes and heat transfer equations or their approximations are the most widely used ones for theoretical investigations of the processes in the two-phase systems. These equations are the results of symmetry of the space-time, i. e. the fulfilment of the fundamental conservation laws was implied in deriving the equations. This fact enables to obtain the significant results in the study the problems of the fluid flows with heat and mass transfer at the thermocapillary interfaces. We focus on the search and investigation of an exact solution of the governing system of differential equations, since the solution inherit basic properties of symmetry of the space-time and of a fluid moving in the space, thereby ensuring feasibility of physical processes described by this solution.

Among possible solutions of the evaporative convection problems are especially highlighted the Ostroumov–Birikh type solutions [1]. They take into account the presence of temperature gradient which can appear both due to evaporation and applied outside or interfacial thermal load. The applicability of such a class of solutions for describing the two-layer flows with diffusive type evaporation at the interface in a horizontal channel is confirmed by a good qualitative agreement between the experimental data and theoretical results [2]. The specific feature of these solutions is that they allow one to test various types of boundary conditions for the vapor concentration and temperature functions, to correctly take into account the influence of the external controlling actions (thermal, mechanical, fluid flow rate etc.) as well as the gravity and thermodiffusion effects [3, 4].

For the first time, the problem of unidirectional two-layer flows induced by the gravity and Marangoni forces was considered in [5]. The first results of the study the flows with evaporation in a bilayer system based on an analogue of the Ostroumov–Birikh solution were presented in [6]. 2D and 3D generalizations of the solution obtained in the framework of the evaporative convection problem in the liquid − gas system with the sharp interface admitting the phase transition were constructed in [7, 8]. The uniform character of evaporation was considered in all the listed works.

In the present paper, an exact solution of the convection equations to describe joint flow of the evaporating liquid and gas-vapor mixture in a horizontal minichannel under conditions of the given gas flow rate and full vapor absorption on the upper channel boundary is constructed. The aim of this work is to take into account an inhomogeneous with respect to the longitudinal coordinate character of evaporation at the interface.

1. Problem statement and form of exact solution

The stationary flow of two viscous incompressible media (of the liquid and gas-vapor mixture) filling the plane channel and having the common thermocapillary interface Γ is considered (Fig. 1). In the Cartesian coordinate system (x, y) the gravity acceleration vector $\mathbf{g}$ has the coordinates $\mathbf{g} = (0, -g)$. The upper and lower boundaries of the channel $y = h$ and $y = -l$ are the rigid walls. The interface remains to be flat, it is given by the equation $y = 0$.

The Oberbeck–Boussinesq approximation of the Navier–Stokes equations is used to describe the flow in each phase. In two-dimensional case the constitutive equations have the following form:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2.1}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + g\,(\beta T + \gamma C), \tag{2.2}$$

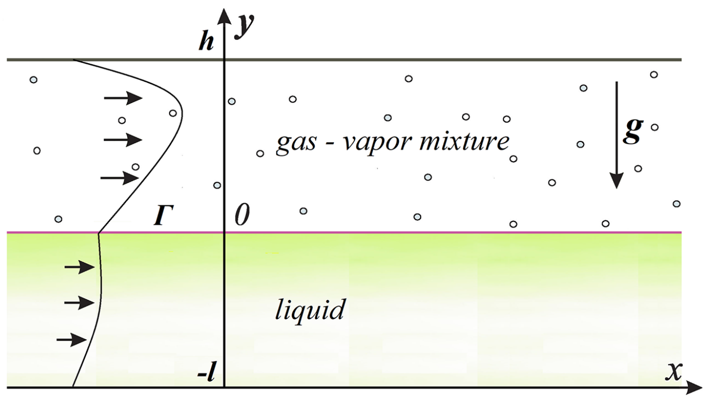

Fig. 1. Flow scheme.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \chi\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \delta\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right)\right). \tag{2.4}$$

The vapor transfer in the gas phase is governed by the convective diffusion equation, which is the result of the Fick's law [9]:

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)\right). \tag{2.5}$$

The terms γC in (2.2) and $\delta \Delta C$ in (2.4) are taken into account by modeling of flows in the gas-vapor layer. In equations (2.1)–(2.5) the following notations are used: u, v are the longitudinal and transversal components of the velocity vector, p is the deviation of pressure from the hydrostatic one, T is the temperature, C is the vapor concentration in background gas, ρ is the density of the liquid and gas (some reference value of the density), ν is the kinematic viscosity coefficient, χ is the heat diffusivity coefficient, D is the coefficient of vapor diffusion in the gas, β is the coefficient of thermal expansion, γ is the concentration coefficient of density, the parameters δ and α characterize the Dufour and Soret effects (the effects of diffusive thermal conductivity and thermodiffusion, correspondingly) [10].

Let the exact solution of the governing equations (2.1)–(2.5) be of a special type, when only the longitudinal velocity component is not equal to zero and depends on the transverse coordinate; functions of temperature and vapor concentration have the linear components with respect to the longitudinal coordinate:

$$\begin{gathered} u_i = u_i(y), \quad v_i = 0, \quad T_i = T_i(x,y) = \left(a_1^i + a_2^i y\right)x + \vartheta_i(y), \\ C = C(x,y) = (b_1 + b_2 y)\,x + \phi(y), \quad p_i = p_i(x,y). \end{gathered} \tag{2.6}$$

Index i denotes characteristics of corresponding fluid: $i = 1$ relates to the liquid in the lower layer, $i = 2$ regards to the gas-vapor mixture filling the upper layer. Parameters a_2^i, b_j ($i = 1, 2$; $j = 1, 2$) are the constants, their values will be determined with the help of boundary conditions. Furthermore, the boundary conditions will dictate certain linking relations for the parameters.

2. Boundary conditions

The boundary conditions will be written subject to the form of exact solution (2.6) of equations (2.1)–(2.5). The no-slip conditions are fulfilled on the fixed impermeable channel walls

$$u_1\,(-l) = 0, \quad u_2\,(h) = 0, \tag{3.1}$$

and the linear temperature distribution is prescribed on these walls

$$T_1\,(x, -l) = A_1 x + \vartheta^-, \quad T_2\,(x, h) = A_2 x + \vartheta^+. \tag{3.2}$$

The condition for vapor concentration on the upper wall is determined by the property of this wall to instantaneously completely absorb the vapor:

$$C\,(x, h) = 0. \tag{3.3}$$

In some real physical cases the vapor absorbtion property is confirmed by a possibility of a freezing out of the vapor. The applicability of boundary condition of such a type in this problem in frame of 3D statement is discussed in [11].

On the thermocapillary interface Γ given by the equation $y = 0$ the kinematic and dynamic conditions as well as the condition of heat balance should be set [12]. The kinematic condition is fulfilled automatically in view of the form of the velocity vector components (see (2.6)). Projection of the dynamic condition on the unit tangential vector to the interface is written as follows:

$$\rho_1\nu_1 u_{1y} = \rho_2\nu_2 u_{2y} - \sigma_T T_x\big|_{y=0}, \tag{3.4}$$

where $\sigma_T > 0$ is the temperature coefficient of the surface tension σ which linearly depends on the temperature, $\sigma = \sigma_0 - \sigma_T(T - T_0)$, $\sigma_0 > 0$ is the characteristic value of the surface tension at a relative temperature T_0. Projection of the dynamic condition on the unit normal vector to the interface leads to the equality

$$p_1 = p_2. \tag{3.5}$$

We demand the fulfilment of continuity conditions for the tangential component of velocity vector and temperature at the interface:

$$u_1 = u_2, \quad T_1 = T_2. \tag{3.6}$$

The continuity of normal component of the velocity vector ensues from the kinematic condition.

The heat transfer condition and the mass balance equation are stated as follows:

$$\kappa_1 T_{1y} - \kappa_2 T_{2y} - \delta\kappa_2 C_y\big|_{y=0} = -\lambda M, \tag{3.7}$$

$$M = -D\rho_2\,(C_y + \alpha T_{2y}|_{y=0})\,. \tag{3.8}$$

The relations include the effects of the thermodiffusion and duffusive thermal conductivity characterized by coefficients α and δ; λ is the latent heat of evaporation. In the present paper, the exact solution is constructed under assumption, that the evaporation mass flow rate of the liquid at the interface linearly depends on the longitudinal coordinate:

$$M = M(x) = M_0 + M_x x. \tag{3.9}$$

The presupposition implies that nonuniform (inhomogeneous) character of phase transition is examined. The evaporation mass flow rate M is one of the important characteristics of the evaporative convection. The positive values of M correspond to evaporation of the liquid into the gas flow; the negative values regards to the vapor condensation.

The saturated vapor concentration is defined with the help of the relation being a sequence of the Clapeyron–Clausius and Mendeleev–Clapeyron equations [6, 7, 13]:

$$C\big|_{y=0} = C_*[1 + \varepsilon(T_2\big|_{y=0} - T_0)]. \tag{3.10}$$

In this equation $\varepsilon = \lambda\mu_0/(RT_0^2)$, μ_0 is the molar mass of the evaporating liquid, R is the universal gas constant, C_* is the saturated vapor concentration at $T_2 = T_0$. Equation (3.10) is valid under assumption of smallness of the dimensionless parameter εT_* (T_* is a characteristic temperature drop), that is provided by moderate values of temperature and temperature drops.

To close the problem statement the condition of a given gas flow rate is assumed to be satisfied:

$$Q = \int_0^h \rho_2 u_2(y)\, dy. \tag{3.11}$$

Used form of the boundary conditions allows one to correctly describe the phase transition of diffusive type. Thus, boundary-value problem (2.1)–(2.5), (3.1)–(3.11) presents the mathematical model to simulate convection in multiphase system under conditions of weak evaporation.

3. The class of the exact solutions

The fulfilment of condition of temperature continuity (3.6) at the interface dictates the following equality: $a_1^i = A$ $(i = 1, 2)$. The value determines the longitudinal temperature gradient presetting the intensity thermal effects on the interface, and as consequence, the intensity of evaporation and surface tension-driven convection.

Deriving the solution of equations (2.1)–(2.5) in the form (2.6) results in the explicit expressions for the required functions which define basic characteristics of the bilayer system (velocity u_i, pressure p_i, temperature T_i in i-th phase and vapor concentration C in gas layer):

$$\begin{aligned}
&u_i(y) = c_3^i + c_2^i y + c_1^i \frac{y^2}{2} + L_3^i \frac{y^3}{6} + L_5^i \frac{y^4}{24},\\
&p_i(x,y) = \left(d_1^i + d_2^i y + d_3^i \frac{y^2}{2}\right) x + c_8^i + K_1^i y + K_2^i \frac{y^2}{2} + K_3^i \frac{y^3}{3} + K_4^i \frac{y^4}{4} +\\
&\qquad + K_5^i \frac{y^5}{5} + K_6^i \frac{y^6}{6} + K_7^i \frac{y^7}{7} + K_8^i \frac{y^8}{8},\\
&T_i(x,y) = \left(A + a_2^i y\right) x + c_5^i + c_4^i y + N_2^i \frac{y^2}{2} + N_3^i \frac{y^3}{6} + N_4^i \frac{y^4}{24} + N_5^i \frac{y^5}{120} + N_6^i \frac{y^6}{720} + N_7^i \frac{y^7}{1008},\\
&C(x,y) = (b_1 + b_2 y) x + c_7 + c_6 y + S_2 \frac{y^2}{2} + S_3 \frac{y^3}{6} + S_4 \frac{y^4}{24} + S_5 \frac{y^5}{120} + S_6 \frac{y^6}{720} + S_7 \frac{y^7}{1008}.
\end{aligned} \tag{3.12}$$

Coefficients L_4^i, L_3^i, S_j, K_m^i $(i = 1, 2;\ j = 2, \ldots, 7;\ m = 1, \ldots, 8)$ are expressed by physical parameters of the problem $g, \beta_i, \nu_i, \chi_i, \rho_i, D, \gamma$, coefficients defining the longitudinal temperature and vapor concentration gradients A, a_2^i, b_i $(i = 1, 2)$, and by integration constants c_j^i $(i = 1, 2;\ j = 1, \ldots, 5; 8)$, c_6, c_7. Exact representations of the listed coefficients are given in Appendix.

4. The common scheme for finding the governing parameters and integration constants

Implementation of boundary conditions (3.1)–(3.11) will lead to a system of equations for calculation of the integration constants c_j^i ($i = 1, 2$; $j = 1, \dots, 5; 8$), c_6, c_7. Determining these constants, the velocity and temperature profiles, the pressure distributions for both fluids and the vapor concentration in the gas are calculated with the help of formulas (3.12).

Below, the algorithm for finding all the unknown parameters and constants is outlined. Let the gas flow rate (3.11) and certain values of the longitudinal temperature gradients A, A_1 (see expression for the temperature functions in (3.12) and boundary conditions (3.2) be given.

(i) In the consequence of the heat transfer and mass balance conditions (3.7), (3.8) at the interface a relationship relating the longitudinal temperature gradients A, A_1 and A_2 is derived. It should be noted that both boundary gradients A_1 and A_2 can be given, then the corresponding relation to calculate the interfacial gradient A is obtained.

(ii) Parameters b_1, b_2, M_x characterizing the flow regime (2.6), (3.9) with nonuniform evaporation are determined with the help of A, A_2 on the basis of (3.3), (3.8), (3.10).

(iii) Solving the system of the linear algebraic equations being a consequence of the no-slip conditions (3.1), dynamic conditions (3.4), (3.5), condition of velocity continuity (3.6) and equality (3.11) defining the gas flow rate, the values of the unknowns $\{c_1^i, c_2^i, c_3^i\}$ ($i = 1, 2$) are calculated.

(iv) Conditions determining the thermal and vapor concentration boundary regimes (3.2), (3.3), conditions at interface setting the temperature continuity (3.6) and saturated vapor concentration (3.10), and heat balance equation (3.7) lead to the system of the linear algebraic equations for calculation $\{c_4^2, c_5^2, c_6\}$ and $\{c_4^1,\ c_5^1,\ c_7\}$.

(v) The value of M_0 will be computed with the help of obtained values c_4^2 and c_6.

Following this algorithm, all the required functions of the form (2.6) and the mass evaporation rate at the interface $M = M(x)$ in the form (3.9) are determined.

5. Concluding remarks with regard to conditions of applicability of the solution

To use the obtained solution for describing convection with evaporation in real physical systems, it should define conditions ensuring the correct application of the approach based on the utilization of the exact solutions of the fluid mechanics equations in the Oberbeck–Boussinesq approximation.

First of all, it must be remembered that the principal limitation for the use of the Oberbeck–Boussinesq approximation is to consider the heat and mass transfer processes occurring under moderate temperature drops. The equations of thermal-concentration convection written in form (2.1)–(2.4) present the "diffusive" laws of the transfer of mass, momentum and energy which adequately govern these processes near the thermodynamical equilibrium state. The moderate temperature drops, in turn, result in small variations of concentration. The latter ensures the correct using of the Fick's law written in form of convection-diffusion equation (2.5) and interface boundary condition (3.10). These requirements concerning quantitative changes in temperature and concentration in the system allow one to consider the processes of phase transition as the diffusive ones, and consequently, to believe that we deal with "weak" evaporation. It means that the phase transitions induced by critical thermal loads as, for example, while boiling are not considered.

The second point is to regard the flows with small velocities. It allows one to suppose that the gas in two-phase systems under study is an incompressible medium. It is worth to noting that for the mini- and microscale fluidic systems this assumption is quite justified [14]. Simultaneously, the requirement concerning the scale of the system is the condition when the Ostroumov–Birikh type solution gives plausible description of all the basic characteristics for a two-phase system with evaporation through the sharp interface [6].

Finally, taking into account the character of dependence of the temperature and vapor concentration functions on the longitudinal coordinate x specified in (2.6), one can conclude that these functions will grow with growth of x. Then, according to the given physical interpretation of solution (2.6), it will give appropriate (physically feasible) results if the convective regimes are considered in the domain of finite size. One should control the values of C function; they cannot be more than 1, since we treat this function as mass fraction of the evaporating component in the background gas. If its values becomes more than 1, it will immediately mean, that the solution gives "purely mathematical solution" of the boundary-value problem under consideration.

Appendix. Formulas for calculating the coefficients in expressions (3.12)

Coefficients L_4^i, L_3^i:

$$L_4^1 = \frac{g\beta_1 a_2^1}{\nu_1}, \quad L_3^1 = \frac{g\beta_1 A}{\nu_1}, \quad L_4^2 = \frac{g}{\nu_2}\left(\beta_2 a_2^2 + \gamma b_2\right), \quad L_3^2 = \frac{g}{\nu_2}\left(\beta_2 A + \gamma b_1\right).$$

Coefficients N_7^i, N_6^i, N_5^i, N_4^i, N_3^i, N_2^i:

$$N_7^1 = \frac{g\beta_1 (a_2^1)^2}{\nu_1\chi_1}, \quad N_6^1 = 5\frac{g\beta_1 A a_2^1}{\nu_1\chi_1}, \quad N_5^1 = \frac{1}{\chi_1}\left(\frac{g\beta_1 (A)^2}{\nu_1} + 3a_2^1 c_1^1\right),$$

$$N_4^1 = \frac{1}{\chi_1}\left(Ac_1^1 + 2a_2^1 c_2^1\right), \quad N_3^1 = \frac{1}{\chi_1}\left(Ac_2^1 + a_2^1 c_3^1\right), \quad N_2^1 = \frac{A}{\chi_1}c_3^1, \quad N_7^2 = B_2\frac{g}{\nu_2}\left(\beta_2 a_2^2 + \gamma b_2\right),$$

$$N_6^2 = \frac{g}{\nu_2}\left[B_1\left(\beta_2 a_2^2 + \gamma b_2\right) + 4B_2\left(\beta_2 A + \gamma b_1\right)\right], \quad N_5^2 = B_1\frac{g}{\nu_2}\left(\beta_2 A + \gamma b_1\right) + 3B_2 c_2^1,$$

$$N_4^2 = B_1 c_1^2 + 2B_2 c_2^2, \quad N_3^2 = B_1 c_2^2 + 2B_2 c_3^2, \quad N_2^2 = B_1 c_3^2.$$

Coefficients S_7, S_6, S_5, S_4, S_3, S_2:

$$S_7 = \frac{g}{\nu_2}\left(\beta_2 a_2^2 + \gamma b_2\right)\left(\frac{b_2}{D} - \alpha B_2\right),$$

$$S_6 = \frac{g}{\nu_2}\left[\left(\frac{b_1}{D} - \alpha B_1\right)\left(\beta_2 a_2^2 + \gamma b_2\right) + 4\left(\frac{b_2}{D} - \alpha B_2\right)\left(\beta_2 A + \gamma b_1\right)\right],$$

$$S_5 = \frac{g}{\nu_2}\left[\left(\frac{b_1}{D} - \alpha B_1\right)\left(\beta_2 A + \gamma b_1\right) + 3\left(\frac{b_2}{D} - \alpha B_2\right)c_1^2\right],$$

$$S_4 = \left(\frac{b_1}{D} - \alpha B_1\right) c_1^2 + 2\left(\frac{b_2}{D} - \alpha B_2\right) c_2^2,$$

$$S_3 = \left(\frac{b_1}{D} - \alpha B_1\right) c_2^2 + \left(\frac{b_2}{D} - \alpha B_2\right) c_3^2, \quad S_2 = \left(\frac{b_1}{D} - \alpha B_1\right) c_3^2.$$

Coefficients d_3^i, d_2^i, d_1^i:

$$d_3^1 = \rho_1 g\beta_1 a_2^1, \quad d_2^1 = \rho_1 g\beta_1 A, \quad d_1^1 = \rho_1\nu_1 c_1^1,$$

$$d_3^2 = \rho_2 g\beta_2 a_2^2 + \rho_2 g\gamma b_2, \quad d_2^2 = \rho_2 g\beta_2 A + \rho_2 g\gamma b_1, \quad d_1^2 = \rho_2\nu_2 c_1^2.$$

Coefficients K_8^i, K_7^i, K_6^i, K_5^i, K_4^i, K_3^i, K_2^i, K_1^i:

$$K_8^1 = \frac{1}{1008}\frac{(g\beta_1 a_2^1)^2\rho_1}{\nu_1\chi_1}, \quad K_7^1 = \frac{1}{144}\frac{(g\beta_1)^2\rho_1}{\nu_1\chi_1} A a_2^1,$$

$$K_6^1 = \frac{1}{120}\frac{g\beta_1\rho_1}{\chi_1}\left(\frac{g\beta_1(A)^2}{\nu_1} + 3a_2^1 c_1^1\right), \quad K_5^1 = \frac{1}{24}\frac{g\beta_1\rho_1}{\chi_1}\left(Ac_1^1 + 2a_2^1 c_2^1\right),$$

$$K_4^1 = \frac{1}{6}\frac{g\beta_1\rho_1}{\chi_1}\left(Ac_2^1 + a_2^1 c_3^1\right), \quad K_3^1 = \frac{1}{2}\frac{g\beta_1\rho_1}{\chi_1} Ac_3^1, \quad K_2^1 = g\beta_1\rho_1 c_4^1, \quad K_1^1 = g\beta_1\rho_1 c_5^1;$$

$$K_8^2 = \frac{1}{1008}\frac{g^2\rho_2}{\nu_2}\left(\beta_2 a_2^2 + \gamma b_2\right)\left(B_2(\beta_2 - \alpha\gamma) + \frac{\gamma b_2}{D}\right),$$

$$K_7^2 = \frac{1}{720}\frac{g^2\rho_2}{\nu_2}[\left(\beta_2 a_2^2 + \gamma b_2\right)\left(B_1(\beta_2 - \alpha\gamma) + \frac{\gamma b_1}{D}\right) +$$

$$+4\left(\beta_2 A + \gamma b_1\right)\left(B_2(\beta_2 - \alpha\gamma) + \frac{\gamma b_2}{D}\right)],$$

$$K_6^2 = \frac{1}{120} g\rho_2\left[\frac{g}{\nu_2}\left(\beta_2 A + \gamma b_1\right)\left(B_1(\beta_2 - \alpha\gamma) + \frac{\gamma b_1}{D}\right) + 3\left(B_2(\beta_2 - \alpha\gamma) + \frac{\gamma b_2}{D}\right) c_1^2\right],$$

$$K_5^2 = \frac{1}{24} g\rho_2\left[\left(B_1(\beta_2 - \alpha\gamma) + \frac{\gamma b_1}{D}\right) c_1^2 + 2\left(B_2(\beta_2 - \alpha\gamma) + \frac{\gamma b_2}{D}\right) c_2^2\right],$$

$$K_4^2 = \frac{1}{6} g\rho_2\left[\left(B_1(\beta_2 - \alpha\gamma) + \frac{\gamma b_1}{D}\right) c_2^2 + \left(B_2(\beta_2 - \alpha\gamma) + \frac{\gamma b_2}{D}\right) c_3^2\right],$$

$$K_3^2 = \frac{1}{2} g\rho_2\left(B_1(\beta_2 - \alpha\gamma) + \frac{\gamma b_1}{D}\right) c_3^2, \quad K_2^2 = g\beta_2\rho_2 c_4^2 + g\gamma\rho_2 c_6^2, \quad K_1^2 = g\beta_2\rho_2 c_5^2 + g\gamma\rho_2 c_7^2.$$

Here, $B_1 = \dfrac{DA - \chi_2\delta b_1}{D\chi_2(1 - \alpha\delta)}$, $B_2 = \dfrac{Da_2^2 - \chi_2\delta b_2}{D\chi_2(1 - \alpha\delta)}$.

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Решение задачи о двухслойном течении с неоднородным испарением на термокапиллярной границе раздела

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Аннотация. В рамках математической модели, учитывающей испарение на межфазной границе и влияние прямого и обратного термодиффузионных эффектов, строится аналог решения Остроумова–Бириха для уравнений термоконцентрационной конвекции. Полученное решение интерпретируется как решение, описывающее установившееся течение испаряющейся жидкости, увлекаемой спутным газопаровым потоком, на рабочем участке плоского горизонтального канала. Приведены точные представления искомых функций. Описан алгоритм определения констант и параметров, которые содержит решение, выписан их явный вид. Решение построено для случая абсорбции пара на верхней стенке канала, которое задаётся граничным условием первого рода для функции концентрации пара. Кратко обсуждается область применимости полученного решения.

Ключевые слова: математическая модель, краевая задача, точное решение, испарительная конвекция.

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The Regularity of the Solutions of Inverse Problems for the Pseudoparabolic Equation

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Abstract. The paper discusses the regularity of the solutions to the inverse problems on finding unknown coefficients dependent on t in the pseudoparabolic equation of the third order with an additional information on the boundary. By the regularity is meant the continuous dependence of the solution on the input data of the inverse problem. The regularity of the solution is proved for two inverse problems of recovering the unknown coefficient in the second order term and the leader term of the linear pseudoparabolic equation.

Keywords: continuous dependence on the input data, a priori estimate, inverse problem, pseudoparabolic equation.

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Introduction

In this paper we discuss inverse problems for the pseudoparabolic diffusion equation

$$(\nu u + \eta M u)_t + k M u + g u = f. \tag{0.1}$$

Here M is an elliptic linear differential operator of the second order in the space variables, $\nu \geqslant 0$ is a constant, the coefficients η and k depends on t, the functions g and f depends on t, x. We establish the regularity of the strong solution of two inverse problems for (0.1) with unknown coefficients η and k dependent on t under the Dirichlet boundary condition and additional integral boundary data akin to the conditions of overdetermination considered in [5,6]. An exact statement of the problems will be given below. In [6], the regularity of the strong solution was investigated for the inverse problem on finding an unknown coefficient $k(t)$ with given constant η and function $g(t,x)$ in the sense that the smoothness of the solution increases with increasing the smoothness of the input data. In this paper by the regularity of the solution is meant its continuous dependence on the input data of the inverse problem. The regularity of solution, as used here, was established for the inverse problem of finding an unknown lower coefficient $g = g(t)$ in equation (0.1) [7].

In [5, 6], following the idea of [9] the existence of the strong solutions of the inverse problems was proved by reducing the inverse problem to an operator equation of the second type for the

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unknown coefficient. It was shown that the operator of this equation is a contraction on a set constructed with the use of the comparison theorems for pseudoparabolic equations.

Applications of such problems deal with the recovery of unknown parameters indicating physical properties of a medium (the heat conductivity, the permeability of a porous medium, the elasticity, the absorption (also known as potential) in the diffusion etc.). An exact statement of the problem will be given below. Since the natural stratum is involved, the parameters in (0.1) should be determined on the basis of the investigation of its behavior under the natural non-steady-state conditions (see [1,12,13] for more details). This leads to the interest in studying the inverse problems for (0.1) and its analogue.

The study of inverse problems for pseudoparabolic equations goes back to 1980 s. The first result [11] refers to the inverse problems of determining a source function f of equation

$$(u + L_1 u)_t + L_2 u = f \tag{0.2}$$

in case $L_1 = L_2$ where L_1 and L_2 are the linear differential operators of the second order in spacial variables. We should mention also the results in [2, 8] concerning with coefficient inverse problems for the linear equation (0.1). In [8], the uniqueness theorem is obtained and an algorithm of determining the coefficients of L_2 is constructed. In [2], the solvability is established for two inverse problems of recovering the unknown coefficients in terms u (the lowest term of $L_2 u$) and u_t of (0.2). In [10], an inverse problem of recovering time-depending right-hand side and coefficients of (0.2) is considered. The values of the solution at separate points are employed as overdetermination conditions. The existence and uniqueness theorems are proven for this problem and the stability estimates of the solution are exposed.

The paper is organized as follows. Section 1 discusses the statement of the inverse problems. In Section 2 the regularity of the solution is investigated for the problem on recovering an unknown coefficient $k(t)$ in the second order term of the equation (0.1). Section 3 is devoted to the regularity of the solution to the problem on identification of the leader coefficient $\eta(t)$ in (0.1).

1. The statement of the problems

Let Ω be a bounded domain in $\mathbf{R}^n$ with a boundary $\partial\Omega \in C^2$, $\overline{\Omega}$ be the closure of Ω. T is an arbitrary real number, $Q_T = \Omega \times (0, T)$ with the lateral surface $S_T = (0, T) \times \partial\Omega$, $\overline{Q}_T$ is the closure of Q_T and the pair (t, x) is a point of Q_T.

From now on we keep the notations: $(\cdot,\cdot)_R$ is the inner product of $\mathbf{R}^n$; $\|\cdot\|$ and $(\cdot,\cdot)$ are the norm and the inner product of $L^2(\Omega)$, respectively; $\|\cdot\|_j$ is the norm of $W_2^j(\Omega)$, $j = 1, 2$; and $\langle\cdot,\cdot\rangle_1$ is the duality relation between $\mathring{W}_2^j(\Omega)$ and $W_2^{-j}(\Omega)$; $\|\cdot\|_{p/2}$ is the norm of $W_2^{p/2}(\partial\Omega)$, $p = 1, 3$.

We introduce a linear differential operator $M = -\mathrm{div}(\mathcal{M}(x)\nabla) + m(x)I$ where $\mathcal{M}(x) \equiv (m_{ij}(x))$ is a matrix of functions $m_{ij}(x)$, $i, j = 1, 2, \dots, n$; I — the identity operator. We also keep the notation

$$\langle Mv_1, v_2\rangle_M = \int_\Omega ((\mathcal{M}(x)\nabla v_1, \nabla v_2)_R + m(x)v_1 v_2)dx$$

for $v_1, v_2 \in W_2^1(\Omega)$ and assume that the following conditions are fulfilled.

I. $m_{ij}(x)$, $\partial m_{ij}/\partial x_l$, $i, j, l = 1, 2, \dots, n$, and $m(x)$ are bounded in Ω. Operator M is elliptic, that is, there exist positive constants m_1 and m_2 such that for all $v \in W_2^1(\Omega)$

$$m_1\|v\|_1^2 \leqslant \langle Mv, v\rangle_M \leqslant m_2\|v\|_1^2. \tag{1.1}$$

II. M is a selfadjoint operator, that is, $m_{ij}(x) = m_{ji}(x)$, $i, j = 1, 2, \dots, n$ for $x \in \Omega$.

In this paper we are studying the inverse problems of recovering unknown coefficients of the equation (0.1) with the initial data

$$(\nu u + \eta M u)|_{t=0} = U_0(x), \tag{1.2}$$

and the boundary condition

$$u|_{\partial\Omega} = \beta(t, x). \tag{1.3}$$

We investigate the regularity of the solutions of two inverse problems.

Problem 1. For $\nu = 1$, given functions $g(t, x)$, $f(t, x)$, $U_0(x)$, $\beta(t, x)$, $\omega(t, x)$, $\varphi(t)$ and a constant η find the pair of unknown functions $\{u, k\}$, $k = k(t)$, satisfying the equation (0.1), the initial data (1.2), the boundary condition (1.3) and the condition of overdetermination

$$\int_{\partial\Omega} \left\{\eta \frac{\partial u_t}{\partial \overline{N}} + k\frac{\partial u}{\partial \overline{N}}\right\}\omega(t, x)ds + k\varphi_1(t) = \varphi_2(t). \tag{1.4}$$

Problem 2. For $\nu = 0$, $k = 1$, given functions $g(t, x)$, $f(t, x)$, $U_0(x)$, $\beta(t, x)$, $\omega(t, x)$, $\varphi(t)$ and real constants μ_1, μ_2, find the pair of unknown functions $\{u, \eta\}$, $\eta = \eta(t)$, satisfying the equation (0.1), the initial data (1.2), the boundary condition (1.3) and the conditions of overdetermination

$$\int_{\partial\Omega} \left\{\left(\eta \frac{\partial u}{\partial \overline{N}}\right)_t + \frac{\partial u}{\partial \overline{N}}\right\}\omega(t, x)ds + (\eta\varphi_1(t))_t = \varphi_2(t), \tag{1.5}$$

$$\eta(0) \int_{\partial\Omega} \frac{\partial u(0, x)}{\partial \overline{N}}\, \omega(0, x)\, ds + \mu_1\eta(0) = \mu_2. \tag{1.6}$$

Here $\dfrac{\partial}{\partial \overline{N}} = (\mathcal{M}(x)\nabla, \mathbf{n})$, $\mathbf{n}$ is the unit outward normal to the boundary $\partial\Omega$.

If $\omega \equiv 1$, then the integral conditions (1.5)–(1.6) means a given flux of a liquid through the surface $\partial\Omega$, for instance, the total discharge of a liquid through the surface of the ground. Similar nonlocal conditions were applied to control problems in [3].

We introduce functions $a(t, x)$, $b(t, x)$ as the solutions of the Dirichlet problems

$$Ma = 0 \quad \text{in } \Omega, \quad a\big|_{\partial\Omega} = \beta(t, x); \qquad Mb = 0 \quad \text{in } \Omega, \quad b\big|_{\partial\Omega} = \omega(t, x), \tag{1.7}$$

$$\Psi(t) = \langle Ma, b\rangle_{1,M}, \qquad F(t, x) = a_t - f(t, x) + g(t, x)a,$$

$$\overline{\Psi} = \max_{t\in[0,T]} \langle Ma, b\rangle_{1,M}\,, \quad \overline{\varphi}_1 = \max_{t\in[0,T]} \varphi_1(t),$$

2. The regularity of the solution to Problem 1

By the strong solution of Problems 1 is meant the pair $\{u, k\} \in C^1([0, T]; W_2^2(\Omega)) \times C([0, T])$ satisfying the equation (0.1) almost everywhere in Q_T and the conditions (1.2)–(1.4) for almost all $(t, x) \in S_T$.

In addition to the notations of Section 2 we introduce the function $h^\eta(t,x)$ as the solution of the Dirichlet problem

$$h^\eta + \eta M h^\eta = 0 \quad \text{in } \Omega, \quad h^\eta\big|_{\partial\Omega} = \omega(t,x), \tag{2.1}$$

and the notations

$$\Phi^\eta(t) = \varphi_2(t) - \frac{\eta}{2}\left\langle Ma_t, h^\eta\right\rangle_{1,M} + (f(t,x) - a_t, h^\eta), \quad \overline{\Phi}^\eta = \max_{t\in[0,T]} \Phi^\eta(t).$$

The existence and uniqueness of the strong solution to Problems 2 is established by the following theorem [6].

Theorem 2.1 *Let the assumptions* I–III *be fulfilled and* η *be a positive constant. Assume that*
(i) $f \in C([0,T]; L^2(\Omega))$, $\beta \in C^1([0,T]; W_2^{3/2}(\partial\Omega))$, $U_0 \in L^2(\Omega)$, $g \in C(\overline{Q}_T)$,
$\omega \in C^1([0,T]; W_2^{3/2}(\partial\Omega))$, $\varphi_1 \in C^1([0,T])$, $\varphi_2 \in C([0,T])$;
(ii) f, U_0, β, ω, φ_1 *are nonnegative and*

$$\int_\Omega h^\eta\, dx \geqslant h_0 = \text{const} > 0, \quad t \in [0,T];$$

(iii) *there exist positive constants* α_i, $i = 0,1,2$, *such that* $\alpha_0,\ \alpha_1 \leqslant 1$, $\alpha_0 + \alpha_1 < 2$,

$$(1-\alpha_0)\,\varphi_1(t) + (1-\alpha_1)\,\Psi(t) \geqslant \alpha_2, \quad t \in [0,T],$$

$$\chi(0) + a(0,x) - U_0(x) \geqslant 0 \quad \textit{for almost all } x \in \Omega,$$

$$g(t,x)\chi(t) + \chi'(t) + F(t,x) \geqslant 0 \quad \textit{for almost all } (t,x) \in Q_T,$$

where

$$\chi(t) = \eta\,(\alpha_0\varphi_1(t) + \alpha_1\Psi(t))\left[\int_\Omega h^\eta\, dx\right]^{-1};$$

(iv) *for any* $t \in [0,T]$

$$\Phi^\eta(t) \geqslant \Phi_0^\eta = \text{const} > 0$$

holds and $g(t,x)$ *satisfies the inequality*

$$\max_{\overline{Q}_T} g(t,x) \leqslant \frac{\Phi_0^\eta}{\eta}\left[\overline{\varphi}_1 + \overline{\Psi} + \eta^{-1}\max_{[0,T]}(a, h^\eta)\right]^{-1} \equiv \frac{k_0}{\eta}.$$

Then Problem 1 has a unique solution $(u,k) \in C^1([0,T]; W_2^2(\Omega)) \times C([0,T])$. *Moreover, the estimates*

$$0 \leqslant u(t,x) \leqslant \chi(t) + a(t,x) \quad \textit{for almost all } (t,x) \in Q_T, \tag{2.2}$$

$$\|u\|_2^2 + \|u_t\|_2^2 \leqslant C_1, \quad t \in [0,T] \tag{2.3}$$

are fulfilled and the coefficient $k(t)$ *satisfies the inequalities*

$$K_0 \leqslant k(t) \leqslant K_1 \tag{2.4}$$

with some positive constants C_1, K_0 *and* K_1.

In the hypotheses of Theorem 2.1 the strong solution of Problem 2 depends continuously on the input data of the problem.

Theorem 2.2 *Let the pair* $\{u^i, k^i\}$ *be the strong solution of Problem 1 with* $\eta > 0$ *and the input data* $\{f_i, g_i, \beta_i, U_0^i, \omega_i, \varphi_1^i, \varphi_2^i\}$ *satisfying the hypotheses of Theorem 2.1,* $i = 1, 2$*. Then the estimates*

$$\|\tilde{k}\|_{C([0,T])} \leqslant C_2\Big\{\frac{1}{\alpha_2}\|\tilde{\varphi}_2\|_{C([0,T])} + K_1\|\tilde{\varphi}_1\|_{C([0,T])} + \|\tilde{U}_0\| + \\ + \max_{t\in[0,T]}\Big[\|\tilde{f}\| + \|\tilde{\beta}\|_{W_2^{1/2}(\partial\Omega)} + \|\tilde{\beta}_t\|_{W_2^{1/2}(\partial\Omega)} + \|\tilde{g}\|_{C(\overline{\Omega})} + \|\tilde{\omega}\|_{W_2^{1/2}(\partial\Omega)}\Big]\Big\}, \tag{2.5}$$

$$\|\tilde{u}\|_{C^1([0,T];W_2^2(\Omega))} \leqslant C_3\Big\{\frac{1}{\alpha_2}\|\tilde{\varphi}_2\|_{C([0,T])} + K_1\|\tilde{\varphi}_1\|_{C([0,T])} + \|\tilde{U}_0\| + \\ \max_{t\in[0,T]}\Big[\|\tilde{f}\| + \|\tilde{\beta}\|_{W_2^{3/2}(\partial\Omega)} + \|\tilde{\beta}_t\|_{W_2^{3/2}(\partial\Omega)} + \|\tilde{g}\|_{C(\overline{\Omega})} + \|\tilde{\omega}\|_{W_2^{1/2}(\partial\Omega)}\Big]\Big\}, \tag{2.6}$$

are valid for the difference $\{\tilde{u}, \tilde{k}\} = \{u^1 - u^2, k^1 - k^2\}$ *with certain positive constants* C_2 *and* C_3 *where* $\tilde{\varphi}_j = \varphi_j^1 - \varphi_j^2$*,* $j = 1, 2$*,* $\tilde{U}_0 = U_0^1 - U_0^2$*,* $\tilde{\beta} = \beta_1 - \beta_2$*,* $\tilde{f} = f_1 - f_2$*,* $\tilde{g} = g_1 - g_2$*,* $\tilde{\omega} = \omega_1 - \omega_2$*.*

Proof. The difference $\{\tilde{u}, \tilde{k}\}$ obeys the relations

$$\begin{cases} \tilde{u}_t + \eta M\tilde{u}_t + k^1(t)M\tilde{u} + g_1\tilde{u} = \tilde{f} - \tilde{g}u^2 - kMu^2, \\ (\tilde{u} + \eta M\tilde{u})\big|_{t=0} = \tilde{U}_0, \quad \tilde{u}\big|_{S_T} = \tilde{\beta}, \end{cases} \tag{2.7}$$

and the condition

$$\int_{\partial\Omega}\left\{\frac{\partial}{\partial\overline{N}}\big[\eta\tilde{u}_t + k^1\tilde{u} + \tilde{k}u^2\big]\omega_1 + \frac{\partial}{\partial\overline{N}}\big[\eta u_t^2 + k^2u^2\big]\tilde{\omega}\right\}dS + \tilde{\varphi}_1k^1 + \varphi_1^2\tilde{k} = \tilde{\varphi}_2. \tag{2.8}$$

Multiplying the first equality of (2.7) by $\tilde{u} - \tilde{a}$ in terms of the scalar product of $L^2(\Omega)$, integration by parts in the second and third term of the left part and in the last term of the right side of the resulting relation gives

$$\frac{1}{2}\frac{\partial}{\partial t}\Big(\|\tilde{u} - \tilde{a}\|^2 + \eta\langle M(\tilde{u} - \tilde{a}), \tilde{u} - \tilde{a}\rangle_1\Big) + k^1(t)\|M(\tilde{u} - \tilde{a})\|^2 + (g_1(\tilde{u} - \tilde{a}), \tilde{u} - \tilde{a}) = \\ = (\tilde{f}, \tilde{u} - \tilde{a}) - (\tilde{g}u^2, \tilde{u} - \tilde{a}) - \tilde{k}\langle Mu^2, \tilde{u} - \tilde{a}\rangle_1.$$

Integrating this equation with respect to t on $(0, \tau)$, $0 < \tau \leqslant T$, and estimating the right part with the help of (1.1), (2.2)–(2.4) and the Cauchy inequality one can obtain the estimate

$$\|\tilde{u} - \tilde{a}\|^2 + \eta\langle M(\tilde{u} - \tilde{a}), \tilde{u} - \tilde{a}\rangle_1 \leqslant \int_0^\tau\left[\frac{2}{K_0m_1}\Big(\|\tilde{f}\|^2 + \|\tilde{a}_t\|^2 + C_1\|\tilde{g}\|^2_{C(\overline{\Omega})}\Big) + \frac{\bar{g}_1}{2}\|\tilde{a}\|^2\right]dt + \\ + (\|\tilde{U}_0\| + \|\tilde{a}_0\|)^2 + C_1m_2\int_0^\tau|\tilde{k}|^2dt \tag{2.9}$$

where $\bar{g}_1 = \|g_1\|_{c(\overline{Q}_T)}$, $\tilde{a}_0 = \tilde{a}(0, x)$. Furthermore, multiplying the first equality of (2.7) by $M\tilde{u}$ in terms of the scalar product of $L^2(\Omega)$, integration by parts in the first term of the left part, integrating the result with respect to t on $(0, \tau)$, $0 < \tau \leqslant T$, and estimating the right part with the help of (1.1), (2.2)–(2.4), (2.9) and the Cauchy inequality we can get the estimate

$$\|\tilde{u}\|_2 \leqslant C_4\left\{\int_0^t\Big[\|\tilde{a}\|^2_{W_2^2(\Omega)} + \|\tilde{a}_t\|^2_{W_2^2(\Omega)} + \|\tilde{f}\|^2 + \|\tilde{g}\|^2_{C(\overline{\Omega})} + |\tilde{k}|^2\Big]d\tau\right\}^{1/2} + \\ + \|\tilde{U}_0\| + c_0\|\tilde{a}\|_{W_2^2(\Omega)}. \tag{2.10}$$

Here c_0 is the constant in the inequality

$$\|\tilde{u} - \tilde{a}\|_2 \leqslant c_0 \|M\tilde{u}\| \tag{2.11}$$

following from the second energy estimate for an elliptic operator [4, Ch. 2]; the constant $C_4 > 0$ depends on η, m_1, m_2, K_0, K_1, T, C_1, $\|g_i\|_{C(\overline{Q}_T)}$, $i = 1, 2$. In a similar manner, multiplying the first equality of (2.7) by $M\tilde{u}_t$ in terms of the scalar product of $L^2(\Omega)$, integrating by parts in the first term of the left part, rearranging the third and fourth terms to the right side of the result we are led to the equation

$$\langle \tilde{u}_t - \tilde{a}_t, M\tilde{u}_t \rangle_1 + \eta \|M\tilde{u}_t\|^2 = (-\tilde{a}_t - k^1(t) M\tilde{u} - g_1 \tilde{u} + \tilde{f} - \tilde{g} u^2 - kMu^2, M\tilde{u}_t)$$

whence it follows by (1.1), (2.2)–(2.4), (2.9)–(2.11) and the Cauchy inequality that

$$\|\tilde{u}_t\|_2 \leqslant C_5 \Big\{ \max_{t \in [0,T]} \Big[\|\tilde{a}\|_{W_2^2(\Omega)} + \|\tilde{a}_t\|_{W_2^2(\Omega)} + \|\tilde{f}\| + \|\tilde{g}\|_{C(\overline{\Omega})} \Big] + \|\tilde{U}_0\| + \\ + |\tilde{k}| + \Big[\int_0^t |\tilde{k}|^2 d\tau \Big]^{1/2} \Big\}. \tag{2.12}$$

The positive constant C_5 depends on η, m_1, m_2, K_0, K_1, T, C_1, C_4 $\|g_i\|_{C(\overline{Q}_T)}$, $i = 1, 2$.

On the other hand, as is shown in [6], following the idea of [9] we can reduce Problem 2 to an equivalent inverse problem with a nonlinear operator equation for $k^i(t)$. Really, let h_i^η be the solution of the problem (2.1) with the boundary data ω_i instead of ω. Multiplying (0.1) for u^i, k^i by $h^\eta(t, x)$ in terms of the inner product in $L^2(\Omega)$, integrating by parts twice, substituting (1.6) into the resulting equation and taking into account (2.8) and the fact that

$$\int_{\partial\Omega} (\eta \beta_{it} + k^i(t) \beta_i) \frac{\partial h_i^\eta}{\partial \overline{N}} ds = -\eta \langle M a_{it}, b^i \rangle_M - (a_{it}, h_i^\eta) + k^i(t) \Psi^i(t) + \frac{k^i(t)}{\eta} (a_i, h_i^\eta),$$

we obtain

$$k^i(t) \Big(\varphi_1^i(t) + \Psi^i(t) + \frac{1}{\eta} (a_i - u^i, h_i^\eta) \Big) = \Phi_i^\eta(t) - \big(g_i u^i, h_i^\eta \big), \quad i = 1, 2, \tag{2.13}$$

where $\Psi^i(t) = \big\langle M a_i, b_i \big\rangle_{1,M}$, $\Phi_i^\eta(t) = \varphi_2^i(t) - \dfrac{\eta}{2} \langle M a_{it}, b_i \rangle_{1,M} + (f_i - a_{it}, h_i^\eta)$, the functions a_i and b_i are the solutions of the problems (1.7) with the boundary data β_i and ω_i instead of β and ω, respectively.

Setting up the difference of the operator equations (2.13) for $i = 1$ and $i = 2$ we are led to the equation

$$\tilde{k}(t) \Big(\varphi_1^1 + \Psi_1 + \frac{1}{\eta} (a_1 - u^1, h_1^\eta) \Big) = \tilde{\Phi}^\eta - \big(g_1 \tilde{u}, h_1^\eta \big) - \big(\tilde{g} u^2, h_1^\eta \big) - \\ - \big(g_2 u^2, \tilde{h}^\eta \big) - k^2 \Big(\tilde{\varphi}_1 + \tilde{\Psi} + \frac{1}{\eta} \big(\tilde{a} - \tilde{u}, h_1^\eta \big) + \frac{1}{\eta} \big(a_2 - u^2, \tilde{h}^\eta \big) \Big),$$

where $\tilde{a} = a_1 - a_2$, $\tilde{\Phi}^\eta = \Phi_1^\eta - \Phi_2^\eta$, $\tilde{h}^\eta = h_1^\eta - h_2^\eta$, $\tilde{\Psi} = \Psi_1 - \Psi_2$. Estimating the right side of this equation with the use of (2.2)–(2.4), (2.9) one can obtain the inequality

$$|\tilde{k}| \leqslant C_6 \Big[\|\tilde{a}_t\|_1 + \|\tilde{a}\| + \|\tilde{b}\|_1 + \|\tilde{h}^\eta\| + \|\tilde{f}\| + \|\tilde{g}\|_{C(\overline{\Omega})} + \|\tilde{U}_0\| + \|\tilde{a}_0\| \Big] + \\ + \frac{1}{\alpha_2} \Big(|\tilde{\varphi}_2| + K_1 |\tilde{\varphi}_1| \Big) + C_7 \int_0^\tau |\tilde{k}|^2 dt, \tag{2.14}$$

where positive constants C_6, C_7 depends on K_0, K_1, η, T, m_1, m_2, C_1, $\|g_i\|_{C(\overline{\Omega})}$, $\overline{\varphi}_1$, $\max_{t\in[0,T]}\left\{\|a_i\|,\|a_{it}\|,\|b_i\|,\|f_i\|\right\}$, $i=1,2$. By Gronwall's lemma and the inequality

$$\|v\|_j \leqslant c_j\|v\|_{W_2^{j-1/2}(\partial\Omega)} \tag{2.15}$$

valid for all $v\in W_2^j(\Omega)$ and an integer $j\geqslant 1$ (see [4, Ch. 2]), (2.14) implies (2.5). Now the estimate (2.6) follows from (2.5), (2.10), (2.12) and (2.15). Theorem is proved. □

3. The regularity of the solution to Problem 2

By the strong solution of Problem 2 is meant the pair $\{u,\eta\}\in C^1([0,T];W_2^2(\Omega))\times C^1([0,T])$ satisfying the equation (0.1) almost everywhere in Q_T and the conditions (1.2), (1.3), (1.5), (1.6) for almost all $(t,x)\in S_T$.

The existence and uniqueness of the strong solution to Problem 2 is established by the following theorem [5].

Theorem 3.1 *Let the assumptions I–II be fulfilled and $\partial\Omega\in C^2$. Assume that*

i) $f\in C([0,T];L^2(\Omega))$, $\beta\in C^1([0,T];W_2^{3/2}(\partial\Omega))$, $U_0\in L^2(\Omega)$, $g\in C(\overline{Q}_T)$, $\omega\in C^1([0,T];W_2^{3/2}(\partial\Omega))$, $\varphi_1\in C^1([0,T])$, $\varphi_2\in C([0,T])$;

ii) f, U_0, β, ω *and* φ_1 *are nonnegative,* $g\leqslant 0$, $\mu_2>0$ *and* $\varphi_1(0)=\mu_1$;

iii) $\Psi(t)\geqslant 0$ *and there exist a positive constant* α *such that*

$$\varphi_1(t)+\Psi(t)\geqslant\alpha,\quad t\in[0,T], \tag{3.1}$$

$$\Phi(t)\equiv\varphi_2(t)-\Psi(t)+(f,b)\geqslant 0.$$

Then Problem 2 has a unique solution $\{u,\eta\}$ in the class

$$V=\left\{\{u,\eta\}\middle|\ u\in C^1([0,T];W_2^2(\Omega)),\ \eta\in C^1([0,T])\right\}.$$

Moreover, there are positive constants η_0 and η_1 such that for all $t\in[0,T]$

$$\eta_0\leqslant\eta(t)\leqslant\eta_1 \tag{3.2}$$

and the estimates

$$|\eta'|\leqslant C_8, \tag{3.3}$$

$$\|u\|_2+\|u_t\|_2\leqslant C_9 \tag{3.4}$$

holds with certain constants C_8 and C_9.

In the hypotheses of Theorem 3.1 the strong solution of Problem 2 depends continuously on the input data of the problem.

Theorem 3.2 *Let the pair $\{u^i,\eta^i\}$ be the strong solution of Problem 2 with the input data $\{f_i,g_i,\beta_i,U_0^i,\omega_i,\varphi_1^i,\varphi_2^i,\mu_1^i,\mu_2^i\}$ satisfying the hypotheses of Theorem 3.1, $i=1,2$. Then the estimates*

$$\|\tilde\eta\|_{C^1([0,T])}\leqslant C_{10}\Big\{\|\tilde\varphi_1\|_{C^1([0,T])}+\|\tilde\varphi_2\|_{C([0,T])}+\|\tilde U_0\|+\|\tilde g\|_{C(\overline{Q}_T)}+|\tilde\mu_2|+$$
$$+\max_{t\in[0,T]}\Big[\|\tilde f\|+\|\tilde\beta\|_{W_2^{1/2}(\partial\Omega)}+\|\tilde\beta_t\|_{W_2^{1/2}(\partial\Omega)}+\|\tilde\omega\|_{W_2^{1/2}(\partial\Omega)}+\|\tilde\omega_t\|_{W_2^{1/2}(\partial\Omega)}\Big]\Big\}, \tag{3.5}$$

$$\|\tilde u\|_{C^1([0,T];W_2^2(\Omega))} \leqslant C_{11}\Big\{\|\tilde\varphi_1\|_{C^1([0,T])} + \|\tilde\varphi_2\|_{C([0,T])} + \|\tilde U_0\| + \|\tilde g\|_{C(\overline\Omega)} + |\tilde\mu_2| + \\ + \max_{t\in[0,T]}\Big[\|\tilde f\| + \|\tilde\beta\|_{W_2^{3/2}(\partial\Omega)} + \|\tilde\beta_t\|_{W_2^{3/2}(\partial\Omega)} + \|\tilde\omega\|_{W_2^{1/2}(\partial\Omega)} + \|\tilde\omega_t\|_{W_2^{1/2}(\partial\Omega)}\Big]\Big\}, \qquad (3.6)$$

are valid for the difference $\{\tilde u,\tilde\eta\} = \{u^1-u^2,\eta^1-\eta^2\}$ *with certain positive constants* C_{10} *and* C_{11} *where again* $\tilde\varphi_j = \varphi_j^1-\varphi_j^2,\ j=1,2,\ \tilde U_0 = U_0^1-U_0^2,\ \tilde\beta = \beta_1-\beta_2,\ \tilde f = f_1-f_2,\ \tilde g = g_1-g_2,$ $\tilde\omega = \omega_1-\omega_2$.

Proof. The difference $\{\tilde u,\tilde\eta\}$ obeys the relations

$$\begin{cases} (\eta_1 M\tilde u)_t + M\tilde u + g_1\tilde u = \tilde f - \tilde g u^2 - (\tilde\eta M u^2)_t, \\ (\eta_1 M\tilde u)\big|_{t=0} = \tilde U_0 - (\tilde\eta M u^2)\big|_{t=0}, \quad \tilde u\big|_{S_T} = \tilde\beta, \end{cases} \qquad (3.7)$$

and the conditions

$$\int_{\partial\Omega}\Big\{\frac{\partial}{\partial\overline N}\Big[(\eta^1\tilde u)_t + (\tilde\eta u^2)_t + \tilde u\Big]\omega_1 + \frac{\partial}{\partial\overline N}\Big[(\eta^2u^2)_t + u^2\Big]\tilde\omega\Big\}ds + (\tilde\eta\varphi_1^1)_t + (\eta^2\tilde\varphi_1)_t = \tilde\varphi_2,$$

$$\int_{\partial\Omega}\Big\{\frac{\partial}{\partial\overline N}\Big[\eta^1\tilde u + \tilde\eta u^2\Big]\omega_1 + \eta^2\frac{\partial u^2}{\partial\overline N}\tilde\omega\Big\}ds\Big|_{t=0} + \tilde\mu_1\eta^1(0) + \mu_1^2\tilde\eta(0) = \tilde\mu_2.$$

Multiplying the first equality of (3.4) by $\exp\left(\int\limits_0^t(\eta^1(\tau))^{-1}d\tau\right)$, integration with respect to t from 0 to $\theta,\ 0<\theta\leqslant T$, and solving the resulting equation for $M\tilde u$ gives

$$M\tilde u = \frac{1}{\eta^1(\theta)}\Big\{\tilde U_0\exp\Big(-\int_0^\theta\frac{d\tau}{\eta^1(\tau)}\Big) + \int_0^\theta(\tilde f - g_1\tilde u - \tilde g u_2)\exp\Big(-\int_t^\theta\frac{d\tau}{\eta^1(\tau)}\Big)dt - \\ - \tilde\eta M u^2 + \int_0^\theta\frac{\tilde\eta}{\eta^1(t)}Mu^2\exp\Big(-\int_t^\theta\frac{d\tau}{\eta^1(\tau)}\Big)dt\Big\}. \qquad (3.8)$$

Furthermore, multiplying this equation by $\tilde u-\tilde a$ in terms of the inner product of $L^2(\Omega)$, integrating by parts in the left side of the resulting equation and estimating the right one with (3.2), (3.4) one can obtain the inequality

$$\|\tilde u\|_1 \leqslant C_{12}\Big\{\|\tilde a\|_{C([0,T];W_2^1(\Omega))} + \|\tilde U_0\| + \int_0^T\big(\|\tilde f\| + \|\tilde g\|_{C(\overline\Omega)}\big)dt + \Big(\int_0^t|\tilde\eta|^2d\theta\Big)\Big\} \qquad (3.9)$$

where the constant $C_{12}>0$ depends on m_1, T, c_0, $\|g_1\|_{C(\overline Q_T)}$ and the constant $m_3>0$ from the inequality

$$\|Mv\|_2 \leqslant m_3\|v\|_2 \qquad (3.10)$$

valid for all $v\in W_2^2(\Omega)$. Now we estimate the left and right sides of (3.8) in the norm of $L^2(\Omega)$ with the use of (2.11), (3.2), (3.4). Taking into account (3.10) we get

$$\|\tilde u\|_2 \leqslant \|\tilde a\|_2 + c_0\|M\tilde u\| \leqslant \|\tilde a\|_2 + C_{13}\Big\{\|\tilde U_0\| + \int_0^\theta\big(\|\tilde f\| + \|\tilde g\|_{C(\overline\Omega)}\big)dt + |\tilde\eta| + \int_0^\theta|\tilde\eta|dt\Big\} + \\ + \bar g_1\int_0^\theta\|\tilde u\|_2dt,$$

whence in accordance with Gronwall's lemma

$$\|\tilde u\|_2 \leqslant C_{14}\Big\{\|\tilde a\|_{C([0,T];W_2^2(\Omega))} + \|\tilde U_0\| + \int_0^T\big(\|\tilde f\| + \|\tilde g\|_{C(\overline\Omega)}\big)dt + \int_0^t|\tilde\eta|d\theta\Big\}. \qquad (3.11)$$

Here the positive constants C_{13} and C_{14} depends on η_0, T, m_3, c_0, $\|g_1\|_{C(\overline{Q}_T)}$.

On the other hand, it is shown in [6] that following the idea of [9] one can reduce Problem 2 to an equivalent inverse problem with a nonlinear operator equation for $\eta^i(t)$. Really, let us multiply (2.4) by b in terms of the inner product of $L^2(\Omega)$ and integrate by parts twice in the second and third terms. In view of (1.5)–(1.7) we have

$$\frac{d}{dt}\left(\eta^i(\varphi_1^i+\Psi^i)\right)-\eta^i\langle Ma_i,b_{it}\rangle_1=\Phi_i-(g_iu^i,b_i)$$

where $\Phi_i=\Phi_i(t)\equiv\varphi_2^i(t)-\Psi_i(t)+(f_i,b_i)$, the functions a_i and b_i are again the solutions of the problems (1.7) with the boundary data β_i and ω_i instead of β and ω, respectively. By (1.5), multiplying this equation by $\zeta_i(t)\equiv\exp\Big(-\int\limits_0^t\langle Ma_i,b_{i\tau}\rangle_1(\varphi_1^i+\Psi_i)^{-1}d\tau\Big)$ and integration with respect to t from 0 to $\theta,\ 0<\theta\leqslant T$ gives

$$\eta^i(\theta)\left(\varphi_1^i(\theta)+\Psi_i(\theta)\right)\zeta_i(\theta)=\eta^i(0)\big(\varphi_1^i(0)+\Psi_i(0)\big)+\int_0^\theta\left[\Phi_i-(g_iu^i,b_i)\right]\zeta_i\,dt,\quad i=1,2.\quad(3.12)$$

Furthermore, we multiply the second relation in (3.7) by $b_i(0,x)=b_i^0(x)$ in terms of the inner product of $L^2(\Omega)$ and integrate by parts twice in the resulting equation. Taking into account (1.3) for $t=0$ and (1.6), we obtain

$$\eta^i(0)\big(\mu_1^i+\Psi_i(0)\big)=\mu_2^i+(U_0^i,b_i^0),\quad i=1,2.\qquad(3.13)$$

Substituting (3.13) into the operator equation (3.12) and setting up the difference of the resulting equations for $i=1$ and $i=2$ we are led to the equality

$$\begin{aligned}\tilde\eta(\theta)\left(\varphi_1^1(\theta)+\Psi_1(\theta)\right)\zeta_1(\theta)=-\eta^2(\theta)\Big[\Big(\tilde\varphi_1(\theta)+\tilde\Psi(\theta)\Big)\zeta_1(\theta)+\left(\varphi_1^2(\theta)+\Psi_2(\theta)\right)\tilde\zeta(\theta)\Big]+\\+\tilde\mu_2+(\tilde U_0,b_1^0)+(U_0^2,\tilde b^0)+\int_0^\theta\Big[\tilde\Phi-(\tilde gu^1,b_1)-(g_2\tilde u,b_1)-(g_2u^2,\tilde b)\Big]\zeta_1\,dt+\\+\int_0^\theta\left[\Phi_2-(g_2u^2,b_2)\right]\tilde\zeta\,dt\qquad(3.14)\end{aligned}$$

where $\tilde\zeta=\zeta_1-\zeta_2,\ \tilde{b^0}=b_1^0-b_2^0,\ \tilde\Phi=\Phi_1-\Phi_2$. Let $y_i(t)=\int\limits_0^t\langle Ma_i,b_{i\tau}\rangle_1(\varphi_1^i+\Psi_i)^{-1}d\tau$, $i=1,2$. By (3.1) and the definition of functions ζ, a and b,

$$|\tilde\zeta|=\Big|\int_{y_1(t)}^{y_2(t)}e^{-y}dy\Big|\leqslant\exp(\max_{i=1,2}|y_i(t)|)\ |y_1-y_2|\leqslant C_{15}\int_0^t\left[\|\tilde a\|_1+\|\tilde b_\tau\|_1+|\tilde\varphi_1|+|\tilde\Psi|\right]d\tau.\quad(3.15)$$

The constant C_{15} depends on m_2, $\|\varphi_1^i\|_{C([0,T])}$, $\max_{t\in[0,T]}\left\{\|a_i\|,\|b_i\|,\|b_{it}\|\right\}$, $i=1,2$. Estimating the right side of the equation (3.14) with regard to (3.2)–(3.4), (3.9) and (3.15) one can obtain the inequality

$$\begin{aligned}|\tilde\eta|\leqslant C_{16}\Big\{\|\tilde\varphi_1\|_{C([0,T])}+\|\tilde a\|_{C([0,T];W_2^1(\Omega))}+\|\tilde b\|_{C^1([0,T];W_2^1(\Omega))}+|\tilde\mu_2|+\|\tilde U_0\|+\\+\|\tilde\varphi_2\|_{C([0,T])}+\|\tilde f\|_{L^2(Q_T)}+\|\tilde g\|_{C(\overline{Q}_T)}\Big\}+C_{17}\int_0^t|\tilde\eta|d\tau\end{aligned}$$

which implies by Gronwall's lemma that

$$|\tilde{\eta}| \leqslant C_{16}e^{C_{17}T}\Big\{\|\tilde{\varphi}_1\|_{C([0,T])} + \|\tilde{a}\|_{C([0,T];W_2^1(\Omega))} + \|\tilde{b}\|_{C^1([0,T];W_2^1(\Omega))} + |\tilde{\mu}_2| + \|\tilde{U}_0\| + \\ + \|\tilde{\varphi}_2\|_{C([0,T])} + \|\tilde{f}\|_{L^2(Q_T)} + \|\tilde{g}\|_{C(\overline{Q}_T)}\Big\}. \tag{3.16}$$

Here the positive constants C_{16} and C_{17} depend on η_0, η_1, T, m_2, C_9 C_{14}, C_{15}, $\|g_i\|_{C(\overline{Q}_T)}$, $\|\varphi_1^i\|_{C([0,T])}$, $\max_{t\in[0,T]}\left\{\|a_i\|, \|a_{it}\|, \|b_i\|, \|b_{it}\|, \|f_i\|\right\}$, $i = 1, 2$.

We are now in a position to obtain the estimates for $\tilde{\eta}'$ and $\tilde{u}_t$. Solving the first equation of (3.7) for $M\tilde{u}_t$ and estimating the right side of the resulting equation with the use of (2.11), (3.2)–(3.4), (3.10), (3.11) yields

$$\|\tilde{u}_t\|_2 \leqslant \|\tilde{a}_t\|_2 + C_{18}\Big\{\|\tilde{U}_0\| + \max_{t\in[0,T]}\left\{\|\tilde{a}\|_2 + \|\tilde{f}\| + |\tilde{\eta}|\right\} + \|\tilde{g}\|_{C(\overline{Q}_T)} + |\tilde{\eta}'|\Big\} \tag{3.17}$$

where the constant $C_{18} > 0$ depends on m_3, η_0, T, c_0, C_8, C_9, C_{14}, $\|g_1\|_{C(\overline{Q}_T)}$. Furthermore, differentiating (3.14) and estimating the right part of the resulting equation with (3.2)–(3.4), (3.9) and (3.15) we are led to the relation

$$|\tilde{\eta}'| = \Big| -\tilde{\eta}\Big[((\varphi_1^1)' + (\Psi_1)')\zeta_1 + (\varphi_1^1 + \Psi_1)\zeta_1'\Big] - (\eta^2)'\Big[(\tilde{\varphi}_1 + \tilde{\Psi})\zeta_1 + (\varphi_1^2 + \Psi_2)\tilde{\zeta}\Big] - \\ - \eta^2\Big[(\tilde{\varphi}_1' + \tilde{\Psi}')\zeta_1 + ((\varphi_1^2)' + \Psi_2')\tilde{\zeta} + (\tilde{\varphi}_1 + \tilde{\Psi})\zeta_1' + (\varphi_1^2 + \Psi_2)\tilde{\zeta}'\Big] + (\Phi_2 - (g_2u^2, b_2))\tilde{\zeta} + \\ + \Big[\tilde{\Phi} - (\tilde{g}u^1, b_1) - (g_2\tilde{u}, b_1) - (g_2u^2, \tilde{b})\Big]\zeta_1\Big|\Big((\varphi_1^1 + \Psi_1)\zeta_1\Big)^{-1} \leqslant C_{19}\Big[\|\tilde{\varphi}_1\|_{C^1([0,T])} + \|\tilde{U}_0\| + \\ + \|\tilde{\varphi}_2\|_{C([0,T])} + \|\tilde{g}\|_{C(\overline{Q}_T)} + \max_{t\in[0,T]}\left\{\|\tilde{f}\| + \|\tilde{a}\|_1 + \|\tilde{a}_t\|_1 + \|\tilde{b}\|_1 + \|\tilde{b}_t\|_1\right\} + |\tilde{\mu}_2|\Big] \tag{3.18}$$

Here the positive constants C_{19} depends on η_0, η_1, T, m_2, C_9 C_{14}, C_{15}, C_{16}, C_{17}, $\|g_i\|_{C(\overline{Q}_T)}$, $\|\varphi_1^i\|_{C^1([0,T])}$, $\max_{t\in[0,T]}\left\{\|a_i\|, \|a_{it}\|, \|b_i\|, \|b_{it}\|, \|f_i\|\right\}$, $i = 1, 2$. By (2.15), the inequalities (3.16) and (3.18) imply (3.5). Now the estimate (3.6) follows from (2.15), (3.11), (3.16)–(3.18). □

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Регулярность решений обратных задач для псевдопараболических уравнений

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Аннотация. В работе обсуждается регулярность решений обратных задач отыскания неизвестного коэффициента, зависящего от времени, в псевдопараболическом уравнении третьего порядка по дополнительной информации о решении на границе. Доказана регулярность решения двух обратных задач восстановления неизвестного коэффициента в члене второго порядка и старшем члене линейного псевдопараболического уравнения.

Ключевые слова: непрерывная зависимость от исходных данных, априорная оценка, обратная задача, псевдопараболическое уравнение.

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Boundary Value Problems for Fourth-Order Sobolev Type Equations

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Abstract. The goal of the article is the study of solvability in the Sobolev spaces of boundary value problems for some classes of Sobolev-type fourth-order linear equations. We will prove that an initial boundary value problems well problems with data both at the initial time moment and the final time moments can be well-posed for the equations under study.

Keywords: Sobolev-type fourth-order differential equation, boundary value problem, existence, uniqueness.

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Introduction

The article is devoted to the study of the solvability of boundary value problems for differential equations

$$D_t^4 u + \sum_{k=0}^{3} A_k D_t^k u = f(x,t) \quad \left(D_t^k = \frac{\partial^k}{\partial t^k}, \quad k = \overline{0,4} \right) \tag{$*$}$$

with operators A_k of the form

$$A_k = \frac{\partial}{\partial x_i}\left(a^{ij,k}(x)\frac{\partial}{\partial x_j}\right) + a_{0,k}(x)$$

(here and below, summation over repeated indices from 1 to n is carried out).

The differential equations $(*)$ are recently attributed to the class of Sobolev-type equations. Various aspects of the theory of Sobolev-type equations are reflected in monographs [1–7] and also in numerous journal articles (it is impossible to mention even a small part of such articles just because they are numerous).

For Sobolev-type differential equations, best studied is the solvability of the Cauchy problem and initial boundary value problems. At the same time, as is shown in [3, 8], in some case, for Sobolev-type equations, simultaneously with initial boundary value problems, other problems can also be well-posed; these include problems with data both at the initial and final time moments. In the present article, for equations $(*)$, we study the solvability both of initial boundary value problems and problems with data at different time moments.

Clarify that the goal of the present article is to prove the solvability of some problem for equations $(*)$ in the classes of regular solutions, i.e., solutions having all weak derivatives in the sense of Sobolev [9–11] occurring in the equation.

Formally, equation $(*)$ with the above operators is a fifth-order equation. The use of the term "fourth-order Sobolev equation" in the title and the article means that the equations under study

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are fourth-order equations with respect to the time (distinguished) variable, which is the leading variable and defines the statements of the problems.

One more remark: Equations ($*$) have model and the simplest form. We will speak of some more general equations and of generalizations of the results at the end of the article.

1. Statements of the Problems

Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with smooth (for simplicity, infinitely differentiable) boundary Γ, Q is the cylinder $\Omega\times(0,T)$ of finite height T, and $S=\Gamma\times(0,T)$ is the lateral boundary of Q. Furthermore, let $a^{ij,k}(x)$, $a_{0,k}(x)$, $i,j=1,\dots,n$, $k=0,\dots,3$, $f(x,t)$ be given functions defined for $x\in\overline{\Omega}$ and $t\in[0,T]$ and let A_k and L be the differential operators whose action at a given function $v(x,t)$ is defined by the equalities

$$A_k v = \frac{\partial}{\partial x_i}\left(a^{ij,k}(x)v_{x_j}\right) + a_{0,k}(x)v,$$

$$Lv = D_t^k v + \sum_{k=0}^{3} A_k D_t^k v.$$

Boundary Value Problem I: *Find a function $u(x,t)$ that is a solution to the equation*

$$Lu = f(x,t) \tag{1}$$

in the cylinder Q such that

$$u(x,t)|_S = 0, \tag{2}$$

$$D_t^k u(x,t)\big|_{t=0,\,x\in\Omega} = 0, \quad k=0,1,2,3. \tag{3}$$

Boundary Value Problem II: *Find a function $u(x,t)$ that is a solution to equation (1) in Q and satisfies conditions (2) and also the condition*

$$D_t^k u(x,t)\big|_{t=0,\,x\in\Omega} = 0, \quad k=0,1,2, \quad D_t^3 u(x,t)\big|_{t=T,\,x\in\Omega} = 0. \tag{4}$$

Boundary Value Problem III: *Find a function $u(x,t)$ that is a solution to equation (1) in Q that satisfies conditions (2) and also the condition*

$$u(x,t)|_{t=0,\,x\in\Omega} = D_t^2 u(x,t)\big|_{t=0,\,x\in\Omega} = D_t u(x,t)|_{t=0,\,x\in\Omega} = D_t^3 u(x,t)\big|_{t=0,\,x\in\Omega} = 0. \tag{5}$$

Boundary Value Problem I is a usual initial boundary value problem for nonstationary differential equations of the fourth order (with respect to time). Boundary Value Problem II is a modified V. N. Vragov's problem (see [12–14]) for fourth-order quasihyperbolic equations. Finally, Boundary Value Problem III is in fact an elliptic boundary value problem.

In the present article, we propose sufficient conditions on the coefficients of (1) new compared to the previous works that guarantee the existence and uniqueness of regular solutions to boundary value problems I, II, or III.

2. Solvability of boundary value Problems I-III

Theorem 1. *Suppose the fulfillment of the conditions*

$$a^{ij,k}(x)\in C^1(\overline{\Omega}), \quad i,j=1,\dots,n, \quad a_{0,k}(x)\in C(\overline{\Omega}), \quad k=0,1,2; \tag{6}$$

$$a^{ij,3}(x)\in C^2(\overline{\Omega}), \quad a^{ij,3}(x)=a^{ji,3}(x), \quad i,j=1,\dots,n, \quad a_{0,3}(x)\in C(\overline{\Omega}), \tag{7}$$

$$-a^{ij,3}(x)\xi_i\xi_j \geqslant m_0|\xi|^2, \quad m_0>0, \quad x\in\overline{\Omega}, \quad \xi\in\mathbb{R}^n. \tag{8}$$

Then, for every function $f(x,t)$ in $L_2(Q)$, Boundary Value Problem I has a solution $u(x,t)$ such that $D_t^k u(x,t)\in L_2(0,T;W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega))$, $k=0,1,2,3$, $D_t^4 u(x,t)\in L_2(Q)$.

Proof. Make use of the method of continuation in a parameter. Let $\lambda \in [0,1]$. Consider the following problem: *Find a function $u(x,t)$ that is a solution to the equation*

$$D_t^4 u + A_3 D_t^3 u + \lambda \sum_{k=0}^{2} A_k D_t^k u = f(x,t) \tag{9}$$

and Q that satisfies conditions (2) and (3). Note that, for $\lambda = 0$, this problem has a solution $u(x,t)$ belonging to the desired class; this follows from the fact that, for $\lambda = 0$, equation (9) is a usual parabolic equation with respect to $u_{ttt}(x,t)$. Furthermore, by the theorem on the method of extension in a parameter (see [15, Chapter III, Sec. 14], the boundary value problem (9), (2), (3) has a regular solution $u(x,t)$ if $f(x,t) \in L_2(Q)$ and problem (9), (2), (3) is solvable in the class of regular solutions for $\lambda = 0$ if all derivatives occurring in (9) are uniformly bounded in $L_2(Q)$.

For proving the desired boundedness, let us first consider the equality

$$\int_0^t \int_\Omega \left[D_\tau^4 u + A_3 D_\tau^3 u + \lambda \sum_{k=0}^{2} A_k D_\tau^k u \right] D_\tau^3 u \, dx \, d\tau = \int_0^t \int_\Omega f D_\tau^3 u \, dx \, d\tau. \tag{10}$$

Integrating by parts, applying Young's inequality and the inequality

$$\int_\Omega w^2(x,t) \, dx \leqslant T \int_0^t \int_\Omega w_\tau^2(x,\tau) \, dx \, d\tau, \tag{11}$$

which is valid for functions $w(x,t)$ vanishing for $t = 0$, and using conditions (6)–(8) and Gronwall's lemma, it is not hard to obtain from (10) the estimate

$$\int_\Omega \left[D_t^3 u(x,t) \right]^2 \, dx + \sum_{i=1}^{n} \int_0^t \int_\Omega \left(D_\tau^3 u_{x_i} \right)^2 \, dx \, d\tau \leqslant C_1 \int_Q f^2 \, dx \, dt, \tag{12}$$

where the constant C_1 is defined only by the functions $a^{ij,k}(x)$, $i,j = 1,\ldots,n$, $a_{0,k}(x)$, $k = 0,1,2,3$, and the number T.

Now, consider the equality

$$-\int_0^t \int_\Omega \left(D_\tau^4 u + A_3 D_\tau^3 u + \lambda \sum_{k=0}^{2} A_k D_\tau^k u \right) A_3 D_\tau^3 u \, dx \, d\tau = -\int_0^t \int_\Omega f A_3 D_\tau^3 u \, dx \, d\tau.$$

Integrating by parts once again, applying Young's inequality, inequality (11), estimate (12), conditions (6)–(8), and also the second main inequality for elliptic operators (see [10, Chapter III, Stc. 8], and Gronwall's lemma, we conclude that solutions $u(x,t)$ to the boundary value problem (9), (2), (3) satisfy the second a priori estimate

$$\sum_{i=1}^{n} \int_\Omega \left[D_t^3 u_{x_i}(x,t) \right]^2 \, dx + \sum_{i,j=1}^{n} \int_0^t \int_\Omega \left(D_\tau^3 u_{x_i x_j} \right)^2 \, dx \, d\tau \leqslant C_2 \int_Q f^2 \, dx \, dt, \tag{13}$$

where the constant C_2 is defined only by the functions $a^{ij,k}(x)$, $a_{0,k}(x)$, $i,j = 1,\ldots,n$, $k = 0,1,2,3$, the domain Ω, and the number T.

Estimates (12) and (13) imply the obvious third estimate

$$\int_0^t \int_\Omega \left(D_\tau^4 u \right)^2 \, dx \, d\tau \leqslant C_3 \int_Q f^2 \, dx \, dt, \tag{14}$$

of solutions $u(x,t)$ to the boundary value problem (9), (2), (3); the constant C_3 in this estimate is again defined only by the functions $a^{ij,k}(x)$, $a_{0,k}(x)$, $i,j=1,\dots,n$, $k=0,1,2,3$, the domain Ω, and the number T.

Estimates (12)–(14) give the desired uniform boundedness over λ in $L_2(Q)$ of all derivatives occurring in (9). As we already said above, this boundedness and the solvability of the boundary value problem (9), (2), (3) for $\lambda=0$ give the solvability of this problem in the desired class also for $\lambda=1$. This exactly means the validity of the theorem.

The theorem is proved. □

Before proving the following theorem on the solvability of Problem I in the class of regular solutions, we formulate an auxiliary assertion on the nonnegativity of the scalar product of a pair of second-order differential operators.

Let A and B be differential operators whose action is defined by the equality

$$Av = \frac{\partial}{\partial x_i}\left(a^{ij}(x)v_{x_j}\right) + a_0(x)v,$$

$$Bv = \frac{\partial}{\partial x_i}\left(b^{ij}(x)v_{x_j}\right) + b_0(x)v.$$

Proposition 1. *Suppose the fulfillment of the conditions*

$$a^{ij}(x)\in C^2(\overline{\Omega}),\quad b^{ij}(x)\in C^2(\overline{\Omega}),\quad a^{ij}(x)=a^{ji}(x),\quad b^{ij}(x)=b^{ji}(x),\quad x\in\overline{\Omega},\quad i,j=1,\dots,n;$$

$$a_0(x)\in C^1(\overline{\Omega}),\quad b_0(x)\in C^1(\overline{\Omega}),\quad a_0(x)\leqslant -a_0<0,\quad b_0(x)\leqslant -b_0<0,\quad x\in\overline{\Omega};$$

$$\exists\,\alpha^i(x):\ \alpha^i(x)\in C(\overline{\Omega}),\quad \alpha^i(x)\geqslant 0,\quad x\in\overline{\Omega},\quad i=1,\dots,n,$$

$$\alpha^i(x)\xi_i^2\leqslant a^{ij}(x)\xi_i\xi_j\leqslant M_0\alpha^i(x)\xi_i^2,\quad x\in\overline{\Omega},\quad \xi\in\mathbb{R}^n;$$

$$|a^{ij}_{x_k}(x)|\leqslant M_1\sqrt{\alpha^i(x)},\quad x\in\overline{\Omega},\quad i,j,k=1,\dots,n;$$

$$a^{ij}(x)\nu_i\nu_j=0\quad \textit{for}\quad x\in\Gamma;$$

$$b^{ij}(x)\xi_i\xi_j\geqslant m_0|\xi|^2,\quad m_0>0,\quad x\in\overline{\Omega},\quad \xi\in\mathbb{R}^n;$$

$$\Big[a_0(x)b^{ij}(x)+b_0(x)a^{ij}(x)+\frac{1}{2}\left(a^{ij}_{x_k}(x)b^{kl}(x)\right)_{x_l}+\frac{1}{2}\left(b^{ij}_{x_k}(x)a^{kl}(x)\right)_{x_l}-$$
$$-\left(a^{il}_{x_k}(x)b^{jk}_{x_l}(x)\right)\Big]\xi_i\xi_j\leqslant 0,\quad x\in\overline{\Omega},\quad \xi\in\mathbb{R}^n;$$

$$a_0(x)b_0(x)+\frac{1}{2}\left(a_{0x_i}(x)b^{ij}(x)\right)_{x_j}+\frac{1}{2}\left(b_{0x_i}(x)a^{ij}(x)\right)_{x_j}\geqslant 0,\quad x\in\overline{\Omega}.$$

Then every function $v(x)\in W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega)$ satisfies the inequality

$$\int_\Omega AvBv\,dx\geqslant 0$$

This assertion is proved in [16].

We say that operators A and B of the above form satisfy the (A,B)–condition if the coefficients of these operators satisfy all conditions of Proposition 1.

Theorem 2. *Suppose the fulfillment of the $(-A_3,-A_2)$–condition and also of the condition*

$$a^{ij,k}(x)\in C^1(\overline{\Omega}),\quad i,j=1,\dots,n,\quad a_{0,k}(x)\in C(\overline{\Omega}),\quad k=0,1. \tag{15}$$

Then, for every function $f(x,t)$ such that $f(x,t)\in L_2(Q)$, $f_t(x,t)\in L_2(Q)$, $f(x,0)=0$ for $x\in\overline{\Omega}$, Boundary Value Problem I has a solution $u(x,t)$ such that $D_t^k u(x,t)\in L_\infty(0,T;W_2^2(\Omega)\cap \overset{\circ}{W}{}_2^1(\Omega))$, $k=0,1,2,3$, $D_t^4u(x,t)\in L_\infty(0,T;L_2(\Omega))$.

Proof. Observe first of all that the $(-A_3, -A_2)$–condition in particular means that $-A_3$ is an elliptic-parabolic operator in $\overline{\Omega}$ and $-A_2$ is an elliptic operator.

Let ε be a positive number. Define operators $A_{3,\varepsilon}$ and L_ε:

$$A_{3,\varepsilon} = A_3 + \varepsilon A_2, \quad L_\varepsilon = L + \varepsilon A_2 D_t^3.$$

Consider the following boundary value problem: *Find a function $u(x,t)$ that is a solution to the equation $L_\varepsilon u = f$ in Q that satisfies conditions (2) and (3).* Obviously, this boundary value problem is Boundary Value Problem I and that it satisfies all conditions of Theorem 1. Moreover, due to the condition $f(x,t) \in L_2(Q)$, $f_t(x,t) \in L_2(Q)$, a solution $u(x,t)$ to this problem satisfies the memberships

$$D_t^k u(x,t) \in L_\infty(0,T; W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)), \quad k = 0,1,2,3,4, \quad D_t^5 u(x,t) \in L_2(Q) \tag{16}$$

(this fact stems from its validity for the "shortened" equation $D_t^4 u + A_{3,\varepsilon} D_t^3 u = f(x,t)$ and the corresponding a priori estimates).

Differentiate the equation $L_\varepsilon u = f(x,t)$ with respect to t (this is possible due to memberships (16)), multiply it by $D_t^4 u(x,t)$, and integrate it over the cylinder $\{x \in \Omega, \quad 0 < \tau < t\}$. Involving the ellipticity of the operators $-A_{3,\varepsilon}$ and $-A_2$, applying Young's inequality, inequality (11), and Gronwall's lemma, we obtain the estimate

$$\varepsilon \int_0^t \int_\Omega \left(A_2 D_\tau^4 u\right)^2 dx\, d\tau + \sum_{i=1}^n \int_\Omega \left[D_t^4 u_{x_i}(x,t)\right]^2 dx + \int_\Omega \left[A_2 D_t^3 u(x,t)\right]^2 dx \leqslant C_4 \int_Q f_t^2 dx\, dt, \tag{17}$$

where the constant C_4 is defined only by the functions $a^{ij,k}(x)$, $a_{0,k}(x)$, $i,j = 1,\ldots,n$, $k = 0,1$, and also the number T.

Let $\{\varepsilon_m\}_{m=1}^\infty$ be a sequence of positive numbers converging to zero and let $\{u_m(x,t)\}_{m=1}^\infty$ be a sequence of solutions to the equation $L_{\varepsilon_m} u = f$ satisfying (2) and (3). Estimate (17), the second main inequality for elliptic operators, and the reflexivity of a Hilbert space mean that there exists a sequence $\{u_{m_l}(x,t)\}_{l=1}^\infty$ and a function $u(x,t)$ that satisfy the following weak convergences as $l \to \infty$ in $L_2(Q)$:

$$\varepsilon_{m_l} A_2 D_t^3 u(x,t) \to 0,$$

$$D_t^4 u_{m_l}(x,t) \to D_t^4 u(x,t),$$

$$A_k D_t^k u_{m_l}(x,t) \to A_k D_t^k u(x,t), \quad k = 0,1,2,3.$$

Obviously, the limit function $u(x,t)$ is a solution to Boundary Value Problem I and this solution still satisfies (17). Therefore, the function $u(x,t)$ is the desired solution to the problem under study.

The theorem is proved. □

Turn to investigating the solvability of Boundary Value Problem II.

The main difference of Boundary Value Problem II from Boundary Value Problem I is that, in its study, it is impossible to use Gronwall's lemma. Gronwall's lemma can be replaced by smallness conditions.

We will give the simplest version of the theorem in the solvability of a Boundary Value Problem II, whose prove involves smallness conditions.

Let operators A_0 and A_1 be defined with the use of the parameter β and the operators $\widetilde{A}_0$ and $\widetilde{A}_1$:

$$A_0 = \beta \widetilde{A}_0, \quad A_1 = \beta \widetilde{A}_1, \quad \widetilde{A}_k = \frac{\partial}{\partial x_i}\left(\widetilde{a}^{ij,k}(x) \frac{\partial}{\partial x_j}\right) + \widetilde{a}_{0i}(x), \quad k = 0,1. \tag{18),}$$

Theorem 3. *Suppose the fulfillment of the conditions*

$$a^{ij,k}(x) \in C^2(\overline{\Omega}), \quad a^{ij,k}(x) = a^{ji,k}(x), \quad i,j = 1,\dots,n, \quad k = 2,3;$$

$$\widetilde{a}^{ij,k}(x) \in C^1(\overline{\Omega}), \quad i,j = 1,\dots,n, \quad \widetilde{a}_{0,k}(x) \in C(\overline{\Omega}), \quad k = 0,1;$$

$$a^{ij,k}(x)\xi_i\xi_j \geqslant m_0|\xi|^2, \quad m_0 > 0, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n, \quad k = 2,3;$$

$$a_{0,k}(x) \in C(\overline{\Omega}), \quad k = 0,1,2,3, \quad a_{0,k}(x) \leqslant 0, \quad k = 2,3.$$

Then there exists a positive number β_0 such that for $|\beta| < \beta_0$ and $f(x,t) \in L_2(Q)$, Boundary Value Problem II has a solution $u(x,t)$ such that $D_t^k u(x,t) \in L_2(0,T; W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega))$, $k = 0,1,2,3$, $D_t^4 u(x,t) \in L_2(Q)$.

Proof. For $\lambda = 0$, Boundary Value Problem II for equation (9) has a solution $u(x,t)$ in the desired class; this follows from the fact that for $\lambda = 0$ equation (9) is an inverse parabolic equation with respect to $D_t^3 u(x,t)$. Further, consider (10). Integrating by parts and estimating the last two summands on the left-hand side (10) from above with the use of (11), we infer that there exists a positive number β_1 such that for $|\beta| < \beta_0$ we have the a priori estimate

$$\sum_{i=1}^{n} \int_Q \left(D_t^3 u_{x_i}\right)^2 dx\,dt \leqslant C_5 \int_Q f^2\,dx\,dt \tag{19}$$

with the constant C_5 defined only by the coefficients of the operators A_k, $k = 0,1,2,3$.

At the next step, consider the equality

$$\int_Q \left[D_t^4 u + A_3 D_t^3 u + \lambda \sum_{k=0}^{2} A_k D_t^k u\right] A_2 D_t^3 u\,dx\,dt = \int_Q f A_2 D_t^3 u\,dx\,dt.$$

Reckoning with the ellipticity of A_2 and A_3 and using the second main inequality for a pair of elliptic operators [10, Chapter III, Sec. 8], it is not hard to show that there exists a number β_0 such that $0 < \beta_0 \leqslant \beta_1$, and for $|\beta| < \beta_0$, for solutions $u(x,t)$ to Boundary Value Problem II for equation (9), estimate (13) holds with some constant C_6 on the right-hand side that is defined only by the coefficients of the operators A_k, $k = 0,1,2,3$, and the domain Ω.

Estimate (14) with the corresponding constant C_7 on the right-hand side obviously follows from the previous estimates.

The obtained estimates of solutions to Boundary Value Problem II for equation (9) and the theorem on the method of continuation in a parameter and give the solvability of Boundary Value Problem II for equation (1) in the desired class.

The theorem is proved. □

Theorem 4. *Suppose the fulfillment of the conditions*

$$a^{ij,k}(x) \in C^2(\overline{\Omega}), \quad a^{ij,k}(x) = a^{ji,k}(x), \quad a_{0,k}(x) \in C(\overline{\Omega}), \quad i,j = 1,\dots,n, \quad k = 0,1,2,3; \tag{20}$$

$$a^{ij,k}(x)\xi_i\xi_j \geqslant m_0|\xi|^2, \quad m_0 > 0, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n, \quad a_{0,k}(x) \leqslant 0, \quad k = 2,3; \tag{21}$$

$$-a^{ij,k}(x)\xi_i\xi_j \geqslant m_1|\xi|^2, \quad m_1 > 0, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n, \quad a_{0,k}(x) \geqslant 0, \quad k = 0,1; \tag{22}$$

$$A_0 = \beta \widetilde{A}_0. \tag{23}$$

Then there is a positive number β_0 such that, for $|\beta| < \beta_0$ and $f(x,t) \in L_2(Q)$, Boundary Value Problem III has a solution $u(x,t)$ such that $D_t^k u(x,t) \in L_2(0,T; W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega))$, $k = 0,1,2,3$, $D_t^4 u(x,t) \in L_2(Q)$.

Proof. Show that solutions $u(x,t)$ to Boundary Value Problem III of the class mentioned in the statement of the theorem satisfy the desired a priori estimates.

Multiply equation (1) by $D_t^2 u(x,t)$. Integrating over Q, applying integration by parts, and using (20)–(22), it is not hard to obtain the first a priori estimate for solutions $u(x,t)$ to Boundary Value Problem III:

$$\int_Q \left[\left(D_t^3 u\right)^2 + \sum_{i=1}^n \left(D_t^2 u_{x_i}\right)^2 \right] dx\,dt \leqslant C_8 \int_Q f^2\,dx\,dt; \tag{24}$$

here the constant C_8 is defined only by the coefficients of the operators A_k, $k=0,1,2,3$.

At the next step, multiply equation (1) by $A_2 D_t^3 u(x,t)$ and integrate it over Q. Using conditions (20)–(23), inequality (11), and also the second main inequality for a pair of elliptic operators, we conclude that there exists a number β_0 such that for $|\beta| < \beta_0$ we have a second estimate

$$\sum_{i,j=1}^n \int_Q \left(D_t^3 u_{x_i x_j}\right)^2\,dx\,dt \leqslant C_9 \int_Q f^2\,dx\,dt; \tag{25}$$

with the constant C_9 defined only by the coefficients of the operators A_k, $k=0,1,2,3$, and the domain Ω.

The last a priori estimate

$$\int_Q \left(D_t^4 u\right)^2\,dx\,dt \leqslant C_{10} \int_Q f^2\,dx\,dt \tag{26}$$

obviously stems of the previous two estimates.

Using estimates (24)–(26) and the method of continuation in a parameter (for example, with the use of the equation

$$D_t^4 u + A_2 D_t^2 u + \lambda(A_3 D_t^3 u + A_1 D_t u + A_0 u) = f(x,t)\big),$$

it is not hard to obtain the desired solvability of Boundary Value Problem III.

The theorem is proved. □

3. Conclusion.

Observe first of all that the conditions of Proposition 1 are fulfilled, for instance, if the numbers a_0 and b_0 are large.

Furthermore, it is not hard to generalize the obtained results to equations more general than (1); for example, to equations with general second-order elliptic operators A_k.

Some of the conditions of the proven theorems can be changed: for example, we can discard the sign-definiteness of the operator A_0 from Theorem 4.

Observe finally that conditions (18) and (23) mean that $\widetilde{A}_1$ and $\widetilde{A}_0$ are fixed operators, whereas the number β is a parameter (namely, a smallness parameter).

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Краевые задачи для уравнений соболевского типа четвертого порядка

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Аннотация. Целью статьи является исследование разрешимости в пространствах Соболева краевых задач для некоторых классов линейных уравнений четвертого порядка соболевского типа. Докажем, что начально-краевые задачи с данными как в начальный момент времени, так и в конечные моменты времени могут быть корректными для исследуемых уравнений.

Ключевые слова: дифференциальное уравнение четвертого порядка соболевского типа, краевая задача, существование, единственность.

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Fast Modelling of Tsunami Wave Propagation using PC with Hardware Computer Code Acceleration

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Abstract. The field programmable gates array (FPGA) microchip is applied to achieve considerable performance gain in simulation of tsunami wave propagation using personal computer. The two-step Mac-Cormack scheme was used for approximation of the shallow water equations. An idea of PC-based tsunami wave propagation simulation is described. Comparison with the available analytic solutions and numerical results obtained with the reference code show that developed approach provides good accuracy in simulations. It takes less then 1 minute to compute 1 hour of the wave propagation in computational domain that contains 3000×2500 nodes. Using the nested greed approach, it is possible to decrease the size of space step from about 300 meters to 10 m. Using the proposed approach, the entire computational process (to calculate the wave propagation from the source area to the coast) takes about 2 min. As an example the distribution of maximal heights of tsunami wave along the coast of the Southern part of Japan is simulated. In particular, the interrelation between maximal wave heights and location of tsunami source is studied. Model sources of size 100×200 km have realistic parameters for this region. It was found that only selected parts of the entire coast line are exposed to tsunami wave with dangerous height. However, the occurrence of extreme tsunami wave heights at some of those areas can be attributed to the local bathymetry. The proposed hardware acceleration to compute tsunami wave propagation can be used for rapid (say, during few minutes) evaluation of danger from tsunami wave for a particular location of the coast.

Keywords: numerical modelling, tsunami wave propagation, computer code acceleration.

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Early warning of dangerous tsunami waves at a particular coastal location is crucially important to reduce human losses and minimize possible impact on economy. Unfortunately, the problem of tsunami early warning after the major offshore earthquake is still unresolved, despite the rather large number of publications on this issue (see, for example, [1]). In the case of the seismic event offshore Japan, tsunami wave approaches the nearest point at the coast in approximately 20 minutes. It means that just a few minutes are available for the analysis to provide the authorities with evaluation of the expected tsunami wave danger. In the case of the strong earthquake the electric power supply may be disrupted, so it would be better not

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to use supercomputer facilities. Advantages of the modern computer architectures to accelerate numerical simulation of tsunami wave propagation can be used. An approach based on the specialized FPGA (Field Programmable Gates Arrays) have been developed and tested. A number of numerical tests demonstrate the advantages of the new fast method for modelling tsunami wave propagation. In the present paper we briefly describe the proposed approach and show the numerical results obtained in the recent years.

Our point is to use just a personal computer (PC) and to achieve performance gain using Graphic Processing Units (GPUs) and FPGAs as co-processors. So, we propose a specific hardware configuration and the corresponding code.

Robust evaluation of tsunami wave danger should be based on the correct process simulation: wave generation, wave propagation, and inundation of dry land. In the study we deal with the stages of wave formation and propagation only to decrease computation time and keep at the same time sufficient accuracy. So, we suppose that the tsunami wave is caused by a certain disturbance of sea surface. From this initial disturbance follows initial conditions for the governing evolution type equations (shallow water equations). The issue of the inundation mapping was not considered. Therefore, we do not compute waves when depth is small (below 5 m) where reflection type boundary conditions are suggested at such depth to estimate the wave height in the near-shore area and to account for reflected waves. On the parts of the boundary which separate our computational domain from the ocean, conditions for free passage of the wave out of the domain are used. There are several application programs to simulate the wave propagation over the real digital bathymetry [2–6].

Among the most popular programs the MOST (Method of Splitting Tsunamis) package of programs should be mentioned. This package is used by the USA NOAA tsunami warning centres to simulate all tsunami phases – generation, propagation, and inundation of the dry land [2, 4]. Simulation of the wave propagation over chosen water area is based on the numerical solution of linear or non-linear shallow water differential equations.

Alternatively, the Mac-Cormack scheme for numerical approximation of the shallow water equations was implemented [7, 8]. Comparison with the exact solutions (in special cases of sea bed relief) shows a very good accuracy of the implemented method [9, 10]. It shows better tracking of the wave front in comparison with the MOST program.

A number of numerical tests where real digital bathymetry of the offshore Japan and Kamchatka Peninsula was used prove that it takes about 50 sec to solve numerically the shallow water equations for the computational domain with 10^7 nodes [11, 12]. Nested grid approach was also tested [13, 14].

1. Formulation of the problem

The referred program MOST (like many other tools) uses the following equivalent form of the shallow water equations which does not take into account such external forces as sea bed friction, Coriolis force and others [4]:

$$
\begin{gathered}
\frac{\partial H}{\partial t} + \frac{\partial uH}{\partial x} + \frac{\partial vH}{\partial y} = 0, \\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial H}{\partial x} = g\frac{\partial D}{\partial x}, \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial H}{\partial y} = g\frac{\partial D}{\partial y},
\end{gathered}
\tag{1}
$$

where $H(x, y, t) = \eta(x, y, t) + D(x, y, t)$ is the entire height of water column, $\eta(x, y, t)$ is the sea surface disturbance (wave height), $D(x, y)$ — depth (which is supposed to be known at all grid points), u and v are components of flow velocity vector, g — acceleration of gravity.

The system of shallow-water equations can be solved using the difference scheme. In this case values of tsunami wave parameters $\eta(x, y, t)$, u, and v are defined in nodes of the regular grid linked to geographical coordinates. To begin with, the initial conditions are set in all grid nodes. For example, everywhere besides the source area, the values of the grid variables η^0_{ij}, u^0_{ij}, v^0_{ij}, $(i = 1, \ldots, N,\ j = 1, \ldots, M)$ are equal to zero. Further, according to the difference scheme that approximates system of differential equations (1), the wave parameters η^n_{ij}, u^n_{ij}, v^n_{ij} at the grid nodes on subsequent time layers $t^n = n\tau$ are calculated. Here, the value of the time step τ is usually determined from the stability condition. This condition requires that wave can not travel more than one spatial step (Δx or Δy) in one time step.

Shallow water equations (1) are approximated at the grid nodes on the n-th time step with the help of explicit two-step Mac-Cormack finite difference scheme of the second order of approximation [15]:

First half-step:

$$\begin{aligned}
&\frac{\hat{H}^{n+1}_{ij} - H^n_{ij}}{\tau} + \frac{H^n_{ij}u^n_{ij} - H^n_{i-1j}u^n_{i-1j}}{\Delta x} + \frac{H^n_{ij}v^n_{ij} - H^n_{ij-1}v^n_{ij-1}}{\Delta y} = 0,\\
&\frac{\hat{u}^{n+1}_{ij} - u^n_{ij}}{\tau} + u^n_{ij}\frac{u^n_{ij} - u^n_{i-1j}}{\Delta x} + v^n_{ij}\frac{u^n_{ij} - u^n_{ij-1}}{\Delta y} + g\frac{\eta^n_{ij} - \eta^n_{i-1j}}{\Delta x} = 0,\\
&\frac{\hat{v}^{n+1}_{ij} - v^n_{ij}}{\tau} + u^n_{ij}\frac{v^n_{ij} - v^n_{i-1j}}{\Delta x} + v^n_{ij}\frac{v^n_{ij} - v^n_{ij-1}}{\Delta y} + g\frac{\eta^n_{ij} - \eta^n_{ij-1}}{\Delta y} = 0.
\end{aligned} \tag{2}$$

Second half-step:

$$\begin{aligned}
&\frac{H^{n+1}_{ij} - (\hat{H}^{n+1}_{ij} + H^n_{ij})/2}{\tau/2} + \frac{\hat{H}^{n+1}_{i+1j}\hat{u}^{n+1}_{i+1j} - \hat{H}^{n+1}_{ij}\hat{u}^{n+1}_{ij}}{\Delta x} + \frac{\hat{H}^{n+1}_{ij+1}\hat{v}^{n+1}_{ij+1} - \hat{H}^{n+1}_{ij}\hat{v}^{n+1}_{ij}}{\Delta y} = 0,\\
&\frac{u^{n+1}_{ij} - (\hat{u}^{n+1}_{ij} + u^n_{ij})/2}{\tau/2} + u^n_{ij}\frac{\hat{u}^{n+1}_{i+1j} - \hat{u}^{n+1}_{ij}}{\Delta x} + v^n_{ij}\frac{\hat{u}^{n+1}_{ij+1} - \hat{u}^{n+1}_{ij}}{\Delta y} + g\frac{\hat{\eta}^{n+1}_{i+1j} - \hat{\eta}^{n+1}_{ij}}{\Delta x} = 0,\\
&\frac{v^{n+1}_{ij} - (\hat{v}^{n+1}_{ij} + v^n_{ij})/2}{\tau} + u^n_{ij}\frac{\hat{v}^{n+1}_{i+1j} - \hat{v}^{n+1}_{ij}}{\Delta x} + v^n_{ij}\frac{\hat{v}^{n+1}_{ij+1} - \hat{v}^{n+1}_{ij}}{\Delta y} + g\frac{\hat{\eta}^{n+1}_{ij+1} - \hat{\eta}^{n+1}_{ij}}{\Delta y} = 0.
\end{aligned} \tag{3}$$

Here $\hat{F}^{n+1}_{ij}$ are intermediate values of wave parameters after the first time half-step.

Usually, the real tsunami wave simulation is performed in the spherical or geodetic coordinate system (λ, ϕ), where λ is the longitude and ϕ is the latitude in arc degrees. Accordingly, the following relations are used to calculate the differences Δx and Δy:

$$\Delta x_{ij} = \frac{\pi(\lambda_{i+1} - \lambda_i)}{180^\circ} R_E \cos(\phi_i), \quad \Delta y_{ij} = \frac{\pi(\phi_{i+1} - \phi_i)}{180^\circ} R_E,$$

where R_E is the Earth radius. This scheme looks similar to the splitting method (with respect to space variables) which is used in the referred MOST program package. Indeed, in order to calculate the values of the sought functions at grid-point $(i, j, n+1)$ the values at 3 points of the previous time step, (i, j, n), $(i-1, j, n)$, and $(i, j-1, n)$ are used during the first half-step in (2), and the values at the points (i, j, n), $(i+1, j, n)$, and $(i, j+1, n)$ are used in the second half-step in (3). Comparison of the known analytic solutions with the numerical solutions shows that the proposed attempt to realize the three-points calculation stencil (Mac–Cormack scheme) seems to be preferable compared to the one from the MOST software package [9, 10]

2. FPGA based calculator

In order to achieve performance gain, the FPGA-based Calculator has been developed.

To employ advantages of the FPGA microchip features, the stream processor architecture was proposed for this algorithm implementation. The proposed Calculator contains several processor elements (PEs). Each elements performs a pipeline with a sequential data stream. "On board" memory contains all the necessary information. The calculation speed-up by FPGA architecture is based on the inner memory (BRAM) access for implementing stencil buffer.

The Calculator architecture allows one to process several nodes in parallel. At the same time, the user can connect a number of PEs to make several iterations. So, the computation pipeline can be optimized considering features of the FPGA microchip in use. The Mac–Cormack finite difference approximation fits very well with the Calculator architecture presented in Fig. 1, processing 1 node at one computer clock cycle.

Calculator based of FPGA microchip Xilinx Virtex-7 VC709 was used for numerical tests (see [7, 8] for details).

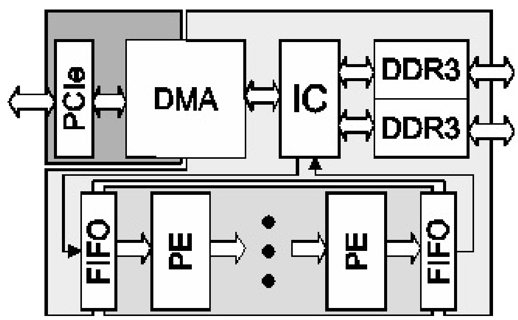

Fig. 1. Calculator architecture. To implement the FPGA algorithm the following architecture of the stream processor was proposed. It consists of processor elements (PE). Such PE executes a version of 2-dimension run, a pipeline with a sequential data stream. In addition to the calculator itself, the processor has memory controllers DDR3, PCIe controller, and DMA module responsible for the interaction between the calculator and the memory of the host computer. Such interaction is arranged as a direct memory access (DMA)

3. Comparison with exact solutions and reference code

In order to verify results obtained by the described method, a number of numerical tests have been carried out. The first test consists of simulation of the tsunami wave propagation from a round source in the area with the sloping bottom topography. The water area of 1000×1000 km was considered. Computational grid has equal spatial steps in both directions, namely, $\Delta x = \Delta y = 1000$ m. The centre of the circular tsunami source with the radius of 50 km was located in the middle of the region (horizontally) 300 km from the lower boundary where the depth was vanishing. The depth is linearly increased according to the formula $D(x, y) = 0.01y$, where y is the distance from the lower boundary of the region. The initial vertical displacement h of the tsunami source is determined by the formula

$$h(r) = 1 + \cos\left(\frac{\pi r}{r_0}\right), \qquad 0 \leqslant r \leqslant r_0. \tag{4}$$

Here, r is the distance to the centre of the source of radius r_0. Thus, in the centre of the source area the initial displacement of the water surface was +2 m. This source generates a circular wave with the height of 0.95 m at 50 km from the centre. It is the wave height that was used as initial circular wave front with the radius of 50 km to estimate the wave amplitude at all points of the region according to the ray approximation [17, 18]. This distribution of tsunami wave amplitude (given in the analytic form) was compared with the distribution obtained with the use of the MOST program and the proposed Mac-Cormack algorithm (Fig. 2).

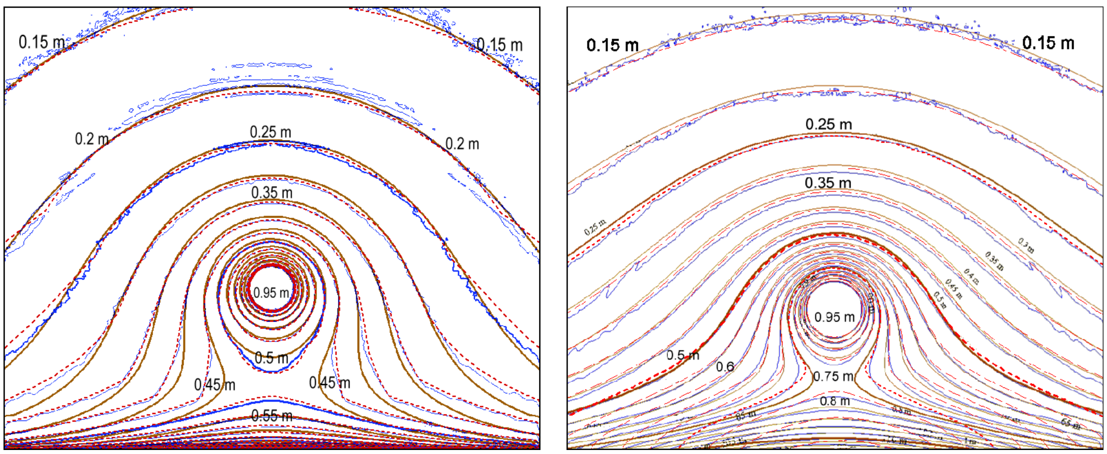

Fig. 2. *Left*: Isolines of maximal wave heights distribution above bottom slope: exact solution of shallow water equations [17, 18] (brown lines), numerical solution with the FPGA based Calculator (red dashed lines), the MOST program (blue lines). The offshore distance is measured along vertical axis relative to the figure bottom boundary. *Right*: Isolines of maximal wave heights distribution above the parabolic bottom topography [16]: the wave-ray solution to shallow water equations (brown lines), numerical solution with the FPGA based Calculator (dashed lines), the MOST program (blue lines). The offshore distance is measured along vertical axis relative to the figure bottom boundary

Fig. 2 (left) shows that at sufficiently large depths (exceeding 500 m) the contours of all three distributions of tsunami height maxima are quite close to each other. The proximity of results of numerical calculations for the two algorithms under consideration is also preserved near the coast. The discrepancy between numerical and wave-ray approach results in the coastal zone is caused by the neglect of the effect of increasing wave height due to reflection from the coast.

Another numerical test is similar to the first one and considers the case of the parabolic bottom relief. Let us consider the same computational area of 1000×1000 km with the computational grid having spatial steps $\Delta x = \Delta y = 1000$ m. The centre of the circular source with 50 km radius is also located in the middle of the area at 300 km from the lower boundary where the depth is equal to zero. The depth increased according to the formula $D(x, y) = 10^{-8}y^2$, where y is the distance from the lower boundary of the region. The initial vertical displacement inside the circular source is determined by (4). Fig. 2 (right) presents the isolines of distributions of tsunami height maxima calculated by the MOST program and the Mac–Cormack algorithm together with the estimates of these maxima obtained in the framework of the ray model [16].

Fig. 2 (right) shows that at sufficiently large depths (more than 200 m) the contours of all three distributions of tsunami height maxima are quite close to each other. Here, the similarity of the results of numerical simulations with two algorithms under consideration is observed up to the coastline (the lower boundary of the region). The discrepancy between numerical and wave-ray approximation results is increased in the coastal zone.

Based on the results of the wave propagation over a bottom slope the correctness of the wave front kinematics modelling is also estimated. Fig. 3 shows the comparison of the wave front position with an interval of 5 minutes obtained with the McCormack scheme with the exact solution of the kinematic problem at the same time points. In order to prevent the points from merging (as it happens at the initial moment) the moments of output of the points of the calculated wave front are taken 3 seconds later than the moment of the corresponding exact solution of the kinematic problem [17, 18]. The wave front positions obtained with the MOST program are not presented since these positions of the front points exactly coincide with the positions obtained with the McCormack scheme.

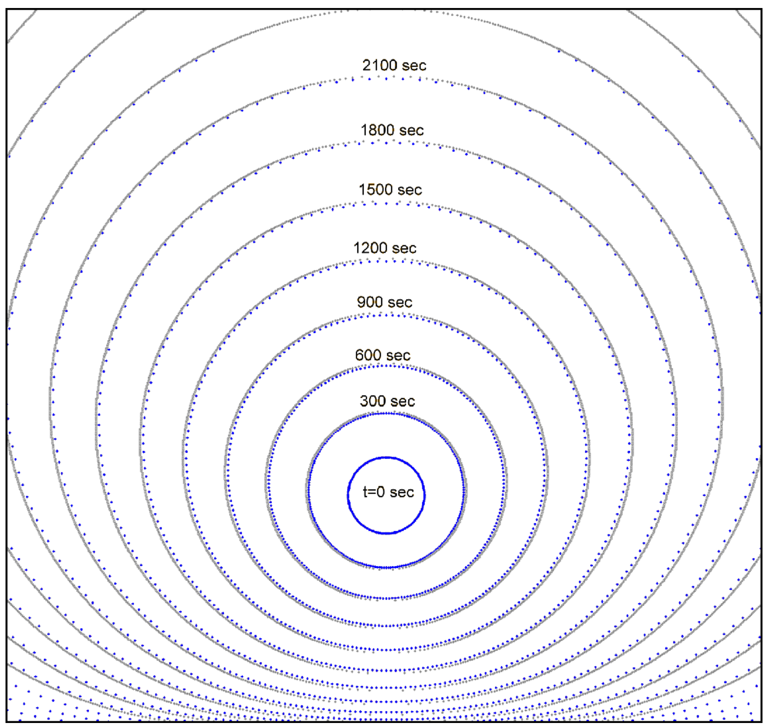

Fig. 3. Comparison of the tsunami wave isochrones over the sloping bottom: numerical experiments with the MOST and Mac-Cormack algorithms (grey lines) and exact solution (blue points)

Additional numerical test was carried out in order to verify the correctness of modelling the reflection of wave from a completely reflecting boundary positioned at 45 degrees to the direction of motion of the flat wave front. Let us consider rectangular 1000×2000 nodes computational domain. A long wave about 1 m height generated by a one-dimensional source parallel to the lower boundary of the region propagates over the region and it is reflected from the inner boundary positioned at 45 degrees to ordinate axis (Fig. 4).

Fig. 4 shows the distribution of the vertical displacement of the water surface in the entire region calculated with the Mac–Cormack scheme (left figure) and the MOST algorithm (right figure). The dark line shows the tsunami height isoline corresponding to the value of 0.4 m. The grey line outlines the water area with the surface displacement less then $-0,4$ m. Both

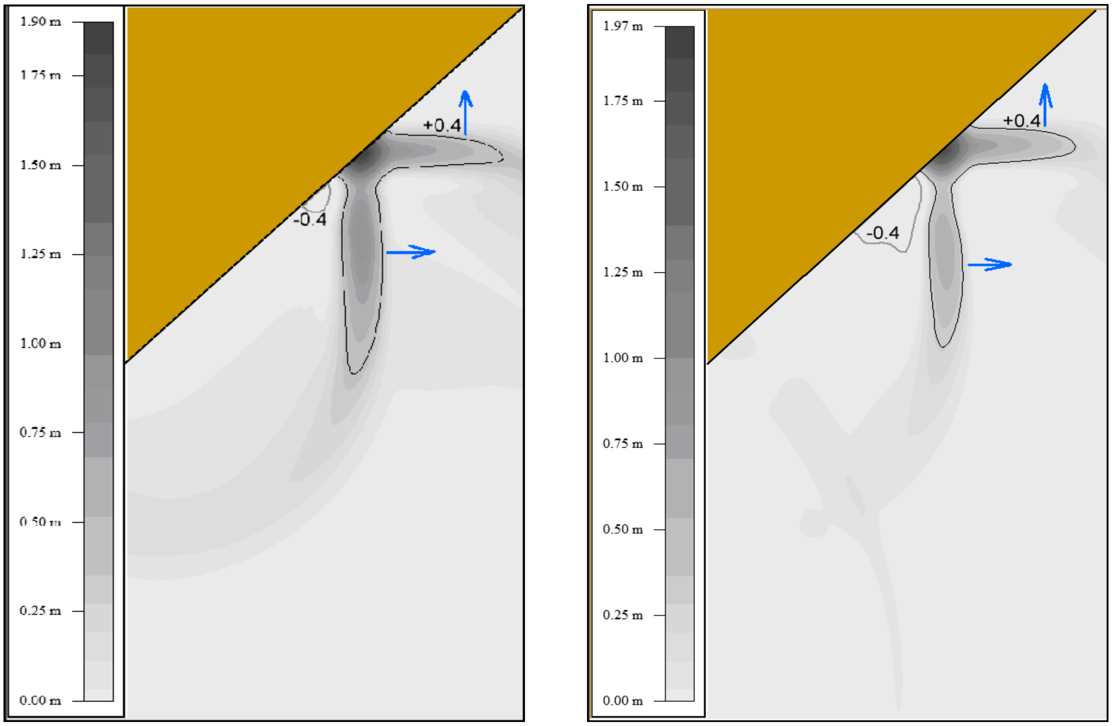

Fig. 4. Water surface after 3,000 sec of wave propagation using the Mac-Cormack scheme with the FPGA (left figure) and the MOST program (right figure). Thin black line indicates the 0.4 m isoline, and the gray line shows −0.4 m isoline

figures confirm the correctness of numerical modelling of the process of wave reflection from a completely reflecting boundary. One can see that the direction of motion of the reflected wave in both cases is orthogonal to the direction of the incident wave.

4. Distribution of maximal wave heights along the shoreline

The acceleration of numerical calculations of the tsunami propagation is required, first of all, by tsunami warning services to fast estimate the expected wave height at various points on the coastline. Therefore, this estimation is required before tsunami wave reaches the shore. The ability of the proposed approach to solve this problem in the area with real bathymetry within a few tens of seconds is demonstrated in this section.

The series of numerical experiments were performed for the areas around Kii Peninsula and Shikoku Island (southern part of Japan). Japanese bathymetric data produced by the Japan Oceanographic Data Center (JODC) (see [19]) were used, and they are presented in Fig. 5.

The above bathymetry and the computational grid have the following characteristics:

(1) Computational domain contains 3000×2496 nodes; **(2)** Grid steps are 0.003 and 0.002 degrees (which means 280.6 and 223 meters, respectively); **(3)** The grid covers the area between 131° and 140° E, 30.01° and 35° N; **(4)** Time step used in computations is equal to 0.5 sec.

The shape of model tsunami sources used in numerical experiments are based on the available geological and geophysical information. The typical for subduction zone seabed displacement area

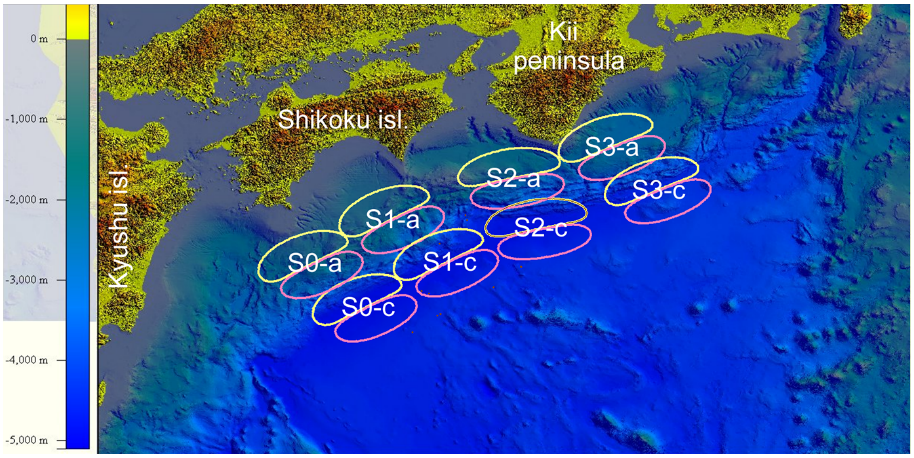

Fig. 5. Digital bathymetry around Kii Peninsula and Shikoku Island (Japan). Positions of model tsunami sources are indicated

for 8.0 M earthquake was approximated by 100×200 km rectangle having maximum height 300 cm. The initial seabed displacement for the model source is shown in Fig. 6.

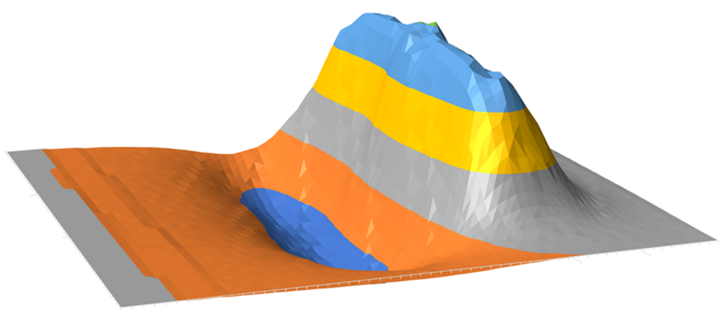

Fig. 6. 3D image of the model tsunami source used in numerical simulation

Positions and shapes of the sources used for tsunami modelling are shown in Fig. 5, where geography of the computational domain is also presented. In this figure only the closest to the coast sources $S_i - a$ $(i = 1, \dots 4)$ and most distant to the coast sources $S_i - c$ $(i = 1, \dots 4)$ are indicated. Their positive parts are shaped by pink colour, and negative parts (water surface depression) are outlined by yellow colour. Intermediate sources $S_i - b$ $(i = 1, \dots 4)$ are located between sources $S_i - a$ and $S_i - c$.

The distributions of the wave height maxima in the entire area generated by some model sources are presented in Fig. 7. In the right part of each drawing the legend for colour-height relation is presented.

As is observed from Fig. 7 (left), the same shape of the initial sea surface displacement causes the tsunami wave heights of up to 6 m at certain areas of the Shikoku Island and the Kyushu Island coasts. At the same time, the Kii Peninsula coast is practically safe with wave heights limited by 0.5 m. Otherwise, the source located opposite Kii peninsula seriously affects only its coast and it has no effect on Shikoku Island (see Fig. 7 (right)).

Distribution of tsunami wave maxima along the shoreline is also important information for

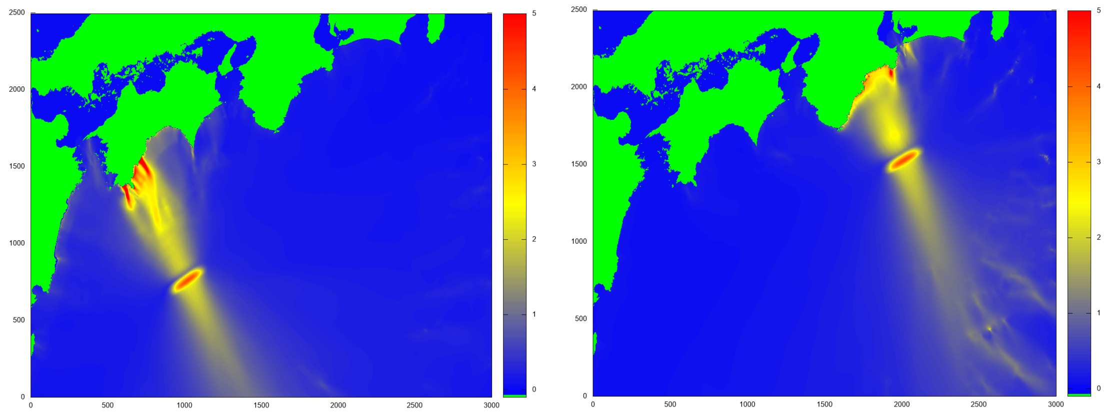

Fig. 7. Calculated height maxima in entire computational domain from tsunami wave generated by the source $S_0 - c$ (left) and $S_3 - b$ (right) (see Fig. 3)

tsunami warning. Figs. 8(A) and 8(D) show such distribution for 6 model sources $S_0 - a, b, c$ and $S_3 - a, b, c$. Numerical experiments were performed for the same shape of the initial see surface displacement (given in Fig. 6). Positions of the model source along the shore and the distance of the model source from the shore were varied. Numbers along the horizontal axis indicate the horizontal indexes of coastal computational grid points.

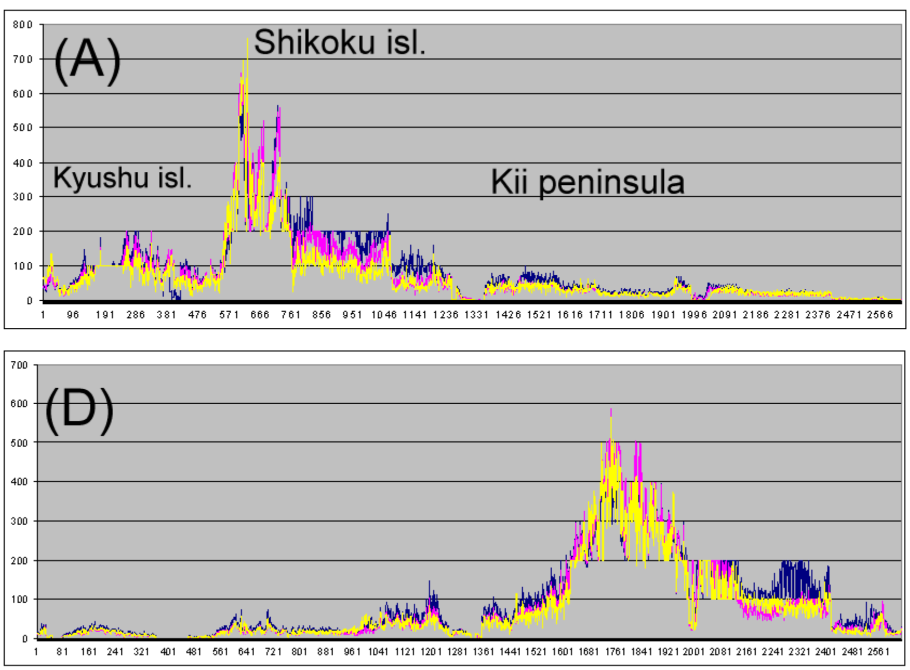

Fig. 8. Distributions of tsunami wave maxima along the shore generated by the sources $S_0 - a, b, c$ (A) and $S_3 - a, b, c$ (D). Wave height maxima: yellow lines — $S_i - a$ sources; blue lines — $S_i - b$ sources; pink lines — $S_i - c$ sources

Let us say few words about digital bathymetry. The grid step 250 m used in our numerical experiments is considered too large these days. However, the size of grid step depends on the goal of simulation. If a detailed evaluation of the expected wave heights along the entire shore line is needed then it is necessary to carry out numerical simulation with the corresponding fine mesh size in the near-coastal regions. It can be done by choosing the sufficiently small grid step in the whole area. However, the number of computational nodes is increased by 2–3 orders. It results in the necessity to extend computational facilities or, alternatively, in a dramatic increase of the CPU time required for simulation. As for the proposed FPGA-based Tsunami Wave Calculator, the available memory resources permits the use of approximately 50 millions computational nodes.

The Calculator with a regular modern PC needs 25 sec to simulate wave propagation from the southern edge of the computational domain shown in Fig. 5 to the shore. The estimated travel time for the wave is 3200 sec. So, realization of Mac-Cormack scheme on FPGA hardware allows one to estimate the expected wave height distribution along the coastline before tsunami arrival.

Conclusion

In order to accelerate the calculation of tsunami wave propagation over the deep water area, a special FPGA based Calculator has been developed. The Mac–Cormack scheme was used for numerical solution of the shallow water equations. Accuracy of the solution obtained with the use of the Calculator was tested by comparison with the known analytic solutions. Similar or even better accuracy is achieved in comparison with the MOST program. These results show the possibility of tsunami danger forecast in the real time mode.

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Быстрое моделирование распространения волны цунами на ПК за счет аппаратного ускорения исполнения кода

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Аннотация. За счет применения микросхемы вентильной матрицы, программируемой пользователем (Field Programmable Gates Array – FPGA), достигается значительное ускорение расчета распространения волн цунами на современном обычном персональном компьютере. Для численной аппроксимации системы уравнений мелкой воды использовалась двухступенчатая схема Мак-Кормака. На базе проведенных численных тестов авторы описывают идею моделирования распространения волн цунами на базе ПК. Проведенное сравнение с известными аналитическими решениями и с эталонным кодом показывает хорошую точность разработанного программного приложения. Расчет одного часа распространения волны занимает меньше 1 минуты на сетке 3000 x 2500 узлов. Используя технологию вложенных сеток, можно перейти от расчетной сетки с шагом 300 м до сетки с шагом 10 м. При использовании предложенного Калькулятора, весь вычислительный процесс (для расчета распространение волны от очага до берега) занимает около 2 мин. Получено распределение максимальных высот волн цунами вдоль побережья южной части Японии. В частности, исследуется зависимость максимальных высот волн от конкретного местоположения источника цунами. Модельный источник размером 100 x 200 км имеет реалистичные параметры для этого географического региона. Результаты численных экспериментов показывают, что только на отдельных участках всей береговой линии наблюдаются опасные амплитуды волн цунами. Наличие аномально высоких волн цунами в некоторых из этих районов могут быть вызваны особенностями локальной батиметрии. Предлагаемое аппаратное ускорение вычисления распространения волны цунами может быть использовано для быстрой (скажем, за несколько минут) оценки опасности цунами для конкретного населенного пункта или промышленного объекта на побережье.

Ключевые слова: численное моделирование, распространение волны цунами, ускорение исполнения программного кода.

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Inverse Problem for Source Function in Parabolic Equation at Neumann Boundary Conditions

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Abstract. The second initial-boundary value problem for a parabolic equation is under study. The term in the source function, depending only on time, is to be unknown. It is shown that in contrast to the standard Neumann problem, for the inverse problem with integral overdetermination condition the convergence of it nonstationary solution to the corresponding stationary one is possible for natural restrictions on the input problem data.

Keywords: parabolic equation, inverse problem, source function, a priori estimate, nonlocal overdetermination condition.

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1. Introduction and preliminaries

We consider the parabolic equation with second-type boundary conditions

$$u_t = \nu u_{xx} + f(t) + g(x,t) \qquad \text{in} \quad Q_T = (0,l) \times [0,T]; \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in (0,l); \tag{2}$$

$$-u_x(0,t) = q_1(t), \quad u_x(l,t) = q_2(t), \tag{3}$$

$$\int_0^l u(x,t)dx = q_3(t), \quad t \in [0,T]. \tag{4}$$

In (1)–(4) the functions $g(x,t)$, $u_0(x)$, $q_i(t), i = 1,2,3$, and positive constants ν, T, l are assumed to be given. The problem of finding a pair $u(x,t)$ and $f(t)$ is called inverse one.

Definition. The pair $f(t)$ and $u(x,t)$ from the class $C[0,T] \times C^{2,1}(Q_T) \cap C^{1,0}(\bar{Q}_T)$, for which equation (1) and conditions (2)–(4) are satisfied, is called a classical solution of the posed inverse problem.

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It is clear that for existence of smooth solution the consistency conditions should be fulfilled. They are the following

$$-u_{0x}(0) = q_1(0), \quad u_{0x}(l) = q_2(0), \quad \int_0^l u_0(x)dx = q_3(0).$$

From the physical point of view the posed problem (1)–(4) allows to describe a motion of viscous fluid in the flat layer with two free boundaries. The function $u(x,t)$ is the velocity in this case, $f(t)$ is unknown pressure gradient, condition (4) means flow rate through the section of layer. The solution of the inverse problem in this case gives an answer on a question: what is pressure gradient needed for providing the given flow rate?

It is necessary to mention that there are many known results concerning to the inverse problems close to the posed problem. Among of them it can be distinguish the coefficient inverse problems (see, e. g. [1–3]), problems with unknown source function [4–6] and problems, where unknown function in the boundary condition occurs [7]. The authors, dealing with finding the source function, usually assume, that this function is included into the equation by multiplicative way (see, for example, [4]).

The overdetermination conditions can be nonlocal integral ones [1,8,9]. One-point and two-point overdetermination conditions are considered in [5,10]. As a rule, in the cited papers and books the existence and uniqueness of solution are proved, some asymptotic methods of solution construction are described. It is common situation when existence and uniqueness of solution are proved in the Sobolev's spaces. Usually, the same questions are considered in the uniform metric for 1-dimensional problems only. Concerning to different kinds of inverse problems and qualitative properties of their solutions we should also mention the monographs authored by Prilepko *et al* [11], Alifanov [12] and Belov [13].

1.1. Some remarks on corresponding direct problem

If the function $f = 0$ in equation (1), and condition (4) is not taken into account, then we deal with standard Neumann problem for the function $u(x,t)$. It is well known that the direct initial boundary problem

$$u_t = \nu\Delta u + g(x,t), \quad x \in \Omega \subset \mathbb{R}^n, \quad t \in [0,T]; \tag{5}$$

$$u(x,0) = u_0(x), \quad x \in \Omega; \qquad \frac{\partial u}{\partial n} = \varphi(x), \quad x \in \partial\Omega, \quad t \in [0,T] \tag{6}$$

has unique solution if the functions $g(x,t)$, $u_0(x)$ and $\varphi(x)$ are smooth ones. The corresponding stationary problem

$$\nu\Delta u^s = -g^s(x), \quad x \in \Omega \subset \mathbb{R}^n, \qquad \frac{\partial u}{\partial n} = \varphi^s(x), \quad x \in \partial\Omega$$

has a countable number of solutions $u^s(x) + const$ if and only if the following condition

$$\frac{1}{\nu}\int_\Omega g^s d\Omega + \int_{\partial\Omega} \varphi^s d\Gamma = 0$$

is fulfilled. For the separation of unique solution it is necessary to give additional functional of $u^s(x)$. For example, it could be $u^s(x_0)$, where $x_0 \in \partial\Omega$.

It should be noted that if $g(x,t) \to g^s(x)$ and $\varphi(x,t) \to \varphi^s(x)$ in the uniform metric at $t \to \infty$ for all $x \in \bar{\Omega}$, then it is not difficult to prove that nonstationary solution $u(x,t)$ does not tend to $u^s(x)$ at $t \to \infty$. We confirm this fact using an example. For the problem in the space $\mathbb{R}^1$ we consider equation (5) with conditions $u_0(x,0) = x,\ u_x(0,t) = u_x(l,t) = 1$ and the right hand side in the form

$$g(x,t) = \frac{\cos \ln M}{M}, \qquad M = 1 + \frac{\nu}{l^2}t.$$

It has the solution

$$u(x,t) = x + \frac{l^2}{\nu} \sin \ln M,$$

which has no a limit at $t \to \infty$ while the corresponding stationary solution is $u^s(x) = x + const$ at $\varphi_1^s = \varphi_2^s = 1$ and $g^s(x) = 0$.

We should also mention that for the Dirichlet and Robin problems for multidimensional linear parabolic equation (5) the sufficient convergence conditions of solution of nonstationary problem to corresponding stationary one are described in [14]. According to the example above, for the Neumann problem there is no such convergence. However, it turn out well to show the convergence of nonstationary solution to the corresponding stationary one for the posed inverse problem (1)–(4).

Below, using the specific features of the considered problem and their 1-dimensionality, we derive sufficient conditions for the initial data for which the nonstationary solution tends to stationary one at $t \to \infty$ in the uniform metric.

2. Analysis of the inverse problem (1)–(4)

Integrating equation (1) by x from 0 till l and using condition (4), the function $f(t)$ can be found in the form

$$f(t) = \frac{1}{l}\left[\frac{\partial q_3}{\partial t} - \nu\left(q_1(t) + q_2(t)\right) - \int_0^l g(x,t)dx\right]. \tag{7}$$

Substitution expression (7) into equation (1) leads to direct problem for the function $u(x,t)$ with conditions (2) and (3). The solution of the obtained problem can be constructed as follows [15]

$$u(x,t) = \int_0^l u_0(y)G(x,y,t)dy + \int_0^t \int_0^l F(y,\tau)G(x,y,t-\tau)\,d\,yd\tau +$$
$$+ \nu \int_0^t q_1(\tau)G(x,0,t-\tau)\,d\tau + \nu \int_0^t q_2(\tau)G(x,l,t-\tau)\,d\tau,$$

where G is the Green's function:

$$G(x,y,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi y}{l}\right)\exp\left(-\frac{\nu n^2\pi^2 t}{l^2}\right), \quad F(x,t) = f(t) + g(x,t).$$

The examples of construction of a priori estimate for the functions presented as series with the Green's functions are given in [16,17]. As it can be observed in those works, the deriving the estimate of the function $|u(x,t)|$, $x \in [0,l]$, $t \in [0,T]$ from this expression is a cumbersome task. We suggest another way and reduce problem (1)–(4) to the axillary problem with the first-type boundary conditions.

Differentiating equation (1) with respect to variable x, we obtain the initial boundary value problem for the function $w(x,t) = u_x(x,t)$:

$$\begin{gathered} w_t = \nu w_{xx} + g_x(x,t), \quad x \in (0,l), \quad t \in [0,T]; \\ w(x,0) = w_0(x) = u_{0x}(x), \quad x \in [0,l]; \\ w(0,t) = -q_1(t), \quad w(l,t) = q_2(t), \quad t \in [0,T]. \end{gathered} \tag{8}$$

The corresponding stationary problem has the solution

$$w^s(x) = -q_1^s + \frac{x}{l}\left[q_1^s + q_2^s + \frac{1}{\nu}\int_0^l g^s(y)dy\right] - \frac{1}{\nu}\int_0^x g^s(y)dy, \tag{9}$$

where q_1^s, q_2^s, $g^s(x)$ are given constants and function respectively.

Let the functions $q_{1,2}(t)$ be known for all $t \geqslant 0$ and the following inequalities be fulfilled

$$|q_j(t)| \leqslant N_j(1+\tau)^{-\alpha}, \quad j=1,2, \quad |g(x,t)| \leqslant N_3(1+\tau)^{-\alpha}, \quad |g_x(x,t)| \leqslant N_4(1+\tau)^{-\alpha} \tag{10}$$

with some positive constants $N_1, \ldots, N_4$ and α for all $x \in [0,l]$, where $\tau = \nu t/l^2$ is the dimensionless time here and below. Then with respect to the results from [14] it can be concluded that the function $w(x,t)$ can be restricted as

$$|w(x,t| \leqslant N_5(1+\tau)^{-\alpha} \tag{11}$$

with constant $N_5 > 0$ at $x \in [0,l]$.

In order to find the stationary solution $u^s(x)$, expression (9) should be integrated

$$u^s(x) = -q_1^s x - \frac{1}{\nu}\left(f^s\frac{x^2}{2} + \int_0^x (x-y)g^s(y)dy\right) + C. \tag{12}$$

Here

$$f^s = -\frac{\nu}{l}\left(q_1^s + q_2^s + \frac{1}{\nu}\int_0^l g^s(y)dy\right), \tag{13}$$

$$C = \frac{q_3^s}{l} + \frac{q_1^s l}{3} - \frac{l q_2^s}{6} + \frac{1}{\nu l}\left(\int_0^l \int_0^x (x-y)g^s(y)dydx - l^2\int_0^l g^s(y)dy\right).$$

After that the stationary solution of the posed inverse problem is constructed, we can start obtaining a priori estimates of the corresponding nonstationary solution.

2.1. A priori estimates of the solution of problem (1)–(4)

It should be noted that if the following conditions are fulfilled

$$\frac{\partial q_3}{\partial t} \to 0, \quad q_j(t) \to q_j^s, \quad g(x,t) \to g^s(x)$$

at $t \to \infty$ and $x \in [0,l]$, then it can be concluded that

$$f(t) \to f^s \quad \text{at} \quad t \to \infty$$

is valid. The function $f(t)$ is from formula (7), and f^s is from (13).

According to the integral mean-value theorem there is the point $x_0 \in (0, l)$ such that $u(x_0, t) = l^{-1} q_3(t)$ (see (4)). That is why for every $x \in [0, l]$, $t \geqslant 0$ it follows that

$$u(x,t) = u(x_0,t) + \int_{x_0}^{x} u_y(y,t)dy = l^{-1}q_3(t) + \int_{x_0}^{x} w(y,t)dy.$$

Using estimate (11), it can be obtained that

$$|u(x,t)| \leqslant l^{-1}|q_3(t)| + \int_0^l |w(y,t|dy \leqslant l^{-1}|q_3(t)| + N_5 l(1+\tau)^{-\alpha}. \tag{14}$$

Let the following inequalities should be fulfilled

$$\begin{gathered} |q_j(t) - q_j^s| \leqslant D_j(1+\tau)^{-\alpha}, \quad j = 1,2,3; \\ \left|\frac{\partial q_3}{\partial t}\right| \to 0,\ t \to \infty, \quad |g(x,t) - g^s(x)| \leqslant D_4(1+\tau)^{-\alpha}, \\ |g_x(x,t) - g_x^s(x)| \leqslant D_5(1+\tau)^{-\alpha} \end{gathered} \tag{15}$$

with constants $D_i > 0\,(i = 1, \ldots, 5)$, $\alpha > 0$ for every $x \in [0, l]$. Then the following estimates can be provided

$$\begin{gathered} |u(x,t) - u^s(x)| \leqslant D_6(1+\tau)^{-\alpha}, \\ |u_x(x,t) - u_x^s| \leqslant D_7(1+\tau)^{-\alpha}, \\ |f(t) - f^s| \leqslant D_8(1+\tau)^{-\alpha}, \end{gathered} \tag{16}$$

where D_6, D_7, D_8 are positive constants.

For the deriving the estimates in (16) it needs to make a change $\tilde{u}(x,t) = u(x,t) - u^s(x)$, $\tilde{w}(x,t) = w(x,t) - w^s(x)$ and $\tilde{f}(t) = f - f^s$ in equation (1) and condition (4). The boundary conditions should be rewritten as $\tilde{q}_j(t) = q_j(t) - q_j^s,\ j = 1,2,3$, and $\tilde{g}(x,t) = g(x,t) - g^s(x)$ in this case. Applying estimates (14) and (11), formulas (7) and (13), using assumptions (15) we derive estimates (16). It concludes that the solution of inverse problem (1)–(4) converges to the corresponding stationary solution (12), (13) in the class $C[0,\infty] \times C^{2,1}(Q_T) \cap C^{1,0}(\bar{Q}_T)$.

Conclusion

For the conclusion some remarks can be made. The first one is following. If the right hand sides of inequalities (15) are restricted by exponent function ($\exp(-\alpha\tau)$, $\alpha > 0$), then the solution of problem (1)–(4) tends to stationary regime $u^s(x)$, f^s (see (12), (13)) with respect to exponent law at $t \to \infty$. The second remark is concerned to question of stabilization. The results obtained can be interpreted as stability of stationary solution (12), (13) if conditions (15) are fulfilled.

The authors were surprised at research of some aspects of solution stabilization in problems on binary mixtures motion that the question on solutions solvability and stability in the problems close to (1)–(4) was not described anywhere in literature. And we were glad to fill this gap in the investigation of such problems.

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Обратная задача определения функции источника для параболического уравнения с краевыми условиями Неймана

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Аннотация. В работе изучается вторая начально-краевая задача для параболического уравнения, когда часть функции источника, зависящая только от времени, неизвестна. Показано, что в отличие от классической задачи Неймана для обратной задачи с интегральным условием переопределения возможна сходимость ее нестационарного решения к соответствующему стационарному при естественных ограничениях на входные данные.

Ключевые слова: параболическое уравнение, обратная задача, функция источника, априорная оценка, нелокальное условие переопределения.

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Analysis of the Boundary Value and Control Problems for Nonlinear Reaction–Diffusion–Convection Equation

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Abstract. The global solvability of the inhomogeneous mixed boundary value problem and control problems for the reaction–diffusion–convection equation are proved in the case when the reaction coefficient nonlinearly depends on the concentration. The maximum and minimum principles are established for the solution of the boundary value problem. The optimality systems are derived and the local stability estimates of optimal solutions are established for control problems with specific reaction coefficients.

Keywords: nonlinear reaction–diffusion–convection equation, mixed boundary conditions, maximum principle, control problems, optimality systems, local stability estimates.

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1. Introduction. Solvability of the boundary value problem

In recent years, there has been an increasing interest in the study of inverse and control problems for models of heat and mass transfer, electromagnetism and acoustics. A number of papers are devoted to the theoretical analysis of these problems, of which we note [1–16]. In these papers, the solvability of boundary value problems, inverse and extremum problems for the specified models was proved, and the questions of uniqueness and stability of their solutions were studied. Related problems for models of complex heat transfer were studied in [17, 18].

This paper which continues a series of papers by the authors [10–14] is devoted to the theoretical analysis of the boundary value and control problems for the nonlinear reaction–diffusion–convection equation, considered under inhomogeneous mixed boundary conditions on the boundary of the domain.

In bounded domain $\Omega \subset \mathbb{R}^3$ with boundary Γ, consisting of two parts Γ_D and Γ_N, the following boundary value problem for nonlinear reaction–diffusion–convection equation is considered:

$$-\mathrm{div}(\lambda(\mathbf{x})\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi + k(\varphi,\mathbf{x})\varphi = f \text{ in } \Omega, \tag{1.1}$$

$$\varphi = \psi \text{ on } \Gamma_D,\ \lambda(\mathbf{x})(\partial\varphi/\partial n + \alpha(\mathbf{x})\varphi) = \chi \text{ on } \Gamma_N. \tag{1.2}$$

Here the function φ means the concentration of the substance, $\mathbf{u}$ is a given vector of velocity, f is a volume density of external sources of substance, $\lambda(\mathbf{x})$ is a diffusion coefficient, function $k(\varphi,\mathbf{x})$ is a reaction coefficient, $\mathbf{x}\in\Omega$. Below we will refer to the problem (1.1), (1.2) for the given functions $\lambda, k, f, \psi, \alpha$ and χ as Problem 1.

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In this paper, we first prove the global solvability of Problem 1 and the nonlocal uniqueness of its solution in the case, when the reaction coefficient $k(\varphi, \mathbf{x})$ is sufficiently arbitrarily depends on both the concentration φ and the spatial variable $\mathbf{x}$, and the nonlinearity $k(\varphi, \mathbf{x})\varphi$ is monotone. Under additional conditions on the functions $\lambda, f, \chi, \alpha, \psi$ and the reaction coefficient k the minimum and maximum principles are established for the concentration φ. Further, a control problem is formulated, in which the role of controls is played by the diffusion coefficient λ, the volume density of external sources of substance f and the density of boundary sources χ and its solvability is proved. For the mentioned problems, with specific reaction coefficients, an optimality system is derived and, based on its analysis a theorem on the local stability estimates of optimal solutions is formulated. This theorem can be proved according to the scheme described in detail in [11–16].

When analyzing the problems under study, we will use the Sobolev functional spaces $H^s(D)$, $s \in \mathbb{R}$. Here D means either a domain Ω, or some subset $Q \subset \Omega$, or part Γ_D of the boundary Γ. By $\|\cdot\|_{s,Q}, |\cdot|_{s,Q}$ and $(\cdot,\cdot)_{s,Q}$ we will denote the norm, seminorm and scalar product in $H^s(Q)$. The norms and scalar products in $L^2(Q)$, $L^2(\Omega)$ or in $L^2(\Gamma_N)$ will be denoted by $\|\cdot\|_Q$ and $(\cdot,\cdot)_Q$, $\|\cdot\|_\Omega$ and $(\cdot,\cdot)$ or $\|\cdot\|_{\Gamma_N}$ and $(\cdot,\cdot)_{\Gamma_N}$, respectively. Let $L^p_+(D) = \{k \in L^p(D) : k \geqslant 0\}$, $p \geqslant 3/2$, $Z = \{\mathbf{v} \in L^4(\Omega)^3 : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v}\cdot\mathbf{n}|_{\Gamma_N} = 0\}$, $H^s_{\lambda_0}(\Omega) = \{h \in H^s(\Omega) : h \geqslant \lambda_0 > 0 \text{ in } \Omega\}$, $s > 3/2$, $\mathcal{T} = \{\varphi \in H^1(\Omega) : \varphi|_{\Gamma_D} = 0\}$. Here and below $\varphi|_{\Gamma_0}$ denotes the trace of a function $\varphi \in H^1(\Omega)$ on the part Γ_0 of the boundary Γ. For any function $\varphi \in \mathcal{T}$ the Friedrichs–Poincaré inequality $\|\nabla\varphi\|^2_\Omega \geqslant \delta_0\|\varphi\|^2_{1,\Omega}$ holds, where positive constant δ_0 does not depend on φ.

Let the following conditions hold:

(i) Ω is a bounded domain in $\mathbb{R}^3$ with boundary $\Gamma \in C^{0,1}$, consisting of closures of two non-intersecting open parts Γ_D and Γ_N ($\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$), and meas $\Gamma_D > 0$;

(ii) $\lambda \in H^s_{\lambda_0}(\Omega)$, $s > 3/2$, $f \in L^2(\Omega)$, $\chi \in L^2(\Gamma_N)$;

(iii) $\mathbf{u} \in Z$, $\psi \in H^{1/2}(\Gamma_D)$, $\alpha \in L^2_+(\Gamma_N)$.

(iv) The function $k : \mathbb{R} \times \Omega \to \mathbb{R}$ is nonnegative. In addition, for any function $v \in H^1(\Omega)$ the embedding $k(v,\cdot) \in L^p_+(\Omega)$ holds for some $p \geqslant 3/2$, independent of v, and on any ball $B_r = \{v \in H^1(\Omega) : \|v\|_{1,\Omega} \leqslant r\}$ of radius r the following inequality holds:

$$\|k(v_1,\cdot) - k(v_2,\cdot)\|_{L^p(\Omega)} \leqslant L_r\|v_1 - v_2\|_{L^4(\Omega)} \ \ \forall v_1, v_2 \in B_r. \tag{1.3}$$

Here the constant L_r depends on r but does not depend on $v_1, v_2 \in B_r$;

(v) $(k(\varphi_1,\cdot)\varphi_1 - k(\varphi_2,\cdot)\varphi_2, \varphi_1 - \varphi_2) \geqslant 0$ for all $\varphi_1, \varphi_2 \in H^1(\Omega)$;

(vi) $\|k(\varphi,\cdot)\|_{L^p(\Omega)} \leqslant A\|\varphi\|^r_{1,\Omega} + B$ for all $\varphi \in H^1(\Omega)$, where number p is defined in (iii), $r \geqslant 0$ is a fixed number, A and B are nonnegative constants.

Let us note that the condition (iv) describes an operator acting from $H^1(\Omega)$ to $L^p(\Omega)$, $p \geqslant 3/2$, allowing to take into account the rather arbitrary dependence of the reaction coefficient k on both the concentration φ and the spatial variable $\mathbf{x}$. Condition (v) means that the nonlinearity $k(\varphi,\cdot)\varphi$ is monotone [19, p. 182], and condition (vi) restricts the growth in φ of the reaction coefficient by a power function with exponent r.

The specified conditions will provide a proof of the solvability of Problem 1 considered under the inhomogeneous Dirichlet condition on the part Γ_D of the boundary Γ. As an example of the function $k(\varphi,\cdot)$ satisfying (iv)–(vi) we give the function $k : \mathbb{R} \times \Omega \to \mathbb{R}$, such that $k(\varphi, \mathbf{x}) = \varphi^2$ for $\mathbf{x} \in Q$ where Q is a subdomain of domain Ω, $k(\varphi, \mathbf{x}) = k_0(\mathbf{x}) \in L^{3/2}_+(\Omega \setminus \overline{Q})$ for $\mathbf{x} \in \Omega \setminus \overline{Q}$.

Let us also remind that, by the Sobolev embedding theorem, the space $H^1(\Omega)$ is embedded into the space $L^s(\Omega)$ continuously at $s \leqslant 6$ and compactly at $s < 6$ and, with a certain constant C_s, depending on s and Ω, we have the estimate

$$\|\varphi\|_{L^s(\Omega)} \leqslant C_s\|\varphi\|_{1,\Omega} \ \ \forall\varphi \in H^1(\Omega). \tag{1.4}$$

The following technical lemma holds (see details in [7]).

Lemma 1.1. *Let, in addition to condition (i)–(iii),* $\mathbf{u} \in Z$, $k_1(\cdot) \in L^p_+(\Omega)$, $p \geqslant 3/2$. *Then the following relations hold:*

$$|(\lambda\nabla\varphi, \nabla\eta)| \leqslant \gamma_s\|\lambda\|_{s,\Omega}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega} \ \forall\, \varphi, \eta \in H^1(\Omega),$$
$$(\lambda\nabla h, \nabla h) \geqslant \lambda_*\|h\|^2_{1,\Omega} \ \forall\, h \in \mathcal{T},\ \lambda_* \equiv \delta_0\lambda_0, \tag{1.5}$$

$$|(\mathbf{u}\cdot\nabla\varphi, \eta)| \leqslant \gamma_1\|\mathbf{u}\|_{L^4(\Omega)^3}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega} \ \forall\, \varphi, \eta \in H^1(\Omega),\ \ (\mathbf{u}\cdot\nabla h, h) = 0 \ \forall\, h \in \mathcal{T}, \tag{1.6}$$

$$|(\chi, \varphi)_{\Gamma_N}| \leqslant \gamma_2\|\chi\|_{\Gamma_N}\|\varphi\|_{1,\Omega} \ \ \forall\, \chi \in L^2(\Gamma_N),\ \varphi \in H^1(\Omega), \tag{1.7}$$

$$|(\lambda\alpha\varphi, \eta)_{\Gamma_N}| \leqslant \gamma_3^s\|\lambda\|_{s,\Omega}\|\alpha\|_{\Gamma_N}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega} \ \ \forall\, \varphi, \eta \in H^1(\Omega), \tag{1.8}$$

$$|(k_1\varphi, \eta)| \leqslant \gamma_p\|k_1\|_{L^p(\Omega)}\|\varphi\|_{1,\Omega}\|\eta\|_{1,\Omega} \ \ \forall\, \varphi, \eta \in H^1(\Omega). \tag{1.9}$$

Here $\lambda_* = \delta_0\lambda_0$, *constants* γ_1 *and* γ_2 *depend on* Ω, *constants* γ_s *and* γ_3^s *depend on* Ω *and* s, γ_p *depends on* Ω *and* p.

Let us multiply the equation (1.1) by $h \in \mathcal{T}$ and integrate over Ω using Green's formulae. Taking into account (1.2), we obtain

$$(\lambda\nabla\varphi, \nabla h)+(k(\varphi, \cdot)\varphi, h)+(\mathbf{u}\cdot\nabla\varphi, h)+(\lambda\alpha\varphi, h)_{\Gamma_N} = (f, h)+(\chi, h)_{\Gamma_N} \ \forall h \in \mathcal{T},\ \varphi|_{\Gamma_D} = \psi. \tag{1.10}$$

Definition 1.1. *The function* $\varphi \in H^1(\Omega)$, *which satisfies (1.10), will be called a weak solution of Problem 1.*

To prove the solvability of Problem 1, we need the following lemma [12].

Lemma 1.2. *Let condition (i) holds. Then for any function* $\psi \in H^{1/2}(\Gamma_D)$ *there exists a function* $\varphi_0 \in H^1(\Omega)$, *such that* $\varphi_0 = \psi$ *on* Γ_D *and with some constant* C_Γ, *depending on* Ω *and* Γ_D, *the estimate* $\|\varphi_0\|_{1,\Omega} \leqslant C_\Gamma\|\psi\|_{1/2,\Gamma_D}$ *holds.*

We represent the solution to Problem 1 as the sum $\varphi = \tilde{\varphi} + \varphi_0$ where φ_0 is a given function from Lemma 1.2 and $\tilde{\varphi} \in \mathcal{T}$ is unknown function. Substituting $\varphi = \tilde{\varphi} + \varphi_0$ in (1.10) we will have

$$(\lambda\nabla\tilde{\varphi}, \nabla h) + (k(\tilde{\varphi} + \varphi_0, \cdot)(\tilde{\varphi} + \varphi_0), h) + (\mathbf{u}\cdot\nabla\tilde{\varphi}, h) + (\lambda\alpha\tilde{\varphi}, h)_{\Gamma_N} =$$
$$= (f, h) + (\chi, h)_{\Gamma_N} - (\lambda\nabla\varphi_0, \nabla h) - (\mathbf{u}\cdot\nabla\varphi_0, h) - (\lambda\alpha\varphi_0, h)_{\Gamma_N} \ \ \forall\, h \in \mathcal{T}. \tag{1.11}$$

Adding the term $-(k(\varphi_0, \cdot)\varphi_0, h)$ to both parts of (1.11), we obtain

$$(\lambda\nabla\tilde{\varphi}, \nabla h) + (k(\tilde{\varphi} + \varphi_0, \cdot)(\tilde{\varphi} + \varphi_0) - k(\varphi_0, \cdot)\varphi_0, h) + (\mathbf{u}\cdot\nabla\tilde{\varphi}, h) + (\lambda\alpha\tilde{\varphi}, h)_{\Gamma_N} =$$
$$\langle l, h\rangle \equiv (f, h)+(\chi, h)_{\Gamma_N} - (\lambda\nabla\varphi_0, \nabla h) - (\mathbf{u}\cdot\nabla\varphi_0, h) - (k(\varphi_0, \cdot)\varphi_0, h) - (\lambda\alpha\varphi_0, h)_{\Gamma_N} \ \forall\, h \in \mathcal{T}. \tag{1.12}$$

Using the Holder inequality, Lemmas 1.1, 1.2, estimate (1.4) and condition (vi), it is easy to show that $l \in \mathcal{T}^*$ and, moreover, the following estimate holds:

$$\|l\|_{\mathcal{T}^*} \leqslant M_l \equiv \|f\|_\Omega + \gamma_2\|\chi\|_{\Gamma_N} + C_\Gamma(\gamma_s\|\lambda\|_{s,\Omega} + \gamma_1\|\mathbf{u}\|_{L^4(\Omega)^3})\|\psi\|_{1/2,\Gamma_D} +$$
$$+C_\Gamma[\gamma_p(AC_\Gamma^r\|\psi\|^r_{1/2,\Gamma_D} + B) + \gamma_3^s\|\lambda\|_{s,\Omega}\|\alpha\|_{\Gamma_N}]\|\psi\|_{1/2,\Gamma_D}. \tag{1.13}$$

Let us introduce the nonlinear operator $A : \mathcal{T} \to \mathcal{T}^*$ by

$$\langle A(\tilde{\varphi}), h\rangle \equiv (\lambda\nabla\tilde{\varphi}, \nabla h) + (k(\tilde{\varphi} + \varphi_0, \cdot)(\tilde{\varphi} + \varphi_0) - k(\varphi_0, \cdot)\varphi_0, h) + (\mathbf{u}\cdot\nabla\tilde{\varphi}, h) +$$
$$+(\lambda\alpha\tilde{\varphi}, h)_{\Gamma_N} \ \ \forall\, \tilde{\varphi},\ h \in \mathcal{T}. \tag{1.14}$$

It is clear that the problem (1.12) is equivalent to the operator equation $A(\tilde{\varphi}) = l$. According to [19, p. 182], to prove the existence of a solution $\tilde{\varphi} \in \mathcal{T}$ of problem (1.12) it suffices to show that: 1) the operator A is monotone on $\mathcal{T}$, that is $\langle A(u)-A(v), u-v\rangle \geqslant 0$ for all $u, v \in \mathcal{T}$; 2) the operator $A : \mathcal{T} \to \mathcal{T}^*$ is continuous and bounded; 3) the operator A is coercive on $\mathcal{T}$.

To prove the monotonicity of the operator A we subtract the relation (1.14) for $\tilde{\varphi} = \tilde{\varphi}_2$ from (1.14) for $\tilde{\varphi} = \tilde{\varphi}_1$ where $\tilde{\varphi}_1$ and $\tilde{\varphi}_2 \in \mathcal{T}$ are arbitrary elements. We obtain

$$\langle A(\tilde{\varphi}_1) - A(\tilde{\varphi}_2), h\rangle = (\lambda\nabla(\tilde{\varphi}_1 - \tilde{\varphi}_2), \nabla h) + (k(\tilde{\varphi}_1 + \varphi_0, \cdot)(\tilde{\varphi}_1 + \varphi_0) - k(\tilde{\varphi}_2 + \varphi_0, \cdot)(\tilde{\varphi}_2 + \varphi_0), h) +$$

$$+(\mathbf{u} \cdot \nabla(\tilde{\varphi}_1 - \tilde{\varphi}_2), h) + (\lambda\alpha(\tilde{\varphi}_1 - \tilde{\varphi}_2), h)_{\Gamma_N} \ \ \forall\, h \in \mathcal{T}. \tag{1.15}$$

For $h = \tilde{\varphi}_1 - \tilde{\varphi}_2$ all terms in the right-hand side of (1.15) are nonnegative due to the properties of the functions $\lambda, \alpha, \mathbf{u}$ indicated in (ii), (iii) and monotonicity of nonlinearity $k(\varphi)\varphi$. Therefore

$$\langle A(\tilde{\varphi}_1) - A(\tilde{\varphi}_2), \tilde{\varphi}_1 - \tilde{\varphi}_2\rangle \geqslant 0 \ \ \forall \tilde{\varphi}_1, \tilde{\varphi}_2 \in \mathcal{T}.$$

To prove the continuity and boundedness of the operator A we rewrite (1.15) in the form

$$\langle A(\tilde{\varphi}_1) - A(\tilde{\varphi}_2), h\rangle = (\lambda\nabla(\tilde{\varphi}_1 - \tilde{\varphi}_2), \nabla h) + (k(\tilde{\varphi}_1 + \varphi_0, \cdot) - k(\tilde{\varphi}_2 + \varphi_0, \cdot), \tilde{\varphi}_1 + \varphi_0, h) +$$

$$+(k(\tilde{\varphi}_2 + \varphi_0, \cdot)(\tilde{\varphi}_1 - \tilde{\varphi}_2), h) + (\mathbf{u} \cdot \nabla(\tilde{\varphi}_1 - \tilde{\varphi}_2), h) + (\lambda\alpha(\tilde{\varphi}_1 - \tilde{\varphi}_2), h)_{\Gamma_N} \ \ \forall h \in \mathcal{T}. \tag{1.16}$$

Using the estimates of Lemma 1.1, the estimates (1.4), (1.9), and condition (iii), from (1.16) we deduce that

$$|\langle A(\tilde{\varphi}_1) - A(\tilde{\varphi}_2), h\rangle| \leqslant (\gamma_s\|\lambda\|_{s,\Omega} + \gamma_p L C_4\|\varphi_1\|_{1,\Omega})\|\tilde{\varphi}_1 - \tilde{\varphi}_2\|_{1,\Omega}\|h\|_{1,\Omega} +$$

$$+(\gamma_p\|k(\tilde{\varphi}_2 + \varphi_0, \cdot)\|_{L^p(\Omega)} + \gamma_1\|\mathbf{u}\|_{L^4(\Omega)^3} + \gamma_3^s\|\lambda\|_{s,\Omega}\|\alpha\|_{\Gamma_N})\|\tilde{\varphi}_1 - \tilde{\varphi}_2\|_{1,\Omega}\|h\|_{1,\Omega} \ \ \forall h \in \mathcal{T}. \tag{1.17}$$

The inequality (1.17) implies the continuity and boundedness of the operator A. Finally, setting $h = \tilde{\varphi}$ in (1.14) and using conditions (ii), (iv), and (1.6), we arrive at the following inequality which implies the coercivity of the operator A:

$$\langle A(\tilde{\varphi}), \tilde{\varphi}\rangle = (\lambda\nabla\tilde{\varphi}, \nabla\tilde{\varphi}) + (k(\tilde{\varphi} + \varphi_0, \cdot)(\tilde{\varphi} + \varphi_0) - k(\varphi_0, \cdot)\varphi_0, \tilde{\varphi}) +$$

$$+(\lambda\alpha\tilde{\varphi}, \tilde{\varphi})_{\Gamma_N} \geqslant \lambda_*\|\tilde{\varphi}\|^2_{1,\Omega} \ \ \forall \tilde{\varphi} \in \mathcal{T}. \tag{1.18}$$

As a result we conclude that the solution $\tilde{\varphi} \in \mathcal{T}$ of the problem (1.11) exists and the estimate $\|\tilde{\varphi}\|_{1,\Omega} \leqslant C_*\|l\|_{\mathcal{T}^*}$, $C_* = \lambda_*^{-1}$ takes place. In this case, the function $\varphi = \varphi_0 + \tilde{\varphi}$ is the desired weak solution to Problem 1 and the following estimate holds:

$$\|\varphi\|_{1,\Omega} \leqslant M_\varphi \equiv C_* M_l + C_\Gamma\|\psi\|_{1/2,\Gamma_D} \ \ (C_* = \lambda_*^{-1}). \tag{1.19}$$

Here the constant M_l was defined in (1.13) and C_Γ is the constant from Lemma 1.2.

Let us show that the solution to Problem 1 is unique. Let φ_1 and $\varphi_2 \in H^1(\Omega)$ be any two solutions to Problem 1. Then their difference $\varphi = \varphi_1 - \varphi_2 \in \mathcal{T}$ satisfies the identity

$$(\lambda\nabla\varphi, \nabla h) + (k(\varphi_1, \cdot)\varphi_1 - k(\varphi_2, \cdot)\varphi_2, h) + (\mathbf{u} \cdot \nabla\varphi, h) + (\lambda\alpha(\varphi_1 - \varphi_2), h)_{\Gamma_N} = 0 \ \forall\, h \in \mathcal{T}.$$

Setting here $h = \varphi$, by virtue of conditions (iii), (v) and (1.6) we arrive at the inequality $\lambda_*\|\varphi\|_{1,\Omega} \leqslant 0$, from which it follows that $\varphi_1 = \varphi_2$ in Ω. This proves the following theorem.

Theorem 1.1. *Let conditions (i)–(vi) hold. Then there exists a unique weak solution $\varphi \in H^1(\Omega)$ of Problem 1 and the estimate (1.19) holds.*

Within the framework of the approach of [20] we prove the maximum and minimum principles for a weak solution φ to Problem 1. To this end, we assume, in addition to (i)–(vi), that the following conditions are satisfied:

(vii) $\psi_{\min} \leqslant \psi \leqslant \psi_{\max}$ a.e. on Γ_D, $f_{\min} \leqslant f \leqslant f_{\max}$ and $\lambda_{\min} \leqslant \lambda \leqslant \lambda_{\max}$ a.e. in Ω, $\alpha_{\min} \leqslant \alpha \leqslant \alpha_{\max}$ and $\chi_{\min} \leqslant \chi \leqslant \chi_{\max}$ a.e. on Γ_N.
Here $\psi_{\min}$, $\psi_{\max}$, $f_{\min}$, $f_{\max}$, $\chi_{\min}$, $\chi_{\max}$ are nonnegative numbers, while $\alpha_{\min}$, $\alpha_{\max}$ and $\lambda_{\min}$, $\lambda_{\max}$ are positive numbers;

Besides, we will assume also that the reaction coefficient k satisfies the following conditions:

(viii) the reaction coefficient k has the form $k = k_1(\varphi)$ where $k_1(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous nonnegative function, satisfying conditions (iv)–(vi), in which one should set $k(\varphi, \cdot) = k_1(\varphi)$, and every of functional with respect to M_1 and m_1 equations

$$k_1(M_1)M_1 = f_{\max} \quad \text{and} \quad k_1(m_1)m_1 = f_{\min} \tag{1.20}$$

has at least one solution.

We set

$$M = \max\{\psi_{\max}, \chi_{\max}/\lambda_{\min}\alpha_{\min}, M_1\}, \quad m = \min\{\psi_{\min}, \chi_{\min}/\lambda_{\max}\alpha_{\max}, m_1\}. \tag{1.21}$$

Theorem 1.2. *Let conditions (i)–(iii), (vii), (viii) hold. Then for the solution $\varphi \in H^1(\Omega)$ of Problem 1 the following maximum and minimum principle holds:*

$$m \leqslant \varphi \leqslant M \text{ a.e. in } \Omega. \tag{1.22}$$

Here the constants m and M are defined in (1.21) where M_1 is a minimum root of the first equation in (1.20) and m_1 is a maximum root of the second equation in (1.20).

Proof. Firstly, we prove the validity of the maximum principle in the form of the estimate $\varphi \leqslant M$ in Ω. For this purpose we introduce a nonnegative function $v = \max\{\varphi - M, 0\}$. From the definition of v it follows that the estimate $\varphi \leqslant M$ holds if and only if $v = 0$ in Ω. We denote by $\Omega_M \subset \Omega$ a measurable subset of Ω, at the points of which the inequality $\varphi > M$ holds, by Γ_M we denote the measurable subset of the part Γ_N, at the points of which the condition $v|_{\Gamma_M} > 0$ is satisfied. Set $\Omega_1 = \Omega \setminus \Omega_M$, $\Gamma_1 = \Gamma_N \setminus \Gamma_M$. From [21, p. 152] and [22] it follows by the definition of the constant M in (1.21) that $v \in \mathcal{T}$, and the following relations hold:

$$v = \varphi - M > 0 \text{ and } \nabla v = \nabla\varphi \text{ in } \Omega_M; \quad v = 0 \text{ and } \nabla v = \mathbf{0} \text{ in } \Omega_1; v|_{\Gamma_1} = 0,$$

$$(\lambda\nabla\varphi, \nabla v) = (\lambda\nabla v, \nabla v)_{\Omega_M} = (\lambda\nabla v, \nabla v), \quad (\mathbf{u}\cdot\nabla\varphi, v) = (\mathbf{u}\cdot\nabla\varphi, v)_{\Omega_M} = (\mathbf{u}\cdot\nabla v, v) = 0.$$

We set $h = v$ in (1.10) at $k(\varphi) = k_1(\varphi)$ and add to both sides of the resulting equality the term $-(k_1(M)M, v)_{Q_M} - (\lambda\alpha M, v)_{\Gamma_M}$. Taking into account the properties of v we obtain

$$(\lambda\nabla v, \nabla v) + (k_1(v + M)(v + M) - k_1(M)M, v)_{Q_M} + (\lambda\alpha v, v)_{\Gamma_M} =$$

$$= (f - k_1(M)M, v)_{Q_M} + (\chi - \lambda\alpha M, v)_{\Gamma_M}. \tag{1.23}$$

From the definition of the constant M in (1.21), relations (1.20) and conditions (ii),(iii), (vi) and (vii) it follows that the right-hand side in (1.23) is non-positive while the second and third terms in the left-hand side are nonnegative. Taking into account this fact and the second inequality in (1.5) from (1.23) we arrive at the estimate $\|v\|^2_{1,\Omega} \leqslant 0$, from which it follows that $v = 0$. This means the validity of the estimate of $\varphi \leqslant M$ in Ω.

To prove the minimum principle in the form of the estimate $\varphi \geqslant m$ in Ω we introduce a non-positive function $w = \min\{\varphi - m, 0\}$ and note that the validity of the minimum principle is equivalent to the condition $w = 0$ in Ω. Let us denote by Ω_m a measurable subset of Ω, at the points of which $\varphi < m$. By Γ_m we denote a measurable subset of the part Γ_N, at the points of which $\varphi|_{\Gamma_m} < m$. Set $\Omega_2 = \Omega \setminus \Omega_m$, $\Gamma_2 = \Gamma_N \setminus \Gamma_m$. By definition of Ω_m and Γ_m we have

$$w = \varphi - m < 0 \text{ and } \nabla w = \nabla\varphi \text{ in } \Omega_m; \quad w = 0 \text{ and } \nabla w = \mathbf{0} \text{ in } \Omega_2, \; w = 0 \text{ on } \Gamma_2.$$

Setting $h = w$ in (1.10) at $k(\varphi) = k_1(\varphi)$ we add to both sides of the resulting relation the term $-(k_1(m)m, w)_{Q_m} - (\lambda\alpha m, w)_{\Gamma_m}$. Taking into account the properties of the function w we obtain

$$(\lambda\nabla w, \nabla w) + (k_1(w+m)(w+m) - k_1(m)m, w)_{Q_m} + (\lambda\alpha w, w)_{\Gamma_m} =$$
$$= (f - k_1(m)m, w)_{Q_m} + (\chi - \lambda\alpha m, w)_{\Gamma_m}. \tag{1.24}$$

From the definition of the constant m in (1.21), (1.20) and conditions (ii), (iii) (v), (vii) it follows that the right-hand side in (1.24) is non-positive while the second and third terms in the left-hand side are nonnegative. Taking into account this fact, from (1.24) we derive that $w = 0$. □

Remark 1.2. For power-law reaction coefficients, the parameters M_1 and m_1 are easily calculated. For example, for $k_1(\varphi) = \varphi^2$, we easily deduce that $M_1 = f_{\max}^{1/3}$, $m_1 = f_{\min}^{1/3}$.

2. Formulation and solvability of control problem

To formulate the control problem we divide the set of initial data of Problem 1 into two groups: a group of fixed data, to which we assign the functions $\mathbf{u}, k(\varphi,\cdot), \alpha$ and ψ, and the control group, to which we assign the functions λ, f and χ, assuming that they can change in some sets K_1, K_2 and K_3 satisfying the condition

(j) $K_1 \subset H^s_{\lambda_0}(\Omega)$, $K_2 \subset L^2(\Omega)$ and $K_3 \subset L^2(\Gamma_N)$ are nonempty convex closed sets.

Define the space $Y = \mathcal{T}^* \times H^{1/2}(\Gamma_D)$. Setting $u = (\lambda, f, \chi)$, $K = K_1 \times K_2 \times K_3$ we introduce the operator $F = (F_1, F_2) : H^1(\Omega) \times K \to Y$ by formulae: $F_2(\varphi) = \varphi|_{\Gamma_D} - \psi$ and

$$\langle F_1(\varphi, u), h\rangle = (\lambda\nabla\varphi, \nabla h) + (k(\varphi,\cdot)\varphi, h) + (\mathbf{u}\cdot\nabla\varphi, h) + (\lambda\alpha\varphi, h)_{\Gamma_N} - (f, h) - (\chi, h)_{\Gamma_N}$$

and rewrite (1.10) in the form $F(\varphi, u) = 0$. Considering this equality as a conditional restriction on the state $\varphi \in H^1(\Omega)$ and control $u \in K$, we introduce the cost functional I and formulate the following conditional minimization problem:

$$\begin{gathered} J(\varphi, u) \equiv \frac{\mu_0}{2} I(\varphi) + \frac{\mu_1}{2}\|\lambda\|^2_{s,\Omega} + \frac{\mu_2}{2}\|f\|^2_{\Omega} + \frac{\mu_3}{2}\|\chi\|^2_{\Gamma_N} \to \inf, \\ F(\varphi, u) = 0, \quad (\varphi, u) \in H^1(\Omega) \times K. \end{gathered} \tag{2.1}$$

We denote by $Z_{ad} = \{(\varphi, u) \in H^1(\Omega) \times K : F(\varphi, u) = 0, J(\varphi, u) < \infty\}$ the set of admissible pairs for the problem (2.1) and suppose that the following condition is satisfied:

(jj) $\mu_0 > 0$, $\mu_i \geqslant 0$, $i = 1, 2, 3$, and K is a bounded set or $\mu_i > 0$, $i = 0, 1, 2, 3$ and functional I is bounded below.

We use the following cost functionals:

$$I_1(\varphi) = \|\varphi - \varphi^d\|^2_Q = \int_Q |\varphi - \varphi^d|^2 d\mathbf{x}, \;\; I_2(\varphi) = \|\varphi - \varphi^d\|^2_{1,Q}. \tag{2.2}$$

Here $\varphi^d \in L^2(Q)$ (or $\varphi^d \in H^1(Q)$) is a given function in some subdomain $Q \subset \Omega$.

Theorem 2.1. *Let, in addition to conditions (i), (iii)–(vi), and (j), (jj), $I : H^1(\Omega) \to \mathbb{R}$ be a weakly semicontinuous below functional and let $Z_{ad} \neq \emptyset$. Then there exists at least one solution $(\varphi, u) \in H^1(\Omega) \times K$ of the control problem (2.1).*

Proof. Let $(\varphi_m, u_m) \in Z_{ad}$ be a minimizing sequence for which the following is true

$$\lim_{m\to\infty} J(\varphi_m, u_m) = \inf_{(\varphi,u)\in Z_{ad}} J(\varphi, u) \equiv J^*.$$

Condition (jj) and Theorem 1.1 yield the following estimates:

$$\|\lambda_m\|_{s,\Omega} \leqslant c_1,\ \|f_m\|_\Omega \leqslant c_2,\ \|\chi_m\|_{\Gamma_N} \leqslant c_3,\ \ \|\varphi_m\|_{1,\Omega} \leqslant c_4 \tag{2.3}$$

where the constants c_i, $i = 1, 2, 3, 4$ don't depend on m.

From the estimates (2.3) and from the condition (j) it follows that there exist weak limits $\lambda^* \in K_1$, $f^* \in K_2$, $\chi^* \in K_3$ and $\varphi^* \in H^1(\Omega)$ of some subsequences of sequences $\{\lambda_m\}$, $\{f_m\}$, $\{\chi_m\}$ and $\{\varphi_m\}$, respectively. Corresponding subsequences will be also denoted by $\{\lambda_m\}$, $\{f_m\}$, $\{\chi_m\}$ and $\{\varphi_m\}$. Moreover, due to the compactness of the embeddings $H^1(\Omega) \subset L^p(\Omega)$ for $p < 6$, $H^{1/2}(\Gamma_N) \subset L^q(\Gamma_N)$ for $q < 4$, $H^s(\Omega) \subset L^\infty(\Omega)$ and $H^{s-1/2}(\Gamma_N) \subset L^\infty(\Gamma_N)$ for $s > 3/2$ we can assume for $m \to \infty$, that

$$\begin{gathered}
\varphi_m \to \varphi^* \text{ weakly in } H^1(\Omega),\ \text{weakly in } L^6(\Omega) \text{ and strongly in } L^s(\Omega),\ s < 6,\\
\varphi_m|_{\Gamma_N} \to \varphi^*|_{\Gamma_N} \text{ weakly in } H^{1/2}(\Gamma_N),\ \text{weakly in } L^4(\Gamma_N) \text{ and strongly in } L^q(\Gamma_N),\ q < 4,\\
f_m \to f^* \text{ weakly in } L^2(\Omega),\ \ \chi_m \to \chi^* \text{ weakly in } L^2(\Gamma_N),\\
\lambda_m \to \lambda^* \text{ weakly in } H^s(\Omega) \text{ and strongly in } L^\infty(\Omega),\\
\lambda_m|_{\Gamma_N} \to \lambda^*|_{\Gamma_N} \text{ weakly in } H^{s-1/2}(\Gamma_N) \text{ and strongly in } L^\infty(\Gamma_N),\ \ s > 3/2.
\end{gathered} \tag{2.4}$$

It is clear, that $F_2(\varphi^*) = 0$. Let us show that $F_1(\varphi^*, u^*) = 0$, that is, that

$$(\lambda^*\nabla\varphi^*, \nabla h) + (k(\varphi^*, \cdot)\varphi^*, h) + (\mathbf{u}\cdot\nabla\varphi^*, h) + (\lambda^*\alpha\varphi^*, h)_{\Gamma_N} = (f^*, h) + (\chi^*, h)_{\Gamma_N}\ \forall\, h \in \mathcal{T}. \tag{2.5}$$

To this end we note that the pair (φ_m, u_m) satisfies the identity

$$\begin{aligned}
(\lambda_m\nabla\varphi_m, \nabla h) + (k(\varphi_m, \cdot)\varphi_m, h) + (\mathbf{u}\cdot\nabla\varphi_m, h) + (\lambda_m\alpha\varphi_m, h)_{\Gamma_N} =\\
= (f_m, h) + (\chi_m, h)_{\Gamma_N}\ \forall\, h \in \mathcal{T}.
\end{aligned} \tag{2.6}$$

Let us pass to the limit in (2.6) as $m \to \infty$. From (2.4) it follows that all linear terms in (2.6) pass into corresponding ones in (2.5).

Let us study the behaviour of nonlinear terms for $m \to \infty$ starting with $(k(\varphi_m, \cdot)\varphi_m, h)$. To prove the convergence

$$(k(\varphi_m, \cdot)\varphi_m, h) \to (k(\varphi^*, \cdot)\varphi^*, h)\ \text{ as } m \to \infty\ \forall\, h \in \mathcal{T} \tag{2.7}$$

it is enough to show that $k(\varphi_m, \cdot)\varphi_m \to k(\varphi^*, \cdot)\varphi^*$ weakly in $L^{6/5}(\Omega)$ as $m \to \infty$. From (1.3) it follows that $k(\varphi_m, \cdot) \to k(\varphi^*, \cdot)$ strongly in $L^{3/2}(\Omega)$, and from (2.4) it follows that $\varphi_m \to \varphi^*$ weakly in $L^6(\Omega)$ as $m \to \infty$. We derive from these properties that $k(\varphi_m, \cdot)\varphi_m \to k(\varphi^*, \cdot)\varphi^*$ weakly in $L^{6/5}(\Omega)$ and therefore (2.7) also holds.

For the term $(\lambda_m\nabla\varphi_m, \nabla h)$ the following equality holds:

$$(\lambda_m\nabla\varphi_m, \nabla h) - (\lambda^*\nabla\varphi^*, \nabla h) = ((\lambda_m - \lambda^*)\nabla\varphi_m, \nabla h) + (\nabla(\varphi_m - \varphi^*), \lambda^*\nabla h). \tag{2.8}$$

Since $\lambda^*\nabla h \in L^2(\Omega)^3$, then from (2.4) it follows that $(\nabla(\varphi_m - \varphi^*), \lambda^*\nabla h) \to 0$ as $m \to \infty$ for all $h \in \mathcal{T}$. Using Holder's inequality, (2.3) and (2.4) we easily deduce for the first term in the right-hand side of (2.8) that

$$|((\lambda_m - \lambda^*)\nabla\varphi_m, \nabla h)| \leqslant \|\lambda_m - \lambda^*\|_{L^\infty(\Omega)}\|\nabla\varphi_m\|_\Omega\|\nabla h\|_\Omega \to 0\ \text{ as } m \to \infty\ \ \forall\, h \in \mathcal{T}.$$

Then from (2.8) we obtain that $(\lambda_m \nabla\varphi_m, \nabla h) \to (\lambda^* \nabla\varphi^*, \nabla h)$ as $m \to \infty\ \forall\, h \in \mathcal{T}$.

Similarly, for the nonlinear term $(\lambda_m \alpha\varphi_m, h)_{\Gamma_N}$ we have that

$$(\lambda_m \alpha\varphi_m, h)_{\Gamma_N} - (\lambda^* \alpha\varphi^*, h)_{\Gamma_N} = ((\lambda_m - \lambda^*)\alpha\varphi_m, h)_{\Gamma_N} + (\lambda^* \alpha(\varphi_m - \varphi^*), h)_{\Gamma_N}. \tag{2.9}$$

Since $\lambda^* \alpha h \in L^{4/3}(\Gamma_N)$ then by virtue of (2.4) $(\varphi_m - \varphi^*, \lambda^* \alpha h)_{\Gamma_N} \to 0$ for all $h \in \mathcal{T}$ as $m \to \infty$. Using Holder inequality, (2.4) and the uniform boundedness of the quantity $\|\varphi_m\|_{L^4(\Gamma_N)}$ for any m, we deduce for the first term in the right-hand side of (2.9), that

$$|((\lambda_m - \lambda^*)\alpha\varphi_m, h)_{\Gamma_N}| \leqslant \|\lambda_m - \lambda^*\|_{L^\infty(\Gamma_N)} \|\alpha\|_{\Gamma_N} \|\varphi_m\|_{L^4(\Gamma_N)} \|h\|_{L^4(\Gamma_N)} \to 0 \text{ as } m \to \infty.$$

To complete the proof notice that the fact $J(\varphi^*, u^*) = J^*$ follows from aforesaid and from the weakly continuity below on $H^1(\Omega) \times H^s(\Omega) \times L^2(\Omega) \times L^2(\Gamma_N)$ of the functional J. □

Remark 2.1. The functionals defined in (2.2) satisfy the conditions of Theorem 2.1.

3. Derivation of the optimality system and stability estimates

The next stage in the study of the control problem (2.1) is the derivation of the optimality system. It provides valuable information about additional properties of optimal solutions for specific reaction coefficients, for example, in the case when $k(\varphi, \cdot) = \varphi^2|\varphi|$. Based on its analysis, one can establish, in particular, the uniqueness and stability of the optimal solutions More details about the method for deriving estimates of local stability of optimal solutions can be found in [11–16].

Based on the theory developed in [11–16] we introduce the space $Y^* = \mathcal{T} \times H^{1/2}(\Gamma_D)^*$ dual of the space Y. It is easy to show that for the case $k(\varphi, \cdot) = \varphi^2|\varphi|$ the Fréchet derivative of the operator $F = (F_1, F_2) : H^1(\Omega) \times K \to Y$ with respect to φ at any point $(\hat\varphi, \hat u) = (\hat\varphi, \hat\lambda, \hat f, \hat\chi)$ is a linear continuous operator $F'_\varphi(\hat\varphi, \hat u) : H^1(\Omega) \to Y$ that maps each element $\tau \in H^1(\Omega)$ into an element $F'_\varphi(\hat\varphi, \hat u)(\tau) = (\hat y_1, \hat y_2) \in Y$. Here the elements $\hat y_1 \in \mathcal{T}^*$ and $\hat y_2 \in H^{1/2}(\Gamma_D)$ are defined by $\hat\varphi, \hat\lambda$ and τ with the help of the following relations:

$$\langle \hat y_1, h\rangle = (\hat\lambda \nabla\tau, \nabla h) + 4(\hat\varphi^2|\hat\varphi|\tau, h) + (\hat\lambda\alpha\tau, h)_{\Gamma_N} + (\mathbf{u}\cdot\nabla\tau, h)\ \ \forall\, h \in \mathcal{T},\ y_2 = \tau|_{\Gamma_D}. \tag{3.1}$$

By $F'_\varphi(\hat\varphi, \hat u)^* : Y^* \to H^1(\Omega)^*$ we denote an operator adjoint of $F'_\varphi(\hat\varphi, \hat u)$.

According to the general theory of smooth-convex extremum problems [23], we introduce an element $\mathbf{y}^* = (\theta, \zeta) \in Y^*$, to which we will refer as to an adjoint state and we will define the Lagrangian $\mathcal{L} : H^1(\Omega) \times K \times Y^* \to \mathbb{R}$ by

$$\mathcal{L}(\varphi, u, \mathbf{y}^*) = J(\varphi, u) + \langle \mathbf{y}^*, F(\varphi, u)\rangle_{Y^* \times Y} \equiv J(\varphi, u) + \langle F_1(\varphi, u), \theta\rangle_{\mathcal{T}^* \times \mathcal{T}} + \langle \zeta, F_2(\varphi, u)\rangle_{\Gamma_D},$$

where $\langle \zeta, \cdot\rangle_{\Gamma_D} = \langle \zeta, \cdot\rangle_{H^{1/2}(\Gamma_D)^* \times H^{1/2}(\Gamma_D)}$.

Since $\hat\varphi^2|\hat\varphi| \in L^2_+(\Omega)$ then from [12] it follows that for any $f \in \mathcal{T}^*$ and $\psi \in H^{1/2}(\Gamma_D)$ there exists a unique solution $\tau \in H^1(\Omega)$ of the linear problem

$$(\hat\lambda \nabla\tau, \nabla h) + 4(\hat\varphi^2|\hat\varphi|\tau, h) + (\hat\lambda\alpha\tau, h)_{\Gamma_N} + (\mathbf{u}\cdot\nabla\tau, h) = \langle f, h\rangle\ \forall\, h \in \mathcal{T},\ \tau|_{\Gamma_D} = \psi. \tag{3.2}$$

Therefore the operator $F'_\varphi(\hat\varphi, \hat u) : H^1(\Omega) \to Y$ is an isomorphism and from [23] the following assertion follows.

Theorem 3.1. *Let, under conditions (i), (iii)–(vi) and (j), (jj), $k(\varphi, \cdot) = \varphi^2|\varphi|$, the functional $I : H^1(\Omega) \to \mathbb{R}$ is continuously differentiable with respect to φ at the point $\hat\varphi$ and let an element*

$(\hat{\varphi},\hat{u}) \in H^1(\Omega)\times K$ be a local minimizer for the problem (2.1). Then there exists a unique Lagrange multiplier (adjoint state) $\mathbf{y}^* = (\theta,\zeta) \in Y^*$, such that the Euler–Lagrange equation $F'_\varphi(\hat{\varphi},\hat{u})^*\mathbf{y}^* = -J'_\varphi(\hat{\varphi},\hat{u})$ in $H^1(\Omega)^*$ takes place which is equivalent to the relation

$$(\hat{\lambda}\nabla\tau,\nabla\theta) + 4(\hat{\varphi}^2|\hat{\varphi}|\tau,\theta) + (\hat{\lambda}\alpha\tau,\theta)_{\Gamma_N} + (\mathbf{u}\cdot\nabla\tau,\theta) + \langle\zeta,\tau\rangle_{\Gamma_D} =$$

$$= -(\mu_0/2)\langle I'_\varphi(\hat{\varphi}),\tau\rangle \;\; \forall\tau\in H^1(\Omega), \tag{3.3}$$

and the minimum principle $\mathcal{L}(\hat{\varphi},\hat{u},\mathbf{y}^*) \leqslant \mathcal{L}(\hat{\varphi},u,\mathbf{y}^*)$ $\forall u \in K$ holds which is equivalent to the inequalities

$$\mu_1(\hat{\lambda},\lambda-\hat{\lambda})_{s,\Omega} + ((\lambda-\hat{\lambda})\nabla\hat{\varphi},\nabla\theta) + ((\lambda-\hat{\lambda})\alpha\hat{\varphi},\theta)_{\Gamma_N} \geqslant 0 \;\forall\lambda\in K_1, \tag{3.4}$$

$$\mu_2(\hat{f},f-\hat{f})_\Omega - (f-\hat{f},\theta) \geqslant 0 \;\; \forall f \in K_2, \tag{3.5}$$

$$\mu_3(\hat{\chi},\chi-\hat{\chi})_{\Gamma_N} - (\chi-\hat{\chi},\theta)_{\Gamma_N} \geqslant 0 \;\; \forall\chi\in K_3. \tag{3.6}$$

The relations (3.3)–(3.6) together with the operator restriction $F(\hat{\varphi},\hat{u}) = 0$ comprise an *optimality system* for problem (2.1). It plays an important role in the study of uniqueness and stability of its solutions.

In conclusion, we formulate a theorem on the local stability of optimal solutions of problem (2.1) for $I(\varphi) = \|\varphi-\varphi^d\|_Q^2$, which is proved according to the scheme proposed in [11].

Theorem 3.2. *Assume that the conditions (i), (iii)–(vi) and (j), (jj) take place and $k(\varphi,\cdot) = \varphi^2|\varphi|$. Let the quadruple $(\varphi_i,\lambda_i,f_i,\chi_i)\in X\times K$ be a solution of the problem (2.1) at $I(\varphi) = \|\varphi-\varphi_i^d\|_Q^2$, which corresponds to a specified function $\varphi_i^d\in L^2(\Omega)$, $i = 1,2$. Let the data of the problem (2.1) or parameters μ_0,μ_1,μ_2 and μ_3 be such that the following condition hold:*

$$\eta_1^2\mu_0 \leqslant (1-\varepsilon)\mu_1, \;\; \eta_2^2\mu_0 \leqslant (1-\varepsilon)\mu_2, \;\; \eta_3^2\mu_0 \leqslant (1-\varepsilon)\mu_3, \tag{3.7}$$

where $\varepsilon\in(0,1)$ is an arbitrary number, the parameters η_k, $k=1,2,3,4$, monotonically depend on the norms of the initial data of the problem (2.1). Then the following local stability estimates hold:

$$\|\lambda_1-\lambda_2\|_{s,\Omega} \leqslant \sqrt{\mu_0/(\varepsilon\mu_1)}(0.5+\eta_4)\|\varphi_1^d-\varphi_2^d\|_Q, \tag{3.8}$$

$$\|f_1-f_2\|_\Omega \leqslant \sqrt{\mu_0/(\varepsilon\mu_2)}(0.5+\eta_4)\|\varphi_1^d-\varphi_2^d\|_Q, \tag{3.9}$$

$$\|\chi_1-\chi_2\|_{\Gamma_N} \leqslant \sqrt{\mu_0/(\varepsilon\mu_3)}(0.5+\eta_4)\|\varphi_1^d-\varphi_2^d\|_Q, \tag{3.10}$$

$$\|\varphi_1-\varphi_2\|_{1,\Omega} \leqslant (\omega_1\sqrt{\mu_0/(\varepsilon\mu_1)} + \omega_2\sqrt{\mu_0/(\varepsilon\mu_2)} + \omega_3\sqrt{\mu_0/(\varepsilon\mu_3)})(0.5+\eta_4)\|\varphi_1^d-\varphi_2^d\|_Q. \tag{3.11}$$

Here $\omega_1 = C_*(\gamma_3^s\|\alpha\|_{\Gamma_N}M_\varphi + \gamma_s M_\varphi)$, $\omega_2 = C_*$, $\omega_3 = \gamma_2 C_*$, where $\lambda_*,\gamma_2,\gamma_3^s,\gamma_s, C_* = 1/\lambda_*$ are the constants from Lemma 1.1 and M_φ is defined in (1.19).

A similar theorem can be formulated and proved for the functional $I_2(\varphi)$ in (2.2). The authors plan to devote a separate paper to a more detailed study of the issues of uniqueness and stability of optimal solutions.

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Анализ краевых задач и задач управления для нелинейного уравнения реакции-диффузии-конвекции

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Аннотация. Доказывается глобальная разрешимость неоднородной смешанной краевой задачи и задач управления для уравнения реакции-диффузии-конвекции в случае, когда коэффициент реакции нелинейно зависит от концентрации. Для решения краевой задачи устанавливаются принципы максимума и минимума. Для задач управления с конкретными коэффициентами реакции выводятся системы оптимальности и устанавливаются оценки локальной устойчивости оптимальных решений.

Ключевые слова: нелинейное уравнение реакции-диффузии-конвекции, смешанные граничные условия, принцип максимума, задачи управления, системы оптимальности, оценки локальной устойчивости.

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On some Inverse Parabolic Problems with Pointwise Overdetermination

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Abstract. We examine well-posedness questions in the Sobolev spaces of inverse problems of recovering coefficients depending on time in a parabolic system. The overdetermination conditions are values of a solution at some collection of points lying inside the domain and on its boundary. The conditions obtained ensure existence and uniqueness of solutions to these problems in the Sobolev classes.

Keywords: parabolic system, inverse problem, pointwise overdetermination, convection-diffusion.

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Dedicated to Yu. Ya. Belov

Introduction

We consider inverse problems with pointwise overdetermination for a parabolic system of the form

$$Lu = u_t + A(t,x,D)u = f(x,t), \quad (t,x) \in Q = (0,T) \times G, \ G \subset \mathbb{R}^n, \tag{1}$$

where

$$A(t,x,D)u = -\sum_{i,j=1}^{n} a_{ij}(t,x)u_{x_j x_j} + \sum_{i=1}^{n} a_i(t,x)u_{x_i} + a_0(t,x)u,$$

G is a bounded domain with boundary $\Gamma \in C^2$, a_{ij}, a_i are matrices of dimension $h \times h$, and u is a vector of length h. The system (1) is supplemented by the initial and boundary conditions

$$u|_{t=0} = u_0, \quad Bu|_S = g, \quad S = (0,T) \times \Gamma, \tag{2}$$

where $Bu = \sum\limits_{i=1}^{n} \gamma_i(t,x)u_{x_i} + \gamma_0(t,x)u$. The overdetermination conditions are as follows:

$$< u(x_i,t), e_i >= \psi_i(t), \quad i = 1,2,\ldots,r, \tag{3}$$

where the symbol $< \cdot,\cdot >$ stands for the inner product in $\mathbb{C}^h$, $\{e_i\}$ is a collection of vectors of unit length and among the points $\{x_i\}$ as well as the vectors $\{e_i\}$ can be coinciding points and

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vectors. The right-hand side is of the form $f = \sum_{i=1}^{m} f_i(x,t)q_i(t) + f_0(x,t)$. The problems is to find the unknowns $q_i(t)$ occurring into the right-hand side and the operator A as coefficients and a solution u to the system (1) satisfying (2) and (3). The conditions (3) generalized the conventional pointwise overdetermination conditions of the form $u(x_i,t) = \psi_i(t)$. In particular, it is possible that only part of the coordinates of the vector u at a point x_i is given. These problems arise of describing heat and mass transfer, diffusion, filtration, and in many other fields (see [1–3]) and they are studied in many articles. First, we should refer to the fundamental articles by A. I. Prilepko and his followers. In particular, an existence and uniqueness theorem for solutions to the problem of recovering the source $f(t,x)q(t)$ with the overdetermination condition $u(x_0,t) = \psi(t)$ (x_0 is a point in G) is established in [4,5]. Similar results are obtained in [6] for the problem of recovering lower-order coefficient $p(t)$ in the equation (1). The Hölder spaces serve as the basic spaces in these articles. The results were generalized in the book [7, Sec. 6.6, Sec. 9.4], where the existence theory for the problems (1)–(3) was developed in an abstract form with the operator A replaced with $-L$, L is generator of an analytic semigroup. The main results employ the assumptions that the domain of L is independent of time and the unknown coefficients occur into the lower part of the equation nonlinearly. Under certain conditions, existence and uniqueness theorems were proven locally in time in the spaces of functions continuously differentiable with respect to time. We note also the article [8], where an existence and uniqueness theorem in the problem of recovering a lower-order coefficient and the right-hand was established with the overdetermination condition $u(x_i,t) = \psi(t)$ (x_i are interior points of G, $i = 1,2$). There are many articles devoted to the problems (1)–(3) in model situations, especially in the case of $n = 1$ (see, for instance, [9–14]). In these articles different collections of coefficients are recovered with the overdetermination conditions of the form (3), in particular, including boundary points x_i. In this case the boundary condition and the overdetermination condition define the Cauchy data at a boundary point. Many results in the case of $n = 1$ are exhibited in [15]. Note the book [16], where the solvability questions for inverse problems with the overdetermination conditions being the values of a solution on some hyperplanes (sections of a space domain) are studied. The problems (1)–(3) were considered in authors' articles in [17, 18], where conditions on the data were weakened in contrast to those in [7, Sec. 9.4] and the solvability questions were treated in the Sobolev spaces. In contrast to the previous results, we examine the case of the points $\{x_i\}$ lying on the boundary of G as well and the special overdetermination conditions (only some combinations of the coordinate of a solution are given). These overdetermination conditions also arise in applications (see [3]). Note that numerical methods for solving the problems (1)–(3) have been developed in many articles (see [2, 3, 19]).

1. Preliminaries

First, we introduce some notations. Let E be a Banach space. Denote by $L_p(G;E)$ (G is a domain in $\mathbb{R}^n$) the space of strongly measurable functions defined on G with values in E and the finite norm $\|\|u(x)\|_E\|_{L_p(G)}$ [20]. We employ conventional notations for the space of continuously differentiable functions $C^k(\overline{G};E)$ and the Sobolev space $W_p^s(Q;E)$, $W_p^s(G;E)$, etc. (see [20, 21]). If $E = \mathbb{C}$ or $E = \mathbb{C}^n$ then the latter space is denoted simply by $W_p^s(G)$. Therefore, the membership $u \in W_p^s(G)$ (or $u \in C^k(\overline{G})$) or $a \in W_p^s(G)$ for a given vector-function $u = (u_1, u_2, \ldots, u_k)$ or a matrix function $a = \{a_{ij}\}_{j,i=1}^{k}$ mean that every of the components u_i (respectively, an entry a_{ij}) belongs to the space $W_p^s(G)$ (or $C^k(\overline{G})$). Given an interval $J = (0,T)$, put $W_p^{s,r}(Q) = W_p^s(J; L_p(G)) \cap L_p(J; W_p^r(G))$, Respectively, we have $W_p^{s,r}(S) = W_p^s(J; L_p(\Gamma)) \cap L_p(J; W_p^r(\Gamma))$. The anisotropic Hölder spaces $C^{\alpha,\beta}(\overline{Q})$ and $C^{\alpha,\beta}(\overline{S})$ are defined by analogy.

The definition of the inclusion $\Gamma \in C^s$ can be found in [22, Chapter 1]. In what follows we assume that the parameter $p > n+2$ is fixed. Let $B_\delta(x_i)$ be a the ball of radius δ centered at x_i (see (3)). The parameter $\delta > 0$ will be referred to as admissible if $\overline{B_\delta(x_i)} \subset G$ for interior points $x_i \in G$, $\overline{B_\delta(x_i)} \cap \overline{B_\delta(x_j)} = \emptyset$ for $x_i \neq x_j$, $i,j = 1,2,\ldots,r$, and, for every point $x_i \in \Gamma$, there exists a neighborhood U (the coordinate neighborhood) about this point and a coordinate system y (local coordinate system) obtained by rotation and translation of the origin from the initial one such that the y_n-axis is directed as the interior normal to Γ at x_i and the equation of the boundary $U \cap \Gamma$ is of the form $y_n = \omega(y')$, $\omega(0) = 0$, $|y'| < \delta_0$, $y' = (y_1, \ldots, y_{n-1})$; moreover, we have $\omega \in C^3(\overline{B'_\delta(0)})$ ($B'_\delta(0) = \{y' : |y'| < \delta\}$) end $G \cap U = \{y : |y'| < \delta, 0 < y_n - \omega(y') < \delta_1\}$, $(\mathbb{R}^n \setminus G) \cap U = \{y : |y'| < \delta, -\delta_1 < y_n - \omega(y') < 0\}$. The numbers δ, δ_1 for a given domain G are fixed and without loss of generality we can assume that $\delta_1 > (M+1)\delta$, with M the Lipschitz constant of the function ω. Assume that $Q^\tau = (0,\tau) \times G$, $G_\delta = \cup_i (B_\delta(x_i) \cap G)$, $Q_\delta = (0,T) \times G_\delta$, $Q_\delta^\tau = (0,\tau) \times G_\delta$, $S_\delta = (0,T) \times \cup_i (B_\delta(x_i) \cap \Gamma)$.

Consider the parabolic system

$$Lu = u_t + A(t,x,D)u = f(t,x), \quad (t,x) \in Q = (0,T) \times G, \ G \subset \mathbb{R}^n, \tag{4}$$

where

$$A(t,x,D)u = -\sum_{i,j=1}^n a_{ij}(t,x) u_{x_j x_j} + \sum_{i=1}^n a_i(t,x) u_{x_i} + a_0(t,x)u,$$

a_{ij}, a_i are matrices of dimension $h \times h$, and u is a vector of length h. The system (4) is supplemented with the initial and boundary conditions (2). We assume that there exists an admissible number $\delta > 0$ such that

$$a_{ij} \in C(\overline{Q}), \quad a_k \in L_p(Q), \quad \gamma_k \in C^{1/2,1}(\overline{S}), \quad a_{ij} \in L_\infty(0,T; W_\infty^1(G_\delta)); \tag{5}$$

$$a_k \in L_p(0,T; W_p^1(G_\delta)), \quad i,j = 1,2,\ldots,n, \ k = 0,1,\ldots,n. \tag{6}$$

The operator L is considered to be parabolic and the Lopatiskii condition holds. State these conditions. Introduce the matrix $A_0(t,x,\xi) = -\sum\limits_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j$ ($\xi \in \mathbb{R}^n$), and assume that there exists a constant $\delta_1 > 0$ such that the roots p of the polynomial

$$\det\big(A_0(t,x,i\xi) + pE\big) = 0$$

(E is the identity matrix) meet the condition

$$\operatorname{Re} p \leqslant -\delta_1 |\xi|^2 \quad \forall \xi \in \mathbb{R}^n \quad \forall (x,t) \in Q. \tag{7}$$

The Lopatinskii condition can be stated as follows: for every point $(t_0, x_0) \in S$ and the operators $A_0(x,t,D)$ and $B_0(x,t,D) = \sum\limits_{i=1}^n \gamma_i(t,x)\partial_{x_i}$, written in the local coordinate system y at this point (the axis y_n is directed as the normal to S and the axes $y_1, \ldots, y_{n-1}$ lie in the tangent plane at (x_0, t_0)), the system

$$\big(\lambda E + A_0(x_0, t_0, i\xi', \partial_{y_n})\big)v(z) = 0, \quad B_0(x_0, t_0, i\xi', \partial_{y_n})v(0) = h_j, \tag{8}$$

where $\xi' = (\xi_1, \ldots, \xi_{n-1})$, $y_n \in \mathbb{R}^+$, has a unique solution $C(\overline{\mathbb{R}}^+)$ decreasing at infinity for all $\xi' \in \mathbb{R}^{n-1}$, $|\arg\lambda| \leqslant \pi/2$, and $h_j \in \mathbb{C}$ such that $|\xi'| + |\lambda| \neq 0$.

We also assume that there exists a constant $\varepsilon_1 > 0$ such that

$$Re\,(-A_0(t,x,\xi)\eta,\eta) \geqslant \varepsilon_1|\xi|^2|\eta|^2 \quad \forall \xi \in \mathbb{R}^n,\ \eta \in \mathbb{C}^h, \tag{9}$$

where the brackets $(\cdot,\cdot)$ denote the inner product in $\mathbb{C}^h$ (see [22, Definition 7, Sec. 8, Ch. 7]). Let

$$\Big|det\Big(\sum_{i=1}^{n}\gamma_i\nu_i\Big)\Big| \geqslant \varepsilon_0 > 0, \tag{10}$$

where ν is the outward unit normal to Γ, ε_0 is a positive constant, and

$$u_0(x) \in W_p^{2-2/p}(G),\ g \in W_p^{k_0,2k_0}(S),\ B(x,0)u_0(x)|_\Gamma = g(x,0)\ \forall x \in \Gamma, \tag{11}$$

where $k_0 = 1/2 - 1/2p$. Fix an admissible $\delta > 0$. Construct functions $\varphi_i(x) \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi_i(x) = 1$ in $B_{\delta/2}(x_i)$ and $\varphi_i(x) = 0$ in $\mathbb{R}^n \setminus B_{3\delta/4}(x_i)$ and denote $\varphi(x) = \sum_{i=1}^{r}\varphi_i(x)$. Additionally it is assumed that

$$\varphi(x)u_0(x) \in W_p^{3-2/p}(G),\ \ \varphi g \in W_p^{k_1,2k_1}(S)\ (k_1 = 1 - 1/2p), \tag{12}$$

$$\Gamma \in C^2,\ \gamma_k \in C^{1,2}(\overline{S_\delta})\ (k = 0,1,2,\ldots,n). \tag{13}$$

The proof of the following theorem can be found in [18].

Theorem 1. *Assume that the conditions* (5)–(13) *hold for some sufficiently small admissible* $\delta > 0$ *and the function* φ, $f \in L_p(Q^\tau)$, $f\varphi \in L_p(0,\tau;W_p^1(G))$, *and* $\tau \in (0,T]$. *Then there exists a unique solution* $u \in W_p^{1,2}(Q^\tau)$ *to the problem* (4), (2). *Moreover,* $\varphi u_t \in L_p\big(0,\tau;W_p^1(G)\big)$ *and* $\varphi u \in L_p\big(0,\tau;W_p^3(G)\big)$. *If* $g \equiv 0$ *and* $u_0 \equiv 0$ *then we have the estimates*

$$\begin{gathered}\|u\|_{W_p^{1,2}(Q^\tau)} \leqslant c\|f\|_{L_p(Q^\tau)},\\ \|u\|_{W_p^{1,2}(Q^\tau)} + \|\varphi u_t\|_{L_p(0,\tau;W_p^1(G))} + \|\varphi u\|_{L_p(0,\tau;W_p^3(G))} \leqslant c\big[\|f\|_{L_p(Q^\tau)} + \|\varphi f\|_{L_p(0,\tau;W_p^1(G))}\big],\end{gathered} \tag{14}$$

where the constant c *is independent of* f, *a solution* u, *and* $\tau \in (0,T]$.

2. Main results

Consider the problem (1)–(3), where

$$A = L_0 - \sum_{k=m+1}^{r} q_k(t)L_k,\ \ L_k u = -\sum_{i,j=1}^{n} a_{ij}^k(t,x)u_{x_jx_j} + \sum_{i=1}^{n} a_i^k(t,x)u_{x_i} + a_0^k(t,x)u,$$

and $k = 0, m+1, m+2, \ldots, r$. The unknowns q_i are sought in the class $C([0,T])$. Construct a matrix $B(t)$ of dimension $r \times r$ with the rows

$$< f_1(t,x_j), e_j >, \ldots, < f_m(t,x_j), e_j >, < L_{m+1}u_0(t,x_j), e_j >, \ldots, < L_r u_0(t,x_j), e_j > .$$

We suppose that

$$\psi_j \in C^1([0,T]),\ \ < u_0(x_j), e_j >= \psi_j(0)\,(j = 1,2,\ldots,r),\ \gamma_l \in C^{1/2,1}(\overline{S}) \cap C^{1,2}(\overline{S_\delta}), \tag{15}$$

$$a_{ij}^k \in C(\overline{Q}) \cap L_\infty(0,T;W_\infty^1(G_\delta)),\ \ a_l^k \in L_p(Q) \cap L_\infty(0,T;W_p^1(G_\delta))\ (i,j = 1,\ldots,n), \tag{16}$$

$$f_i \in L_p(Q) \cap L_\infty(0,T;W_p^1(G_\delta)) \ (i=0,1,\ldots,m), \tag{17}$$

foe some admissible $\delta > 0$, $p > n+2$, and $k = 0, m+1, \ldots, r$, $l = 0, 1, \ldots n$;

$$a_i^k(t,x_l), f_i(t,x_l) \in C([0,T]) \tag{18}$$

for all possible values of i, k, l. We also need the condition

(C) there exists a number $\delta_0 > 0$ such that

$$|\det B(t)| \geqslant \delta_0 \ \text{ a.e. on } (0,T).$$

Note that the entries of the matrix B belong to the class $C([0,T])$. Consider the system

$$\psi_{jt}(0) + < L_0 u_0(0,x_j), e_j > - < f_0(0,x_j), e_j >=$$
$$= \sum_{k=1}^{m} q_{0k} < f_k(0,x_j), e_j > + \sum_{k=m+1}^{m_1} q_{0k} < L_k u_0(0,x_j), e_j >, \ j = 1, \ldots, r, \tag{19}$$

where the vector $\vec{q}_0 = (q_{01}, q_{02}, \ldots, q_{0r})$ is unknown. Under the condition (C), this system is uniquely solvable. Let $A_1 = L_0 - \sum\limits_{k=m+1}^{r} q_{0k} L_k$. Now we can state our main result.

Theorem 2. *Let the conditions* (9)–(13), (C), (15)–(18) *hold. Moreover, we assume that the conditions* (7), (8) *are fulfilled for the operator* $\partial_t + A_1$. *Then there exists a number* $\tau^0 \in (0,T]$ *such that, on the interval* $(0,\tau^0)$, *there exists a unique solution* $(u, q_1, q_2, \ldots, q_r)$ *to the problem* (1)–(3) *such that* $u \in L_p(0,\tau^0;W_p^2(G))$, $u_t \in L_p(Q^{\tau^0})$, $q_i(t) \in C([0,\tau^0])$, $i = 1, \ldots, r$. *Moreover,* $\varphi u \in L_p(0,\tau^0;W_p^3(G_\delta))$, $\varphi u_t \in L_p(0,\tau^0;W_p^1(G_\delta))$.

Proof. First, we find a solution to the problem

$$\Phi_t + A_1\Phi = f_0 + \sum_{k=1}^{m} q_{0i} f_i \ \ ((x,t) \in Q), \ \ \Phi|_{t=0} = u_0(x), \ \ B\Phi|_S = g. \tag{20}$$

By Theorem 1, $\Phi \in W_p^{1,2}(Q)$, $\varphi\Phi_t \in L_p(0,T;W_p^1(G))$, $\varphi\Phi \in L_p(0,T;W_p^3(G))$. As a consequence of Theorem III 4.10.2 in [24] and embedding theorems [20, Theorems 4.6.1,4.6.2.], we infer $\varphi\Phi \in C([0,T];W_p^{3-2/p}(G)) \subset C([0,T];C^{3-2/p-n/p}(\overline{G}))$. Hence, $\varphi\Phi \in C([0,T];C^2(G))$ after a possible change on a set of zero measure. The equations (20) and (18) imply that $\Phi_t(t,x_j) \in C([0,T])$. Note that this function is defined, since every summand in (20) with the weight φ belongs to $L_p(0,T;W_p^1(G)) \subset C^\alpha(\overline{G};L_p(0,T))$ $(\alpha \leqslant 1 - n/p)$ (see the embedding theorems in [25] and the arguments below). Multiply the equation (20) scalarly by e_j and take $x = x_j$. We obtain the equality

$$< \Phi_t(0,x_j), e_j > + < L_0 u_0(0,x_j), e_j > - < f_0(0,x_j), e_j >=$$
$$= \sum_{k=1}^{m} q_{0k} < f_k(0,x_j), e_j > + \sum_{k=m+1}^{r} q_{0k} < L_k u_0(0,x_j), e_j >, \ j = 1, \ldots, r. \tag{21}$$

The relations (19) and (21) imply that $< \Phi_t(0,x_j), e_j >= \psi_{jt}(0)$. After the change of variables $\vec{q} = \vec{q}_0 + \vec{q}_1$ and $u = w + \Phi$ in (1), we arrive at the problem

$$Lw = w_t + A_1 w - \sum_{k=m+1}^{r} q_{1k} L_k w = \sum_{i=1}^{m} f_i q_{1i} + \sum_{i=m+1}^{r} q_{1i} L_i \Phi = F, \ \ w|_{t=0} = 0, \ \ Bw|_S = 0, \tag{22}$$

$$< w(t,x_j), e_j >= \tilde{\psi}_j(t) = \psi_j(t) - < \Phi(t,x_j), e_j > \in C^1([0,T]), \quad \tilde{\psi}_j(0) = \tilde{\psi}_{jt}(0) = 0. \tag{23}$$

Fixing the vector $\vec{q}_1 = (q_{11}, \ldots, q_{1r}) \in C([0,\tau])$ and determining a solution w to the problem (22) on $(0,\tau)$, we construct a mapping $w = w(\vec{q}_1) = L^{-1}F$. Demonstrate that there exists $R_0 > 0$ such that, for $\vec{q}_1 \in B_{R_0}$, the problem

$$Lv = g, \ \ v|_{t=0} = 0, \ \ Bv|_S = 0 \tag{24}$$

for every $g \in H_\tau$ и $\tau \in (0,T]$ has a unique solution in the class $v \in W_p^{1,2}(Q^\tau)$, $\varphi v_t \in L_p(0,\tau; W_p^1(G))$, $\varphi v \in L_p(0,\tau; W_p^3(G))$ satisfying the estimate

$$\|v\|_{W_p^{1,2}(Q^\tau)} + \|\varphi v_t\|_{L_p(0,\tau;W_p^1(G))} + \|\varphi v\|_{L_p(0,\tau;W_p^3(G))} \leqslant c\|g\|_{H_\tau} \tag{25}$$

where the constant c is independent of τ and the vector $\vec{q}_1 \in B_{R_0}$ and the space H_τ is endowed with the norm

$$\|f\|_{H_\tau} = \|f\|_{L_p(Q^\tau)} + \|\varphi f\|_{L_p(0,\tau;W_p^1(Q))}.$$

In accord with Theorem 1, the problem

$$L_{01}v = v_t + A_1 v = g, \ \ v|_{t=0} = 0, \ \ Bv|_S = 0$$

for every $g \in H_\tau$ has a unique solution such that $v \in W_p^{1,2}(Q^\tau)$, $\varphi v_t \in L_p(0,\tau; W_p^1(G))$, $\varphi v \in L_p(0,\tau; W_p^3(G))$ and

$$\|v\|_{W_p^{1,2}(Q^\tau)} + \|\varphi v_t\|_{L_p(0,\tau;W_p^1(G))} + \|\varphi v\|_{L_p(0,\tau;W_p^3(G))} \leqslant c_1\|g\|_{H_\tau}, \tag{26}$$

where the constant c_1 is independent of τ. In this case the question of solvability of the problem (24) is reduced to the same question for the equation

$$f - \sum_{i=m+1}^{r} q_{1i} L_i L_{01}^{-1} f = g, \tag{27}$$

where $f = L_{01}v$. We have the estimate

$$\Big\| - \sum_{i=m+1}^{r} q_{1i} L_i v \Big\|_{H_\tau} \leqslant c\|\vec{q}_1\|_{C([0,\tau])} \big(\|v\|_{W_p^{1,2}(Q^\tau)} + \|\varphi v_t\|_{L_p(0,\tau;W_p^1(G))} + \|\varphi v\|_{L_p(0,\tau;W_p^3(G))}\big), \tag{28}$$

where the constant c depends on the coefficients of the operators L_k in Q and is independent of τ and $\vec{q}_1$. Indeed, the following estimate is obvious

$$\Big\| - \sum_{k=m+1}^{r} q_{1k} L_k v \Big\|_{H_\tau} \leqslant \|\vec{q}_1\|_{C([0,\tau])} \sum_{k=m+1}^{r} \|L_k v\|_{H_\tau}. \tag{29}$$

Estimate the quantity $\|L_k v\|_{H_\tau}$. To this aim, we estimate the norms of every of the summands in this quantity. For example, estimate the norm

$$\|a_{ij}^k v_{x_i x_j}\|_{H_\tau} \leqslant c_0\big(\|a_{ij}^k v_{x_i x_j}\|_{L_p(Q^\tau)} + \sum_{l=1}^{n} \|\varphi (a_{ij}^k v_{x_i x_j})_{x_l}\|_{L_p(Q^\tau)}\big) \leqslant$$

$$\leqslant c_1\big(\|v\|_{L_p(0,\tau;W_p^2(G))} + \|\varphi v\|_{L_p(0,\tau;W_p^3(G))}\big) + \sum_{l=1}^{n} \|\varphi a_{ijx_l}^k v_{x_i x_j}\|_{L_p(Q^\tau)}, \tag{30}$$

where the constant c_1 depends on the norms $\|a^k_{ij}\|_{L_\infty(Q)}$. The last summand here is estimated as follows:

$$\sum_{l=1}^{n}\|\varphi a^k_{ijx_l}v_{x_ix_j}\|_{L_p(Q^\tau)}\leqslant c_2\big(\|\varphi v\|_{L_p(0,\tau;W^2_\infty(G))}+\|v\|_{L_p(0,\tau;W^1_\infty(G))}\big)\leqslant$$
$$\leqslant c_3\big(\|\varphi v\|_{L_p(0,\tau;W^3_p(G))}+\|v\|_{L_p(0,\tau;W^2_p(G))}\big),\quad (31)$$

where the constant c_2 depends on the norms $\|\nabla a^k_{ij}\|_{L_p(0,T;L_\infty(G_\delta))}$. Thus, we infer

$$\|a^k_{ij}v_{x_ix_j}\|_{H_\tau}\leqslant c_4\big(\|v\|_{L_p(0,\tau;W^2_p(G))}+\|\varphi v\|_{L_p(0,\tau;W^3_p(G))}\big),\quad (32)$$

where the constant c_4 is independent of τ. Similarly, we derive that

$$\|a^k_iv_{x_i}\|_{H_\tau}\leqslant c_0\big(\|a^k_iv_{x_i}\|_{L_p(Q^\tau)}+\sum_{l=1}^{n}\|\varphi(a^k_iv_{x_i})_{x_l}\|_{L_p(Q^\tau)}\big)\leqslant$$
$$\leqslant c_1\big(\|\nabla v\|_{L_\infty(Q^\tau)}+\|\varphi v\|_{L_p(0,\tau;W^2_p(G))}\big),\quad (33)$$

where the constant c_1 depends on the norms of $a^k_i, a^k_{ix_l}$ in $L_p(Q)$ and the norms of a^k_i in $L_\infty(Q_\delta)$. However (see Lemma 3.3 in [22]), we have

$$\|\nabla v\|_{L_\infty(Q^\tau)}\leqslant c_1\|v\|_{W^{1,2}_p(Q^\tau)},$$

where the embedding constant is independent of τ. Summing the estimates obtained we justify (28). Using (28) and the estimate of Theorem 1, we conclude that

$$\Big\|\sum_{i=m+1}^{r}q_{1i}L_iL_{01}^{-1}f\Big\|_{H_\tau}\leqslant c_2\|\vec{q}_1\|_{C([0,\tau])}\|f\|_{H_\tau},\quad (34)$$

where c_2 is independent of τ and $\vec{q}_1\in B_{R_0}$. Let $R_0=1/2c_2$. In this case $c_2\|\vec{q}_1\|_{C([0,\tau])}\leqslant 1/2$ and thereby the equation (27) has a unique solution satisfying the estimate $\|f\|_{H_\tau}\leqslant 2\|g\|_{H_\tau}$, which along with Theorem 1 ensures (25).

Assume that w is a solution to the problem (22), (23). Take $x=x_j$ in (22) and multiply the equation scalarly by e_j. The traces of all function occurring into the equation exist. First, our conditions for coefficients and embedding theorems yield $\varphi w\in C([0,T];C^2(\overline{G}))$ (see the above arguments for the function Φ). Second, as we have indicated above, every of the summands in (22) with the weight φ belongs to $L_p(0,T;W^1_p(G))\subset C^\alpha(\overline{G};L_p(0,T))$ ($\alpha\leqslant 1-n/p$) (see embedding theorems in [25]). We arrive at the system

$$<\tilde{\psi}_{jt},e_j>+<A_1w(t,x_j),e_j>-\sum_{i=m+1}^{r}q_{1i}<L_iw(t,x_j),e_j>=$$
$$=\sum_{i=1}^{m}<f_i(t,x_j),e_j>q_{1i}(t)+\sum_{i=m+1}^{r}q_{1i}<L_i\Phi(t,x_j),e_j>\ \ (j=1,2,\dots,r),\quad (35)$$

which can be rewritten in the form

$$\tilde{B}\vec{q}_1=\vec{\psi}+R(\vec{q}_1),$$

where coordinates of the vectors $\vec{\psi}$ and $R(\vec{q}_1)$ agree with the functions $< \tilde{\psi}_{jt}, e_j >$ and $< A_0 w(t,x_j), e_j > - \sum_{i=m+1}^{r} q_{1i} < L_i w(t,x_j), e_j >$ ($w = w(\vec{q}_1)$); respectively, j-th row of the matrix $\tilde{B}(t)$ of dimension $r \times r$ is written as

$$< f_1(t,x_j), e_j >, \ldots, < f_m(t,x_j), e_j >, < L_{m+1}\Phi(t,x_j), e_j >, \ldots, < L_r\Phi(t,x_j), e_j >,$$

where $j = 1, \ldots, r$. This matrix differs from B by the entries $< L_i\Phi(t,x_j), e_j >$. It is easy to prove that this matrix is nondegenerate as well on some segment $[0, \tau_0]$. Indeed, the embedding theorems (see Lemma 3.3 of Chapter 1 in [22]) imply that $\nabla\Phi, \Phi_{x_i x_j} \in C^{\beta/2,\beta}(\overline{Q_{\delta/2}})$ for $\beta < 1 - (n+2)/p$ and all i, j and, therefore,

$$| < L_k\Phi(t,x_j) - L_k u_0(t,x_j), e_j > | \leqslant \sum_{i,k=1}^{n} \sup_{t\in[0,T]} \|a_{ik}^k(t,x_j)\| |\Phi_{x_k x_i}(t,x_j) - u_{0x_k x_i}(x_j)| +$$

$$+ \sum_{i=1}^{n} \sup_{t\in[0,T]} \|a_i^k(t,x_j)\| |\Phi_{x_i}(t,x_j) - u_{0x_i}(x_j)| + \sup_{t\in[0,T]} \|a_0^k(t,x_j)\| |\Phi(t,x_j) - u_0(x_j)| \leqslant ct^{\beta/2},$$

on $[0,T]$, where, by the norm of a matrix (for example, $\|a_i^k(t,x_j)\|$), we mean the norm of the corresponding linear operator $a_i^k(t,x_j) : \mathbb{C}^h \to \mathbb{C}^h$. Taking the condition (C) into account, we can say that there exists $\tau_0 > 0$ such that

$$|\det\tilde{B}(t)| \geqslant \delta_0/2 \quad \forall t \leqslant \tau_0. \tag{36}$$

We thus obtain the integral equation

$$\vec{q}_1 = \tilde{B}^{-1}\vec{\psi} + R_0(\vec{q}_1), \quad R_0(\vec{q}_1) = \tilde{B}^{-1}R(\vec{q}_1), \tag{37}$$

where the operator $R_0(\vec{q}_1) : C([0,\tau]) \to C([0,\tau])$ ($\tau \leqslant \tau_0$) is bounded. Check the conditions of the fixed point theorem. Denote $R_{0\tau} = 2\|\tilde{B}^{-1}\vec{\psi}\|_{C([0,\tau])}$. Let $\vec{q}_{01}, \vec{q}_{02}$ be two vectors of length r with coordinates q_i^j ($i = 1, 2, \ldots, r$, $j = 1, 2$) lying in the ball $B_{R_0} = \{\vec{q} : \|\vec{q}\|_{C([0,\tau])} \leqslant R_0\}$. The functions $w_1 = w(\vec{q}_{01}), w_2 = w(\vec{q}_{02})$ are solutions to the equation (22) satisfying homogeneous initial and boundary conditions. Let $v = w_1 - w_2$. We infer

$$Lv = v_t + A_1 v - \sum_{i=m+1}^{r} q_i^2 L_i v = \sum_{i=1}^{m} f_i(q_i^1 - q_i^2) + \sum_{i=m+1}^{r} (q_i^1 - q_i^2) L_i w_1, \quad v = w_1 - w_2. \tag{38}$$

In view of (23) and the definition of $R_{0\tau}$, $R_{0\tau} \to 0$ as $\tau \to 0$. Hence, there exists a parameter $\tau_1 \leqslant \tau_0$ such that, for $\tau \leqslant \tau_1$, $R_{0\tau} \leqslant R_0$. Let $R = R_{0\tau_1}$. We now derive that there exists a parameter $\tau^0 \leqslant \tau_1$ such that the equation (37) has a unique solution in the ball B_R of the space $C([0,\tau^0])$. Take $\tau \leqslant \tau_1$. Let $\vec{q}_{01}, \vec{q}_{02} \in B_R$. We have

$$\|R_0(\vec{q}_{01}) - R_0(\vec{q}_{02})\|_{C([0,\tau])} \leqslant c_1\|R(\vec{q}_{01}) - R(\vec{q}_{02})\|_{C([0,\tau])} \leqslant$$
$$\leqslant c_2 \sum_{j=1}^{r} (\|L_0 v(t,x_j)\|_{C([0,\tau])} + \sum_{i=m+1}^{r} \|q_i^2 L_i v(t,x_j)\|_{C([0,\tau])}) \leqslant$$
$$\leqslant c_3 \sum_{j=1}^{r} (\|L_0 v(t,x_j)\|_{C([0,\tau])} + \sum_{i=m+1}^{r} \|L_i v(t,x_j)\|_{C([0,\tau])}), \tag{39}$$

where v is a solution to the problem (38). Note that

$$\|L_k v(t,x_j)\|_{C([0,\tau])} \leqslant c\tau^{\beta}(\|\varphi\nabla v\|_{W_p^{1,2}(Q^\tau)} + \|v\|_{W_p^{1,2}(Q^\tau)}), \tag{40}$$

where the constant c is independent of $\tau \in (0,T]$ and $\beta > 0$. Validate this inequality. In view of the conditions on the coefficients a_{il}^k, $a_{il}^k(t,x_j) \in C([0,T])$. Fix an arbitrary $s \in (n/p, 1-2/p)$. The embedding $W_p^s(G_{\delta/2}) \subset C(\overline{G_{\delta/2}})$ [20, Theorems 4.6.1,4.6.2.] yields

$$\|a_{il}^k(t,x_j)v_{x_ix_l}(t,x_j)\|_{C([0,\tau])} \leqslant c\|v_{x_ix_l}(t,x_j)\|_{C([0,\tau])} \leqslant c_1\|v_{x_kx_l}(t,x)\|_{L_\infty(0,\tau;W_p^s(G_{\delta/2}))} \leqslant \leqslant c_2\|\nabla v(t,x)\|_{L_\infty(0,\tau;W_p^{1+s}(G_{\delta/2}))}. \quad (41)$$

Next, we employ the interpolation inequality (see [20])

$$\|v\|_{W_p^{s_0}(G)} \leqslant c\|v\|^{\theta}_{W_p^{s_1}(G)}\|v\|^{1-\theta}_{W_p^{s_2}(G)}, \quad s_1 < s_0 < s_2,\ \theta s_1 + (1-\theta)s_2 = s_0 \quad (42)$$

and the inequality

$$\|g\|_{L_\infty(0,\tau;E)} \leqslant \tau^{(p-1)/p}\|g_t\|_{L_p(0,\tau;E)}, \quad \forall g \in W_p^1(0,\tau;E),\ g(0)=0, \quad (43)$$

resulting from the Newton-Leibnitz formula. Here E is a Banach space. We obtain that

$$\|\nabla v(t,x)\|_{L_\infty(0,\tau;W_p^{1+s}(G_{\delta/2}))} \leqslant c\|\nabla v(t,x)\|^{\theta}_{L_\infty(0,\tau;W_p^{2-2/p}(G_{\delta/2}))}\|\nabla v(t,x)\|^{(1-\theta)}_{L_\infty(0,\tau;L_p(G_{\delta/2}))} \leqslant \leqslant c_1\tau^{(1-\theta)(p-1)/p}(\|\varphi\nabla v\|_{W_p^{1,2}(Q)} + \|v\|_{W_p^{1,2}(Q)}),\ (2-2/p)\theta = 1+s. \quad (44)$$

Here we have used the inequality

$$\|\nabla v(t,x)\|_{L_\infty(0,\tau;W_p^{2-2/p}(G_{\delta/2}))} \leqslant c\|\nabla v(t,x)\|_{W_p^{1,2}(G_{\delta/2}))}, \quad (45)$$

where the constant c is independent of τ (in the class of functions vanishing at $t=0$). Estimate the lower-order summands of the form $a_i^kv_{x_i}(t,x_j)$, $a_0^kv(t,x_j)$ in $L_iu(t,x_j)$. We conclude that ($s \in (n/p, 1-2/p)$, $(2-2/p)\theta_1 = 1+s$)

$$\|a_i^kv_{x_i}(t,x_j)\|_{C([0,\tau])} \leqslant c\|v_{x_i}(t,x_j)\|_{C([0,\tau])} \leqslant c_1\|v(t,x)\|_{L_\infty(0,\tau;W_p^{1+s}(G_{\delta/2}))} \leqslant \leqslant \|v(t,x)\|^{\theta_1}_{L_\infty(0,\tau;W_p^{2-2/p}(G_{\delta/2}))}\|v(t,x)\|^{1-\theta_1}_{L_\infty(0,\tau;L_p(G_{\delta/2}))} \leqslant c_2\tau^{(1-\theta_1)(p-1)/p}\|v\|_{W_p^{1,2}(Q^\tau)}. \quad (46)$$

We have used the estimate (45) applied to v rather than ∇v. The second summand is estimated similarly. The estimates (39)–(46) ensure that

$$\|R_0(\vec{q}_{01}) - R_0(\vec{q}_{02})\|_{C([0,\tau])} \leqslant c_4\tau^{\beta}(\|\varphi\nabla v(t,x)\|_{W_p^{1,2}(Q^\tau)} + \|v(t,x)\|_{W_p^{1,2}(Q^\tau)}), \quad (47)$$

where the constant c_4 is independent of τ and $\beta = \min(1-\theta, (1-\theta_1)(p-1)/p)$. Since v is a solution to the problem (38) and $\tau \leqslant \tau_1$, we can employ (25) and obtain that

$$\|\varphi\nabla v(t,x)\|_{W_p^{1,2}(Q^\tau)} + \|v(t,x)\|_{W_p^{1,2}(Q^\tau)} \leqslant c\Big\|\sum_{i=1}^{m} f_i(q_i^1 - q_i^2) + \sum_{i=m+1}^{r}(q_i^1 - q_i^2)L_iw_1\Big\|_{H_\tau}, \quad (48)$$

where the constant c is independent of τ. Every of the functions w_1, w_2 is a solution to the problem (22), where the right-hand side contains the components of the vector $\vec{q}_{01}$ or $\vec{q}_{02}$. The estimate (25) yields

$$\|\varphi\nabla w_j(t,x)\|_{W_p^{1,2}(Q^\tau)} + \|w_j(t,x)\|_{W_p^{1,2}(Q^\tau)} \leqslant c\Big\|\sum_{i=1}^{m} f_iq_i^j + \sum_{i=m+1}^{r} q_i^jL_i\Phi\Big\|_{H_\tau}. \quad (49)$$

The estimate (48), (49) and the conditions on the coefficients imply that

$$\|\varphi\nabla w_j(t,x)\|_{W_p^{1,2}(Q^\tau)} + \|w_j(t,x)\|_{W_p^{1,2}(Q^\tau)} \leqslant c_1(R). \tag{50}$$

$$\|\varphi\nabla v(t,x)\|_{W_p^{1,2}(Q^\tau)} + \|v(t,x)\|_{W_p^{1,2}(Q^\tau)} \leqslant c_2\|\vec{q}_{01} - \vec{q}_{02}\|_{C([0,\tau])}, \tag{51}$$

where the constant c_i are independent of τ. In turn, these estimates and those in (47) validate the estimate

$$\|R_0(\vec{q}_{01}) - R_0(\vec{q}_{02})\|_{C([0,\tau])} \leqslant c_5\tau^\beta\|\vec{q}_{01} - \vec{q}_{02}\|_{C([0,\tau])} \tag{52}$$

with a constant c_5 independent of τ. Choose a parameter $\tau^0 \leqslant \tau_1$ such that $c_5(\tau^0)^\beta \leqslant 1/2$. The fixed point theorem ensures solvability of the equation (37) in the ball B_R.

Show that w satisfies the overdetermination conditions (23). Multiply the equation (22) scalarly by e_j and take $x = x_j$ in the equation. We obtain the equality

$$\begin{aligned} &< w(t,x_j), e_j >_t + < L_0 w(t,x_j), e_j > - \sum_{i=m+1}^{r} q_i < L_i w(t,x_j), e_j >= \\ &= \sum_{i=1}^{m} < f_i(t,x_j), e_j > q_i(t) + \sum_{i=m+1}^{r} q_i < L_i\Phi(t,x_j), e_j >, \quad j = 1,2,\ldots,r, \end{aligned} \tag{53}$$

Subtracting this equality from (35), we obtain that $\tilde{\psi}_{j_t} - < w(t,x_j), e_j >_t = 0$. Integrating this equality from 0 to t, we derive that $\tilde{\psi}_j(t) - < w(t,x_j), e_j >= 0$, since the agreement conditions imply that $\tilde{\psi}_j(0) = 0$, $< w(0,x_j), e_j >= 0$. Thus, we infer $\tilde{\psi}_j(t) =< w(t,x_j), e_j >$ and the equality (23) holds. □

In the case of the unknown lower-order coefficients, the results can be reformulated in a more convenient form. In this case the operator A is assumed to be representable in the form

$$A = L_0 - \sum_{i=m+1}^{r} q_i(t) l_i, \quad L_0 u = -\sum_{i,j=1}^{n} a_{ij}(t,x) u_{x_j x_j} + \sum_{i=1}^{n} a_i(t,x) u_{x_i} + a_0(t,x)u,$$
$$l_i u = \sum_{j=1}^{n} b_{ij}(t,x) u_{x_j} + b_{i0}(t,x)u. \tag{54}$$

Moreover, the rows of the matrix $B(t)$ of dimension $r \times r$ are as follows:

$$< f_1(t,x_i), e_i >, \ldots, < f_m(t,x_i), e_i >, < l_{m+1} u_0(t,x_i), e_i >, \ldots, < l_r u_0(t,x_i), e_i > .$$

We suppose that

$$\psi_j \in W_p^1(0,T), \quad < u_0(x_j), e_j >= \psi_j(0), \quad j = 1,2,\ldots,r, \tag{55}$$

$$f_i, b_{kj} \in L_\infty(0,T; W_p^1(G_\delta)) \cap L_\infty(0,T; L_p(G)), \quad f_0 \in L_p(Q) \cap L_p(0,T; W_p^1(G_\delta)), \tag{56}$$

for some admissible $\delta > 0$, where $i = 1,\ldots,m$, $j = 0,1,\ldots,n$, $k = m+1,\ldots,r$. The remaining coefficients satisfy the conditions

$$a_{ij} \in C(\overline{Q}), \quad a_k \in L_p(Q), \quad \gamma_k \in C^{1/2,1}(\overline{S}) \cap C^{1,2}(\overline{S_\delta}), \quad a_{ij} \in L_\infty(0,T; W_\infty^1(G_\delta)); \tag{57}$$

$$a_k \in L_p(Q) \cap L_p(0,T; W_p^1(G_\delta)), \; i,j = 1,2,\ldots,n, \; k = 0,1,\ldots,n. \tag{58}$$

The corresponding theorem is stated in the following form.

Theorem 3. *Assume that the parabolicity condition and the Lopatinskii condition* (7), (8) *for the operator* $\partial_t + L_0$*, the conditions* (9)–(13), (55)–(58), (C) *for some admissible* $\delta > 0$ *and* $p > n + 2$ *hold. Then, for some* $\gamma_0 \in (0, T]$*, on the interval* $(0, \gamma_0)$*, there exists a unique solution* $(u, q_1, q_2, \ldots, q_r)$ *to the problem* (1)–(3) *such that* $u \in L_p(0, \gamma_0; W_p^2(G))$*,* $u_t \in L_p(Q^{\gamma_0})$*,* $\varphi u \in L_p(0, \gamma_0; W_p^3(G))$*,* $\varphi u_t \in L_p(0, \gamma_0; W_p^1(G))$*,* $q_i(t) \in L_p(0, \gamma_0)$*,* $i = 1, \ldots, r$.

The proof is omitted, since it is quite similar to that of the previous theorem.

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О некоторых классах параболических обратных задач с точечным переопределением

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Аннотация. В работе рассматривается вопрос о корректности в пространствах Соболева обратных задач о восстановлении коэффициентов параболической системы, зависящих от времени. В качестве условий переопределения рассматриваются значения решения в некотором наборе точек области, лежащих как внутри области, так и на ее границе. Приведены условия, гарантирующие существование и единственность решений задачи в классах Соболева.

Ключевые слова: параболическая система, обратная задача, конвекция-диффузия, точечное переопределение.

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On the Uniqueness of the Classical Solutions of the Radial Viscous Fingering Problems in a Hele-Shaw Cell

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Abstract. In [9, 10] we established the existence of classical solutions to two-phase and one-phase radial viscous fingering problems, respectively, in a Hele-Shaw cell by the parabolic regularization and by vanishing the coefficient of the derivative with respect to time in a parabolic equation. In this paper we show the uniqueness of such solutions to the respective problems.

Keywords: classical solution, unique existence, radial viscous fingering.

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Dedicated to the memory of Yu. Ya. Belov (1944–2019)

1. Introduction and preliminaries

Viscous fingering occurs in the flow of two immiscible, viscous fluids between the plates of a Hele-Shaw cell ([3]). Due to pressure gradients or gravity, the initially planar interface separating the two fluids undergoes a Saffman–Taylor instability ([5]), and develops finger-like structure (see also [4] and the literatures therein).

In [9,10] we established the existence of solutions belonging to the standard Hölder spaces for two-phase and one-phase radial viscous fingering problems in a Hele–Shaw cell, without surface tension effect, by the parabolic regularization and by vanishing the coefficient of the derivative with respect to time in parabolic equations (cf. [1, 2]). However, our results in [9, 10] are only the existence of the solutions because of the sub-sequential limiting procedure.

The aim of this paper is to prove the uniqueness of such solutions to the respective problems.

This paper consists of three sections. In the rest of this section, we give a brief formulation of the problem in the two-phase case that we discuss. In Section 2, we give a proof of the uniqueness of the classical solution to the two-phase problem, and in Section 3 to the one-phase problem.

1.1. Formulation of the two-phase problem

The motion of a slow quasistationary displacement of a fluid by another fluid in a Hele–Shaw cell is described by

$$\nabla \cdot \mathbf{v}_i = 0, \quad \mathbf{v}_i = -M_i \nabla p_i \quad \text{in} \;\; \Omega_i(t), \; t > 0 \; (i = 1, 2). \tag{1.1}$$

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Here $M_i = b^2/12\mu_i$ is mobility; μ_i is the fluid viscosity; b is the width of two plates; $\mathbf{v}_i$ is the velocity vector field in the fluid and p_i is the pressure ($i = 1$ and 2 for the displacing and the displaced fluid, respectively). For a radial fingering problem it is sufficient to consider (1.1) under the following geometric situation:

$$\Omega_1(t) = \left\{ x \in \mathbb{R}^2 \;\middle|\; R_* < |x| < R(t) + \zeta\left(\frac{x}{|x|}, t\right) \right\},$$
$$\Omega_2(t) = \left\{ x \in \mathbb{R}^2 \;\middle|\; R(t) + \zeta\left(\frac{x}{|x|}, t\right) < |x| < R^* \right\},$$

where R_* is the radius of the hole through which the displacing fluid is injected or driven by suction at a flow rate $Q(t)$, R^* is the radius of the Hele–Shaw cell occupied by the displaced fluid, $R(t)$ is the time-dependent unperturbed radius satisfying

$$\pi R(t)^2 = \pi R_0^2 + \int_0^t Q(\tau)\, \mathrm{d}\tau, \quad R_0 \equiv R(0) > R_*,$$

and ζ is the perturbed radius.

The boundary and initial conditions for (1.1) are as follows:

$$\begin{cases} \mathbf{v}_1 \cdot \mathbf{n} = \dfrac{Q(t)}{2\pi R_*} \quad \text{on } \Gamma_*,\ t > 0, \qquad p_2 = p_e \quad \text{on } \Gamma^*,\ t > 0, \\ \mathbf{v}_1 \cdot \mathbf{n} = \mathbf{v}_2 \cdot \mathbf{n} = V_n, \quad p_1 = p_2 \quad \text{on } \Gamma(t),\ t > 0, \end{cases} \tag{1.2}$$

$$\begin{cases} \mathbf{v}_i|_{t=0} = \mathbf{v}_i^0, \quad p_i = p_i^0 \quad \text{on} \quad \Omega_i(0) \equiv \Omega_i \quad (i = 1, 2), \\ \zeta|_{t=0} = \zeta^0 \in (R_* - R_0, R^* - R_0) \quad \text{on} \quad \Gamma(0) \equiv \Gamma, \end{cases} \tag{1.3}$$

where $\Gamma_* = \{x \in \mathbb{R}^2 \mid |x| = R_*\}$, $\Gamma(t) = \left\{x \in \mathbb{R}^2 \;\middle|\; |x| = R(t) + \zeta(x/|x|, t)\right\}$, $\Gamma^* = \{x \in \mathbb{R}^2 \mid |x| = R^*\}$; V_n is the normal velocity of the interface $\Gamma(t)$; $\mathbf{n}$ is the unit normal vectors, outward to Γ_* or to $\Gamma(t)$ in the direction from $\Omega_1(t)$ to $\Omega_2(t)$; p_e is the surface pressure acting on Γ^*.

Our two-phase problem is to find $(\mathbf{v}_i, p_i)$ ($i = 1, 2$) and ζ satisfying (1.1)–(1.3), which is reduced to find (p_1, p_2) and ζ satisfying

$$\begin{cases} \Delta p_i = 0 \quad \text{in } \Omega_i(t),\ t > 0\ (i = 1, 2), \\ -M_1 \nabla p_1 \cdot \mathbf{n} = \dfrac{Q(t)}{2\pi R_*} \quad \text{on } \Gamma_*,\ t > 0, \quad p_2 = p_e \quad \text{on } \Gamma^*,\ t > 0, \\ -M_1 \nabla p_1 \cdot \mathbf{n} = -M_2 \nabla p_2 \cdot \mathbf{n} = V_n, \quad p_1 = p_2 \quad \text{on } \Gamma(t),\ t > 0, \\ p_i \,\Big|_{t=0} = p_i^0 \quad \text{on } \Omega_i\ (i = 1, 2), \quad \zeta \,\Big|_{t=0} = \zeta^0 \quad \text{on } \Gamma. \end{cases} \tag{1.4}$$

As the compatibility conditions p_1^0 and p_2^0 are assumed to satisfy

$$\begin{cases} \Delta p_i^0 = 0 \quad \text{in} \quad \Omega_i \quad (i = 1, 2), \\ -M_1 \nabla p_1^0 \cdot \mathbf{n} = \dfrac{Q(0)}{2\pi R_*} \quad \text{on} \quad \Gamma_*, \quad p_2^0 = p_e|_{t=0} \quad \text{on} \quad \Gamma^*, \quad p_1^0 = p_2^0 \quad \text{on} \quad \Gamma. \end{cases} \tag{1.5}$$

In polar coordinates (r,θ) problem (1.4) is written as

$$\left\{\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_1}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 p_1}{\partial\theta^2}=0\quad(r\in(R_*,R(t)+\zeta),\ \theta\in[0,2\pi),\ t>0)\,,\\
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_2}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 p_2}{\partial\theta^2}=0\quad(r\in(R(t)+\zeta,R^*),\ \theta\in[0,2\pi),\ t>0)\,,\\
&M_1\frac{\partial p_1}{\partial r}\Big|_{r=R_*}=-\frac{Q(t)}{2\pi R_*}\quad p_2\Big|_{r=R^*}=p_e\quad(\theta\in[0,2\pi),\ t>0)\,,\\
&-\frac{\partial}{\partial t}(R(t)+\zeta)=M_1\left(\frac{\partial p_1}{\partial r}-\frac{1}{r^2}\frac{\partial\zeta}{\partial\theta}\frac{\partial p_1}{\partial\theta}\right)=M_2\left(\frac{\partial p_2}{\partial r}-\frac{1}{r^2}\frac{\partial\zeta}{\partial\theta}\frac{\partial p_2}{\partial\theta}\right),\quad p_1=p_2\\
&\qquad\qquad(r=R(t)+\zeta,\ \theta\in[0,2\pi),\ t>0)\,,\\
&p_1\Big|_{t=0}=p_1^0\quad\big(r\in(R_*,R_0+\zeta^0),\ \theta\in[0,2\pi)\big)\,,\\
&p_2\Big|_{t=0}=p_2^0\quad\big(r\in(R_0+\zeta^0(\theta),R^*),\ \theta\in[0,2\pi)\big)\,,\\
&\zeta\Big|_{t=0}=\zeta^0\quad(\theta\in[0,2\pi))\,.
\end{aligned}\right.\tag{1.6}$$

Now let us transform the free boundary problem (1.6) into the problem on fixed domains. Introduce the transformations from $\Omega_1(t)=\{R_*<r<R(t)+\zeta(\theta,t),\ 0\leqslant\theta<2\pi\}$ onto $\Omega_1=\big\{R_*<r'<R_0+\zeta^0(\theta'),\ 0\leqslant\theta'<2\pi\big\}$ by the change of the variables $r'=\dfrac{R_0+\zeta^0-R_*}{R+\zeta-R_*}\times$ $\times\,(r-R_*)+R_*,\ \theta'=\theta,\ t'=t$, and $\Omega_2(t)=\{R(t)+\zeta(\theta,t)<r<R^*,\ 0\leqslant\theta<2\pi\}$ onto $\Omega_2=\big\{R_0+\zeta^0(\theta')<r'<R^*,\ 0\leqslant\theta'<2\pi\big\}$ by $r'=\dfrac{R_0+\zeta^0-R^*}{R+\zeta-R^*}(r-R^*)+R^*,\ \theta'=\theta,\ t'=t$. Moreover, by letting $p_i(r,\theta,t)=p_i'(r',\theta',t')$ $(i=1,2)$, $\zeta(\theta,t)=\zeta'(\theta',t')$, and by omitting the *primes* for simplicity, problem (1.6) takes the form

$$\left\{\begin{aligned}
&\mathcal{L}^i_\zeta p_i=0\quad\text{in}\ \ \Omega_i,\ t>0\quad(i=1,2),\\
&\frac{\partial p_1}{\partial r}=-\frac{Q(t)}{2\pi R_*M_1}\frac{R+\zeta-R_*}{R_0+\zeta^0-R_*}\quad\text{on}\ \ \Gamma_*\equiv\{r=R_*,\ \theta\in[0,2\pi]\},\ t>0,\\
&p_2=p_e\quad\text{on}\ \ \Gamma^*\equiv\{r=R^*,\ \theta\in[0,2\pi]\},\ t>0,\\
&\frac{\partial\zeta}{\partial t}-b_2^1(\zeta)\frac{\partial p_1}{\partial r}-b_1^1(\zeta)\frac{\partial p_1}{\partial\theta}-b_2^2(\zeta)\frac{\partial p_2}{\partial r}-b_1^2(\zeta)\frac{\partial p_2}{\partial\theta}=-\frac{Q(t)}{2\pi R},\\
&b_2^1(\zeta)\frac{\partial p_1}{\partial r}+b_1^1(\zeta)\frac{\partial p_1}{\partial\theta}=b_2^2(\zeta)\frac{\partial p_2}{\partial r}+b_1^2(\zeta)\frac{\partial p_2}{\partial\theta},\quad p_1=p_2\\
&\qquad\qquad\text{on}\ \Gamma\equiv\{r=R_0+\zeta^0,\theta\in[0,2\pi]\},\ t>0,\\
&p_i\Big|_{t=0}=p_i^0\quad\text{on}\ \ \Omega_i\ (i=1,2),\quad\zeta\Big|_{t=0}=\zeta^0\quad\text{on}\ \ [0,2\pi].
\end{aligned}\right.\tag{1.7}$$

Here $\mathcal{L}^i_\zeta\equiv\mathcal{L}^i_\zeta(r,\theta;\partial/\partial r,\partial/\partial\theta)$ is a Laplace operator represented by the composite change of variables of polar coordinates (r,θ) and the mapping from $\Omega_i(t)$ to Ω_i $(i=1,2)$, and

$$b_2^j(\zeta)=\frac{M_j}{2}\left[\left(1+\frac{1}{(R_0+\zeta^0)^2}\left(\frac{\partial\zeta}{\partial\theta}\right)^2\right)\frac{R_0+\zeta^0-R_*}{R+\zeta-R_*}-\frac{1}{(R_0+\zeta^0)^2}\frac{\partial\zeta}{\partial\theta}\frac{\mathrm{d}\zeta^0}{\mathrm{d}\theta}\right],$$

$$b_1^j(\zeta)=-\frac{M_j}{2}\,\frac{1}{(R_0+\zeta^0)^2}\,\frac{\partial\zeta}{\partial\theta}\quad(j=1,2).$$

In detail, see [9].

We consider problem (1.7) in the standard Hölder spaces, $C^{l+\alpha}(\bar\Omega)$, $C^{l+\alpha,(l+\alpha)/2}_{x,t}(\bar Q_T)$ ($\bar Q_T\equiv\bar\Omega\times[0,T]$; $\Omega\subset\mathbb{R}^n$ $(n\in\mathbb{N})$, a domain; T, any positive number; $l\geqslant 0$, an integer; $\alpha\in(0,1)$) with the norms:

$$|u|^{(\alpha)} = |u|^{(0)} + \langle u\rangle^{(\alpha)}, \quad |u|^{(0)} = \sup_{(x,t)\in\bar{Q}_T} |u(x,t)|, \quad \langle u\rangle^{(\alpha)} = \langle u\rangle_x^{(\alpha)} + \langle u\rangle_t^{(\alpha/2)},$$

$$\langle u\rangle_x^{(\alpha)} \equiv \sup_{x,y\in\bar{\Omega},\, t\in[0,T]} \frac{|u(x,t)-u(y,t)|}{|x-y|^\alpha}, \quad \langle u\rangle_t^{(\alpha)} \equiv \sup_{x\in\bar{\Omega},\, t,t'\in[0,T]} \frac{|u(x,t)-u(x,t')|}{|t-t'|^\alpha}.$$

We also use the semi-norm

$$[u]^{(\alpha,\beta)} \equiv \sup_{\substack{x,y\in\bar{\Omega}\\ t,t'\in[0,T]}} \frac{|u(x,t)-u(y,t)-u(x,t')+u(y,t')|}{|x-y|^\alpha\,|t-t'|^\beta} \quad (\alpha,\ \beta\in(0,1)),$$

and introduce the Banach spaces $E^{k+\alpha}(\bar{Q}_T)$ $(k=0,1,2)$ which are the completion of infinitely differential functions in respective norms

$$\|u\|_\alpha = \|u\|_{E^\alpha(\bar{Q}_T)} \equiv E^{\alpha,\alpha/2}[u] = |u|^{(0)} + \langle u\rangle^{(\alpha)} + [u]^{(\alpha,\alpha/2)},$$

$$D^{\alpha,\alpha}[u] = |u|^{(0)} + \langle u\rangle_x^{(\alpha)} + \langle u\rangle_t^{(\alpha)} + [u]^{(\alpha,\alpha)},$$

$$\|u\|_{k+\alpha} = \|u\|_{E^{k+\alpha}(\bar{Q}_T)} = E^{\alpha,\alpha/2}[\mathrm{D}_x^k u] + \sum_{j=0}^{k-1} D^{\alpha,\alpha}[\mathrm{D}_x^j u] \quad \left(\mathrm{D}_x^k = \sum_{|j|=k} \frac{\partial^j}{\partial x^j},\ k=1,2\right);$$

$$\hat{E}^{2+\alpha}(\bar{Q}_T) = \left\{u \mid \|u\|_{\hat{E}^{2+\alpha}(\bar{Q}_T)} < \infty\right\}, \quad \|u\|_{\hat{E}^{2+\alpha}(\bar{Q}_T)} = \|u\|_{2+\alpha} + \left\|\frac{\partial u}{\partial t}\right\|_{1+\alpha}.$$

The function spaces on a smooth manifold Γ in $\mathbb{R}^n$ are defined with the help of partition of unity and of local maps.

2. Uniqueness of the solution to problem (1.7)

Our main result for two-phase problem (1.7) is as follows:

Theorem 2.1. *Let $T>0$ and $\alpha\in(0,1)$. Assume that $(p_1^0,p_2^0,\zeta^0)\in C^{3+\alpha}(\bar{\Omega}_1)\times C^{3+\alpha}(\bar{\Omega}_2)\times C^{4+\alpha}([0,2\pi])$ satisfy the compatibility conditions, $\partial p_1^0/\partial r - \partial p_2^0/\partial r > 0$ on Γ, $Q\in C^\alpha([0,T])$ and $p_e\in C_{\theta,t}^{3+\alpha,(3+\alpha)/2}(\Gamma_T^*)$ with $\partial p_e/\partial t|_{t=0}=0$. Then there exists $T_0^*>0$ depending on the data of the problem such that problem (1.7) has a unique solution $(p_1,p_2,\zeta)\in E^{2+\alpha}(\bar{Q}_{1,T_0^*})\times E^{2+\alpha}(\bar{Q}_{2,T_0^*})\times \hat{E}^{2+\alpha}(\Gamma_{T_0^*})$ except for the extension of ζ^0 to $[0,2\pi]\times[0,T]$ satisfying*

$$\|p_1\|_{E^{2+\alpha}(\bar{Q}_{1,T_0^*})} + \|p_2\|_{E^{2+\alpha}(\bar{Q}_{2,T_0^*})} + \|\zeta\|_{\hat{E}^{2+\alpha}(\Gamma_{T_0^*})} \leqslant C. \tag{2.1}$$

In [9] we showed the existence of the solution to problem (1.7) on some time interval $[0,T_0]$ $(0<T_0\leqslant T)$ in the form

$$\begin{cases} p_1 = p_1^* + p_1^0 + \dfrac{r-R_*}{R+\bar{\zeta}-R_*}\dfrac{\partial p_1^0}{\partial r}\zeta^*, \quad p_2 = p_2^* + p_2^0 + \dfrac{r-R^*}{R+\bar{\zeta}-R^*}\dfrac{\partial p_2^0}{\partial r}\zeta^*, \\ \zeta = \zeta^* + \bar{\zeta} \end{cases} \tag{2.2}$$

by the parabolic regularization and by vanishing the coefficient of the derivative with respect to time in a parabolic equation. Here $\bar{\zeta}\in C_{\theta,t}^{4+\alpha,(4+\alpha)/2}([0,2\pi]\times[0,T])$ is an extension of ζ^0 such that $(\bar{\zeta},\partial\bar{\zeta}/\partial t,\partial^2\bar{\zeta}/\partial t^2)|_{t=0} = (\zeta^0,\ \partial\zeta/\partial t,\partial^2\zeta/\partial t^2)|_{t=0}$ whose right hand side are obtained from the fourth equation in (1.7) and its derivative in t at $t=0$.

Then (1.7) becomes

$$
\begin{cases}
\mathcal{L}_*^i p_i^* = \Phi_i \quad \text{in } \Omega_i,\ t>0\ (i=1,2), \\
\dfrac{\partial p_1^*}{\partial r} = \Psi_* \quad \text{on } \Gamma_*,\ t>0, \quad p_2^* = \Psi^* \quad \text{on } \Gamma^*,\ t>0, \\
\dfrac{\partial \zeta^*}{\partial t} - b_2^1(\bar\zeta)\dfrac{\partial p_1^*}{\partial r} - b_1^1(\bar\zeta)\dfrac{\partial p_1^*}{\partial \theta} - b_2^2(\bar\zeta)\dfrac{\partial p_2^*}{\partial r} - b_1^2(\bar\zeta)\dfrac{\partial p_2^*}{\partial \theta} = \Psi_1 + \Psi_2, \\
b_2^1(\bar\zeta)\dfrac{\partial p_1^*}{\partial r} + b_1^1(\bar\zeta)\dfrac{\partial p_1^*}{\partial \theta} - b_2^2(\bar\zeta)\dfrac{\partial p_2^*}{\partial r} - b_1^2(\bar\zeta)\dfrac{\partial p_2^*}{\partial \theta} = -\Psi_1 + \Psi_2, \\
p_1^* - p_2^* + d(\bar\zeta)\zeta^* = \Psi_3 \quad \text{on } \Gamma,\ t>0, \\
p_i^*\,\big|_{t=0} = 0 \quad \text{on } \Omega_i\ (i=1,2), \quad \zeta^*\,\big|_{t=0} = 0 \quad \text{on } [0,2\pi].
\end{cases}
\tag{2.3}
$$

Here $\mathcal{L}_*^i$ is the principal part of $\mathcal{L}_\zeta^i$ with ζ replaced by $\bar\zeta$,

$$
\begin{aligned}
&\Phi_i = \Phi_i(p_i^*, \zeta^*) = -\mathcal{L}_\zeta^i p_i + \mathcal{L}_*^i p_i^* \quad (i=1,2), \quad \Psi^* = p_e - p_2^0, \\
&\Psi_* = \Psi_*(\zeta^*) = -\frac{\partial}{\partial r}\left(p_1^0 + \frac{r-R_*}{R+\bar\zeta-R_*}\frac{\partial p_1^0}{\partial r}\right)\zeta^* - \frac{R+\zeta-R_*}{R_0+\zeta^0-R_*}\frac{Q(t)}{2\pi R_* M_1}, \\
&\Psi_j = \Psi_j\left(p_1^*, p_2^*, \zeta^*\right) = b_2^j(\zeta)\frac{\partial p_j}{\partial r} + b_1^j(\zeta)\frac{\partial p_j}{\partial \theta} - b_2^j(\bar\zeta)\frac{\partial p_j^*}{\partial r} - b_1^j(\bar\zeta)\frac{\partial p_j^*}{\partial \theta} - \frac{Q(t)}{4\pi R} - \frac{1}{2}\frac{\partial\bar\zeta}{\partial t} \\
&\qquad (j=1,2), \\
&\Psi_3 = p_2^0 - p_1^0, \quad d(\bar\zeta) = \frac{R_0+\zeta^0-R_*}{R+\bar\zeta-R_*}\frac{\partial p_1^0}{\partial r} - \frac{R_0+\zeta^0-R^*}{R+\bar\zeta-R^*}\frac{\partial p_2^0}{\partial r}
\end{aligned}
$$

with (p_1, p_2, ζ) replaced by (2.2).

If problem (2.3) admits a unique solution on some time interval $[0, T_0^*]$ $(0 < T_0^* \leqslant T_0)$, then the limit process holds for the full sequence, not the subsequence, on $[0, T_0^*]$, so that the proof of Theorem 2.1 is completed.

In what follows we shall prove the uniqueness of solution to problem (2.3).

Let (p_1^*, p_2^*, ζ^*) and $(p_1^{**}, p_2^{**}, \zeta^{**})$ be two solutions of (2.3) satisfying

$$
\|p_1^\dagger\|_{E^{2+\alpha}(\bar Q_{1,T_0})} + \|p_2^\dagger\|_{E^{2+\alpha}(\bar Q_{2,T_0})} + \|\zeta^\dagger\|_{\hat E^{2+\alpha}(\Gamma_{T_0})} \leqslant C_1 \quad (\dagger = *,\ **). \tag{2.4}
$$

To the end, as the same way as in [9] it is essential to consider the following four model problems in the whole- and half-spaces:

$$
\mathcal{L}u = f \ \ \text{in } \mathbb{R}^2,\ t>0,\ \ u\,\big|_{t=0} = 0; \tag{2.5}
$$

$$
\mathcal{L}u = f \quad (x_1\in\mathbb{R},\ x_2>0,\ t>0), \quad u\,\big|_{x_2=0} = 0, \quad u\,\big|_{t=0} = 0; \tag{2.6}
$$

$$
\mathcal{L}u = f \quad (x_1\in\mathbb{R},\ x_2>0,\ t>0), \quad \frac{\partial u}{\partial x_2}\,\Big|_{x_2=0} = 0, \quad u\,\big|_{t=0} = 0; \tag{2.7}
$$

$$
\begin{cases}
\mathcal{L}u^+ = 0 \ \ (x_1\in\mathbb{R}, x_2>0, t>0), \quad \mathcal{L}u^- = 0 \ \ (x_1\in\mathbb{R}, x_2<0, t>0), \\
\dfrac{\partial\varrho}{\partial t} - b^+\dfrac{\partial u^+}{\partial x_2} - b^-\dfrac{\partial u^-}{\partial x_2}\,\Big|_{x_2=0} = g_1, \quad -b^+\dfrac{\partial u^+}{\partial x_2} + b^-\dfrac{\partial u^-}{\partial x_2}\,\Big|_{x_2=0} = g_2, \\
-u^+ + u^- + d\varrho\,\big|_{x_2=0} = g_3, \quad (u^+, u^-, \varrho)\,\big|_{t=0} = 0.
\end{cases}
\tag{2.8}
$$

Moreover, it suffices to assume that $\mathcal{L} = \Delta$, and $b^\pm$ and d are positive constants, and set $b \equiv (2db^+b^-)/(b^+ + b^-)$ in (2.5)–(2.8).

In estimating the difference $(p_1^*, p_2^*, \zeta^*) - (p_1^{**}, p_2^{**}, \zeta^{**})$, we trace a proof in [9] with the help of a fundamental solution and Green functions of (2.5)–(2.8) instead of Γ_ε, G_ε, N_ε and Z_ε in [9]. It is clear that the solutions to problems (2.5)–(2.7) for $\mathcal{L} = \Delta$ are given by

$$u(x,t) = \int_{\mathbb{R}^2} \Gamma_0(x-y) f(y,t)\,\mathrm{d}y, \quad u(x,t) = \int_{\mathbb{R}^2} G_0(x-y) f(y,t)\,\mathrm{d}y,$$
$$u(x,t) = \int_{\mathbb{R}^2} N_0(x-y) f(y,t)\,\mathrm{d}y,$$

respectively, where $\Gamma_0(x) = -\log|x|/(2\pi)$, $G_0(x_1,x_2) = \Gamma_0(x_1,x_2) - \Gamma_0(x_1,-x_2)$, and $N_0(x_1,x_2) = \Gamma_0(x_1,x_2) + \Gamma_0(x_1,-x_2)$. Whereas, the solution $(u^+, u^-, \varrho) = (\mathcal{FL})^{-1}[(\tilde{u}^+, \tilde{u}^-, \tilde{\varrho})]$ of problem (2.8) is represented by virtue of Green function

$$Z_0(x_1,t) = (\mathcal{FL})^{-1}[\tilde{Z}_0] = (\mathcal{FL})^{-1}\left[\frac{1}{s+b|\xi|}\right]:$$
$$\tilde{u}^+ = (\mathcal{FL})[u^+] = \tilde{v}^+(\xi,s)\,\mathrm{e}^{-|\xi|x_2} \quad (x_2>0), \quad \tilde{u}^- = (\mathcal{FL})[u^-] = \tilde{v}^-(\xi,s)\,\mathrm{e}^{|\xi|x_2} \quad (x_2<0),$$
$$\tilde{v}^+ = \frac{1}{(b^++b^-)\,|\xi|}\,\tilde{g}_2 - \frac{b^-}{b^++b^-}\,\tilde{g}_3 + \tilde{Z}_0\left(\frac{db^-}{b^++b^-}\,\tilde{g}_1 - \frac{b-db^-}{b^++b^-}\,\tilde{g}_2 + \frac{bb^-}{b^++b^-}\,|\xi|\tilde{g}_3\right),$$
$$\tilde{v}^- = \frac{1}{(b^++b^-)\,|\xi|}\,\tilde{g}_2 + \frac{b^+}{b^++b^-}\,\tilde{g}_3 + \tilde{Z}_0\left(-\frac{db^+}{b^++b^-}\,\tilde{g}_1 - \frac{b-db^+}{b^++b^-}\,\tilde{g}_2 - \frac{bb^+}{b^++b^-}\,|\xi|\tilde{g}_3\right),$$
$$\tilde{\varrho} = \tilde{Z}_0\left(\tilde{g}_1 - \frac{b^+-b^-}{b^++b^-}\,\tilde{g}_2 + \frac{2b^+b^-}{b^++b^-}\,|\xi|\tilde{g}_3\right).$$

Here $\tilde{u} = (\mathcal{FL})[u]$ is the Fourier transformation in x_1 and Laplace transformation in t of u, and $(\mathcal{FL})^{-1}[\tilde{u}]$ is its inverse transformation.

Lemma 2.2. *When $b=1$, we have the following estimates of Z_0:*

$$|Z_0(x_1,t)| \leqslant C_2\,\frac{1}{\sqrt{x_1^2+t^2}}, \quad \left|\frac{\partial}{\partial t}\,Z_0(x_1,t)\right| + \left|\frac{\partial}{\partial x_1}\,Z_0(x_1,t)\right| \leqslant C_2\,\frac{1}{x_1^2+t^2},$$
$$\left|\frac{\partial^2}{\partial t\partial x_1}\,Z_0(x_1,t)\right| + \left|\frac{\partial^2}{\partial x_1^2}\,Z_0(x_1,t)\right| \leqslant C_2\,\frac{1}{(x_1^2+t^2)^{3/2}}.$$

Using Lemma 2.2, we estimate these solutions of (2.5)–(2.8) in the same way as in [9] (cf. [1,2]). For a general domain by using the regularizer method for elliptic system ([6–8]), we finally obtain the estimate of $(p_1^*, p_2^*, \zeta^*) - (p_1^{**}, p_2^{**}, \zeta^{**})$ with the help of Young's and interpolation inequalities and (2.4):

$$\|p_1^* - p_1^{**}\|_{E^{2+\alpha}(\bar{Q}_{1,t})} + \|p_2^* - p_2^{**}\|_{E^{2+\alpha}(\bar{Q}_{2,t})} + \|\zeta^* - \zeta^{**}\|_{\hat{E}^{2+\alpha}(\Gamma_t)} \leqslant \tag{2.9}$$
$$\leqslant C_3\left(\sum_{i=1}^{2}\left(\|\Phi_i(p^*,\zeta^*) - \Phi_i(p^{**},\zeta^{**})\|_{E^\alpha(\bar{Q}_{i,t})} + \|\Psi_i(p^*,\zeta^*) - \Psi_i(p^{**},\zeta^{**})\|_{E^{1+\alpha}(\bar{\Gamma}_t)}\right) + \right.$$
$$\left. + \|\Psi_*(\zeta^*) - \Psi_*(\zeta^{**})\|_{E^{1+\alpha}(\bar{\Gamma}_{*t})}\right) \leqslant$$
$$\leqslant C_3\left(\beta + C_\beta t^\chi F(4C_1)\right)\left(\|p_1^* - p_1^{**}\|_{E^{2+\alpha}(\bar{Q}_{1,t})} + \|p_2^* - p_2^{**}\|_{E^{2+\alpha}(\bar{Q}_{2,t})} + \|\zeta^* - \zeta^{**}\|_{\hat{E}^{2+\alpha}(\Gamma_t)}\right)$$

for any $t \in (0,T_0)$ and any $\beta > 0$, where C_β is a positive constant depending on β non-increasingly, χ is a constant depending on α, $F(\cdot)$ is a polynomial in its argument.

Now choosing first $\beta = 1/(4C_3)$, and then

$$T_0^* = \min\left\{T_0, \left(\frac{1}{4C_3C_\beta F(4C_1)}\right)^{1/\chi}\right\},$$

we conclude from (2.9) that the solution to problem (2.3) is unique on $[0, T_0^*]$.

3. Uniqueness of the solution to the one-phase problem

Our next main result for one-phase problem is as follows:

Theorem 3.1. *Let $T > 0$ and $\alpha \in (0,1)$. Assume that $(p^0, \zeta^0) \in C^{3+\alpha}(\bar{\Omega}) \times C^{4+\alpha}([0,2\pi])$ satisfy the compatibility conditions, $\partial p^0/\partial r < 0$ on Γ, $Q \in C^\alpha([0,T])$ and $p_e \in C_{\theta,t}^{3+\alpha,(3+\alpha)/2}(\Gamma_T^*)$ with $\partial p_e/\partial t|_{t=0} = 0$. Then there exists $T_0^* > 0$ depending on the data of the problem such that one-phase problem has a unique solution $(p, \zeta) \in E^{2+\alpha}(\bar{Q}_{T_0^*}) \times \hat{E}^{2+\alpha}(\Gamma_{T_0^*})$ except for the extension of ζ^0 to $[0,2\pi] \times [0,T]$ satisfying*

$$\|p\|_{E^{2+\alpha}(\bar{Q}_{T_0^*})} + \|\zeta\|_{\hat{E}^{2+\alpha}(\Gamma_{T_0^*})} \leqslant C'. \tag{3.1}$$

Like the two-phase problem we transform the one-phase problem into just the same equations as (2.3). In [10] the existence of the solution (p^*, ζ^*) (cf. (2.2)) to one-phase problem on some time interval $[0, T_0]$ $(0 < T_0 < T)$ was shown.

Let (p^*, ζ^*) and (p^{**}, ζ^{**}) be two solutions satisfying

$$\|p^\dagger\|_{E^{2+\alpha}(\bar{Q}_{T_0})} + \|\zeta^\dagger\|_{\hat{E}^{2+\alpha}(\Gamma_{T_0})} \leqslant C_1' \quad (\dagger = *,\ **). \tag{3.2}$$

For one-phase case the essential model problems are the same as (2.5), (2.7), (2.8) with (u^+, u^-, ϱ) replaced by $(u, 0, \varrho)$.

$$\mathcal{L}u = f \quad \text{in } \mathbb{R}^2,\ t > 0, \quad u\big|_{t=0} = 0; \tag{3.3}$$

$$\mathcal{L}u = f \quad (x_1 \in \mathbb{R},\ x_2 > 0,\ t > 0)\,, \quad \frac{\partial u}{\partial x_2}\Big|_{x_2=0} = 0, \quad u\big|_{t=0} = 0; \tag{3.4}$$

$$\begin{cases} \mathcal{L}u = 0 \ \ (x_1 \in \mathbb{R}, x_2 > 0, t > 0)\,, \\ \dfrac{\partial \varrho}{\partial t} - b\dfrac{\partial u}{\partial x_2}\Big|_{x_2=0} = g_1, \quad u - d\varrho\big|_{x_2=0} = g_2, \quad (u,\varrho)\big|_{t=0} = 0. \end{cases} \tag{3.5}$$

The solution $(u, \varrho) = (\mathcal{F}\mathcal{L})^{-1}[(\tilde{u}, \tilde{\varrho})]$ of problem (3.5) is represented by virtue of Green function Z_0 (cf. in Sec. 2):

$$\tilde{u} = \tilde{v}(\xi, s)\,\mathrm{e}^{-|\xi|x_2} \ \ (x_2 > 0), \quad \tilde{v} = d\,\tilde{\varrho} + g_2, \quad \tilde{\varrho} = \tilde{Z}_0\,(\tilde{g}_1 - b\,d\,|\xi|\,\tilde{g}_2)\,.$$

Just in the same way as the two-phase problem, we can estimate the solutions of (3.3)–(3.5) with the help of Lemma 2.2, and for a general domain the regularizer method for elliptic system leads to the estimate of $(p^*, \zeta^*) - (p^{**}, \zeta^{**})$:

$$\begin{aligned} &\|p^* - p^{**}\|_{E^{2+\alpha}(\bar{Q}_t)} + \|\zeta^* - \zeta^{**}\|_{\hat{E}^{2+\alpha}(\Gamma_t)} \leqslant \\ &\leqslant C_3'\left(\beta 7 + C_{\beta'}t^{\chi'}F'(4C_1')\right)\left(\|p^* - p^{**}\|_{E^{2+\alpha}(\bar{Q}_t)} + \|\zeta^* - \zeta^{**}\|_{\hat{E}^{2+\alpha}(\Gamma_t)}\right) \end{aligned} \tag{3.6}$$

for any $t \in (0, T_0)$ and any $\beta' > 0$, where $C_{\beta'}$ is a positive constant depending on β' non-increasingly, χ' is a constant depending on α, $F'(\cdot)$ is a polynomial of its argument.

Now choosing first $\beta' = 1/(4C_3')$, and then

$$T_0^* = \min\left\{T_0, \left(\frac{1}{4C_3'C_{\beta'}F'(4C_1')}\right)^{1/\chi}\right\},$$

we conclude from (3.6) that the solution to one-phase problem is unique on $[0, T_0^*]$.

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О единственности классических решений задач радиальной вязкой пальцеобразной структуры в ячейке Хеле-Шоу

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Аннотация. В [9, 10] мы установили существование классических решений двухфазной и однофазной задач радиальной вязкой аппликатуры соответственно в ячейке Хеле-Шоу параболической регуляризацией и обращением в нуль коэффициента производной по времени в параболическом уравнении. В этой статье мы показываем единственность таких решений соответствующих задач.

Ключевые слова: классическое решение, уникальное наличие, радиальная вязкая пальцеобразная структура.

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On the Solvability of the Identification Problem for a Source Function in a Quasilinear Parabolic System of Equations in Bounded and Unbounded Domains

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Abstract. The paper considers the problem of identification for a source function in one of two equations of parabolic quasilinear system. The case of Cauchy data in an unbounded domain and the case of boundary conditions of the first kind in a rectangular domain are considered. The question of the existence and uniqueness of the solution is studied. The proof uses a differential level splitting method known as the weak approximation method. The solution is obtained on a small time interval in the class of sufficiently smooth bounded functions.

Keywords: inverse problem, quasilinear equations system, source function determination, weak approximation method, small parameter.

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This work is dedicated to the memory of our teacher, Doctor of Physical and Mathematical Sciences, Yuri Yakovlevich Belov, who was a recognized and famous specialist in inverse problems of mathematical physics. A number of his latest papers were devoted to the study of solvability of inverse problems for parabolic-elliptic semi-evolutionary systems of differential equations. To study the existence of solutions to such inverse problems, Yu. Belov suggested using the well-known ε-approximation method. The essence of the method for a system of parabolic and elliptic equations, for example, is to replace the elliptic equation with a parabolic one, which contains a small parameter ε at the time derivative. The method was proposed by N. N. Yanenko who suggested replacing the Navier-Stokes equations of a viscous incompressible fluid with equations of the Cauchy-Kovalevskaya type with a small parameter ε. Thus, as ε approaches to zero, the approximating equations become the original ones.

The main part of the mentioned works by Yu. Ya Belov is devoted to the study of linear parabolic-elliptic systems. The case when the unknown component of the source vector function is in an equation that does not contain a small parameter, under the first and second boundary conditions were studied in [1, 2]. In [3], a one-dimensional system is considered, in which the unknown component of the source vector function is in the equation containing the ε parameter. In the case when the components of the vector function are unknown in each equation of the system, the Cauchy problem and the first boundary value problem are investigated in [4].

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The research scheme in these works, as a rule, assumes that we first have to investigate the solvability of the auxiliary approximating problem (in an unbounded domain with Cauchy data and / or in a bounded domain with boundary conditions of the first or second kind), since these problems are nonclassical problems for loaded equations, and investigate the necessary properties of solutions that obviously depend on the parameter ε. And then the second step is to obtain estimates that will guarantee the convergence of the sequence of solutions of the approximating problem to the solution of the original problem as ε approaches zero.

In this paper, a quasilinear system of two parabolic equations with one unknown coefficient of the source function is considered. The question of a solution existence to this problem is studied. This is a model problem in which the authors set the goal of working out the splitting algorithm and obtaining a priori estimates for quasilinear systems, which is much more complicated than in the linear case. It is also important to note that the system under consideration contains a small fixed parameter $\varepsilon > 0$, which does not affect the study of the question of a solution existence, but allows using this system subsequently as an approximating model for the problem of identifying the source function in a quasilinear parabolic-elliptic system.

1. Formulation of the problem and reduction it to the direct problem

Consider in the strip $G_{[0,T]} = \{(t,x) \mid 0 \leqslant t \leqslant T,\, x \in E_1\}$ the problem of determining real-valued functions $\big(u(t,x), v(t,x), r(t)\big)$, satisfying the system of equations

$$\begin{cases} u_t(t,x) + a_{11}(t)u(t,x) + a_{12}(t)v(t,x) = \mu_1 u_{xx}(t,x) + v(t,x)u_x(t,x) + r(t)f(t,x), \\ \varepsilon v_t(t,x) + a_{21}(t)u(t,x) + a_{22}(t)v(t,x) = \mu_2 v_{xx}(t,x) + u(t,x)v_x(t,x) + g(t,x), \end{cases} \tag{1}$$

where $\varepsilon \in (0,1]$ is a *const*, with initial conditions

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \tag{2}$$

and the over determination condition

$$u(t,x^0) = \varphi(t), \tag{3}$$

where $\varphi(t)$ is a given function on $[0,T]$, $0 \leqslant t \leqslant T$, x^0 is a fixed point.

System (1), for example, can be an approximation of the parabolic-elliptic system of equations

$$\begin{cases} \tilde{u}_t(t,x) + a_{11}(t)\tilde{u}(t,x) + a_{12}(t)\tilde{v}(t,x) = \mu_1 \tilde{u}_{xx}(t,x) + \tilde{v}(t,x)\tilde{u}_x(t,x) + \tilde{r}(t)f(t,x), \\ \qquad a_{21}(t)\tilde{u}(t,x) + a_{22}(t)\tilde{v}(t,x) = \mu_2 \tilde{v}_{xx}(t,x) + \tilde{u}(t,x)\tilde{v}_x(t,x) + g(t,x). \end{cases}$$

Note that the study of the behavior of the solution when ε approaches zero is beyond the scope of this study, and in our work ε is a nonnegative fixed constant.

Let the functions $a_{ij}(t)$, $i,j = 1,2$, be defined on $[0,T]$ and let the functions $f(t,x)$, $g(t,x)$ be defined on $G_{[0,T]}$. Let μ_1, $\mu_2 > 0$ be given constants.

Let the relationship

$$\big|f(t,x^0)\big| \geqslant \delta > 0, \quad t \in [0,T] \quad (\delta \ \text{ is a const}) \tag{4}$$

hold.

Assume that the following consistency condition is fulfilled

$$u_0(x^0) = \varphi(0). \tag{5}$$

Reduce the inverse problem (1)–(2) to an auxiliary direct problem. In system (1) we set $x = x^0$:

$$\begin{cases} \varphi'(t) + a_{11}(t)\varphi(t) + a_{12}(t)v(t,x^0) = \mu_1 u_{xx}(t,x^0) + v(t,x_0)u_x(t,x^0) + r(t)f(t,x^0), \\ \varepsilon v_t(t,x^0) + a_{21}(t)\varphi(t) + a_{22}(t)v(t,x^0) = \mu_2 v_{xx}(t,x^0) + \varphi(t)v_x(t,x^0) + g(t,x^0). \end{cases} \tag{6}$$

From (6) we obtain

$$r(t) = \frac{\psi(t) + a_{12}v(t,x^0) - \mu_1 u_{xx}(t,x^0) - v(t,x_0)u_x(t,x^0)}{f(t,x^0)}. \tag{7}$$

where $\psi(t) = \varphi'(t) + a_{11}(t)\varphi(t)$ is known.

Substituting expression for $r(t)$ in (1) we obtain the following direct problem:

$$\begin{cases} u_t(t,x) + a_{11}(t)u(t,x) + a_{12}(t)v(t,x) = \mu_1 u_{xx}(t,x) + v(t,x)u_x(t,x) + \\ + \dfrac{\psi(t) + a_{12}(t)v(t,x^0) - \mu_1 u_{xx}(t,x^0) - v(t,x^0)u_x(t,x^0)}{f(t,x^0)} f(t,x), \\ \varepsilon v_t(t,x) + a_{21}(t)u(t,x) + a_{22}(t)v(t,x) = \mu_2 v_{xx}(t,x) + u(t,x)v_x(t,x) + g(t,x), \end{cases} \tag{8}$$

$$u(0,x) = u_0(x), \tag{9}$$

$$v(0,x) = v_0(x). \tag{10}$$

2. Proof of solvability of the problem (1)–(3)

To prove the existence of a solution to the auxiliary problem (1)–(3), we use the weak approximation method [5, 6]. We split the problem (8)–(10) and linearize it by shifting in time by $\dfrac{\tau}{4}$.

$$\begin{cases} u_t^\tau(t,x) = 4\mu_1 u_{xx}^\tau(t,x), \\ \varepsilon v_t^\tau(t,x) = 4\mu_2 v_{xx}^\tau(t,x), \quad j\tau < t \leqslant \left(j + \dfrac{1}{4}\right)\tau, \end{cases} \tag{11}$$

$$\begin{cases} u_t^\tau(t,x) + 4a_{11}(t)u^\tau(t,x) = 0, \\ \varepsilon v_t^\tau(t,x) + 4a_{22}(t)v^\tau(t,x) = 0, \quad \left(j + \dfrac{1}{4}\right)\tau < t \leqslant \left(j + \dfrac{1}{2}\right)\tau, \end{cases} \tag{12}$$

$$\begin{cases} u_t^\tau(t,x) = 4v^\tau\left(t - \dfrac{\tau}{4}, x\right)u_x^\tau(t,x), \\ \varepsilon v_t^\tau(t,x) = 4u^\tau\left(t - \dfrac{\tau}{4}, x\right)v_x^\tau(t,x), \quad \left(j + \dfrac{1}{2}\right)\tau < t \leqslant \left(j + \dfrac{3}{4}\right)\tau, \end{cases} \tag{13}$$

$$\begin{cases} u_t^\tau(t,x) + 4a_{12}(t)v^\tau(t - \dfrac{\tau}{4}, x) = \\ = 4\dfrac{\psi(t) + a_{12}(t)v^\tau(t - \frac{\tau}{4}, x^0) - \mu_1 u_{xx}^\tau(t - \frac{\tau}{4}, x^0) - v^\tau(t - \frac{\tau}{4}, x^0)u_x^\tau(t - \frac{\tau}{4}, x^0)}{f(t,x^0)} f(t,x), \\ \varepsilon v_t^\tau(t,x) + 4a_{21}(t)u^\tau\left(t - \dfrac{\tau}{4}, x\right) = 4g(t,x), \quad \left(j + \dfrac{3}{4}\right)\tau < t \leqslant (j+1)\tau, \end{cases} \tag{14}$$

$$u^{\tau}(t,x)|_{t\leqslant 0} = u_0(x), \tag{15}$$

$$v^{\tau}(t,x)|_{t\leqslant 0} = v_0(x). \tag{16}$$

Here $j = 0, 1, \ldots, N-1;\ \tau N = T$.

Concerning the input data, assume that they are sufficiently smooth, have all continuous derivatives occurring in the next lower relations of (17)–(19) and satisfy them:

$$|a_{ij}(t)| \leqslant C, \quad i = 1, 2, \quad j = 1, 2, \tag{17}$$

$$\left|\frac{\partial^k}{\partial x^k} f(t,x)\right| + \left|\frac{\partial^k}{\partial x^k} F(t,x)\right| + \left|\frac{d^k}{dx^k} u_0(x)\right| + \left|\frac{d^k}{dx^k} v_0(x)\right| \leqslant C, \quad k = 0, \ldots, p+6, \tag{18}$$

$$|\varphi(t)| + \left|\varphi^{'}(t)\right| \leqslant C, \quad (t,x) \in G_{[0,T]}. \tag{19}$$

Below, for convenience, we consider some proofs assuming that the constant C is greater than 1 and that the constant $p \geqslant 6$ is an even number.

For the solution $u^{\tau}(t,x), v^{\tau}(t,x)$ of the split linearized problem (9)–(12) are obtained a priori estimates uniform in τ for $j = 0, 1, \ldots, p+1,\ k = 0, 1, \ldots, p,\ (t,x) \in G_{[0,T]}$

$$\left|\frac{\partial^{p+4}}{\partial x^{p+4}} u^{\tau}_t(t,x)\right| + \left|\frac{\partial^{p+4}}{\partial x^{p+4}} v^{\tau}_t(t,x)\right| \leqslant C, \quad (t,x) \in G_{[0,t^*]}, \tag{20}$$

where t^* does not depend on τ and depends on ε.

By virtue of the (20), the theorem of Arzela [7] and the convergence theorem of the weak approximation method [6], it follows that the limit functions $u(t,x), v(t,x)$ for $\tau \to 0$ are a solution to the direct problem (8)–(10), and $u(t,x), v(t,x)$ and $r(t,x)$ defined by relation (7) are solutions of problem (1), (2).

The uniqueness of the found solution is proved in a standard way, by obtaining estimates showing that the difference of two possible solutions in $G_{[0,t^*]}$ is equal to zero.

The following theorem gives sufficient conditions for the existence and uniqueness of a solution.

Theorem 2.1. *Let the conditions* (4), (5), (17)–(19) *hold. Then there exists a unique solution* $u(t,x), v(t,x), r(t)$ *of problem* (1)–(3) *in the class*

$$Z(t^*) = \Big\{u(t,x), v(t,x), r(t) | u(t,x) \in C^{1,p+4}_{t,x}(G_{[0,t^*]}), v(t,x) \in C^{1,p+4}_{t,x}(G_{[0,t^*]}), \\ r(t) \in C([0,t^*])\Big\},$$

and the following relations hold

$$\sum_{k=0}^{p+4}\left(\left|\frac{\partial^k}{\partial x^k} u(t,x)\right| + \left|\frac{\partial^k}{\partial x^k} v(t,x)\right|\right) + ||r(t)||_{C^1[0,t^*]} + \left|\frac{\partial}{\partial t} u(t,x)\right| + \left|\frac{\partial}{\partial t} v(t,x)\right| \leqslant C(\varepsilon), \\ (t,x) \in G_{[0,t^*]}. \tag{21}$$

where

$$C^{1,p+4}_{t,x}(G_{[0,t^*]}) = \left\{u(t,x) | u_t \in C(G_{[0,t^*]}), \frac{\partial^k}{\partial x^k} u \in C(G_{[0,t^*]}), k = 0, \ldots, p+4\right\}.$$

Obviously, the solution depends on the constant ε, just as the constant $C(\varepsilon)$ depends on ε and the input data.

3. Periodicity

In the domain $Q_{t^*} = \{(t,x) \mid 0 < t < t^*,\ 0 < x < l\}$ consider the boundary value problem

$$\begin{cases} u_t(t,x) + a_{11}(t)u(t,x) + a_{12}(t)v(t,x) = \mu_1 u_{xx}(t,x) + v(t,x)u_x(t,x) + r(t)f(t,x), \\ \varepsilon v_t(t,x) + a_{21}(t)u(t,x) + a_{22}(t)v(t,x) = \mu_2 v_{xx}(t,x) + u(t,x)v_x(t,x) + g(t,x), \end{cases} \tag{22}$$

ε *is a const*, $\varepsilon \in (0,1]$,

$$u(0,x) = u_0(x), \quad x \in [0,l], \tag{23}$$

$$v(0,x) = v_0(x), \quad x \in [0,l], \tag{24}$$

$$u(t,0) = u(t,l) = v(t,0) = v(t,l) = 0, \quad t \in [0,t^*], \tag{25}$$

$$u(t,x^0) = \varphi(t), \quad 0 < x_0 < l, \tag{26}$$

$$u_0(x^0) = \varphi(0). \tag{27}$$

Let us extend the functions $u_0(x), v_0(x), f(t,x), g(t,x)$ to the segment $[-l,l]$:

$$u_0(x) = -u_0(-x), \quad \text{for} - l \leqslant x < 0,$$
$$v_0(x) = -v_0(-x), \quad \text{for} - l \leqslant x < 0.$$

Then we continue the functions from $[-l,l]$ to $\Re$ in a periodic manner.

Extend the functions $f(t,x)$ and $g(t,x)$ from $[0,t^*] \times [0,l]$ to $[0,t^*] \times \Re$ to periodic and odd in x functions.

Note that the functions $u_0(x), v_0(x), f(t,x), g(t,x)$, according to the construction method, satisfy the conditions:

$$u_0(-x) = -u_0(x), \quad u_0(l-x) = -u_0(l+x), \tag{28}$$

$$v_0(-x) = -v_0(x), \quad v_0(l-x) = -v_0(l+x), \tag{29}$$

$$f(t,-x) = -f(t,x), \quad f(t,l-x) = -f(t,l+x), \tag{30}$$

$$g(t,-x) = -g(t,x), \quad g(t,l-x) = -g(t,l+x), \tag{31}$$

The functions $u_0(x), v_0(x), f(t,x), g(t,x)$ continued in this way are used as the input data for the Cauchy problem

$$\begin{cases} u_t(t,x) + a_{11}(t)u(t,x) + a_{12}(t)v(t,x) = \mu_1 u_{xx}(t,x) + v(t,x)u_x(t,x) + r(t)f(t,x), \\ \varepsilon v_t(t,x) + a_{21}(t)u(t,x) + a_{22}(t)v(t,x) = \mu_2 v_{xx}(t,x) + u(t,x)v_x(t,x) + g(t,x), \end{cases} \tag{32}$$

ε *is const*, $\varepsilon \in (0,1]$,

$$u(0,x) = u_0(x), \quad x \in (-\infty,+\infty), \tag{33}$$

$$v(0,x) = v_0(x), \quad x \in (-\infty,+\infty). \tag{34}$$

Split the problem (32)–(34):

$$\begin{cases} u_t^\tau(t,x) = 4\mu_1 u_{xx}^\tau(t,x), \\ \varepsilon v_t^\tau(t,x) = 4\mu_2 v_{xx}^\tau(t,x), \quad j\tau < t \leqslant \left(j + \dfrac{1}{4}\right)\tau, \end{cases} \tag{35}$$

$$\begin{cases} u^\tau_t(t,x) + 4a_{11}(t)u^\tau(t,x) = 0, \\ \varepsilon v^\tau_t(t,x) + 4a_{22}(t)v^\tau(t,x) = 0, \quad \left(j+\dfrac{1}{4}\right)\tau < t \leqslant \left(j+\dfrac{1}{2}\right)\tau, \end{cases} \tag{36}$$

$$\begin{cases} u^\tau_t(t,x) = 4v^\tau\left(t-\dfrac{\tau}{4},x\right)u^\tau_x(t,x), \\ \varepsilon v^\tau_t(t,x) = 4u^\tau\left(t-\dfrac{\tau}{4},x\right)v^\tau_x(t,x), \quad \left(j+\dfrac{1}{2}\right)\tau < t \leqslant \left(j+\dfrac{3}{4}\right)\tau, \end{cases} \tag{37}$$

$$\begin{cases} u^\tau_t(t,x) + 4a_{12}(t)v^\tau\left(t-\dfrac{\tau}{4},x\right) = \\ = 4\dfrac{\psi(t) + a_{12}(t)v^\tau\big(t-\frac{\tau}{4},x^0\big) - \mu_1 u^\tau_{xx}\big(t-\frac{\tau}{4},x^0\big) - v^\tau\big(t-\frac{\tau}{4},x^0\big)u^\tau_x\big(t-\frac{\tau}{4},x^0\big)}{f(t,x^0)}f(t,x), \\ \varepsilon v^\tau_t(t,x) + 4a_{21}(t)u^\tau\left(t-\dfrac{\tau}{4},x\right) = 4g(t,x), \quad \left(j+\dfrac{3}{4}\right)\tau < t \leqslant (j+1)\,\tau, \end{cases} \tag{38}$$

$$u^\tau(0,x) = u_0(x), \tag{39}$$

$$v^\tau(0,x) = v_0(x). \tag{40}$$

Let $u^\tau(t,x)$, $v^\tau(t,x)$ be a solution to the split problem. Let us show that $u^\tau(t,x)$, $v^\tau(t,x)$ satisfy the conditions

$$u^\tau(t,-x) = -u^\tau(t,x), \quad u^\tau(t,l-x) = -u^\tau(t,l+x), \tag{41}$$

$$v^\tau(t,-x) = -v^\tau(t,x), \quad v^\tau(t,l-x) = -v^\tau(t,l+x). \tag{42}$$

At the first fractional step, using the integral representation, we obtain

$$u^\tau(t,x) = \int_{-\infty}^{+\infty} u_0(\xi)\frac{1}{4\sqrt{\pi t\mu_1}}e^{-\frac{(x-\xi)^2}{12\mu_1 t}}d\xi. \tag{43}$$

$$v^\tau(t,x) = \int_{-\infty}^{+\infty} v_0(\xi)\frac{1}{4\sqrt{\pi t\mu_2}}e^{-\frac{(x-\xi)^2}{12\mu_2 t}}d\xi. \tag{44}$$

Let us check the first conditions from (41) and (42)

$$u^\tau(t,-x) + u^\tau(t,x) = \int_{-\infty}^{+\infty} u_0(\xi)\frac{1}{4\sqrt{\pi t\mu_1}}(e^{-\frac{(x-\xi)^2}{12\mu_1 t}} + e^{-\frac{(x+\xi)^2}{12\mu_1 t}})d\xi. \tag{45}$$

$$v^\tau(t,-x) + v^\tau(t,x) = \int_{-\infty}^{+\infty} v_0(\xi)\frac{1}{4\sqrt{\pi t\mu_2}}(e^{-\frac{(x-\xi)^2}{12\mu_2 t}} + e^{-\frac{(x+\xi)^2}{12\mu_2 t}})d\xi. \tag{46}$$

The integrand changes sign when ξ is replaced by $-\xi$, therefore, the integrals are equal to 0. The second conditions from (41) and (42) are verified similarly by replacing $\eta = l-\xi$ the variable of integration.

At the second fractional step, $u^\tau(t,x)$, $v^\tau(t,x)$ have the form

$$u^\tau(t,x) = u^\tau\left(\frac{\tau}{4},x\right)e^{4\int\limits_{\frac{\tau}{4}}^{t} a_{11}(\eta)d\eta}, \quad \frac{\tau}{2} < t \leqslant \frac{\tau}{4}, \tag{47}$$

$$v^\tau(t,x) = v^\tau\left(\frac{\tau}{4},x\right)e^{4\int\limits_{\frac{\tau}{4}}^{t} a_{22}(\eta)d\eta}, \quad \frac{\tau}{2} < t \leqslant \frac{\tau}{4}. \tag{48}$$

Consequently,

$$u^\tau(t,-x)+u^\tau(t,x)=\left(u^\tau\left(\frac{\tau}{4},-x\right)+u^\tau\left(\frac{\tau}{4},x\right)\right)e^{4\int\limits_{\frac{\tau}{4}}^{t}a_{11}(\eta)d\eta},\quad \frac{\tau}{2}<t\leqslant\frac{\tau}{4}, \tag{49}$$

$$v^\tau(t,-x)+v^\tau(t,x)=\left(v^\tau\left(\frac{\tau}{4},-x\right)+v^\tau\left(\frac{\tau}{4},x\right)\right)e^{4\int\limits_{\frac{\tau}{4}}^{t}a_{22}(\eta)d\eta},\quad \frac{\tau}{2}<t\leqslant\frac{\tau}{4}. \tag{50}$$

The conditions (41) and (42) follows from the first fractional step.

At the third fractional step, we use Lemma 1.

Lemma 1. *Let the function $u(t,x)$ be a solution to the equation $u_t=a(t,x)u_x$ in the domain $D=\{(t,x)|t_0<t<t_1,x\in\Re\}$ with the initial condition $u(t_0,x)=u_0(x)$. Let the function (a,t,x) satisfy the Lipschitz condition in x and the relations*

$$a(t,c+x)=-a(t,c-x),\quad u_0(c+x)=u_0(c-x),\quad c\ \textit{is a const}$$

hold. Then the function $u(t,x)$ satisfies the relation $u(t,c+x)=-u(t,c-x)$.

The proof of Lemma 1 is presented in [8].

Where do we get the fulfillment of the conditions (41) and (42).

At the fourth fractional step, we get

$$\begin{aligned}u^\tau(t,x)=u^\tau\Big(\frac{3\tau}{4},x\Big)+4\int_{\frac{3\tau}{4}}^{t}(a_{12}(\eta)v^\tau\Big(\eta-\frac{\tau}{4},x\Big)+\\+\frac{\psi(\eta)+a_{12}(t)v^\tau(\eta-\frac{\tau}{4},x^0)-\mu_1u^\tau_{xx}(\eta-\frac{\tau}{4},x^0)-v^\tau(t-\frac{\tau}{4},x^0)u^\tau_x(t-\frac{\tau}{4},x^0)}{f(\eta,x^0)}f(\eta,x))d\eta,\end{aligned} \tag{51}$$

$$v^\tau(t,x)=v^\tau\Big(\frac{3\tau}{4},x\Big)+4\frac{1}{\varepsilon}\int_{\frac{3\tau}{4}}^{t}\Big(a_{21}(\eta)u^\tau\Big(\eta-\frac{\tau}{4},x\Big)+g(\eta,x)\Big)\,d\eta, \tag{52}$$

Let us check the first conditions from (41) and (42)

$$\begin{aligned}u^\tau(t,-x)+u^\tau(t,x)=u^\tau\Big(\frac{3\tau}{4},x\Big)+u^\tau\Big(\frac{3\tau}{4},-x\Big)+\\+4\int_{\frac{3\tau}{4}}^{t}(a_{12}(\eta)\Big(v^\tau\Big(\eta-\frac{\tau}{4},x\Big)+v^\tau\Big(\eta-\frac{\tau}{4},-x\Big)\Big)+\\+\frac{\psi(\eta)+a_{12}(t)v^\tau(\eta-\frac{\tau}{4},x^0)-\mu_1u^\tau_{xx}(\eta-\frac{\tau}{4},x^0)-v^\tau(t-\frac{\tau}{4},x^0)u^\tau_x(t-\frac{\tau}{4},x^0)}{f(\eta,x^0)}\times\\\times(f(\eta,x)+f(\eta,-x)))d\eta=0,\end{aligned} \tag{53}$$

$$\begin{aligned}v^\tau(t,-x)+v^\tau(t,x)=v^\tau\Big(\frac{3\tau}{4},x\Big)+v^\tau\Big(\frac{3\tau}{4},-x\Big)+\\+4\frac{1}{\varepsilon}\int_{\frac{3\tau}{4}}^{t}\Big(a_{21}(\eta)\Big(u^\tau\Big(\eta-\frac{\tau}{4},x\Big)+u^\tau\Big(\eta-\frac{\tau}{4},-x\Big)\Big)+\big(g(\eta,x)+g(\eta,-x)\big)\Big)\,d\eta=0.\end{aligned} \tag{54}$$

The second conditions from (41) and (42) are obviously also satisfied.

We have proved that the conditions (41) and (42) are satisfied at the zero integer step. Arguing in the same way at the next steps, we obtain that the conditions (41) and (42) are satisfied for all $t \in [0, t^*]$. Substituting $x = 0$ in (41) and (42), we get

$$u^\tau(t,0) = u^\tau(t,l) = 0, \quad t \in [0, t^*] \tag{55}$$

$$v^\tau(t,0) = v^\tau(t,l) = 0, \quad t \in [0, t^*]. \tag{56}$$

Theorem 3.1. *Let conditions* (28)-(31) *and the conditions of Theorem 1 hold. The components* u, v *of the solution* (u, v, r) *to problem* (1)–(3) *are periodic functions in the variable* x *with period* $2l$ *and satisfy*

$$\frac{\partial^{2m} u(t,0)}{\partial x^{2m}} = \frac{\partial^{2m+1} u(t,l)}{\partial x^{2m}} = \frac{\partial^{2m+1} v(t,0)}{\partial x^{2m}} = \frac{\partial^{2m+1} v(t,l)}{\partial x^{2m}} = 0, \quad m = 0, 1, \ldots, \frac{p+4}{2}. \tag{57}$$

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О разрешимости задачи идентификации функции источника в квазилинейной параболической системе уравнений в ограниченных и неограниченных областях

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Аннотация. В работе рассматривается задача идентификации функции источника в одном из двух уравнений квазилинейной системы двух параболических уравнений. Рассматривается случай данных Коши в неограниченной области, а также случай краевых условий первого рода в прямоугольной области. Изучен вопрос существования и единственности решения. Для доказательства используется метод расщепления на дифференциальном уровне, известный как метод слабой аппроксимации. Решение получено на малом временном интервале в классе достаточно гладких ограниченных функций.

Ключевые слова: обратная задача, система квазилинейных уравнений, определение функции источника, метод слабой аппроксимации, малый параметр.

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Distribution of Zones of Elastic and Plastic Deformation Appearing in a Layer under Compression by Two Rigid Parallel Plates

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Abstract. A problem of distribution of zones of elastic and plastic deformation appearing in a layer of elasto-plastic material under compression by two rigid parallel plates, for the case of plane strain state with Tresca – Saint-Venant yield criterion is solved. The technique based on application of conservation laws is used to solve the problem.

Keywords: layer compression, elasto-plastic problem, conservation laws, plane strain state.

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It this work, a problem of compression of a layer of elasto-plastic material between two rigid parallel plates is considered, and a distribution of zones of elastic and plastic deformation appearing in the layer for the case of a plane strain state is obtained. An elasto-plastic problem is solved to find zones of deformation. As it is known, the complexity of such problems is in finding of an elasto-plastic boundary separating regions of elastic and plastic deformation. An overview of elasto-plastic problems and the methods of their solving are presented in [1]. One should note that the methods of functions of the complex variable theory are the main tool of solving of these problems.

In the present work, a technique based on construction of conservation laws for differential equations is used to solve the considered problem [2–4]. The proposed technique has been applied successfully to find solutions of elasto-plastic problems of the torsion of rods and the bending of uniform cross-section beams [5–6], and to solve the problem of construction of elasto-plastic boundary in a deformed rectangular plate weakened by holes [7].

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1. Problem statement

We study a layer of elasto-plastic material of the length l and the width h compressed between two rigid parallel plates (Fig. 1).

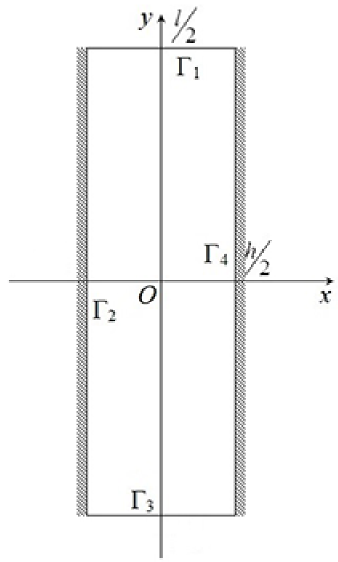

Fig. 1. Layer of Material Compressed Between Two Rigid Plates

The contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ bounding the layer is given as follows:

$$\begin{aligned} &\Gamma_1 : y = \frac{l}{2}, \quad \frac{h}{2} < x < -\frac{h}{2}; \qquad \Gamma_2 : x = -h/2, \quad \frac{l}{2} < y < -\frac{l}{2}; \\ &\Gamma_3 : y = -\frac{l}{2}, \quad -\frac{h}{2} < x < \frac{h}{2}; \qquad \Gamma_4 : x = h/2, \quad -\frac{l}{2} < y < \frac{l}{2}. \end{aligned} \tag{1}$$

It worth noting that for further numerical calculations the following values of the layer size are chosen: $l = 0.1$ m and $h = 0.02$ m.

The layer is compressed along the axis Ox. The boundaries Γ_1 and Γ_3 are free from external loading. It is supposed that the contour Γ is in plastic state.

One should solve an elasto–plastic problem for the domain bounded by Γ to determine zones of elastic and plastic deformation. In the case of plane strain state, components of the stress tensor σ_x, σ_y, τ satisfy the equilibrium equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0, \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial x} = 0. \tag{2}$$

The compatibility equation holds in elastic domain:

$$\Delta(\sigma_x + \sigma_y) = 0. \tag{3}$$

The yield criterion of Tresca – Saint-Venant for the case of plane strain state on the contour Γ has the form:

$$(\sigma_x - \sigma_y)^2 + 4\tau^2 = 4k^2, \tag{4}$$

where k is a constant of material plasticity.

The boundary conditions for the layer are written as:

$$\sigma_x n_1 + \tau n_2 = X, \quad \sigma_y n_2 + \tau n_1 = Y. \tag{5}$$

Here X, Y are the components of external force and n_1, n_2 are the components of normal vector to the contour Γ.

Taking into consideration the last equations and the yield criterion (4), one can obtain the following values of normal and tangential stresses on Γ: on the boundaries Γ_1, Γ_3

$$\sigma_x = \pm 2k, \quad \sigma_y = 0, \quad \tau = 0 \tag{6}$$

and on the boundaries Γ_2, Γ_4

$$\sigma_x = -2k, \quad \sigma_y = 0, \quad \tau = 0. \tag{7}$$

2. Solving the problem using conservation laws

A technique based on application of conservation laws is used to solve this elasto-plastic problem. One can find the description of the technique in details in [7].

Solution of the problem consists of three main steps.

First, the Laplace equation $\Delta F = 0$ with boundary conditions $F|_\Gamma = \sigma_x + \sigma_y$ is solved (where σ_x, σ_y are functions from (4)–(5), $F = \sigma_x + \sigma_y$ is a harmonic function from (3)).

Let $\sigma_x = 2k$ on Γ_1, Γ_3 (see (6)), then one gets $F|_{\Gamma_1,\Gamma_3} = 2k$. Also, from (7), $F|_{\Gamma_2,\Gamma_4} = -2k$.

Further on, the finite element method is employed to find the values of the function F in every point (x_0, y_0) of region bounded by the contour Γ.

Second, values of the functions σ_x, τ are found in each point (x_0, y_0) of the region applying the formulae obtained using the conservation laws:

$$\sigma_x(x_0, y_0) = \frac{1}{2\pi}\int_\Gamma (\omega_1^1\sigma_x + \omega_1^2\tau + f_1)dy - (-\omega_1^2\sigma_x + \omega_1^1\tau + g_1)dx, \tag{8}$$

where $f_1 = 0,\ g_1 = \displaystyle\int \omega_1^2 d_y F$ and $\omega_1^1 = \dfrac{x - x_0}{(x-x_0)^2 + (y-y_0)^2}$, $\omega_1^2 = -\dfrac{y - y_0}{(x-x_0)^2 + (y-y_0)^2}$;

$$\tau(x_0, y_0) = \frac{1}{2\pi}\int_\Gamma (\omega_2^1\sigma_x + \omega_2^2\tau + f_2)dy - (-\omega_2^2\sigma_x + \omega_2^1\tau + g_2)dx. \tag{9}$$

Here $f_2 = 0,\ g_2 = \displaystyle\int \omega_2^2 d_y F$; $\omega_2^1 = \dfrac{y - y_0}{(x-x_0)^2 + (y-y_0)^2}$, $\omega_2^2 = \dfrac{x - x_0}{(x-x_0)^2 + (y-y_0)^2}$.

Note that a detailed derivation of formulae (8–9) is given in [7].

From the definition of F one can get values of the function σ_y in points (x_0, y_0) of the region bounded by Γ:

$$\sigma_y(x_0, y_0) = F - \sigma_x(x_0, y_0). \tag{10}$$

The third step of solving of the problem is in the yield criterion (4) verification in all inner points of the considered region. If the stresses in a point (x_0, y_0) satisfy the condition

$$(\sigma_x - \sigma_y)^2 + 4\tau^2 < 4k^2,$$

the point belongs to the elastic zone. If the inequality is not fulfilled, the point (x_0, y_0) gets into the zone of plastic deformation.

Fig. 2 shows distribution of points forming zones of elastic and plastic deformation in the compressed layer.

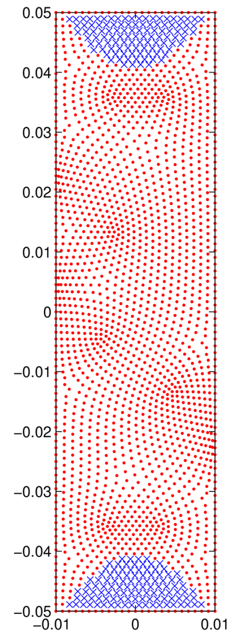

Fig. 2. Distribution of Zones of Elastic (×) and Plastic (·) Deformation in the Layer under Compression

Conclusion

A problem of compression of an elasto-plastic layer between two rigid parallel plates is considered for the case of plane strain state. Zones of plastic and elastic deformations are obtained using the technique based on the conservation laws application. Distribution of the zones of deformation is given using finite element method.

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Распределение областей упругих и пластических деформаций, возникающих при сжатии слоя двумя жесткими параллельными плитами

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Аннотация. В работе решена задача о распределении областей упругого и пластического деформирования, возникающих в слое упруго-пластического материала, сжимаемого двумя жесткими параллельными плитами, для случая плоского деформированного состояния с условием текучести Треска – Сен-Венана. При решении задачи была использована методика, основанная на применении законов сохранения.

Ключевые слова: сжатие слоя, упруго-пластическая задача, законы сохранения, плоское деформированное состояние.

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The Problem of Determining of the Source Function and of the Leading Coefficient in the Many-dimensional Semilinear Parabolic Equation

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Abstract. We consider the problem of determining the source function and the leading coefficient in a multidimensional semilinear parabolic equation with overdetermination conditions given on two different hypersurfaces. The existence and uniqueness theorem for the classical solution of the inverse problem in the class of smooth bounded functions is proved. A condition is found for the dependence of the upper bound of the time interval, in which there is a unique solution to the inverse problem, on the input data.

Keywords: inverse problem, overdetermination conditions, semilinear multidimensional parabolic equation, Cauchy problem, weak approximation method, input data, identification of coefficients.

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Introduction

The purpose of this work is to investigate the unique solvability of the problem of determining the source function and the coefficient at the second derivative in the spatial variable in a multidimensional semilinear parabolic equation with Cauchy data and overdetermination conditions, given on two different hypersurfaces. The unique solvability in classes of smooth bounded functions of various inverse problems of determining two coefficients of semilinear parabolic equations, different from the inverse problem considered in this article, was studied, for example in [1–3].

Using the overdetermination conditions, the initial inverse problem is reduced to the direct auxiliary Cauchy problem for the nonlinear loaded equation. The solvability of the direct problem is proved, for this purpose rather smooth input data and the method of weak approximation are used [4, 5]. The solution of the original inverse problem is written out explicitly through the solution of the direct problem. On this basis, the existence and uniqueness theorem for the classical solution of the inverse problem in the class of smooth bounded functions is proved for $t^* \in (0, T]$, $T > 0$, $T -$ const. The condition for the dependence of t^* on the constants of the sufficiently smooth input data is formulated.

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1. Statement of the problem

We consider in $G_{[0,T]} = \{(t,x,z) \big| \, 0 \leqslant t \leqslant T, \, x \in E_n, \, z \in E_1\}$ the Cauchy problem

$$\frac{\partial u}{\partial t} = L_x(u) + a(t,x)u_{zz} + \beta_1(t,x)u_z + \beta_2(t,x)u^2 + b(t,x)f(t,x,z), \tag{1}$$

$$u(0,x,z) = u_0(x,z), \quad (x,z) \in E_{n+1}. \tag{2}$$

Here $L_x(u) = \sum\limits_{i,j=1}^{n} \alpha_{ij} \dfrac{\partial^2 u}{\partial x_i \partial x_j} + \sum\limits_{i=1}^{n} \alpha_i \dfrac{\partial u}{\partial x_i}$, the functions $u_0(x,z)$, $f(t,x,z)$ are given in E_{n+1} and $G_{[0,T]}$ respectively, the coefficients $\alpha_{ij}(t)$, $\alpha_i(t)$, $i,j = \overline{1,n}$, $\beta_1(t,x)$, $\beta_2(t,x)$ are continuously differentiable real-valued functions of the variable t, and t, x respectively, $0 \leqslant t \leqslant T$, $T > 0$, $T -$ const, E_n is the n-dimensional Euclidean space, $n \geqslant 1$, $n \in \mathbf{N}$.

Let be $\alpha_{ij}(t) = \alpha_{ji}(t)$ and the relation $\sum\limits_{i,j=1}^{n} \alpha_{ij}\xi_i\xi_j > 0 \;\; \forall \xi \in E_n \backslash \{0\}$, $t \in [0,T]$ is true.

The coefficients $a(t,x)$, $b(t,x)$ and the solution $u(t,x,z)$ of (1), (2) are unknown.

We assume that overdetermination conditions are given on two different hypersurfaces $z = d_1(t)$ and $z = d_2(t)$:

$$u(t,x,d_1(t)) = \phi(t,x), \quad u(t,x,d_2(t)) = \psi(t,x), \tag{3}$$

where $(t,x) \in \Pi_{[0,T]}$, $\Pi_{[0,T]} = \{(t,x) | \, 0 \leqslant t \leqslant T, \, x \in E_n\}$; $d_1(t)$, $d_2(t)$ are continuously differentiable functions of the variable t, $d_1(t) \neq d_2(t)$; $\phi(t,x)$, $\psi(t,x)$ are given functions satisfying the matching conditions

$$\phi(0,x) = u_0(x, d_1(0)), \quad \psi(0,x) = u_0(x, d_2(0)), \tag{4}$$

where $x \in E_n$.

The solution of the inverse problem (1)–(3) in $G_{[0,t^*]}$, $0 < t^* \leqslant T$, is a triple of functions $u(t,x,z)$, $a(t,x)$, $b(t,x)$, that satisfies relations (1)–(3). Below we consider classical (sufficiently smooth) solutions.

2. The transition from an inverse problem to a direct problem

We reduce the problem (1)–(3) to some auxiliary direct problem. Let be $z = d_1(t)$, $z = d_2(t)$ in (1) and in view of (3), we obtain

$$P = a(t,x)u_{zz}|_{z=d_1(t)} + b(t,x)f(t,x,d_1(t)),$$
$$Q = a(t,x)u_{zz}|_{z=d_2(t)} + b(t,x)f(t,x,d_2(t)),$$

where

$$P = P(t,x) = F_1 - (\beta_1(t,x) + d_1'(t))u_z|_{z=d_1(t)}, \quad Q = Q(t,x) = F_2 - (\beta_1(t,x) + d_2'(t))u_z|_{z=d_2(t)},$$

$$F_1 = \phi_t(t,x) - L_x(\phi(t,x)) - \beta_2(t,x)\phi^2(t,x), \quad F_2 = \psi_t(t,x) - L_x(\psi(t,x)) - \beta_2(t,x)\psi^2(t,x).$$

Using the Cramer's method, we find:

$$a(t,x) = \frac{Pf(t,x,d_2(t)) - Qf(t,x,d_1(t))}{u_{zz}|_{z=d_1(t)} f(t,x,d_2(t)) - f(t,x,d_1(t)) u_{zz}|_{z=d_2(t)}},$$
$$b(t,x) = \frac{Qu_{zz}|_{z=d_1(t)} - Pu_{zz}|_{z=d_2(t)}}{u_{zz}|_{z=d_1(t)} f(t,x,d_2(t)) - f(t,x,d_1(t)) u_{zz}|_{z=d_2(t)}}. \quad (5)$$

We denote:

$$N_1 = N_1(t,x) = Pf(t,x,d_2(t)) - Qf(t,x,d_1(t)),$$
$$N_2 = N_2(t,x) = u_{zz}|_{z=d_1(t)} f(t,x,d_2(t)) - f(t,x,d_1(t)) u_{zz}|_{z=d_2(t)}, \quad (6)$$
$$N_3 = N_3(t,x) = Qu_{zz}|_{z=d_1(t)} - Pu_{zz}|_{z=d_2(t)}.$$

Then, substituting (5) into (1), we turn to the following problem:

$$u_t = L_x(u) + \frac{N_1}{N_2} u_{zz}(t,x,z) + \beta_1(t,x) u_z(t,x,z) + \beta_2(t,x) u^2(t,x,z) + \frac{N_3}{N_2} f(t,x,z), \quad (7)$$

$$u(0,x,z) = u_0(x,z). \quad (8)$$

We introduce the cutoff function $S_\delta(y) \in C^4(E_1)$, with the following properties:

$$S_\delta(y) \geqslant \frac{\delta}{3} > 0, \ S_\delta(y) = \begin{cases} y, \ y \geqslant \dfrac{\delta}{2}, \\ \chi(y), \ \dfrac{\delta}{3} < y < \dfrac{\delta}{2}, \\ \dfrac{\delta}{3}, \ y \leqslant \dfrac{\delta}{3}, \end{cases} \quad (9)$$

where $y \in E_1$, $\delta = const$, $\chi(y) \in C^4(E_1)$.

We replace in (7) N_1 and N_2 by $S_{\delta_1}(N_1(t,x))$, $S_{\delta_2}(N_2(t,x))$ respectively, we obtain

$$u_t = L_x(u) + \frac{S_{\delta_1}(N_1(t,x))}{S_{\delta_2}(N_2(t,x))} u_{zz} + \beta_1(t,x) u_z + \beta_2(t,x) u^2 + \frac{N_3(t,x)}{S_{\delta_2}(N_2(t,x))} f(t,x,z). \quad (10)$$

We assume that the input data are sufficiently smooth and it has all the continuous derivatives contained in the following relation

$$\Big| D_x^\gamma \frac{\partial^k}{\partial z^k} \frac{\partial^g}{\partial t^g} f(t,x,z) \Big| + \Big| D_x^\gamma \frac{\partial^k}{\partial z^k} u_0(x,z) \Big| + \Big| D_x^\gamma \frac{\partial^g}{\partial t^g} \beta_1(t,x) \Big| + \Big| D_x^\gamma \frac{\partial^g}{\partial t^g} F_s(t,x) \Big| + \big| \frac{d^{s_1}}{dt^{s_1}} d_s(t) \big| \leqslant C,$$
$$k = \overline{0, 10 - 2|\gamma|}, \ |\gamma| \leqslant 4, \ g = 0,1, \ s = 1,2, \ s_1 = 1,2. \quad (11)$$

Here $(t,x,z) \in G_{[0,T]}$, $\gamma = (\gamma_1, \dots, \gamma_n)$ is multi-index, $|\gamma| = \sum_{i=0}^{n} \gamma_i$, $D_x^\gamma = \dfrac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}}$, C is a constant more than one. Generally speaking, constants C here and throughout are different.

Let us suppose that the following conditions are true

$$N_1(0,x) = P(0,x) f(0,x,d_2(0)) - Q(0,x) f(0,x,d_1(0)) \geqslant \delta_1, \quad (12)$$
$$N_2(0,x) = \frac{\partial^2 u_0(x, d_1(0))}{\partial z^2} f(0,x,d_2(0)) - \frac{\partial^2 u_0(x, d_2(0))}{\partial z^2} f(0,x,d_1(0)) \geqslant \delta_2,$$

where $(t,x) \in \Pi_{[0,T]}$, δ_1, $\delta_2 > 0$, δ_1, $\delta_2 = const$, and

$$P(0,x) = \phi_t(0,x) - L_x(\phi(0,x)) - (\beta_1(0,x) + d_1'(0)) u_{0z}|_{z=d_1(0)} - \beta_2(0,x) \phi^2(0,x),$$

$$Q(0,x) = \psi_t(0,x) - L_x(\psi(0,x)) - (\beta_1(0,x) + d_2'(0)) u_{0z}|_{z=d_2(0)} - \beta_2(0,x) \psi^2(0,x).$$

Let us prove the existence of a solution of the auxiliary direct problem (10), (8).

3. Solvability of the direct problem

We apply the method of weak approximation [4,5] to prove the existence of the solution of the problem (10), (8). We split the problem and linearize it by a $\frac{\tau}{3}$ time shift in the nonlinear terms

$$u_t^\tau = 3L_x(u^\tau), \quad n\tau < t \leqslant \Big(n+\frac{1}{3}\Big)\tau, \tag{13}$$

$$u_t^\tau = 3\bigg(\frac{S_{\delta_1}(N_1^\tau(t,x))}{S_{\delta_2}(N_2^\tau(t,x))}u_{zz}^\tau + \beta_1(t,x)u_z^\tau\bigg), \quad \Big(n+\frac{1}{3}\Big)\tau < t \leqslant \Big(n+\frac{2}{3}\Big)\tau, \tag{14}$$

$$u_t^\tau \;=\; 3(\beta_2(t,x)u^\tau u^\tau\Big(t-\frac{\tau}{3}\Big) + \frac{N_3^\tau(t,x)}{S_{\delta_2}(N_2^\tau(t,x))}f(t,x,z)), \quad \Big(n+\frac{2}{3}\Big)\tau \;<\; t \;\leqslant\; (n+1)\tau, \tag{15}$$

$$u^\tau(0,x,z) = u_0(x,z), \quad x\in E_n, z\in E_1. \tag{16}$$

Here $n=0,1,\ldots,N-1$, $\tau N=T$, $N>0$, $N\in\mathbf{Z}$, $u^\tau = u^\tau(t) = u^\tau(t,x,z)$,

$$N_1^\tau = N_1^\tau(t,x) = P^\tau f(t,x,d_2(t)) - Q^\tau f(t,x,d_1(t)),$$
$$N_2^\tau = N_2^\tau(t,x) = u_{zz}^\tau\Big(t-\frac{\tau}{3},x,d_1(t)\Big)f|_{z=d_2(t)} - f|_{z=d_1(t)}u_{zz}^\tau\Big(t-\frac{\tau}{3},x,d_2(t)\Big),$$
$$N_3^\tau = N_3^\tau(t,x) = Q^\tau u_{zz}^\tau\Big(t-\frac{\tau}{3},x,d_1(t)\Big) - P^\tau u_{zz}^\tau\Big(t-\frac{\tau}{3},x,d_2(t)\Big),$$
$$P^\tau = F_1 - (\beta_1(t,x)+d_1'(t))u_z^\tau\Big(t-\frac{\tau}{3},x,d_1(t)\Big), \quad Q^\tau = F_2 - (\beta_1(t,x)+d_2'(t))u_z^\tau\Big(t-\frac{\tau}{3},x,d_2(t)\Big).$$

We introduce the notation

$$U^{\tau,t_0}(t) = \sum_{k=0}^{10} U_k^{\tau,t_0}(t), \tag{17}$$

$$U_k^{\tau,t_0}(t) = \sup_{t_0<\xi\leqslant t}\ \sup_{x\in E_n,z\in E_1}\left|\frac{\partial^k}{\partial z^k}u^\tau(\xi,x,z)\right|, \qquad U_k(0) = \sup_{x\in E_n,z\in E_1}\left|\frac{\partial^k}{\partial z^k}u_0(x,z)\right|, \tag{18}$$

$$U_k^{\tau,t_0}(t_0) = \sup_{x\in E_n,z\in E_1}\left|\frac{\partial^k}{\partial z^k}u^\tau(t_0,x,z)\right|,\ t\in\Big(t_0,\Big(n+\frac{p}{3}\Big)\tau\Big], \quad t_0\in\Big[0,\Big(n+\frac{p}{3}\Big)\tau\Big),\ t>t_0,\ p=1,2,3. \tag{19}$$

The functions $U_k^{\tau,t_0}(t)$, $U_k^{\tau,t_0}(t_0)$, $U_k(0)$ are nonnegative and non-decreasing on each half-open interval $(n\tau,(n+1)\tau]$.

Let us prove the priori estimates guaranteeing the compactness of a set of solutions $\{u^\tau(t,x,z)\}$ of the problem (13)–(16).

Let the half-interval $(n\tau,(n+1)\tau]$ be n-th time step, where $n=0,1,\ldots,N-1$.

We consider the zero integer step ($n=0$).

At the first fractional step ($p=1$), we obtain the following estimate for the solution u^τ of problem (13), (16), due to (11) and the maximum principle [6]

$$|u^\tau(\xi,x,z)| \leqslant \sup_{x\in E_n,z\in E_1}|u_0(x,z)|, \quad 0<\xi\leqslant\frac{\tau}{3}. \tag{20}$$

We obtain the following estimates using differentiating the equation (13), (16) with respect to z from one to ten times, respectively, due to (11) and the maximum principle [6]

$$\left|\frac{\partial^k}{\partial z^k}u^\tau(\xi,x,z)\right| \leqslant \sup_{x\in E_n, z\in E_1}\left|\frac{\partial^k}{\partial z^k}u_0(x,z)\right|, \quad k=\overline{1,10}, \quad 0<\xi\leqslant\frac{\tau}{3}. \tag{21}$$

We obtain the following estimate from (20), (21) through (17), (18)

$$U^{\tau,0}(t)\leqslant U(0), \qquad 0<t\leqslant\frac{\tau}{3}. \tag{22}$$

At the second fractional step ($p=2$), we obtain the following estimate for the solution of equation (14) with initial data $u^\tau\left(\frac{\tau}{3},x,z\right)$ due to (11), (9) and the maximum principle [6]

$$U^{\tau,\frac{\tau}{3}}(t)\leqslant U^{\tau,\frac{\tau}{3}}\left(\frac{\tau}{3}\right), \qquad \frac{\tau}{3}<t\leqslant\frac{2\tau}{3}. \tag{23}$$

Collectively, on the first and second fractional steps, due to (22), (23) we get

$$U^{\tau,0}(t)\leqslant U(0), \quad 0<t\leqslant\frac{2\tau}{3}. \tag{24}$$

At the third fractional step ($p=3$), integrating the equation (15) with $t\in\left(\frac{2\tau}{3},\xi\right]$, $\frac{2\tau}{3}<\xi\leqslant\tau$, we receive the equality

$$u^\tau(\xi)=u^\tau\left(\frac{2\tau}{3}\right)+3\int_{\frac{2\tau}{3}}^{\xi}(\beta_2(\eta,x)u^\tau(\eta)u^\tau\left(\eta-\frac{\tau}{3},x,z\right)+\frac{N_3^\tau(\eta,x)}{S_{\delta_2}(N_2^\tau(\eta,x))}f(\eta,x,z))d\eta.$$

The last relation implies the inequality

$$|u^\tau(\xi)|\leqslant\left|u^\tau\left(\frac{2\tau}{3}\right)\right|+3\int_{\frac{2\tau}{3}}^{\xi}(|\beta_2(\eta,x)|u^\tau(\eta)|\left|u^\tau\left(\eta-\frac{\tau}{3}\right)\right|+\frac{|N_3^\tau(\eta,x)|}{|S_{\delta_2}(N_2^\tau(\eta,x))|}|f(\eta,x,z)|)d\eta,$$

where $\dfrac{2\tau}{3}<\xi\leqslant t\leqslant\tau$.

Since this inequality holds for all x,z we replace the functions of the integral terms by their exact upper bounds with respect to $x\in E_n, z\in E_1$, and then replace the function $|u^\tau|$, on the left-hand side of the inequality by $\sup\limits_{x\in E_n z\in E_1}|u^\tau|$ considering (17)–(19) we obtain

$$\begin{gathered}U_0^{\tau,\frac{2\tau}{3}}(t)\leqslant U_0^{\tau,\frac{2\tau}{3}}\left(\frac{2\tau}{3}\right)+C\int_{\frac{2\tau}{3}}^{t}\left(U_0^{\tau,\frac{2\tau}{3}}(\eta)U_0^{\tau,\frac{2\tau}{3}}\left(\eta-\frac{\tau}{3}\right)+U_2^{\tau,\frac{2\tau}{3}}\left(\eta-\frac{\tau}{3}\right)+\right.\\ \left.+U_2^{\tau,\frac{2\tau}{3}}\left(\eta-\frac{\tau}{3}\right)U_1^{\tau,\frac{2\tau}{3}}\left(\eta-\frac{\tau}{3}\right)\right)d\eta.\end{gathered} \tag{25}$$

Further, in the same way, differentiating equations (15) with respect to z from one to 10 times, similarly to the second fractional step, we get

$$\begin{gathered}U_k^{\tau,\frac{2\tau}{3}}(t)\leqslant U_k^{\tau,\frac{2\tau}{3}}\left(\frac{2\tau}{3}\right)+C\int_{\frac{2\tau}{3}}^{t}\sum_{q=0}^{k}\left(U_{k-q}^{\tau,\frac{2\tau}{3}}(\eta)U_q^{\tau,\frac{2\tau}{3}}\left(\eta-\frac{\tau}{3}\right)+U_2^{\tau,\frac{2\tau}{3}}\left(\eta-\frac{\tau}{3}\right)+\right.\\ \left.+U_2^{\tau,\frac{2\tau}{3}}\left(\eta-\frac{\tau}{3}\right)U_1^{\tau,\frac{2\tau}{3}}\left(\eta-\frac{\tau}{3}\right)\right)d\eta, \quad k=\overline{1,10}.\end{gathered} \tag{26}$$

Adding (25) and (26), by virtue of (17) we receive

$$U^{\tau,\frac{2\tau}{3}}(t) \leqslant U^{\tau,\frac{2\tau}{3}}\left(\frac{2\tau}{3}\right) + C\int_{\frac{2\tau}{3}}^{t} \left(U^{\tau,\frac{2\tau}{3}}(\eta)U^{\tau,\frac{2\tau}{3}}\left(\eta-\frac{\tau}{3}\right) + U^{\tau,\frac{2\tau}{3}}\left(\eta-\frac{\tau}{3}\right) + \right.$$

$$\left. + U^{\tau,\frac{2\tau}{3}}\left(\eta-\frac{\tau}{3}\right)U^{\tau,\frac{2\tau}{3}}\left(\eta-\frac{\tau}{3}\right)\right)d\eta$$

or

$$U^{\tau,\frac{2\tau}{3}}(t) \leqslant U^{\tau,\frac{2\tau}{3}}\left(\frac{2\tau}{3}\right) + C\int_{\frac{2\tau}{3}}^{t} \left(U^{\tau,\frac{2\tau}{3}}(\eta)U^{\tau,\frac{2\tau}{3}}\left(\frac{2\tau}{3}\right) + U^{\tau,\frac{2\tau}{3}}\left(\frac{2\tau}{3}\right) + \right.$$

$$\left. + U^{\tau,\frac{2\tau}{3}}\left(\frac{2\tau}{3}\right)U^{\tau,\frac{2\tau}{3}}\left(\frac{2\tau}{3}\right)\right)d\eta,$$

where $C \geqslant 1$-constant, independent of τ.

To the last inequality we apply the Gronwall lemma [7], then

$$U^{\tau,\frac{2\tau}{3}}(t) \leqslant \left(U^{\tau,\frac{2\tau}{3}}\left(\frac{2\tau}{3}\right)+1\right)e^{2C\tau(U^{\tau,\frac{2\tau}{3}}(\frac{2\tau}{3})+1)} - 1, \quad \frac{2\tau}{3} < t \leqslant \tau.$$

Consequently, from (24) and last inequality at the zero whole step the following estimate holds

$$U^{\tau,0}(t) \leqslant (U(0)+1)e^{2C\tau(U(0)+1)} - 1, \quad 0 < t \leqslant \tau.$$

Repeating similar arguments at the first whole step, we obtain

$$U^{\tau,\tau}(t) \leqslant (U^{\tau,\tau}(\tau)+1)e^{2(U^{\tau,\tau}(\tau)+1)C\tau} - 1, \quad \tau < t \leqslant 2\tau.$$

Assuming that τ is sufficiently small and the inequality $e^{2(U(0)+1)C\tau} \leqslant 2$ holds, at the zero and first whole steps we get

$$U^{\tau,0}(t) \leqslant (U(0)+1)e^{6(U(0)+1)C\tau} - 1, \quad 0 < t \leqslant 2\tau.$$

Analogous reasoning, at the $n-$th whole step $(n < N)$ we obtain

$$U^{\tau,n\tau}(t) \leqslant (U^{\tau,n\tau}(n\tau)+1)e^{2C\tau(U^{\tau,n\tau}(n\tau)+1)} - 1, \quad n\tau < t \leqslant (n+1)\tau.$$

Consequently, at n whole steps, we getting

$$U^{\tau,0}(t) \leqslant (U(0)+1)e^{2(2n+1)(U(0)+1)C\tau} - 1, \quad 0 < t \leqslant (n+1)\tau.$$

Hence, following estimate is true

$$U^{\tau,0}(t) \leqslant (U(0)+1)e^{2(U(0)+1)Ct_*} - 1, \quad 0 < t \leqslant t_*,$$

where t_* satisfies the inequality

$$e^{2(U(0)+1)Ct_*} \leqslant 2. \tag{27}$$

Here $U(0) = \sum\limits_{k=0}^{10} \sup\limits_{x\in E_n, z\in E_1} \left|\frac{\partial^k}{\partial z^k}u_0(x,z)\right|$, C is constant depends of C, δ_1, δ_2 from (11), (12).

And, therefore, taking into account the notation (17), (18) uniformly with respect to τ

$$\left|\frac{\partial^k}{\partial z^k}u^\tau(t,x,z)\right| \leqslant C, \quad k = \overline{0,10}, \quad (t,x,z) \in G_{[0,t_*]}. \tag{28}$$

After differentiating problem (13)–(16) with respect to x_i, x_j, x_l and x_m, we obtain equations that can be regarded as linear with coefficients uniformly bounded in τ. Arguing by analogy and considering (28), we obtain estimates uniformly with respect to τ

$$\left| D_x^\gamma \frac{\partial^k}{\partial z^k} u^\tau(t,x,z) \right| \leqslant C, \quad k = \overline{0, 10-2|\gamma|}, \quad |\gamma| \leqslant 4, \quad (t,x,z) \in G_{[0,t_*]}. \tag{29}$$

We obtain from (29) and (13)–(16) uniformly with respect to τ

$$|u_t^\tau(t,x,z)| \leqslant C, \quad (t,x,z) \in G_{[0,t_*]}.$$

We differentiate equations (13)–(16) once with respect to z. By (29), the right-hand side of the equations obtained is uniformly bounded in τ, and consequently the left-hand side is also uniformly bounded in τ

$$|u_{tz}^\tau(t,x,z)| \leqslant C, \quad (t,x,z) \in G_{[0,t_*]}.$$

By analogy, uniformly with respect to τ

$$\left| \frac{\partial^k}{\partial z^k} D_x^\lambda u_t^\tau(t,x,z) \right| \leqslant C, \quad k = \overline{0,4}, \quad |\lambda| \leqslant 2, \quad (t,x,z) \in G_{[0,t_*]}.$$

Thus, the following estimate holds uniformly with respect to τ for $(t,x,z) \in G_{[0,t_*]}$

$$\begin{gathered} \left| \frac{\partial}{\partial t} \frac{\partial^k}{\partial z^k} D_x^\lambda u^\tau(t,x,z) \right| + \left| \frac{\partial}{\partial x_i} \frac{\partial^k}{\partial z^k} D_x^\lambda u^\tau(t,x,z) \right| + \left| \frac{\partial}{\partial z} \frac{\partial^k}{\partial z^k} D_x^\lambda u^\tau(t,x,z) \right| \leqslant C, \\ k = \overline{0,4}, \quad |\lambda| \leqslant 2. \end{gathered} \tag{30}$$

The estimate (29) implies the uniform boundedness in τ of the family $\left\{ D_x^\gamma \frac{\partial^k}{\partial z^k} u^\tau \right\}$ in $G_{[0,t_*]}$, and from (29), (30) their equicontinuity in t, x and z is equicontinuous in $G_{[0,t_*]}$. Therefore, for any fixed γ, k, $|\gamma| \leqslant 2$, $k = \overline{0,4}$, by the Arzela theorem [8] the set $\left\{ D_x^\gamma \frac{\partial^k}{\partial z^k} u^\tau \right\}$ is compact in $C(G^M_{[0,t_*]})$, $M > 0$ is an integer, $G^M_{[0,t_*]} = \{(t,x,z) | t \in [0,T], |x| \leqslant M, |z| \leqslant M\}$.

In a diagonal way, we choose a subsequence $\{u^\tau\}$ (we do not change the notation) converging together with the corresponding derivatives with respect to x and z to some function u in $G_{[0,t_*]}$, and uniformly in each $G^M_{[0,t_*]}$. The function u is continuous, has derivatives of the corresponding order in x and z that are continuous in $G_{[0,t_*]}$, and satisfies the initial data (2) and inequality

$$\left| D_x^\beta \frac{\partial^k}{\partial z^k} u(t,x,z) \right| \leqslant C, \;\; k = \overline{0,4}, \;\; |\beta| \leqslant 2, \;\; (t,x,z) \in G_{[0,t_*]}. \tag{31}$$

Since $D_x^\gamma \frac{\partial^k}{\partial z^k} u^\tau \underset{\tau \to 0}{\rightrightarrows} D_x^\gamma \frac{\partial^k}{\partial z^k} u$ on $G^M_{[0,t_*]}$ $\forall M > 0$, $|\gamma| \leqslant 2$, $k = \overline{0,4}$ and the inequality (31) is satisfied, then we can prove that the proof is similar to the proof of Theorem 2.4.1 (see Sec. 2.4. One theorem of the weak approximation method [4]) that the function u is a solution of the problem (10), (8) in $G^M_{[0,t_*]}$ for any fixed M, and since M is arbitrary, then also in $G_{[0,t_*]}$.

The function $u(t,x,z)$ belongs to the class

$$\begin{gathered} C^{1,2,4}_{t,x,z}(G_{[0,t_*]}) = \Big\{ f_1(t,x,z) | \frac{\partial^g}{\partial t^g} f_1 \in C(G_{[0,t_*]})), D_x^\beta \frac{\partial^k}{\partial z^k} f_1 \in C(G_{[0,t_*]}), \\ |\beta| \leqslant 2, k = \overline{0,4}, g = 0,1 \Big\}. \end{gathered} \tag{32}$$

The estimate (31) is true. In order to prove the existence of a solution of problem (7), (8), it is necessary to remove the cutoff functions in equation (10). For this, we prove that for $(t,x)\in\Pi_{[0,t^*]}$,

$$N_1(t,x)\geqslant\frac{\delta_1}{2},\quad N_2(t,x)\geqslant\frac{\delta_2}{2}.$$

We differentiate the expressions for $N_1(t,x)$, $N_2(t,x)$ ($N_1(t,x)$, $N_2(t,x)$ in (6)) with respect to t,

$$\begin{gathered}M_1(t,x)=(N_1(t,x))'_t=P'_tf(t,x,d_2(t))+P(f'_t(t,x,d_2(t))+f'_z(t,x,d_2(t))d'_2(t))-\\-Q'_tf(t,x,d_1(t))-Q(f'_t(t,x,d_1(t))+f'_z(t,x,d_1(t))d'_1(t)),\\M_2(t,x)=(N_2(t,x))'_t=(u_{zzt}(t,x,d_1(t))+u_{zzz}(t,x,d_1(t))d'_1(t))f(t,x,d_2(t))+\\+u_{zz}(t,x,d_1(t))(f'_t(t,x,d_2(t))+f'_z(t,x,d_2(t))d'_2(t))-(f'_t(t,x,d_1(t))+\\+f'_z(t,x,d_1(t))d'_1(t))u_{zz}(t,x,d_2(t))-f(t,x,d_1(t))(u_{zz}(t,x,d_2(t)))'_t,\end{gathered}\tag{33}$$

where

$$\begin{gathered}P'_t=\phi_{tt}-L_{xt}(\phi(t,x))-\beta_{1t}(t,x)u_z(t,x,d_1(t))-\beta_1(t,x)(u_{zt}(t,x,d_1(t))+\\+u_{zz}(t,x,d_1(t))d'_1(t))-\beta_{2t}(t,x)\phi^2(t,x)-2\beta_2(t,x)\phi(t,x)\phi_t(t,x)-(u_{zt}(t,x,d_1(t))+\\+u_{zz}(t,x,d_1(t))d'_1(t))d'_1(t)-u_z(t,x,d_1(t))d''_1(t),\\Q'_t=\psi_{tt}-L_{xt}(\psi(t,x))-\beta_{1t}(t,x)u_z(t,x,d_2(t))-\beta_1(t,x)(u_{zt}(t,x,d_2(t))+\\+u_{zz}(t,x,d_2(t))d'_2(t))-\beta_{2t}(t,x)\psi^2(t,x)-2\beta_2(t,x)\psi(t,x)\psi_t(t,x)-(u_{zt}(t,x,d_2(t))+\\+u_{zz}(t,x,d_2(t))d'_2(t))d'_2(t)-u_z(t,x,d_2(t))d''_2(t),\end{gathered}$$

$$L_{xt}(\phi(t,x))=\sum_{i,j=1}^n\Big((\alpha_{ij})'\frac{\partial^2\phi}{\partial x_i\partial x_j}+\alpha_{ij}\frac{\partial\phi}{\partial x_i\partial x_j\partial t}\Big)+\sum_{i=1}^n\Big((\alpha_i)'\frac{\partial\phi}{\partial x_i}+\alpha_i\frac{\partial^2\phi}{\partial x_i\partial t}\Big),$$
$$L_{xt}(\psi(t,x))=\sum_{i,j=1}^n\Big((\alpha_{ij})'\frac{\partial^2\psi}{\partial x_i\partial x_j}+\alpha_{ij}\frac{\partial\psi}{\partial x_i\partial x_j\partial t}\Big)+\sum_{i=1}^n\Big((\alpha_i)'\frac{\partial\psi}{\partial x_i}+\alpha_i\frac{\partial^2\psi}{\partial x_i\partial t}\Big).$$

By virtue of (11), (31)

$$|M_1(t,x)|\leqslant K_1,\quad |M_2(t,x)|\leqslant K_2,\tag{34}$$

here K_1, K_2 are constants depending on δ_1,δ_2,C.

We integrate expressions (33) with respect to t in the range from 0 to t, we obtain

$$N_1(t,x)=N_1(0,x)+\int_0^tM_1(\eta,x)d\eta,\quad N_2(t,x)=N_2(0,x)+\int_0^tM_2(\eta,x)d\eta.$$

By virtue of (12), (34) $N_1(t,x)\geqslant\delta_1-K_1t$, $N_2(t,x)\geqslant\delta_2-K_2t$

$$N_1(t,x)\geqslant\frac{\delta_1}{2},\quad N_2(t,x)\geqslant\frac{\delta_2}{2},\quad t\in[0,t^*].\tag{35}$$

By the definition of the cutoff function (9) and (35), we obtain $S_{\delta_1}(N_1(t,x))=N_1(t,x)$, $S_{\delta_2}(N_2(t,x))=N_2(t,x)$ with $t\in[0,t^*]$, when $t^*=\min\Big(t_*,\dfrac{\delta_1}{2K_1},\dfrac{\delta_2}{2K_2}\Big)$.

Thus, in equation (10), the cutoffs are removed. The function $u(t,x,z)$ satisfies equation (7). The coefficients $a(t,x)$ and $b(t,x)$ can be written in the form (5).

Thus, we have proved the existence of a solution $u(t,x,z)$ of the direct problem (7), (8) in the class $C^{1,2,4}_{t,x,z}(G_{[0,t^*]})$. It is proved

Theorem 1. *Let conditions* (9), (11), (12) *are satisfied. Then there exists a solution* $u(t,x,z)$ *of the problem* (7), (8) *in the class* $C^{1,2,4}_{t,x,z}(G_{[0,t^*]})$ *satisfying* (31). *The constant* $t^* = \min\Big(t_*, \dfrac{\delta_1}{2K_1}, \dfrac{\delta_2}{2K_2}\Big)$, *where* t_* *satisfies the inequality* (27), *the constants* K_1, K_2 *depend on* C, δ_1, δ_2, *from the relations* (11), (12).

4. The existence and uniqueness of a classical solution of the inverse problem

Let us prove that the triple of functions $u(t,x,z), a(t,x), b(t,x)$ are the solution of the inverse problem (1)–(3), where $a(t,x)$ and $b(t,x)$ are defined in (5). Since $u(t,x,z)$ is the solution of the direct problem (7), (8), substituting $u(t,x,z)$, $a(t,x)$, $b(t,x)$ in (1), we obtain the correct identity.

According to (11), (31) from (5), (7), we obtain that the triple of functions $u(t,x,z)$, $a(t,x)$, $b(t,x)$ belongs to the class

$$Z(t^*) = \big\{u(t,x,z), a(t,x), b(t,x) | u \in C^{1,2,4}_{t,x,z}(G_{[0,t^*]}),\\ a(t,x), b(t,x) \in C^{0,2}_{t,x}(\Pi_{[0,t^*]})\big\},$$

and satisfies the inequalities

$$\sum_{|\beta|\leqslant 2}\sum_{k=0}^{4}\Big|D^\beta_x \frac{\partial^k}{\partial z^k} u(t,x,z)\Big| \leqslant C, \quad (t,x,z)\in G_{[0,t^*]}, \tag{36}$$

$$\sum_{|\beta|\leqslant 2}\Big|D^\beta_x a(t,x)\Big| + \sum_{|\beta|\leqslant 2}\Big|D^\beta_x b(t,x)\Big| \leqslant C, \quad (t,x)\in \Pi_{[0,t^*]}. \tag{37}$$

The class $C^{1,2,4}_{t,x,z}(G_{[0,t^*]})$ is defined in (32), and

$$C^{0,2}_{t,x}(\Pi_{[0,t^*]}) = \{a_1(t,x) | D^\beta_x a_1(t,x) \in C(\Pi_{[0,t^*]}), |\beta|\leqslant 2\}.$$

Using conditions (4) and equation (1), we can prove that the overdetermination conditions (3) are satisfied.

The existence in the class $Z(t^*)$ of the solution $u(t,x,z)$, $a(t,x)$, $b(t,x)$ of the problem (1)–(3) satisfying relations (1)–(3) is proved.

The uniqueness of the solution to problem (1)–(3) is proved by a standard method: the difference between the two solution to problem (1)–(3) that obey (36), (37) is shown to vanish.

Thus, it is proved

Theorem 2. *Let us conditions* (4), (11), (12) *are satisfied. Then there exists a unique solution* *u(t,x,z), a(t,x), b(t,x) of problem* (1)–(3) *in the class* $Z(t^*)$ *satisfying relations* (36), (37). *The constant* $t^* = \min\Big(t_*, \dfrac{\delta_1}{2K_1}, \dfrac{\delta_2}{2K_2}\Big)$, *where* t_* *satisfies the inequality* (27), *the constants* K_1, K_2 *of the temple from* C, δ_1, δ_2, *from relations* (11), (12).

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Задача определения функции источника и старшего коэффициента в полулинейном многомерном параболическом уравнении

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Аннотация. Рассматривается задача определения функции источника и коэффициента при второй производной по пространственной переменной в многомерном полулинейном параболическом уравнении с условиями переопределения, заданными на двух различных гиперповерхностях. Доказана теорема существования и единственности классического решения обратной задачи в классе гладких ограниченных функций. Найдено условие зависимости верхней границы временного отрезка, в котором существует и единственно решение обратной задачи, от входных данных.

Ключевые слова: обратная задача, условия переопределения, полулинейное многомерное параболическое уравнение, задача Коши, метод слабой аппроксимации, входные данные, определение коэффициентов.

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Two-layer Stationary Flow in a Cylindrical Capillary Taking into Account Changes in the Internal Energy of the Interface

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Abstract. The problem of two-dimensional stationary flow of two immiscible incompressible binary mixtures in a cylindrical capillary in the absence of mass forces is investigated. The mixtures are contacted through a common the interface on which the total energy condition is taken into account. The temperature and concentration in the mixtures are distributed according to a quadratic law, which is in good agreement with the velocity field of the type Hiemenz. The resulting conjugate boundary value problem is nonlinear and inverse with respect to the pressure gradients along the axis of the cylindrical capillary. The tau-method (a modification of the Galerkin method) was applied to this problem, which showed the possibility of the existence of two solutions. It is shown that the obtained solutions with a decrease in the Marangoni number converge to the solutions of the problem of the creeping flow of binary mixtures. When solving the model problem for small Marangoni numbers, it is found that the effect of the increments of the internal energy of the interfacial surface significantly affects the dynamics of flows of mixtures in layers.

Keywords: binary mixture, interface, internal energy, inverse problem, pressure gradient, thermal Marangoni number.

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Introduction

The specifics of the phenomena occurring at the interface of liquids are related to the existence of the energy and entropy of the surface phase, which are excessive in relation to the bulk phases in the transition layer [1]. However, the energy exchange between the bulk and surface phases has not been sufficiently studied. For ordinary liquids at room temperature, the effect of changes in the internal energy of the interfacial surface on the formation of heat fluxes, temperature fields, and velocities in its vicinity is insignificant in relation to viscous friction and heat transfer . However, at sufficiently high temperatures, when the viscosity and thermal conductivity of ordinary liquids are significantly reduced, as well as for liquids with reduced viscosity (for example, for some cryogenic liquids), the effect of the internal energy increments of the interfacial surface is significant [3].

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In this paper, we consider a mathematical model describing the two-dimensional stationary thermodiffusion motion of two immiscible incompressible binary mixtures in a cylindrical capillary in the absence of mass forces. The mixtures are contacted through a common interface on which the total energy condition is taken into account. In this geometry, the mechanism of influence of changes in the surface internal energy on the dynamics of binary mixtures is investigated. Without taking into account the effects of thermal diffusion, such a model was studied in the works [4, 5].

1. Statement of the problem

We consider a two-dimensional stationary axisymmetric flow of two immiscible incompressible binary mixtures in a cylindrical tube of radius R_2, the temperature of which is maintained constant. Binary mixture occupy the field: $\Omega_1 = \{0 \leqslant r \leqslant R_1, |z| < \infty\}$ and $\Omega_2 = \{R_1 \leqslant r \leqslant R_2, |z| < \infty\}$, where r, z are the radial and axial cylindrical coordinates. Here $r = R_1 = \text{const}$ is the total interface of binary mixtures, $r = R_2 = \text{const}$ is the solid wall. The values related to the regions Ω_1 and Ω_2 are denoted by indexes 1 and 2, respectively. The area of Ω_1 is called the core, and the area Ω_2 is an interlayer or film. It is assumed that its characteristic transverse size is small by compared to the radius of the core, $R_2 - R_1 \ll R_1$. Such a geometry corresponds, for example, to the case of displacement of the liquid that originally filled the capillary by another liquid.

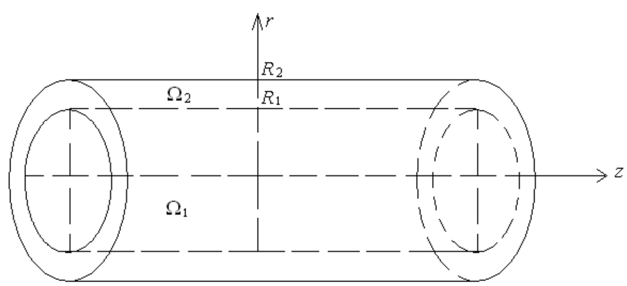

Fig. 1. The scheme of the flow region

Binary mixture is characterized by constant thermal conductivities k_j, specific heat capacities c_{pj}, dynamic viscosities μ_j, densities ρ_j; let $\chi_j = k_j/\rho_j c_{pj}$ is the thermal conductivity, $\nu_j = \mu_j/\rho_j$ is the kinematic viscosity (here and further, $j = 1, 2$). The influence of gravity is not taken into account, which may be justified, for example, if the tube it is quite narrow to the capillaries.

The system of equations of motion, continuity, internal energy balance and concentration transfer has the following form [6]:

$$\begin{gathered}
u_j u_{jr} + w_j u_{jz} + \frac{1}{\rho_j}\, p_{jr} = \nu_j \left(\Delta u_j - \frac{u_j}{r^2}\right), \\
u_j w_{jr} + w_j w_{jz} + \frac{1}{\rho_j}\, p_{jz} = \nu_j \Delta w_j, \\
u_{jr} + \frac{u_j}{r} + w_{jz} = 0, \\
u_j \theta_{jr} + w_j \theta_{jz} = \chi_j \Delta \theta_j, \\
u_j c_{jr} + w_j c_{jz} = d_j \Delta c_j + \alpha_j d_j \Delta \theta_j,
\end{gathered} \tag{1}$$

where u_j, w_j are projections of the velocity vector on the r, z axis of the cylindrical coordinate system; p_j is the pressure in the layers; θ_j, c_j are deviations of temperature and concentration from their equilibrium values; d_j, α_j are the diffusion and thermal diffusion coefficients, respectively; $\Delta = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + \partial^2/\partial z^2$ is the Laplace operator.

The linear dependence of the interfacial tension coefficient on temperature and concentration is assumed:

$$\sigma(\theta, c) = \sigma_0 - æ_1(\theta - \theta_0) - æ_2(c - c_0). \tag{2}$$

Here $æ_1 > 0$ is the temperature coefficient, $æ_2$ is the concentration coefficient of the surface tension (normally $æ_2 < 0$, because the surface tension increases with increasing concentration); θ_0, c_0 are the temperature and concentration on the interfacial surface in the as balance.

The solution to the problem is sought in a special form:

$$\begin{gathered} u_j = u_j(r), \quad w_j = z v_j(r), \quad p_j = p_j(r, z), \\ \theta_j = a_j(r) z^2 + b_j(r), \quad c_j = h_j(r) z^2 + g_j(r). \end{gathered} \tag{3}$$

A solution of the form (3) is called a solution of the type Hiemenz [7], in which the velocity field is linear with respect to the transverse coordinate. Thus, the temperature θ_j takes an extreme value at the point $z = 0$: the maximum at $a_j(r) < 0$ and the minimum at $a_j(r) > 0$. We get a similar interpretation for the concentration c_j, only instead of $a_j(r)$ the function $h_j(r)$ is considered.

After substituting the special form (3) into the equations of motion (1) we will have the following system with unknown functions $u_j(r)$, $v_j(r)$, $p_j(r)$, $a_j(r)$, $b_j(r)$, $h_j(r)$, $g_j(r)$:

$$u_j u_{jr} + \frac{1}{\rho_j} p_{jr} = \nu_j \Big(u_{jrr} + \frac{1}{r} u_{jr} - \frac{u_j}{r^2} \Big), \tag{4}$$

$$z(u_j v_{jr} + v_j^2) + \frac{1}{\rho_j} p_{jz} = \nu_j z \Big(v_{jrr} + \frac{1}{r} v_{jr} \Big), \tag{5}$$

$$u_{jr} + \frac{u_j}{r} + v_j = 0, \tag{6}$$

$$u_j a_{jr} + 2 v_j a_j = \chi_j \Big(a_{jrr} + \frac{1}{r} a_{jr} \Big), \tag{7}$$

$$u_j b_{jr} = \chi_j \Big(b_{jrr} + \frac{1}{r} b_{jr} + 2a_j \Big), \tag{8}$$

$$u_j h_{jr} + 2 v_j h_j = d_j \Big(h_{jrr} + \frac{1}{r} h_{jr} \Big) + \alpha_j d_j \Big(a_{jrr} + \frac{1}{r} a_{jr} \Big), \tag{9}$$

$$u_j g_{jr} = d_j \Big(g_{jrr} + \frac{1}{r} g_{jr} + 2h_j \Big) + \alpha_j \, d_j \Big(b_{jrr} + \frac{1}{r} b_{jr} + 2a_j \Big). \tag{10}$$

From the equations (4), (5), we express the pressure gradients (p_{jr}, p_{jz}):

$$p_{jr} = \rho_j \nu_j \Big(u_{jrr} + \frac{1}{r} u_{jr} - \frac{u_j}{r^2} \Big) - \rho_j \, u_j \, u_{jr}, \tag{11}$$

$$p_{jz} = z \Big[\rho_j \nu_j \Big(v_{jrr} + \frac{1}{r} v_{jr} \Big) - \rho_j (u_j \, v_{jr} + v_j^2) \Big], \tag{12}$$

Conditions for the compatibility of the equations (11), (12) are satisfied identically: $p_{jrz} = p_{jzr} = 0$. It follows that the pressure in the layers will be restored by the formula:

$$p_j = -\rho_j f_j \frac{z^2}{2} + s_j(r), \tag{13}$$

where the derivative of the variable r from the functions $s_j(r)$ is exactly the right-hand side of the equation (11). Integrating this equation, we obtain for the functions $s_j(r)$ the following view:

$$s_j(r) = \rho_j \nu_j \Big(u_{jr} + \frac{1}{r} u_j \Big) - \frac{1}{2} \rho_j \, u_j^2 + s_{j0}, \quad s_{j0} \equiv \text{const}. \tag{14}$$

In turn, the functions $v_j(r)$ are defined from the equation:

$$u_j v_{jr} + v_j^2 = \nu_j\Big(v_{jrr} + \frac{1}{r}\, v_{jr}\Big) + f_j, \tag{15}$$

where $f_j \equiv \text{const}$. The flow in the layers is induced by the longitudinal pressure gradients f_j. These are unknown constants that are subject to by definition. Therefore, the problem is reversed.

On a solid wall $r = R_2$, the boundary conditions are satisfied:

$$\begin{gathered} u_2(R_2) = 0, \quad v_2(R_2) = 0, \quad a_2(R_2) = a_{20}, \quad b_2(R_2) = b_{20}, \\ h_{2r}(R_2) + \alpha_2 a_{2r}(R_2) = 0, \quad g_{2r}(R_2) + \alpha_2 b_{2r}(R_2) = 0, \end{gathered} \tag{16}$$

with the given constants a_{20}, b_{20}. Note that when $a_{20} < 0$ the wall temperature has a maximum value at the point $z = 0$, and for $a_{20} > 0$ — minimal.

On the interface $r = R_1$, given the dependence (2), we will have the following conditions:

$$cu_1(R_1) = u_2(R_1), \quad v_1(R_1) = v_2(R_1), \tag{17}$$

$$\begin{gathered} a_1(R_1) = a_2(R_1), \quad b_1(R_1) = b_2(R_1), \\ h_1(R_1) = h_2(R_1), \quad g_1(R_1) = g_2(R_1), \end{gathered} \tag{18}$$

$$\mu_2 v_{2r}(R_1) - \mu_1 v_{1r}(R_1) = -2æ_1 a_1(R_1) - 2æ_2 h_1(R_1), \tag{19}$$

$$d_1[h_{1r}(R_1) + \alpha_1 a_{1r}(R_1)] = d_2[h_{2r}(R_1) + \alpha_2 a_{2r}(R_1)], \tag{20}$$

$$\begin{gathered} k_2 a_{2r}(R_1) - k_1 a_{1r}(R_1) = æ_1 a_1(R_1) v_1(R_1), \\ k_2 b_{2r}(R_1) - k_1 b_{1r}(R_1) = æ_1 b_1(R_1) v_1(R_1). \end{gathered} \tag{21}$$

The relation (21) is called the *energy condition on the interface* of two binary mixtures [8–10]. It means that the jump in the heat flow in the direction of the normal to the surface section $r = R_1$ is compensated by a change in the internal energy of this surface. In turn, this change is associated with both a change in temperature (and with it the specific internal energy) and a change in the area of the interface.

For a complete statement of the problem to the relations (17)–(21), it is necessary to add the boundedness of the functions on the axis of the cylindrical capillary at $r = 0$:

$$\begin{gathered} |u_1(0)| < \infty, \quad |v_1(0)| < \infty, \quad |s_1(0)| < \infty, \quad |a_1(0)| < \infty, \\ |b_1(0)| < \infty, \quad |h_1(0)| < \infty, \quad |g_1(0)| < \infty. \end{gathered} \tag{22}$$

2. Transformation to a problem in dimensionless variables

For what follows, it is essential that the equations (6), (7), (9), (15) are independent of the others and form a closed subsystem for defining the functions $v_j(r)$, $a_j(r)$, $h_j(r)$ and the constants f_j $(j = 1, 2)$. After solving it, the functions $b_j(r)$, $g_j(r)$ are found from the equations (8), (10), and $s_j(r)$ is uniquely restored by the formula (14). If we integrate the continuity equation (6) and exclude functions $u_j(r)$ in the equations (7), (9), (15) with given the conditions of boundedness (22) and sticking on a solid wall (16), the problem is reduced to the conjugate boundary value problem of finding only the functions $v_j(r)$, $a_j(r)$, $h_j(r)$ and the constants f_j. We introduce dimensionless variables and functions by equalities:

$$\begin{gathered} \xi = \frac{r}{R_1}\,, \quad R = \frac{R_2}{R_1} > 1\,, \quad V_j = \frac{R_1^2 v_j}{\mathrm{Ma}\,\nu_1}\,, \\ A_j = \frac{a_j}{a_{20}}\,, \quad H_j = \frac{h_j}{c_0}\,, \quad F_j = \frac{R_1^4 f_j}{\mathrm{Ma}\,\nu_1^2}\,, \end{gathered} \tag{23}$$

where a_{20}, c_0 are the characteristic temperature and concentration.

As the defining parameters of the problem under consideration, we choose the following:

$$\begin{gathered}\mathrm{Ma} = \frac{æ_1 a_{20} R_1^3}{\mu_2 \nu_1}, \quad \mathrm{Mc} = \frac{æ_2 c_0\, R_1^3}{\mu_2 \nu_1}, \quad \mathrm{Pr}_j = \frac{\nu_j}{\chi_j}, \quad \mathrm{Sc}_j = \frac{\nu_j}{d_j}, \quad \mathrm{Sr}_j = \frac{\alpha_j a_{20}}{c_0}, \\ \mu = \frac{\mu_1}{\mu_2}, \quad \nu = \frac{\nu_1}{\nu_2}, \quad k = \frac{k_1}{k_2}, \quad d = \frac{d_1}{d_2}, \quad \mathrm{M} = \frac{\mathrm{Mc}}{\mathrm{Ma}} = \frac{æ_2 c_0}{æ_1 a_{20}}.\end{gathered} \tag{24}$$

Here Ma is the thermal Marangoni number, Mc is the concentration Marangoni number, Pr_j are the Prandtl numbers, Sc_j are the Schmidt numbers, Sr_j are the Soret numbers.

After de-dimensionalization, we obtain a nonlinear inverse boundary value problem in the domain with respect to the spatial variable ξ, which, for $j = 1$ varies between 0 and 1, and when $j = 2$ — in the range from 1 to R.

For $0 < \xi < 1$ we will have:

$$K_1(V_1, F_1) \equiv V_{1\xi\xi} + \frac{1}{\xi} V_{1\xi} + \frac{\mathrm{Ma}}{\xi} V_{1\xi} \int_0^\xi x V_1(x)\, dx - \mathrm{Ma} V_1^2 + F_1 = 0, \tag{25}$$

$$S_1(V_1, A_1) \equiv \frac{1}{\mathrm{Pr}_1} \left(A_{1\xi\xi} + \frac{1}{\xi} A_{1\xi} \right) + \frac{\mathrm{Ma}}{\xi} A_{1\xi} \int_0^\xi x V_1(x)\, dx - 2\mathrm{Ma} A_1 V_1 = 0; \tag{26}$$

$$\begin{gathered} T_1(V_1, A_1, H_1) \equiv \frac{1}{\mathrm{Sc}_1} \left(H_{1\xi\xi} + \frac{1}{\xi} H_{1\xi} \right) + \frac{\mathrm{Sr}_1}{\mathrm{Sc}_1} \left(A_{1\xi\xi} + \frac{1}{\xi} A_{1\xi} \right) + \\ + \frac{\mathrm{Ma}}{\xi} H_{1\xi} \int_0^\xi x V_1(x)\, dx - 2\mathrm{Ma} H_1 V_1 = 0. \end{gathered} \tag{27}$$

For $1 < \xi < R$, we have:

$$K_2(V_2, F_2) \equiv \frac{1}{\nu} \left(V_{2\xi\xi} + \frac{1}{\xi} V_{2\xi} \right) - \frac{\mathrm{Ma}}{\xi} V_{2\xi} \int_\xi^R x V_2(x)\, dx - \mathrm{Ma} V_2^2 + F_2 = 0, \tag{28}$$

$$S_2(V_2, A_2) \equiv \frac{1}{\mathrm{Pr}_2 \nu} \left(A_{2\xi\xi} + \frac{1}{\xi} A_{2\xi} \right) - \frac{\mathrm{Ma}}{\xi} A_{2\xi} \int_\xi^R x V_2(x)\, dx - 2\mathrm{Ma} A_2 V_2 = 0; \tag{29}$$

$$\begin{gathered} T_2(V_2, A_2, H_2) \equiv \frac{1}{\mathrm{Sc}_2 \nu} \left(H_{2\xi\xi} + \frac{1}{\xi} H_{2\xi} \right) + \frac{\mathrm{Sr}_2}{\mathrm{Sc}_2 \nu} \left(A_{2\xi\xi} + \frac{1}{\xi} A_{2\xi} \right) - \\ - \frac{\mathrm{Ma}}{\xi} H_{2\xi} \int_\xi^R x V_2(x)\, dx - 2\mathrm{Ma} H_2 V_2 = 0. \end{gathered} \tag{30}$$

Then, on a solid wall $\xi = R$, the conditions are met:

$$V_2(R) = 0, \quad A_2(R) = 1, \quad H_{2\xi}(R) + \mathrm{Sr}_2 A_{2\xi}(R) = 0. \tag{31}$$

On the interface $\xi = 1$:

$$V_1(1) = V_1(1), \quad \int_0^1 x V_1(x)\, dx = 0, \quad \int_1^R x V_2(x)\, dx = 0, \tag{32}$$

$$A_1(1) = A_2(1), \quad H_1(1) = H_2(1), \tag{33}$$

$$V_{2\xi}(1) - \mu V_{1\xi}(1) = -2A_1(1) - 2\mathrm{M} H_1(1), \tag{34}$$

$$d(H_{1\xi}(1) + \mathrm{Sr}_1 A_{1\xi}(1)) = H_{2\xi}(1) + \mathrm{Sr}_2 A_{2\xi}(1), \tag{35}$$

$$A_{2\xi}(1) - kA_{1\xi}(1) = EA_1(1)V_1(1), \tag{36}$$

where $E = æ_1^2 a_{20} R_1^2/\mu_2 k_2$ is a parameter that determines the effect of the internal energy of the interface on the dynamics of the movement of liquids inside the layers.

On the axis of symmetry, the conditions of boundedness are set:

$$|V_1(0)| < \infty, \quad |A_1(0)| < \infty, \quad |H_1(0)| < \infty. \tag{37}$$

Remark. The integral redefinition conditions in (32), meaning the flow closure conditions, are necessary to find the unknown longitudinal pressure gradients F_j in the layers of binary mixtures, $j = 1, 2$.

3. Solving of the conjugate problem for small Marangoni numbers

We will assume that the thermal Marangoni number Ma $\ll 1$ (*a creeping motion*), and Ma $\sim$ Mc, that is, the thermal and concentration effects on the interface $\xi = 1$ of the same order. Formally decomposing the functions V_j, A_j, H_j in a series of Ma, we obtain for the first approximation the problem (25)–(27), (28)–(30) with Ma $= 0$. In the equations of momentum, energy, and concentration transport, the convective terms are discarded. As for the nonlinear boundary condition (36), it is remains unchanged. To do this, we must assume that $E = O(1)$.

Then the conjugate inverse boundary value problem for small Marangoni numbers becomes linear:

$$V_{1\xi\xi} + \frac{1}{\xi}V_{1\xi} = -F_1, \tag{38}$$

$$A_{1\xi\xi} + \frac{1}{\xi}A_{1\xi} = 0, \tag{39}$$

$$H_{1\xi\xi} + \frac{1}{\xi}H_{1\xi} = 0, \quad 0 < \xi < 1; \tag{40}$$

$$V_{2\xi\xi} + \frac{1}{\xi}V_{2\xi} = -F_2\nu, \tag{41}$$

$$A_{2\xi\xi} + \frac{1}{\xi}A_{2\xi} = 0, \tag{42}$$

$$H_{2\xi\xi} + \frac{1}{\xi}H_{2\xi} = 0, \quad 1 < \xi < R; \tag{43}$$

with the boundary conditions (31)–(37).

Common solutions of systems (38)–(43) are easily found (the boundedness conditions (37) are taken into account):

$$V_1(\xi) = C_1 - \frac{F_1}{4}\xi^2, \quad A_1(\xi) = C_2, \quad H_1(\xi) = C_3; \tag{44}$$

$$V_2(\xi) = C_4 + C_5 \ln\xi - \frac{F_2\nu}{4}\xi^2, \quad A_2(\xi) = C_6 + C_7 \ln\xi, \quad H_2(\xi) = C_8 + C_9 \ln\xi, \tag{45}$$

with the constants $C_1, \ldots, C_9$, which are determined from the boundary conditions (31)–(36). Exactly,

$$\begin{gathered} C_1 = \frac{F_1}{8}, \quad C_2 = C_6 = \frac{8}{8 - EF_1 \ln R}, \quad C_3 = C_8, \\ C_4 = \frac{2F_2\nu - F_1}{8}, \quad C_5 = \frac{2F_2\nu(R^2-1) + F_1}{8\ln R}, \quad C_7 = \frac{EF_1}{EF_1\ln R - 8}, \quad C_9 = -\mathrm{Sr}_2 C_7. \end{gathered} \tag{46}$$

As for the constant C_3, from the boundary condition (34) it is defined as follows:

$$C_3 = \frac{F_2\nu - F_1\mu - 4C_2 - 2C_5}{4\mathrm{M}}. \tag{47}$$

But such a representation for C_3 makes it difficult to further search for the pressure gradients F_1, F_2 along the layers when solving the inverse boundary value problem. On the other hand, this constant can be found if you set the average concentration over the cross section $z = 0$, so $\int\limits_0^1 \xi H_1(\xi)\, d\xi = 0$. From where we get that $C_3 = 0$ and, therefore, $C_8 = 0$.

The pressure gradients F_1, F_2 are related by the relation $F_2 = F_1 N(R)$, where the function $N(R)$ is defined by the formula:

$$N(R) = \frac{R^2 - 2\ln R - 1}{2\nu(R^2-1)[(R^2+1)\ln R - R^2 + 1]}. \tag{48}$$

In addition, the functions $U_j(\xi)$are recovered from the continuity equation (6):

$$\begin{gathered} U_1(\xi) = \frac{F_1}{16}\xi(\xi-1)(\xi+1)\,, \\ U_2(\xi) = \frac{F_1}{16\xi}[(R^2-\xi^2)(8C_4 - 4C_5 - F_2\nu(R^2+\xi^2)) + 8C_5(R^2\ln R - \xi^2\ln\xi)]. \end{gathered} \tag{49}$$

If the expression for the constant C_3 from (47) vanishes, then after some calculations a quadratic equation arises with respect to the unknown pressure gradient F_1:

$$EL(R)\ln R\, F_1^2 - 8L(R)F_1 - 128\ln R = 0, \tag{50}$$

where $L(R)$ is defined by the formula:

$$L(R) = 4\nu\ln R(\rho - N(R)) + 2\nu N(R)(R^2-1) + 1. \tag{51}$$

Of interest are the cases related to the number of solutions of the equation (50).

1. If $E = 0$, we get the equation: $-8L(R)F_1 - 128\ln R = 0$, which has a unique solution $F_1 = -16\ln R/L(R)$. The pressure gradient F_2 is easily determined from the ratio (48).

2. If $R \to 1$, then we have the equation: $-8L(R)F_1 = 0$, which has the only solution $F_1 = 0$. Here it is taken into account that the function $L(R)$ takes positive values on the interval $(1, +\infty)$. Then it follows from (48) that $F_2 = 0$. The equality of the pressure gradients to zero means that there is no source of motion of the mixtures in both layers. Thus the mixtures are at rest.

Next, we find the discriminant of the quadratic equation (50):

$$D = 64L(R)(L(R) + 8E\ln^2 R), \tag{52}$$

depending on the sign of which the equation has a different number of roots.

3. If $D > 0$, we get: $E > -L(R)/8\ln^2 R$. In this case, the square equation has two roots:

$$F_1^{1,2} = \frac{4L(R) \pm 4\sqrt{L^2(R) + 8EL(R)\ln^2 R}}{EL(R)\ln R}. \tag{53}$$

4. The discriminant vanishes at $E = -L(R)/8\ln^2 R, (L(R) \neq 0)$. Then the equation will have a unique solution: $F_1 = -32\ln R/L(R)$. Note, what is the expression $L(R)/8\ln^2 R > 0$ when $R \in (1, +\infty)$. Therefore, the parameter E takes negative values. This is possible with $a_{20} < 0$, since E depends on this parameter.

5. The negative sign of the discriminant corresponds to the condition: $E < -L(R)/8\ln^2 R$, which is equivalent to the absence of real roots of the square equation.

Thus, the number of solutions to the equation (50) depends more on the parameter E. In other words, the energy of interfacial heat transfer has a significant effect on the processes occurring in the contacting liquids.

4. Model problem

We present the quantitative results of solving the problem for the model system formic acid (mixture 1) — transformer oil (mixture 2). According to the tabular data, the physical constants are as follows:

$$\mu_1 = 1.78 \cdot 10^{-3} \frac{\mathrm{kg}}{\mathrm{m \cdot s}}, \quad \mu_2 = 198.1 \cdot 10^{-4} \frac{\mathrm{kg}}{\mathrm{m \cdot s}}, \quad \nu_1 = 1.46 \cdot 10^{-6} \frac{\mathrm{m}^2}{\mathrm{s}}, \quad \nu_2 = 22.5 \cdot 10^{-6} \frac{\mathrm{m}^2}{\mathrm{s}},$$
$$\chi_1 = 1.057 \cdot 10^{-7} \frac{\mathrm{m}^2}{\mathrm{s}}, \quad \chi_2 = 7.55 \cdot 10^{-8} \frac{\mathrm{m}^2}{\mathrm{s}}, \quad k_1 = 0.267 \frac{\mathrm{Wt}}{\mathrm{m \cdot K}}, \quad k_2 = 0.1106 \frac{\mathrm{Wt}}{\mathrm{m \cdot K}},$$
$$\sigma_0 = 37.58 \cdot 10^{-3} \frac{\mathrm{N}}{\mathrm{m}}, \quad æ_1 = 1.2826 \cdot 10^{-4} \frac{\mathrm{N}}{\mathrm{m \cdot K}}.$$

The following parameter values were also used: $R = 1.5$, $R_1 = 10^{-9}\,\mathrm{m}$, $E = 0.7\,(a_{20} > 0)$. As a result of the calculations, two solutions were obtained for the longitudinal pressure gradients in the layers: $F_1^1 = -1.78305$, $F_2^1 = -71.22054$ and $F_1^2 = 29.96938$, $F_2^2 = 1197.06399$. It can be seen that for the second solution, the gradient values in both mixtures are too high, which is unphysical.

Fig. 2–4 demonstrates the function $V_j(\xi)$ and the velocity profile $U_j(\xi)$ depending on the various defining parameters of the model.

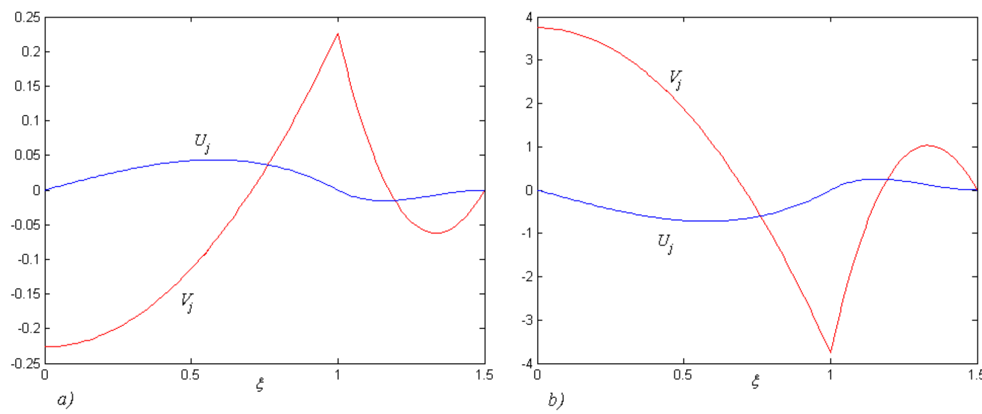

Fig. 2. The behavior of the function $V_j(\xi)$ and the velocity profile $U_j(\xi)$: a) for the first solution, b) for the second solution

Fig. 2 shows the functions $V_j(\xi)$, $U_j(\xi)$, corresponding to the two solutions $\{F_1^1, F_2^1\}$ and $\{F_1^2, F_2^2\}$.

Fig. 3 shows that as the parameter E increases, the values of the functions $V_j(\xi)$, $U_j(\xi)$ in absolute value decreases significantly. You can choose such values of E, at which the model problem will have a single solution. So, for $E = 0$ $(a_{20} = 0)$ we get: $F_1 = -1.89641$, $F_2 = -76.27046$. By $E \approx -2.6$ $(a_{20} = -3.46 \cdot 10^{23})$ we have: $F_1 = -3.79282$, $F_2 = -152.54093$.

The increase of the parameter R is strongly influenced by the velocity profile $U_j(\xi)$ and the function $V_j(\xi)$. Fig. 4 shows that the absolute values of the functions increase. This is due to the fact that for a fixed R_1, the radius of the outer cylinder increases, since $R = R_2/R_1$. It is also important to trace how the change in the radius of the inner cylinder R_1 affects the flow pattern in the layers. It turned out that with the growth of R_1, the values of the functions $V_j(\xi)$, $U_j(\xi)$ in absolute value decreases. This is due to the fact that with an increase in the radius of the inner cylinder at fixed R and E, the influence of a constant temperature set on the surface of the outer cylindrical tube weakens.

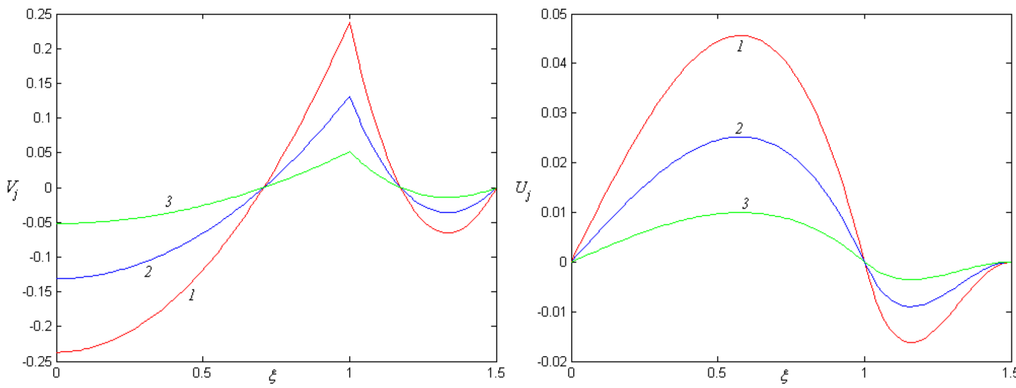

Fig. 3. The dependence of the functions $V_j(\xi)$, $U_j(\xi)$ on the parameter E: $1 - E = 0.05$, $2 - E = 0.2$, $3 - E = 0.7$

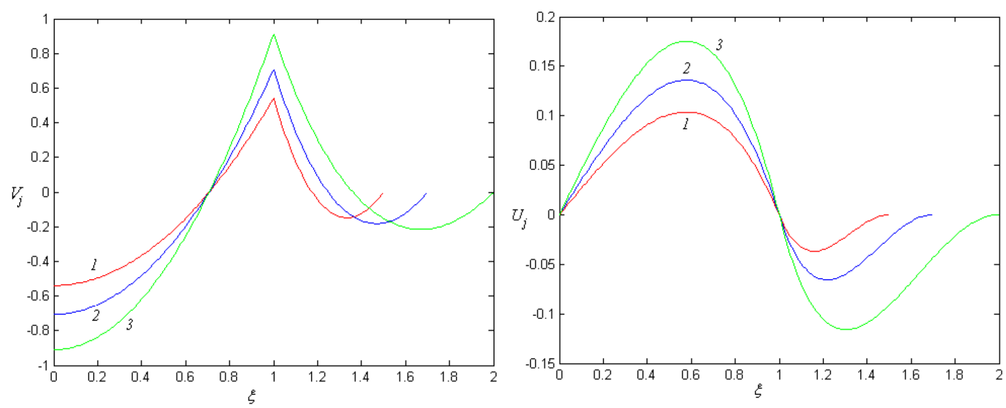

Fig. 4. The dependence of the functions $V_j(\xi)$, $U_j(\xi)$ on the parameter R: $1 - R = 1.5$, $2 - R = 1.7$, $3 - R = 2.0$

Fig. 5 shows the "temperature" and "concentration" functions $A_j(\xi)$, $H_j(\xi)$, corresponding to the first solution $\{F_1^1, F_2^1\}$. In the first layer, these functions are constant. In the second layer $A_j(\xi)$ increases and $H_j(\xi)$ decreases, which corresponds to the phenomenon of abnormal thermal diffusion.

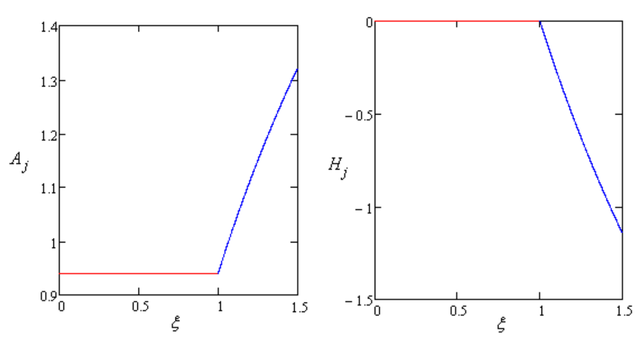

Fig. 5. The behavior of functions $A_j(\xi)$, $H_j(\xi)$ in the case of the first solution

Thus, the effect of changes in the internal energy of the interfacial surface on the two-layer flow of two immiscible binary mixtures in a cylindrical capillary is studied. It is found that with an increase in the parameter E, which is responsible for the influence of changes in the surface internal energy on the dynamics of liquids in layers, the absolute values of the functions $V_j(\xi)$, $U_j(\xi)$ decreases.

5. Derivation of a finite-dimensional system of nonlinear algebraic equations

To solve the nonlinear problem (25)–(37), the tau method is used, which is a modification of the Galerkin method [11]. For the future, it is essential to replace the variables: $\xi' = \xi$ with $j = 1$ and $\xi' = (\xi - R)/(1 - R)$ when $j = 2$ and re-assign $\xi' \leftrightarrow \xi$. An approximate solution is sought in the form of sums:

$$V_j^n(\xi) = \sum_{l=0}^{n} V_j^l R_k^{(0,1)}(\xi), \quad A_j^n(\xi) = \sum_{l=0}^{n} A_j^l R_k^{(0,1)}(\xi), \quad H_j^n(\xi) = \sum_{l=0}^{n} H_j^l R_k^{(0,1)}(\xi), \tag{54}$$

где $R_k^{(0,1)}(\xi)$ are the shifted Jacobi polynomials. In general, they are defined in terms of the Jacobi polynomials $P_k^{(\alpha,\beta)}(y)$ as follows ($\alpha > -1, \beta > -1$) [12]:

$$R_k^{(\alpha,\beta)}(y) = P_k^{(\alpha,\beta)}(2y - 1), \quad y \in [0, 1]. \tag{55}$$

Coefficients V_j^l, A_j^l, H_j^l and constants F_j are found from the Galerkin approximation system, namely:

$$\int_0^1 K_j(V_j^n, F_j) R_m^{(0,1)}(\xi)\, \xi \, d\xi = 0, \tag{56}$$

$$\int_0^1 S_j(V_j^n, A_j^n) R_m^{(0,1)}(\xi)\, \xi \, d\xi = 0, \tag{57}$$

$$\int_0^1 T_j(V_j^n, A_j^n, H_j^n) R_m^{(0,1)}(\xi)\, \xi \, d\xi = 0, \quad m = 0, \ldots, n-2, \quad j = 1, 2. \tag{58}$$

It follows from the integral redefinition conditions of (32) that $V_1^0 = V_2^0 = 0$.

The boundary conditions are transformed as follows:

$$\sum_{l=0}^{n} (-1)^l V_2^l = 0, \quad \sum_{l=0}^{n} (-1)^l A_2^l = 1, \tag{59}$$

$$\sum_{l=1}^{n} (-1)^{l-1} l(l+1)(l+2)[H_2^l + \mathrm{Sr}_2 A_2^l] = 0. \tag{60}$$

$$\sum_{l=0}^{n} V_1^l = \sum_{l=0}^{n} V_2^l, \quad \sum_{l=0}^{n} A_1^l = \sum_{l=0}^{n} A_2^l, \quad \sum_{l=0}^{n} H_1^l = \sum_{l=0}^{n} H_2^l, \tag{61}$$

$$\sum_{l=1}^{n} l(l+2)(V_2^l - \mu V_1^l) = -2 \sum_{l=0}^{n} (A_1^l + \mathrm{M} H_1^l). \tag{62}$$

$$d \sum_{l=1}^{n} l(l+2)[H_1^l + \mathrm{Sr}_1 A_1^l] = \sum_{l=1}^{n} l(l+2)[H_2^l + \mathrm{Sr}_2 A_2^l], \tag{63}$$

$$\sum_{l=1}^{n} l(l+2)(A_2^l - k A_1^l) = -E \sum_{l=0}^{n} A_1^l \sum_{l=0}^{n} V_1^l. \tag{64}$$

Verbose output finite-dimensional system galerkins approximations for the coefficients V_j^l, A_j^l, H_j^l, $l = 0, \dots, n$, $j = 1, 2$, and also the calculation of definite integrals from different product of shifted Jacobi polynomials are present in the work [13].

As a result, the system of integro-differential equations are converted to a closed system of nonlinear algebraic equations unknown coefficients V_j^l, A_j^l, H_j^l and gradients of pressure F_j, where $l = 0, \dots, n$, $j = 1, 2$. Its solution was used Newton's method with a given accuracy $\varepsilon = 10^{-5}$. As an initial approximation, the results obtained in solving the model problem were taken.

Applied to a nonlinear inverse boundary value problem (25)–(37) the tau-method showed the possibility of existence of two solutions for the longitudinal pressure gradients and, accordingly, for the rest of the desired functions of the problem. Calculations were performed for $n = 10, 12$ in Galerkin approximations. As the number of n increases, a rapid increase in the accuracy of the solution is detected.

Fig. 6 shows the dependence of the functions $V_j(\xi)$, $U_j(\xi)$ on different values of the thermal Marangoni number, obtained for the first solution: $F_1 = -1.78355$, $F_2^1 = -71.73149$. We conclude that the solutions found with a decrease in the Marangoni number converge to solutions of the problem of the creeping flow of binary mixtures.

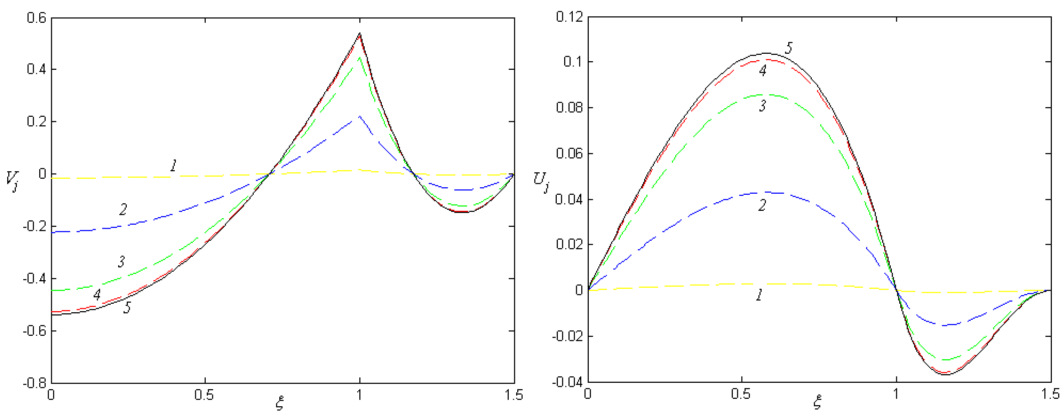

Fig. 6. The dependence of the functions $V_j(\xi)$, $U_j(\xi)$ of the thermal Marangoni number: 1 — Ma = 15, 2 — Ma = 3, 3 — Ma = 0.5, 4 — Ma = 0.28, 5 — a creeping current

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Двухслойное стационарное течение в цилиндрическом капилляре с учетом изменения внутренней энергии поверхности раздела

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Аннотация. Изучена задача о двумерном стационарном течении двух несмешивающихся несжимаемых бинарных смесей в цилиндрическом капилляре в отсутствие массовых сил. Смеси контактируют через общую поверхность раздела, на которой учитывается полное энергетическое условие. Температура и концентрация в смесях распределены по квадратичному закону, что хорошо согласуется с полем скоростей типа Хименца. Возникающая сопряженная краевая задача является нелинейной и обратной относительно градиентов давлений вдоль оси цилиндрического капилляра. К этой задаче применен тау-метод (модификация метода Галеркина), который показал возможность существования двух решений. Показано, что полученные решения с уменьшением числа Марангони сходятся к решениям задачи о ползущем течении бинарных смесей. При решении модельной задачи при малых числах Марангони установлено, что влияние приращений внутренней энергии межфазной поверхности существенно сказывается на динамике течения смесей в слоях.

Ключевые слова: бинарная смесь, поверхность раздела, внутренняя энергия, обратная задача, градиент давления, тепловое число Марангони.

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The Splitting Algorithm in Finite Volume Method for Numericai Solving of Navier-Stokes Equations of Viscous Incompressible Fluids

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Abstract. For the numerical solution of the Navier-Stokes equations, written in an integral form, an implicit of finite-volume algorithm is proposed, which is a generalization of previously proposed differences schemes. Using the integral form of equations allowed to ensure its conservatism, and the technology of splitting — the economy of the algorithm. The numerical test of the algorithm on the exact solution, in problems about the viscosity flow in the cavern with a moving lid and the current of the heated walls of the channel, confirmed the sufficient accuracy of the algorithm and its effectiveness. The work is presented in the issue of the memory of Prof. Yu. Ya. Belov.

Keywords: Navier-Stokes equations, viscous flows, finite-volume method, splitting algorithms.

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Introduction

The Navier–Stokes equations of a viscous incompressible fluid are the basic model for solving various classes of problems in hydromechanics [1, 2]. They are nonlinear and their solutions contain areas of high gradients, boundary layers, separation zones, etc., which imposes additional difficulties in their study. Therefore, the problem of constructing economical numerical algorithms for solving the Navier–Stokes equations is still relevant today. Some approaches for constructing finite-difference and finite-volumes schemes are given, for example, in [3–11]. When solving multidimensional problems, including those in curvilinear and multiply connected domains, the method of finite volumes [2, 8, 9], based on the approximation of equations in integral form, may turn out to be more convenient. It has the property of conservatism, the approximation of the equations in it, is constructed for each cell, the shape of which is easier to adapt to the boundaries of the region.

The use of explicit schemes in solving the Navier–Stokes equations leads to large expenditures of computer resources, especially in the multidimensional case, due to strict restrictions on the ratio of the temporal and spatial steps of the grid. Implicit unfactorized algorithms are also uneconomical due to the need to invert large matrices (see, for example, [5, 11]). An alternative to this approach is the splitting and factorization methods [3], which make it possible to reduce the solution of a multidimensional problem to the solution of its one-dimensional analogs or simpler problems. In [11], a difference scheme was proposed for solving the Navier–Stokes equations for a viscous incompressible fluid in physical variables "velocity, pressure", based on the method

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of splitting into physical processes and spatial directions. This made it possible to simplify the implementation of the algorithm.

Below we generalize it to the finite volume method. The properties of the algorithm in terms of the accuracy of calculations and the rate of convergence are investigated when finding a stationary solution by the established method. The algorithm was tested on the solution of the Poisson flow problem, which has an exact solution, on the problems of fluid flow in a square cavity with a moving cover and flow in a rectangle with heated walls. The results obtained illustrate the capabilities of the proposed method and allow us to conclude about its effectiveness.

1. Initial equations. Algorithm for solving the Navier–Stokes equations

When studying the flows of a viscous incompressible fluid, taking into account the effects of heat conduction, one usually uses models described by the Navier–Stokes equations of an incompressible fluid, supplemented by the heat conduction equation. Let us consider them in gas-dynamic variables "velocity–pressure" in the form of a system of integral conservation laws in gas-dynamic variables [1]:

$$\boldsymbol{M}\frac{\partial}{\partial t}\int_V \boldsymbol{U}dV + \oint_S(\boldsymbol{W}\vec{n})ds = \int_V \boldsymbol{F}dV, \tag{1}$$

$$\frac{\partial}{\partial t}\int_V TdV + \oint_S(W_T\vec{n})ds = 0, \tag{2}$$

where $\boldsymbol{V}$ the volume of the computational domain, S its boundary, $\vec{n}$ the external normal to the boundary area, $\boldsymbol{U}, T$ the vector of the sought functions and temperature, $\boldsymbol{W}$, W_T are matrices composed of columns of flows at the boundaries of the volume, μ , k the coefficients of viscosity and thermal conductivity, g the acceleration coefficient, and $\boldsymbol{F}$ the force of gravity. The algorithm will be presented using the example of two-dimensional equations in Cartesian coordinates written in dimensionless form in the absence of external forces. Then:

$$\boldsymbol{U}=\begin{bmatrix}p\\v_1\\v_2\end{bmatrix},\quad \boldsymbol{W}=\begin{bmatrix}v_1 & v_2\\ v_1^2+p-\sigma_1^1 & v_1v_2-\sigma_2^1\\ v_1v_2-\sigma_1^2 & v_2^2+p-\sigma_2^2\end{bmatrix},\quad \boldsymbol{F}=\begin{bmatrix}0\\0\\d\end{bmatrix},\quad \boldsymbol{M}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}, \tag{3}$$

$$\sigma_j^i=\mu\frac{\partial v_i}{\partial x_j},\ \sigma_j^3=k\frac{\partial T}{\partial x_j},\ \boldsymbol{W}_T=\begin{bmatrix}v_1T-\sigma_1^3 & v_2T-\sigma_2^3\end{bmatrix},\ d=agT,\ \mu=const.$$

Let's set the grid step $\tau = T/N$, where N is the number of time steps. $\boldsymbol{U}$, T the functions will be set at the nodes i, j of the cell, and the flows $\boldsymbol{W}$ at the boundaries of the cells at the nodes $i \pm 1/2, j$ and $i, j \pm 1/2$ (Fig. 1).

We introduce the averaging of the sought functions over the elementary volume

$$V_{i,j}=\omega,\quad \boldsymbol{U}_{i,j}=\frac{1}{\omega}\int_\omega \boldsymbol{U}\partial\omega,\quad T_{i,j}=\frac{1}{\omega}\int_\omega T\partial\omega,\quad \boldsymbol{F}_{i,j}=\frac{1}{\omega}\int_\omega \boldsymbol{F}\partial\omega,$$

and we approximate the integral operators in cells by grid operators by the formulas

$$\frac{\partial}{\partial t}\int_V \boldsymbol{U}d\omega \approx \omega\frac{\boldsymbol{U}^{n+1}-\boldsymbol{U}^n}{\tau},\quad \oint_S(\boldsymbol{W}\boldsymbol{S_n})dS \approx \Omega = \sum_{m=1}^{2}\Delta_m(\boldsymbol{W_m}S),$$

$$\oint_S(W_T\vec{n})ds \approx \sum_{m=1}^{2}\Delta_m(W_{Tm}S),$$

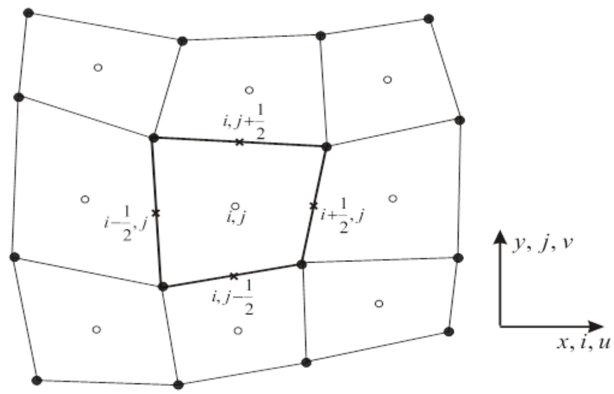

Fig. 1.

where

$$S\boldsymbol{W} = \begin{bmatrix} v_1S_1 + v_2S_2 \\ (v_1^2 + p - \sigma_1^1)S_1 + (v_1v_2 - \sigma_1^2)s_2 \\ (v_1v_2 - \sigma_1^2)S_1 + (v_1^2 + p - \sigma_2^2)S_2 \end{bmatrix},$$

$$\Delta_m(\boldsymbol{W}S)_m = [S_{m+1/2}(\boldsymbol{W}_{m+1} + \boldsymbol{W}_m) - S_{m-1/2}(\boldsymbol{W}_m + \boldsymbol{W}_{m-1})]/2$$

is the flows through opposite faces of a cell, $m = 1$ corresponding to index i, $m = 2$ index j. We construct an algorithm for solving the system of equations (1), (2), at first for equations (1), assuming that the temperature value is known. Finite-dimensional scheme with weights

$$\boldsymbol{M}\frac{\boldsymbol{U}^{n+1} + \boldsymbol{U}^n}{\tau} + \frac{1}{\omega}\sum_{m=1}^{2}\Delta_m(\alpha\boldsymbol{W}_m^{n+1} + \beta\boldsymbol{W}_m^n)S\boldsymbol{n} = \boldsymbol{F} \tag{4}$$

approximates the original equations (1), (3) with order $O(\tau^2 + h^2)$ for $\alpha > 0.5$ and at $\alpha \neq 0$, it is nonlinear. Here $h = \omega^{1/2}$. Operators σ_m^l on the boundaries contain directional derivatives with respect to x_m, that cannot coincide with the direction of the cell faces, so we introduce parameterization $x_i = x_i(q_j),\ q_j = q_j(x_i)$ where $0 \leqslant q_j \leqslant 1$. Then:

$$\frac{\partial}{\partial x_j} = \sum_{l=1}^{2} z_j^l \frac{\partial}{\partial q_l}, \quad z_j^l = \frac{\partial q_l}{\partial x_j}, \quad \sigma_j^i = \mu\sum_{l=1}^{2} z_j^l \frac{\partial v_i}{\partial q_l} \tag{5}$$

and in new variables σ_j^i contain derivatives with respect to normals q_1 and tangents q_2 to the cell boundaries ω. We approximate them at the nodes $i \pm 1/2$ or $j \pm 1/2$ by symmetric operators $\sigma_j^i = \mu\sum\limits_{m=1}^{2} z_j^m \Delta_m v_i$, $\Delta_{m\pm1/2}v = \pm(v_{m\pm1j} - v_m)$ and linearize the vector $\boldsymbol{W}^{n+1}$ with respect to $\boldsymbol{U}$ and known values μ, z_m^l:

$$\boldsymbol{W}_m^{n+1} = \boldsymbol{W}_m^n + \tau\frac{\partial \boldsymbol{W}_m^n}{\partial \boldsymbol{U}}\frac{\partial \boldsymbol{U}^n}{\partial t} + O(\tau^2) = \boldsymbol{W}_m^n + \tau B_m\frac{\boldsymbol{U}^{n+1} - \boldsymbol{U}^n}{\tau} + \ldots,$$

where $B_m = \begin{bmatrix} 0 & S_1 & S_2 \\ S_1 & V - t_{mk} & 0 \\ S_2 & 0 & V - t_{mk} \end{bmatrix}$, $t_{mk} = \mu\sum\limits_{k=1}^{2} z_m^k S_k \Delta_k,\ V = v_1S_1 + v_2S_2$ the projection of the velocity vector $\boldsymbol{V}$ times the normal to the face area. To construct economical algorithms, we introduce an operator B_m in which only derivatives with respect to the normals are stored in the coefficients t_{mk}, and represent it in the form of a splitting into physical processes:

$$B_m = B_{1m} + B_{2m}, \quad B_{1m} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & V - t_m & 0 \\ 0 & 0 & V - t_m \end{bmatrix}, \quad B_{2m} = \begin{bmatrix} 0 & S_1 & S_2 \\ S_1 & 0 & 0 \\ S_2 & 0 & 0 \end{bmatrix},$$

$$t_m = \mu z_m^m S_m \Delta_m$$

the finite-volume scheme:

$$\overline{C}\frac{\boldsymbol{U}^{n+1} - \boldsymbol{U}^n}{\tau} = -\frac{1}{\omega}\boldsymbol{\Omega}^n + \boldsymbol{F}^n, \quad \overline{C} = \boldsymbol{M} + \frac{\tau\alpha}{\omega}\sum_{m=1}^{2}\Delta_m(B_{1m}^n + B_{2m}^n), \tag{6}$$

$$\Omega^n = \sum_{m=1}^{2}\Delta_m(\boldsymbol{W}^n S\boldsymbol{n})$$

linear, but approximates the original equations with order $O(\tau + h^2)$. Note that the matrix $\boldsymbol{M}$ is degenerated, which does not allow standard factorization methods. However, there is a special splitting of the operator $\overline{C}$ in the physical process and spatial directions (see [11]), in which it can be factorized in the form:

$$\overline{C} = C + O(\tau), \quad C = \prod_{m=1}^{2}(I + d\Delta_m B_{1m}^n) \cdot (\boldsymbol{M} + d\sum_{m=1}^{2}\Delta_m B_{2m}^n), \quad d = \frac{\tau\alpha}{\omega}$$

Then the scheme of approximate factorization:

$$C\frac{\boldsymbol{U}^{n+1} - \boldsymbol{U}}{\tau} = -\frac{1}{\omega}\boldsymbol{\Omega}^n + \boldsymbol{F}^n \tag{7}$$

or it equivalent scheme in fractional steps:

$$\xi^n = -\frac{1}{\omega}\boldsymbol{\Omega}^n + \boldsymbol{F}^n, \quad (I + d\Delta_1 B_{11}^n)\xi^{n+1/3} = \xi^n, \quad (I + d\Delta_2 B_{12}^n)\xi^{n+2/3} = \xi^{n+1/3}, \tag{8}$$

$$(\boldsymbol{M} + d\sum_{m=1}^{2}\Delta_m B_{2m}^n)\xi^{n+1} = \xi^{n+2/3}, \quad \boldsymbol{U}^{n+1} = \boldsymbol{U}^n + \tau\xi^{n+1}$$

Where $\xi = (\xi_p, \xi_1, \xi_2)^T$ the residuals of the solution at fractional steps approximate Eqs. (1), (2) with the same order as (6). The values ξ^n are computed explicitly. At the first ($m = 1$) and second ($m = 2$) fractional steps of the equation scheme

$$[1 + d\Delta_m(V - t_m)]\xi_l^{n+m/3} = \xi_l^{n+(m-1)/3}(l = 1, 2)$$

are solved by scalar sweeps independently for each component of the velocity residual $\xi_l^{n+m/3}$, and $\xi_p^{n+m/3} = \xi_p^n$. At the third fractional step of scheme (8), the system of equations is solved

$$\begin{gathered} d[\Delta_1 S_1 \xi_1^{n+1} + \Delta_2 S_2 \xi_2^{n+1}] = \xi_p^{n+2/3}, \quad \xi_1^{n+1} = \xi_1^{n+2/3} - d\Delta_1 S_1 \xi_p^{n+1}, \\ \xi_2^{n+1} = \xi_2^{n+2/3} - d\Delta_2 S_2 \xi_p^{n+1}. \end{gathered} \tag{9}$$

Eliminating the velocity components from the continuity equation, we obtain the equation for it

$$\Delta\xi_p^{n+1} = f, \tag{10}$$

where $\Delta = \Delta_{11} + \Delta_{22}$, $\Delta_{mm} = \Delta_m S_m \Delta_m S_m$, $f = \left[\sum\limits_{m=1}^{2} \Delta_m S_m \xi_m^n\right] / d - \xi_p^n / d^2$. The solution to the Poisson equation can be obtained by various iterative algorithms [12], for example, by the iterative approximate factorization scheme

$$(I - \tau_0\Delta_{11})(I - \tau_0\Delta_{22})(\xi^{\nu+1} - \xi^{\nu})/\tau_0 = \Delta(\xi_p^{n+1/3})^{\nu} - f$$

or an equivalent scheme in fractional steps

$$(I + \tau_0\Delta_{11})\eta_1 = \Delta(\xi_p^{n+1/3})^{\nu} - f, \quad (I - \tau_0\Delta_{22})\eta = \eta_1, \quad \xi^{\nu+1} = \xi^{\nu} + \tau_0\eta.$$

Realized with fractional steps, also by scalar sweeps. The solution is carried out until the iterations converge, i.e., until the condition $\Delta\xi^{\nu} - f = O(\tau_0 h^2)$ is met in all cells. Then, from (9), the new values of the velocity residuals ξ_l^{n+1} are clearly found. The new values of the functions $\boldsymbol{U}$ are explicitly calculated from (8) and, if necessary, the calculation process is repeated.

2. Algorithm for solving the heat equation

We approximate the integral operators in (2) by the grid operators

$$\frac{\partial}{\partial t}\int_V T d\omega \approx \omega\frac{T^{n+1} - T^n}{\tau},$$

$$\oint_S (W_T\vec{n})ds \approx \sum_{m=1}^{2} \Delta_m[(v_1 - k\sigma_1^3)S_1 + (v_2 T - k\sigma_2^3)S_2]T$$

and, like scheme (7), we consider the finite-volume scheme of approximate factorization

$$\prod_{m=1}^{2}[I + d\Delta_m(V - t_m)]\frac{T^{n+1} - T^n}{\tau} = -\frac{1}{\omega}\sum_{k=1}^{2}\Delta_m W_T^n \tag{11}$$

or the equivalent of a scheme in fractional steps

$$\begin{gathered} \xi_T^n = -\frac{1}{\omega}\textstyle\sum_{m=1}^{2}\Delta_m W_T^n, \quad [I + d\Delta_1(V - t_1)]\xi_T^{n+1/2} = \xi_T^n, \\ [I + d\Delta_2(V - t_2)]\xi_T^{n+1} = \xi_T^{n+1/2}, \quad T^{n+1} = T^n + \tau\xi_T^{n+1}. \end{gathered} \tag{12}$$

Here $t_m = k(z_1^m S_1 + z_2^m S_2)\Delta_m$, $\sigma_m^3 = k\sum\limits_{k=1}^{2} z_m^k \Delta_k T$, $W_T^n = [VT - (\sigma_1^3 S_1 + \sigma_2^3 S_2)]$. It approximates the thermal conductivity equation (2), (5) with order $O(\tau^2 + h^2)$ when $\alpha = 0.5 + O(\tau)$. The values ξ_T^n are calculated explicitly, then the equations are solved at fractional steps by scalar sweeps in each direction. The new temperature values are calculated explicitly from the last equation in scheme (12). This completes one step of the calculations and, if necessary, the process continues to find a solution at subsequent points in time.

3. Examples of numerical calculations

The proposed algorithm was tested on a number of problems. Testing was carried out on three problems, the numerical solution of which was obtained by different authors and different numerical algorithms (see, for example, [5, 7–9]). This made it possible to compare and evaluate the properties of the algorithm. When carrying out numerical calculations, equations (2), (3) were reduced to dimensionless form [1]. This led to the appearance in the equations of the dimensionless parameters of the Reynolds numbers Re, Rayleigh Ra and Prandtl Pr, respectively: $Re = 1/\mu$, $Ra = ag$, $Pr = \mu/k$. Then the characteristic size of length $L \approx 1$, speed $|v| = 1$, and pressure $p = 1$ (it is set up to a constant).

In the first problem, the stationary flow of a viscous incompressible fluid in a channel was investigated in the framework of the Navier Stokes model (1) where $F = 0$. Its solution is reduced to the Poiseuille flow, the exact solution of which is: $v_2 = 0$, $v_1 = 1 - x_2^2$, $p = 1 - 2\mu x_1$.

In the computational domain, a square grid with a number of cells $J = I \times I$ was used. The velocity and pressure $v_2 = 0$, $v_1 = 1 - x_2^2$, $p = 1$ were set at the channel inlet, and $v_1 = v_2 = 0$ for the adhesion conditions on the channel walls. At the initial moment of time, constant values of velocity and pressure were set inside the region. The stationary solution of the problem was found by the establishment method. The establishment criterion was set in the form:

$$\max |p^{n+1} - p^n| \leqslant K(\tau\omega), \quad K \approx 0.1 - 1.0$$

for all interior points. Since the sought functions are specified at the centers of the cells, and the flows are determined at the boundaries of the cells, the implementation of the algorithm requires the introduction of near-boundary dummy cells and the specification of functions in them. On the upper and lower walls of the channel at dummy points, the velocity components are set equal in magnitude, but opposite in sign, and at the input, their values are set equal to those at the input. To evaluate the accuracy of the algorithm and to estimate the rate of convergence of the solution to the stationary one, we performed calculations on grids with different numbers of cells. Tab. 1 shows estimates of the errors of solutions

Table 1

τ	$h_1 = h_2$	Δp	Δv_1	Δv_2
0.1	0.1	0.008817	0.009947	0.006017
0.05	0.05	0.002204	0.002487	0.001505
0.025	0.025	0.000551	0.000622	0.000376
0.0125	0.0125	0.000138	0.000155	0.000094

As follows from the calculation results, an increase in the number of cells (decrease in grid steps) by 2 times in each direction leads to a decrease in the error by 4 times, which confirms the second order of accuracy of the algorithm. The number of iterations before stopping depends on the initial guess and the number of nodes J. Their typical number is given in Tab. 2 on a $J = 80 \times 80$ grid at various values of the viscosity coefficients μ.

Table 2

μ	0.01	0.025	0.001
iterations	1111	2731	3652

In the second problem, the fluid flow in a square cavity with a moving cover in the absence of gravity was studied. No-slip conditions $v_1 = v_2 = 0$ were set on the stationary walls of the channel. At the initial moment of time $v = 0$. At $t > 0$, the lid begins to move at a constant speed (Fig. 2).

The stationary solution of the problem was found by the establishment method. The calculations were carried out on a sequence of grids at various values of the Reynolds number Re.

Fig. 3 shows the distribution of the longitudinal and transverse velocity components on various grids at $Re = 3 \cdot 10^3$.

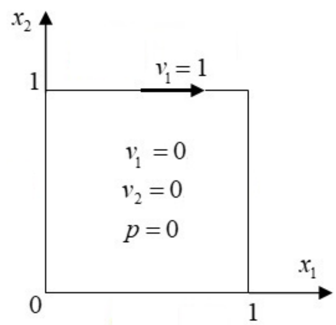

Fig. 2

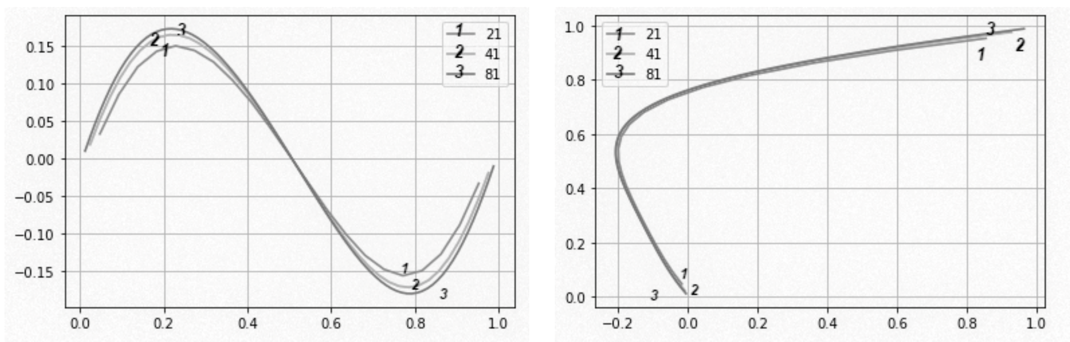

Fig. 3

The convergence of solutions is observed with an increase in the number of grid steps. Its further refinement practically did not lead to differences in values on the 81×81 grid. At numbers $Re > 10^2$, a vortex appears in the cavity, the center of which is shifted to the right, and two small vortices at the corners of the cavity. With an increase in Re, the angular vortices increase, their intensity increases, which follows from theoretical estimates and calculations using other algorithms [11, 13–15]. A typical flow pattern is shown in Fig. 4.

Comparison of the results obtained with the calculations in [5, 7–9] shows the visual coincidence of the flow fields.

In the third problem, some results of calculations of fluid flows in a closed flat cavity with heating of one of the sides are presented. The system of Navier–Stokes equations (1) was supplemented by the equation for temperature (2), and a term of the form $d = RaT$ was added to the equation of motion. At $t = 0$, the liquid was assumed to be stationary, and the adhesion conditions were set at the boundaries of the region. On the left and right walls of the region, $T = 1$ and $T = 0$ respectively, and on the upper and lower walls, according to a linear law $T = 1 - x_1$. Due to the temperature difference in the region, a rotational motion of the liquid occurs, its intensity is determined by the numbers Re and Ra. The numerical solution of the problem was found according to schemes (7), (11) on various grids. The calculations for various values of the parameters of the problem and comparison with the calculations [13–15] showed the identity of the solutions obtained. For example, in Fig. 5 streamlines for $Ra = 1$ $Re = 3 \cdot 10^2 (a)$ and $10^3 (b)$ are shown.

A large central region of the circulation flow ("central vortex") and secondary "corner vortices" in the lateral part of the cavity are distinguished. Note the increase in the vortex velocity with the increase in the Re numbers. A change in the Pr numbers for fixed Re and Ra also leads to a significant rearrangement of the flow pattern, and a change in the Ra numbers in a wide range of parameters has little effect on convection, as follows from calculations by other authors (see [11, 13–15]).

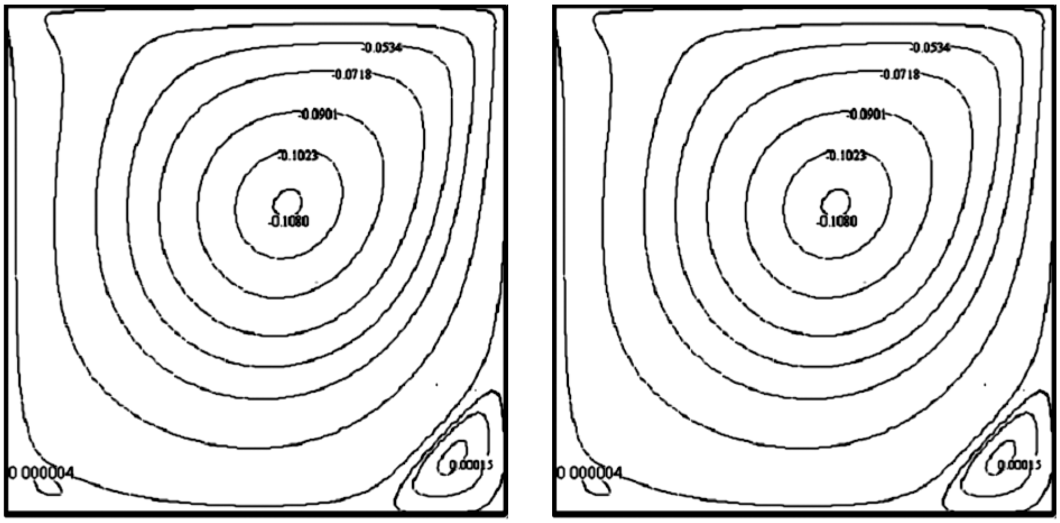

Fig. 4

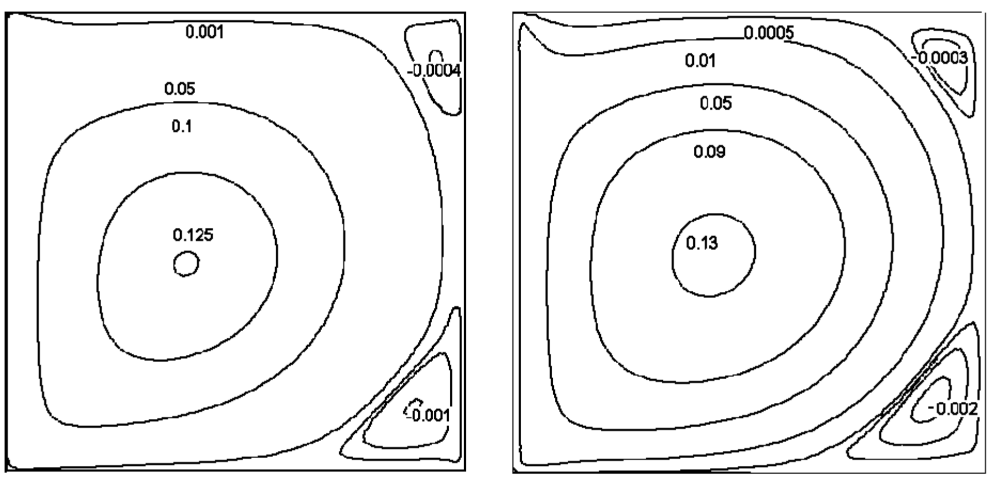

Fig. 5

Conclusion

The paper proposes a generalization of the finite difference splitting scheme for the numerical solution of the Navier–Stokes equations of a viscous incompressible fluid to the finite volume method. It has been tested for solving a number of problems (Poiseuille flows, in a cavity with a moving cover, and in a square region with heated sides). The performed comparisons in terms of the accuracy of the algorithm and the rate of convergence when finding a stationary solution by the establishmed method showed the efficiency of the algorithm and its sufficient accuracy.

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Алгоритм расщепления в методе конечных объемов для численного решения уравнений Навье-Стокса вязкой несжимаемой жидкости

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Аннотация. Для численного решения уравнений Навье–Стокса, записанных в интегральной форме, предложен неявный конечно-объемный алгоритм, являющийся обобщением предложенных ранее разносных схем. Использование интегральной формы уравнений позволило обеспечить его консервативность, а технологии расщепления — экономичность алгоритма. Проведена численная апробация алгоритма на точном решении, в задачах о течении жидкости в каверне с движущейся крышкой и течении с подогревом стенок канала, подтвердившая достаточную точность алгоритма и его эффективность. Работа представлена в выпуск памяти профессора Ю. Я. Белова.

Ключевые слова: уравнения Навье-Стокса, вязкие течения, конечно-объемный метод, алгоритмы расщепления.

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Inverse Problems of Finding the Lowest Coefficient in the Elliptic Equation

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Abstract. The article is devoted to the study of problems of finding the non-negative coefficient $q(t)$ in the elliptic equation

$$u_{tt} + a^2\Delta u - q(t)u = f(x,t)$$

($x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$, $t \in (0,T)$, $0 < T < +\infty$, Δ — operator Laplace on $x_1, \dots, x_n$). These problems contain the usual boundary conditions and additional condition (spatial integral overdetermination condition or boundary integral overdetermination condition). The theorems of existence and uniqueness are proved.

Keywords: elliptic equation, unknown coefficient, spatial integral condition, boundary integral condition, existence, uniqueness.

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The problems studied in this work belong to the class of nonlinear inverse coefficient problems for elliptic differential equations.

Various aspects of the theory of linear and nonlinear inverse coefficient problems for differential equations are well covered in the world literature — see, for example, monographs [1–8], articles [9–19]. Directly for elliptic equations inverse coefficient problems were studied in [15–19] (a more detailed bibliography can be found in [17]).

The nonlinear inverse coefficient problems for elliptic equations studied in this work, the results obtained in it will be essentially differ either in the formulations (in particular, in the given redefinition conditions), or in the results from the statements and results from the works of predecessors.

The problems studied in this work have a model form. More general cases and also possible generalization of the obtained results will be discussed at the end of the article.

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1. Statement of the problems

Let Ω be a bounded domain of variables $(x_1, \dots, x_n)$ of space $\mathbb{R}^n$, Γ is the boundary of Ω. We assume that Γ is a compact infinitely differentiable manifold. Next, Q is a cylinder $\Omega \times (0, T)$ of finite height T, $S = \Gamma \times (0, T)$ is the lateral surface of Q. Let $f(x,t)$, $u_0(x)$, $u_1(x)$, $N(x)$ and $\mu(t)$ be given functions defined for $x \in \overline{\Omega}$, $t \in [0, T]$; let a be given positive number.

Inverse Problem I: *Find functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation*

$$u_{tt} + a^2 \Delta u - q(t) u = f(x,t) \tag{1}$$

provided that $u(x,t)$ satisfies the conditions

$$u(x,0) = u_0(x), \quad u(x,T) = u_1(x), \quad x \in \Omega; \tag{2}$$

$$u(x,t)|_S = 0, \tag{3}$$

$$\int_\Omega N(x) u(x,t)\, dx = \mu(t), \quad t \in (0,T). \tag{4}$$

Inverse Problem II: *Find functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation (1) provided that $u(x,t)$ satisfies (2), (4) and also the condition*

$$\left. \frac{\partial u(x,t)}{\partial \nu} \right|_S = 0. \tag{5}$$

Inverse Problem III: *Find functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation (1) provided that $u(x,t)$ satisfies (2), (5), and also the condition*

$$\int_\Gamma N(x) u(x,t)\, ds_x = \mu(t). \tag{6}$$

In Inverse Problems I and II conditions (2) and (3), (2) and (5) are the conditions of a correct boundary value problem for second-order differential elliptic equation in a cylinder Q, whereas condition (4) is space-integral overdetermination condition. In Inverse Problem III conditions (2) and (5) are also the conditions of a correct boundary value problem for second-order differential elliptic equations, whereas condition (6) is an boundary-integral overdetermination condition.

All constructions and arguments in this paper will be carried out using the Lebesgue spaces L_p and Sobolev spaces W_p^l. The necessary information about the functions from these spaces can be found in the books [20–22].

The goal of this article is to prove the existence and uniqueness of regular solutions to the problems under study, that is, of solutions having all the weak derivatives in the sense of Sobolev involved in the equation.

2. Solvability of the inverse Problems I и II

Perform some auxiliary constructions for Inverse Problem I. Given the function $w(x,t)$, we define the function $\Phi(t; w)$: $\Phi(t; w) = a^2 \int\limits_\Omega N(x) \Delta w(x,t)\, dx$.

Put $v_0(x,t) = \dfrac{t}{T} u_1(x) + \dfrac{T-t}{T} u_0(x)$, $\qquad f_1(x,t) = f(x,t) - a^2 \Delta v_0(x,t)$,

$$f_0(t) = \int_\Omega N(x) f_1(x,t)\, dx, \quad \varphi(t) = \frac{1}{\mu(t)}, \quad \psi(t) = \varphi(t)[\mu''(t) - f_0(t)],$$

$$f_2(x,t) = f_1(x,t) + [\psi(t) + \varphi(t)\Phi(t;v_0)]v_0(x,t).$$

Consider the boundary value problem: *Find a function $v(x,t)$ that is a solution to equation*

$$v_{tt} + a^2\Delta v - [\varphi(t)\Phi(t; v+v_0) + \psi(t)]v = f_2(x,t) + \varphi(t)v_0(x,t)\Phi(t;v) \tag{1'}$$

and satisfies condition

$$v(x,0) = v(x,T) = 0, \quad x \in \Omega, \tag{2'}$$

and also the condition (3). Using a solution $v(x,t)$ of this boundary value problem we can establish the solvability of the inverse problem I.

Integro-differential equation (1′) is called loaded equation [23, 24].

Put $\varphi_0 = \max\limits_{[0,T]} |\varphi(t)|, \psi_0 = \min\limits_{[0,T]} \psi(t)$, $N_1 = 2\sum\limits_{i=1}^{n}\int\limits_{\Omega}\left[u_{0x_i}^2 + u_{1x_i}^2\right] dx$, $N_2 = \dfrac{1}{2}\varphi_0^2 N_1 T \|N\|^2_{L_2(\Omega)}$, $N_3 = \sum\limits_{i=1}^{n}\int\limits_{Q} f_{2x_i}^2 \, dx\, dt$, $N_4 = \dfrac{2N_3}{a^2(1-N_2)}$, $N_5 = a^2\|N\|_{L_2(\Omega)}\left(TN_4\right)^{1/2} + |\Phi(0,u_0)| + |\Phi(0,u_1)|$.

Theorem 2.1. *Suppose the fulfillment of conditions*

$$N(x) \in L_2(\Omega), \quad \mu(t) \in C^2([0,T]); \quad f(x,t) \in L_2(0,T; \overset{\circ}{W}{}_2^1(\Omega)) \cap L_\infty(0,T; L_2(\Omega)),$$

$$u_0(x) \in W_2^3(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega), \quad u_1(x) \in W_2^3(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega), \quad \Delta u_0(x) = \Delta u_1(x) = 0 \quad \textit{for} \quad x \in \Gamma;$$

$$\varphi_0 > 0, \quad \psi_0 > 0, \quad N_2 < 1, \quad N_5 \leqslant \frac{\psi_0}{\varphi_0};$$

$$\mu(0) = \int_\Omega N(x)u_0(x)\,dx, \quad \mu(T) = \int_\Omega N(x)u_1(x)\,dx.$$

Then the inverse problem I has a solution $\{u(x,t), q(t)\}$ such that $u(x,t) \in W_2^2(Q)$, $\Delta u(x,t) \in W_2^1(Q)$, $q(t) \in L_\infty([0,T])$, $q(t) \geqslant 0$ for $t \in [0,T]$.

Proof. We establish the solvability of the boundary value problem (1′), (2′), (3) in the space $W_2^2(Q)$. We use the regularization method and method of cut-off functions.

Let γ be a number from the interval $\left(0, \frac{\varphi_0}{\psi_0}\right]$. Define the cut-off function $G_\gamma(\xi)$:

$$G_\gamma(\xi) = \begin{cases} \xi, & \text{if} \quad |\xi| \leqslant \gamma, \\ \gamma, & \text{if} \quad \xi \geqslant \gamma, \\ -\gamma, & \text{if} \quad \xi \leqslant -\gamma. \end{cases}$$

Next, let ε be a positive number. Consider the boundary value problem: *find a function $v(x,t)$ that is a solution to equation*

$$v_{tt} + a^2\Delta v - [\psi(t) + \varphi(t)G_\gamma(\Phi(t; v+v_0))]v - \varepsilon\Delta^2 v = f_2(x,t) + \varphi(t)v_0(x,t)\Phi(t;v) \tag{$1'_\varepsilon$}$$

in the cylinder Q and satisfies conditions (2′) *and* (3′) *and also the condition*

$$\Delta v(x,t)|_S = 0. \tag{7}$$

Show that for a fixed number ε this problem has a solution belonging to $W_2^{4,2}(Q)$. Let's use the fixed point method.

Let $w(x,t)$ be a function from the space $W_2^{4,2}(Q)$. Consider the boundary value problem: *find a function $v(x,t)$ that is a solution to equation*

$$v_{tt} + a^2\Delta v - [\psi(t) + \varphi(t)G_\gamma(\Phi(t; w+v_0))]v - \varepsilon\Delta^2 v = f_2(x,t) + \varphi(t)v_0(x,t)\Phi(t;v) \tag{$1'_{\varepsilon,w}$}$$

in the cylinder Q and satisfies conditions $(2')$, *(3), (7).*

This problem is the first boundary value problem for linear loaded quasi-elliptic equation. Using method of continuation in the parameter (see [25]), it is not difficult to establish its solvability in the space $W_2^{4,2}(Q)$.

Let λ be a number from the segment $[0,1]$. Consider the boundary value problem: *find a function $v(x,t)$ that is a solution to equation*

$$v_{tt} + a^2\Delta v - [\psi(t) + \varphi(t)G_\gamma(\Phi(t; w + v_0))]v - \varepsilon\Delta^2 v = f_2(x,t) + \lambda\varphi(t)v_0(x,t)\Phi(t;v) \quad (1'_{\varepsilon,w,\lambda})$$

in the cylinder Q and satisfies conditions $(2')$, *(3) and (7).* For $\lambda = 0$, this problem is solvability in the space $W_2^{4,2}(Q)$ (this is not difficult to prove using the classical Galerkin method with the choice of a special basis [21]). Next, all possible solutions $v(x,t)$ of the boundary value problem $(1'_{\varepsilon,w,\lambda})$, $(2')$, (3), (7) at a fixed ε satisfy estimate

$$\left(1 - \frac{\varphi_0^2 N_1 T}{2}\|N\|^2_{L_2(\Omega)}\right)\int_Q (\Delta v_t)^2\,dx\,dt + \frac{a^2}{2}\sum_{i=1}^n \int_Q (\Delta v_{x_i})^2\,dx\,dt +$$

$$+\varepsilon\int_Q \left(\Delta^2 v\right)^2\,dx\,dt \leqslant C_1\int_Q f_2^2\,dx\,dt, \tag{8}$$

with the constant C_1 defined only by ε. In order to prove this estimate we multiply the equation $(1'_{\varepsilon,w,\lambda})$ by the function $-\Delta^2 v$ and integrate on the cylinder Q. Using $\psi(t) + \varphi(t)G_\gamma(\Phi(t; w + v_0)) \geqslant 0$ and applying Hölder's and Young's inequality, and also inequality

$$\int_\Omega [\Delta v(x,t)]^2\,dx \leqslant T\int_Q [\Delta v_t(x,t)]^2\,dx\,dt$$

we obtain the estimate (8). From estimate (8) and the second main inequality for elliptic operator (see [21]) it follows that all possible solutions $v(x,t)$ to boundary value problem $(1'_{\varepsilon,w,\lambda})$, $(2')$, (3), (7) for a fixed ε satisfy the a priori estimate

$$\|v\|_{W_2^{4,2}(Q)} \leqslant C_2\|f_2\|_{L_2(Q)} \tag{9}$$

with the constant C_2 defined only by the domain Ω, the functions $\mu(t)$, $N(x)$, $u_0(x)$ and $u_1(x)$ and numbers a, T,ε. According to the theorem on the method of continuation in a parameter [25, ch. III, Sec. 14], solvability of the boundary value problem $(1'_{\varepsilon,w,0})$, $(2')$, (3), (7) in $W_2^{4,2}(Q)$ and estimate (9) imply that the problem $(1'_{\varepsilon,w})$, $(2')$, (3), (7) has a solution $v(x,t)$ lying in the space $W_2^{4,2}(Q)$.

Held arguments signify that the boundary value problem $(1'_{\varepsilon,w})$, $(2')$, (3), (7) generates the operator A, taking the space $W_2^{4,2}(Q)$ to itself: $A(w) = v$. We show that for the operator A, all the conditions of Schauder's fixed point theorem are satisfied.

Observe first of all that from the estimate (9) it follows that the operator A takes a closed ball of radius $R_0 = C_2\|f_2\|_{L_2(Q)}$ of space $W_2^{4,2}(Q)$ to itself.

We now show that the operator A will be continuous on a closed ball of radius R_0 of the space $W_2^{4,2}(Q)$.

Let $\{w_m(x,t)\}_{m=1}^\infty$ be a sequence of functions from this ball converging in the space $W_2^{4,2}(Q)$ to the function $\overline{w}(x,t)$. Let $v_m(x,t)$, $\overline{v}(x,t)$ be images of functions $w_m(x,t)$ and $\overline{w}(x,t)$ under action of the operator A. There are equalities

$$v_{mtt} - \overline{v}_{tt} + a^2\Delta(v_m - \overline{v}) - \varepsilon\Delta^2(v_m - \overline{v}) - [\varphi(t)G_\gamma(\Phi(t; w_m + v_0)) + \psi(t)](v_m - \overline{v}) =$$

$$= \varphi(t)[G_\gamma(\Phi(t; w_m + v_0)) - G_\gamma(\Phi(t; \overline{w} + v_0))]\overline{v} + \varphi(t)v_0(x,t)\Phi(t; v_m - \overline{v}), \quad (x,t) \in Q,$$

$$v_m(x,0) - \overline{v}(x,0) = v_m(x,T) - \overline{v}(x,T) = 0, \quad x \in \Omega,$$

$$v_m(x,t) - \overline{v}(x,t)|_S = \Delta(v_m(x,t) - \overline{v}(x,t))|_S = 0.$$

These equalities mean that the functions $v_m(x,t) - \overline{v}(x,t)$ are solutions to the first boundary value problem for the linear quasi-elliptic "loaded" equation $(1'_{\varepsilon,w})$. Note that the function $G_\gamma(\xi)$ satisfies the Lipschitz condition and $\overline{v}(x,t) \in W_2^{4,2}(Q)$. Repeating the proof of the estimate (9) and applying the Holder's inequality, we get inequality

$$\|v_m - \overline{v}\|_{W_2^{4,2}(Q)} \leqslant C_3 \|w_m - \overline{w}\|_{L_2(Q)} \tag{10}$$

with constant C_3, defined by the functions $\mu(t)$, $N(x)$, $u_0(x)$ and $u_1(x)$, as well as the numbers a, T, and ε. From this inequality and from the convergence of the sequence $\{w_m(x,t)\}_{m=1}^\infty$ in space $W_2^{4,2}(Q)$ to the function $\overline{w}(x,t)$ it follows that the sequence $\{v_m(x,t)\}_{m=1}^\infty$ converges in the same space to the function $\overline{v}(x,t)$. This means that the operator A is continuous on a closed ball of radius R_0 of the space $W_2^{4,2}(Q)$.

We show that the operator A is compact on a closed ball of radius R_0 of the space $W_2^{4,2}(Q)$.

Let $\{w_m(x,t)\}_{m=1}^\infty$ be a family of functions from this ball. Let $\{v_m(x,t)\}_{m=1}^\infty$ be a family of images of functions $w_m(x,t)$ under the action of the operator A. Boundedness of families $\{w_m(x,t)\}_{m=1}^\infty$ in the space of $W_2^{4,2}(Q)$ and the classical embedding theorems [20–22] imply that there is a subsequence $\{w_{m_k}(x,t)\}_{k=1}^\infty$, strongly convergent in the space $L_2(Q)$. Repeating for the difference $v_{m_k}(x,t) - v_{m_{k+l}}(x,t)$ proof of the estimate (10), it is easy to obtain, that the sequence $\{v_{m_k}(x,t)\}_{k=1}^\infty$ is the fundamental in the space $W_2^{4,2}(Q)$. And this means that the operator A is compact on the ball of radius R_0 of the space $W_2^{4,2}(Q)$.

So, the operator A takes a ball of radius R_0 of the space $W_2^{4,2}(Q)$ to itself. The operator A is continuous and compact on this ball. According to Schauder's theorem, in the indicated ball there is at least one function $v(x,t)$, for which holds $A(v) = v$. This function $v(x,t) \in W_2^{4,2}(Q)$ is solution of the boundary value problem $(1'_\varepsilon)$, $(2')$, (3), (7). Show that the solutions $v(x,t)$ satisfy a priori estimates uniform in ε.

Consider the equality

$$-\int_Q \{v_{tt} + a^2\Delta v - [\psi(t) + \varphi(t)G_\gamma(\Phi(t; v + v_0))]v - \varepsilon\Delta^2 v\}\Delta^2 v \, dx \, dt =$$

$$= -\int_Q f_2 \Delta^2 v \, dx \, dt - \int_Q \varphi(t)v_0(x,t)\Phi(t; v)\Delta^2 v \, dx \, dt.$$

Integrating by parts and applying the Cauchy–Bunyakovsky and Young inequalities, we conclude that this equality implies the estimate

$$(1 - N_2)\int_Q (\Delta v_t)^2 \, dx \, dt + \frac{a^2}{2}\sum_{i=1}^n \int_Q (\Delta v_{x_i})^2 \, dx \, dt \leqslant N_3^{1/2}\left(\sum_{i=1}^n \int_Q (\Delta v_{x_i})^2 \, dx \, dt\right)^{1/2}.$$

It is easy to show that there are estimates

$$\sum_{i=1}^n \int_Q (\Delta v_{x_i})^2 \, dx \, dt \leqslant \frac{4N_3}{a^4}, \quad \int_Q (\Delta v_t)^2 \, dx \, dt \leqslant \frac{2N_3}{a^2(1 - N_2)} = N_4.$$

Summing up the inequalities and also using the second main inequality for elliptic operators, we obtain that solutions solutions$v(x,t)$ to the boundary value problem $(1'_\varepsilon)$, $(2')$, (3), (7) satisfy the estimates

$$|\Phi(t; v + v_0)| \leqslant N_5, \tag{11}$$

$$\|v\|_{W_2^2(Q)} + \|\Delta v\|_{W_2^1(Q)} + \sqrt{\varepsilon}\|\Delta^2 v\|_{L_2(Q)} \leqslant C_4, \tag{12}$$

with the constant C_4 в (12) defined by the functions $\mu(t)$, $N(x)$, $u_0(x)$ и $u_1(x)$, and the numbers a и T.

The estimate (12) and the reflexivity of a Hilbert space imply that there exist sequences $\{\varepsilon_m\}_{m=1}^\infty$ of positive numbers and $\{v_m(x,t)\}_{m=1}^\infty$ of solutions to the boundary value problem $(1'_{\varepsilon_m})$, $(2')$, (3), (7) and also a function $v(x,t)$ such that, as $m \to \infty$, the convergences

$$\varepsilon_m \to 0, \qquad v_m(x,t) \to v(x,t) \quad \text{strongly in} \quad W_2^2(Q),$$

$$\varepsilon_m \Delta^2 v_m(x,t) \to 0 \quad \text{weakly in} \quad L_2(Q)$$

hold.

Obviously, the limit function $v(x,t)$ will be a solution to the boundary value problem $(1'_0)$, $(2')$, (3), and due to estimate (12) for this solution will be the inclusions $v(x,t) \in W_2^2(Q)$, $\Delta v(x,t) \in W_2^1(Q)$.

Let us fix the number γ: $\gamma = \dfrac{\psi_0}{\varphi_0}$. Let us define the functions $u(x,t)$ and $q(t)$:

$$u(x,t) = v(x,t) + v_0(x,t), \quad q(t) = \psi(t) + \varphi(t)\Phi(t;u).$$

Estimate (11) and the inequality from the condition of the theorem for the number N_5 mean that the equality $G_\gamma(\Phi(t;u)) = \Phi(t;u)$ holds, and that $q(t) \geqslant 0$ $\forall t \in [0,T]$. Obviously, the functions $u(x,t)$ and $q(t)$ will be related in the cylinder Q by equation (1). Let's show that for the function $u(x,t)$ the overdetermination condition (4) will be satisfied.

We multiply equation (1) by the function $N(x)$ and integrate over the domain Ω. Taking into account the form of the functions $\varphi(t)$, $\psi(t)$, $\Phi(t;u)$ and consistency conditions for of the functions $u_0(x)$, we obtain that the function $\alpha(t)$ satisfies the problem

$$\alpha''(t) - q(t)\alpha(t) = 0, \qquad \alpha(0) = \alpha(T) = 0. \tag{13}$$

Since $q(t) \geqslant 0$, then $\alpha(t) \equiv 0$. This means that the function $u(x,t)$ satisfies the overdetermination condition (4). The theorem is proved. □

The study of the solvability of the inverse problem II differs only in insignificant details from the above study of the solvability of the inverse problem I.

Let

$$N_6 = \sqrt{2}\varphi_0 \left(\int_\Omega \left[u_0^2(x) + u_1^2(x)\right] dx \right)^{1/2} \|N\|_{L_2(\Omega)},$$

$$N_7 = \frac{aT^{1/2}}{(1-N_6)^{1/2}} \left(\sum_{i=1}^n \int_\Omega N_{x_i}^2\, dx \right)^{1/2} \|f_2\|_{L_2(Q)} + |\Phi(0,u_0)| + |\Phi(0,u_1)|.$$

Theorem 2.2. *Suppose the fulfillment of conditions*

$$N(x) \in W_2^1(\Omega), \quad \mu(t) \in C^2([0,T]); \quad f(x,t) \in L_2(0,T; W_2^1(\Omega)),$$

$$u_0(x) \in W_2^3(\Omega), \quad u_1(x) \in W_2^3(\Omega); \quad \varphi_0 > 0, \quad \psi_0 > 0, \quad N_6 < 1, \quad N_7 \leqslant \frac{\psi_0}{\varphi_0},$$

$$\mu(0) = \int_\Omega N(x)u_0(x)\,dx, \quad \mu(T) = \int_\Omega N(x)u_1(x)\,dx.$$

Then inverse problem II has a solution $\{u(x,t), q(t)\}$ *such that* $u(x,t)\in W_2^2(Q)$, $q(t)\in L_\infty([0,T])$, $q(t) \geqslant 0$ *for* $t \in [0,T]$.

3. Solvability of the inverse Problems III

We introduce the function $\Phi_1(t;w)$: $\Phi_1(t;w) = a^2 \int\limits_\Gamma N(x)\Delta w(x,t)\,ds_x$, where $w(x,t)$ is some given function.

Introduce the notations $F_0(t) = \int\limits_\Gamma N(x)f(x,t)\,ds_x$, $\psi_1(t) = \varphi(t)[\mu''(t) - F_0(t)]$,

$$\widetilde{f}_2(x,t) = f_1(x,t) + [\psi_1(t) + \varphi(t)\Phi_1(t;v_0)]v_0(x,t).$$

Consider the boundary value problem: *Find a function* $v(x,t)$ *that is a solution to equation*

$$v_{tt} + a^2\Delta v - [\varphi(t)\Phi_1(t;v+v_0) + \psi(t)]v = \widetilde{f}_2(x,t) + \varphi(t)v_0(x,t)\Phi_1(t;v) \tag{14}$$

in the cylinder Q *and satisfies conditions* $(2')$ *and* (5). A solution $v(x,t)$ to this problem will provide an opportunity construct the required solution to the inverse problem III.

The function $w(x) \in W_2^1(\Omega)$ satisfies the inequality

$$\int_\Gamma w^2(x)\,ds_x \leqslant c_0 \int_\Omega \left[w^2(x) + \sum_{i=1}^n w_{x_i}^2(x)\right]\,dx \tag{15}$$

with a constant c_0 determined only by the domain Ω (see [20, 21]).

Let us specify again that the function $v_0(x,t)$ satisfies the inequality

$$\sum_{i=1}^n \int_\Omega v_{0x_i}^2(x,t)\,dx \leqslant 2\sum_{i=1}^n \int_\Omega \left[u_{0x_i}^2(x) + u_{1x_i}^2(x)\right]\,dx. \tag{16}$$

As before, we define the required constants:

$$\psi_1 = \min_{[0,T]} \psi_1(t), \qquad N_8 = \left(\sum_{i=1}^n \int_Q \widetilde{f}_{2x_i}^2\,dx\,dt\right)^{1/2}, \quad N_9 = \frac{c_0\varphi_0^2 N_1 T^2}{4a^2}\|N\|_{L_2(\Gamma)}^2,$$

$$N_{10} = \frac{c_0\varphi_0^2 N_1}{4a^2}, \quad N_{11} = \frac{2N_8}{a^2 - 2N_9}, \quad N_{12} = \frac{N_8 N_{11}}{1 - N_{10}},$$

$$N_{13} = \sqrt{2}\varphi_0 \int_\Omega \left[(\Delta u_0)^2 + (\Delta u_1)^2\right]\,dx\left(N_9 N_{11}^2 + N_{10}N_{12}\right)^{1/2} + \left(\int_Q \left(\Delta\widetilde{f}_2\right)^2\,dx\,dt\right)^{1/2},$$

$$N_{14} = a^2\|N\|_{L_2(\Gamma)}^2 (c_0 T)^{1/2}\left[N_{12} + \frac{N_{13}^2}{a^2}\right]^{1/2} + |\Phi_1(0;u_0)| + |\Phi_2(0;u_1)|.$$

Theorem 3.1. *Suppose the fulfillment of conditions* $N(x) \in L_2(\Gamma)$, $\mu(t) \in C^2([0,T])$;

$$f(x,t) \in L_2(0,T;W_2^2(\Omega)), \quad u_0(x) \in W_2^3(\Omega), \quad u_1(x) \in W_2^3(\Omega);$$

$$\frac{\partial f(x,t)}{\partial \nu} = \frac{\partial \Delta u_0(x)}{\partial \nu} = \frac{\partial \Delta u_1(x)}{\partial \nu} = 0 \quad \textit{for} \quad x \in \Gamma;$$

$$\varphi_0 > 0, \quad \psi_1 > 0, \quad N_{10} < 1, \quad a^2 - 2N_9 > 0, \quad N_{14} \leqslant \frac{\psi_1}{\varphi_0}.$$

$$\mu(0) = \int_\Gamma N(x) u_0(x)\, dx, \quad \mu(T) = \int_\Gamma N(x) u_1(x)\, dx.$$

Then inverse problem III has a solution $\{u(x,t), q(t)\}$ *such that* $u(x,t) \in W_2^2(Q)$, $\Delta u(x,t) \in W_2^1(Q)$, $q(t) \in L_\infty([0,T])$, $q(t) \geqslant 0$ *for* $t \in [0,T]$.

Proof. Let γ be a number from the interval $\left(0, \frac{\varphi_0}{\psi_1}\right]$, $\varepsilon > 0$.

Consider the boundary value problem: *Find a function* $v(x,t)$ *that is a solution to equation*

$$v_{tt} + a^2 \Delta v - [\psi_1(t) + \varphi(t) G_\gamma(\Phi_1(t; v + v_0))] v - \varepsilon \Delta^2 v = \widetilde{f}_2(x,t) + \varphi(t) v_0(x,t) \Phi_1(t;v) \quad (14'_\varepsilon)$$

in the cylinder Q *and satisfies conditions* $(2')$, (5), *and*

$$\left.\frac{\partial(\Delta v)}{\partial \nu}\right|_S = \left.\frac{\partial(\Delta^2 v)}{\partial \nu}\right|_S = 0. \quad (17)$$

Using the fixed point method and the method of continuation in a parameter, it is easy to show that for a fixed ε and for satisfying the conditions of the theorem, this problem has a solution $v(x,t)$ such that $v(x,t) \in W_2^2(Q)$, $\Delta^2 v(x,t) \in L_2(Q)$. Let us show that the function $v(x,t)$ satisfies a priori estimates uniform in ε.

Multiply equation $(14'_\varepsilon)$ by the function $\Delta^2 v(x,t)$ and integrate over the cylinder Q. We obtain the equality

$$\int_Q (\Delta v_t)^2\, dx\, dt + a^2 \sum_{i=1}^n \int_Q (\Delta v_{x_i})^2\, dx\, dt + \varepsilon \int_Q \left(\Delta^2 v\right)^2\, dx\, dt +$$

$$+ \int_Q [\psi_1(t) + \varphi(t) G_\gamma(\Phi_1(t; v+v_0))] (\Delta v)^2\, dx\, dt = \sum_{i=1}^n \int_Q \widetilde{f}_{2x_i} \Delta v_{x_i}\, dx\, dt -$$

$$- \sum_{i=1}^n \int_Q \varphi(t) \Phi_1(t;v) v_{0x_i} \Delta v_{x_i}\, dx\, dt. \quad (18)$$

Let us introduce the notation: $I_1 = \int\limits_Q (\Delta v_t)^2\, dx\, dt$, $\quad I_2 = \sum\limits_{i=1}^n \int\limits_Q (\Delta v_{x_i})^2\, dx\, dt$.

Taking into account the notation introduced above and using the Young's and Cauchy - Bunyakovsky's inequalities it is easy from the equality (18) go to inequality

$$I_1 + \frac{a^2}{2} I_2 \leqslant N_8 I_2^{1/2} + \frac{\varphi_0^2 N_1}{2a^2} \int_0^T \Phi_1^2(t;v)\, dt. \quad (19)$$

There is an estimate

$$\int_0^T \Phi_1^2(t;v)\, dt \leqslant N_9 I_2 + N_{10} I_1. \quad (20)$$

Summing up, we obtain a consequence of inequalities (19) и (20):

$$I_1 + \frac{a^2}{2} I_2 \leqslant N_8 I_2^{1/2} + N_9 I_2 + N_{10} I_1. \tag{21}$$

Elementary calculations allow us to derive from (21) the estimate

$$I_2 \leqslant N_{11}^2, \tag{22}$$

and further, the estimate

$$I_1 \leqslant N_{12}. \tag{23}$$

Equality (18) and estimates (22), (23) imply the boundedness of the first term on the left side of($14'_\varepsilon$):

$$\varepsilon \int_Q v_{tt}^2 \, dx \, dt \leqslant C_5. \tag{24}$$

Here the constant C_5 is determined by the functions $f(x,t)$, $u_0(x)$, $u_1(x)$, $N(x)$, $\mu(t)$, numbers a and T (the exact value of the number C_5 is not important).

Multiply equation ($14'_\varepsilon$) by the function $\Delta^2 v(x,t)$ and integrate over the cylinder Q. We obtain the equality

$$\sum_{i=1}^n \int_Q (\Delta v_{x_i t})^2 \, dx \, dt + a^2 \int_Q \left(\Delta^2 v\right)^2 \, dx \, dt +$$

$$+ \sum_{i=1}^n \int_Q [\phi_1(t) + \varphi(t) G_\gamma(\Phi_1(t; v + v_0))] (\Delta v_{x_i})^2 \, dx \, dt + \varepsilon \int_Q \left(\Delta^3 v\right)^2 \, dx \, dt =$$

$$= \int_Q \Delta \widetilde{f}_2 \Delta^2 v \, dx \, dt + \int_Q \varphi(t) \Phi_1(t; v) \Delta v_0 \Delta^2 v \, dx \, dt. \tag{25}$$

An inequality similar to the inequality (18) holds:

$$\int_\Omega [\Delta v_0(x,t)]^2 \, dx \leqslant 2 \int_\Omega \left[(\Delta u_0)^2 + (\Delta u_1)^2\right] \, dx.$$

Using this inequality, Hölder's inequality and estimates (20), (22), (23), we obtain from (25) the inequality:

$$a^2 \int_Q \left(\Delta^2 v\right)^2 \, dx \, dt \leqslant \left(\int_Q (\Delta^2 v)^2 \, dx \, dt\right)^{1/2} \left[\left(\int_Q \left(\Delta \widetilde{f}_2\right)^2 \, dx \, dt\right)^{1/2} + \right.$$

$$\left. + \sqrt{2} \varphi_0 \left(\int_\Omega \left[(\Delta u_0)^2 + (\Delta u_1)^2\right] \, dx\right)^{1/2} \left(N_9 N_{11}^2 + N_{10} N_{12}\right)^{1/2}\right].$$

This inequality and again from equality (25) imply the estimates

$$\int_Q \left(\Delta^2 v\right)^2 \, dx \, dt \leqslant \frac{N_{13}^2}{a^4}, \tag{26}$$

$$\varepsilon \int_Q \left(\Delta^3 v\right)^2 \, dx \, dt + \sum_{i=1}^n \int_Q (\Delta v_{x_i t})^2 \, dx \, dt \leqslant \frac{N_{13}^2}{a^2}. \tag{27}$$

Estimates (22)–(24), (27), estimates for solutions of elliptic equations (see [21]) and also the reflexivity of a Hilbert space imply that there exist sequences $\{\varepsilon_m\}_{m=1}^\infty$ of positive numbers and

$\{v_m(x,t)\}_{m=1}^{\infty}$ to the boundary value problems ($14'_{\varepsilon_m}$), ($2'$), (5), (17) and also a function $v(x,t)$ such that, as $m \to \infty$, the convergences

$$\varepsilon_m \to 0, \qquad v_m(x,t) \to v(x,t) \quad \text{weakly in} \quad W_2^2(Q),$$
$$\Delta v_m(x,t) \to \Delta v(x,t) \quad \text{weakly in} \quad W_2^1(Q),$$
$$\Delta v_m(x,t) \to \Delta v(x,t) \quad \text{strongly in} \quad L_2(\Gamma),$$
$$\varepsilon_m \Delta^3 v_m(x,t) \to 0 \quad \text{weakly in} \quad L_2(Q)$$

hold. The limit function $v(x,t)$ satisfies the equation

$$v_{tt} + a^2\Delta v - [\psi_1(t) + \varphi(t)G_\gamma(\Phi_1(t; v+v_0))]v = \widetilde{f}_2(x,t) + \varphi(t)v_0(x,t)\Phi_1(t;v),$$

and the conditions ($2'$), (5). The function $v(x,t)$ belongs to $W_2^2(Q)$ and $\Delta v(x,t) \in W_2^1(Q)$, $\Delta^2 v(x,t) \in L_2(Q)$, $\Delta v_{x_i t}(x,t) \in L_2(Q)$, $i = 1,\dots,n$. The following inequalities

$$|\Phi_1(t; v+v_0)| \leqslant |\Phi_1(t;v)| + |\Phi_1(t;v_0)| \leqslant a^2\|N\|_{L_2(\Gamma)} \left(\int_\Gamma (\Delta v)^2\, ds\right)^{1/2} + |\Phi(0;u_0)| + |\Phi(0;u_1)| \leqslant$$
$$\leqslant a^2 c_0^{1/2}\|N\|_{L_2(\Gamma)} \left[\int_\Omega (\Delta v)^2\, dx + \sum_{i=1}^n \int_\Omega (\Delta v_{x_i})^2\, dx\right]^{1/2} + |\Phi(0;u_0)| + |\Phi(0;u_1)| \leqslant$$
$$\leqslant a^2 (c_0T)^{1/2}\|N\|_{L_2(\Gamma)} \left[\int_Q (\Delta v_t)^2\, dx\,dt + \sum_{i=1}^n \int_Q (\Delta v_{x_i t})^2\, dx\right]^{1/2} + |\Phi(0;u_0)| +$$
$$+|\Phi(0;u_1)| \leqslant a^2(c_0T)^{1/2}\|N\|_{L_2(\Gamma)}\left[\frac{N_{13}^2}{a^2} + N_{12}\right]^{1/2} + |\Phi(0;u_0)| + |\Phi(0;u_1)| = N_{14} \tag{28}$$

hold.

Let $\gamma = \dfrac{\psi_1}{\varphi_0}$. Due to the condition $N_{14} \leqslant \dfrac{\psi_1}{\varphi_0}$ it follows from (28) that $G_\gamma(\Phi_1(t;v+v_0)) = \Phi_1(t; v+v_0)$. Let us define the functions $u(x,t)$ и $q(t)$:

$$u(x,t) = v(x,t) + v_0(x,t), \quad q(t) = \psi_1(t) + \varphi(t)\Phi_1(t;u).$$

It is these functions that give the required solution to the inverse problem III (which is shown as in the proof of Theorem 2.1). The theorem is proved. □

4. Uniqueness of solutions

The following theorems give conditions under which the inverse problems I – III can only have one solution.

Let $W_{R_0} = \left\{ v(x,t):\ v(x,t) \in W_2^2(Q), \quad \operatorname*{vrai\,max}_{[0,T]} \left(\int\limits_\Omega v^2(x,t)\, dx\right) \leqslant R_0 \right\}$.

Theorem 4.1. *Let $\{u_1(x,t), q_1(t)\}$, $\{u_2(x,t), q_2(t)\}$ be two solutions of the inverse problem I such that $u_i(x,t) \in W_{R_0}$, $q_i(t) \in L_\infty([0,T])$, $q_i(t) \geqslant 0$ for $t \in [0,T]$, $i = 1,2$. Suppose the fulfillment of the conditions*

$$N(x) \in L_2(\Omega), \quad \mu(t) \in C^2([0,T]), \quad f(x,t) \in L_\infty(0,T; L_2(\Omega)); \quad \varphi_0 > 0, \quad \varphi_0 R_0^{1/2}\|N\|_{L_2(\Omega)} < 1.$$

Then the functions $u_1(x,t)$ and $u_2(x,t)$ coincide almost everywhere in Q, the functions $q_1(t)$ and $q_2(t)$ coincide for almost all t from the segment $[0,T]$.

Proof. The function $w(x,t) = u_1(x,t) - u_2(x,t)$ satisfies the following problem

$$w_{tt} + a^2\Delta w - q_1(t)w = \varphi(t)\Phi(t;w)u_2, \quad (x,t) \in Q;$$

$$w(x,0) = w(x,T) = 0, \quad x \in \Omega;$$

$$w(x,t)|_S = 0.$$

We multiply the equation by the function $\Delta w(x,t)$ and integrate over the cylinder Q. Taking into account the nonnegativity of the function $q_1(t)$ and the boundary conditions, applying Hölder's inequality, we obtain the inequality $\int\limits_Q (\Delta w)^2\,dx\,dt \leqslant 0$. This inequality implies that the functions $u_1(x,t)$ and $u_2(x,t)$ coincide almost everywhere in Q. But then the functions $q_1(t)$ и $q_2(t)$ coincide for almost of all t from the segment $[0,T]$. The theorem is proved. □

Theorem 4.2. *Let $\{u_1(x,t), q_1(t)\}$, $\{u_2(x,t), q_2(t)\}$ be two solutions of the inverse problem II such that $u_i(x,t) \in W_{R_0}$, $q_i(t) \in L_\infty([0,T])$, $q_i(t) \geqslant 0$ for $t \in [0,T]$, $i = 1,2$. Suppose the assumptions of Theorem 2.2 are fulfilled. Then the functions $u_1(x,t)$ and $u_2(x,t)$ coincide almost everywhere in Q, the functions $q_1(t)$ and $q_2(t)$ coincide for almost all t from the segment $[0,T]$.*

The proof of this theorem is quite similar to the proof of Theorem 4.1.

Let

$$\widetilde{W}_{R_0} = \left\{ v(x,t):\ v(x,t) \in W_2^2(Q), \Delta v(x,t) \in W_2^1(Q), \underset{[0,T]}{\text{vrai}\max} \left(\sum_{i=1}^{n} \int_\Omega v_{x_i}^2(x,t)\,dx \right) \leqslant R_0 \right\}.$$

Theorem 4.3. *Let $\{u_1(x,t), q_1(t)\}$, $\{u_2(x,t), q_2(t)\}$ be two solutions of the inverse problem III such that $u_i(x,t) \in \widetilde{W}_{R_0}$, $q_i(t) \in L_\infty([0,T])$, $q_i(t) \geqslant 0$ for $t \in [0,T]$, $i = 1,2$. Suppose the fulfillment of the conditions*

$$N(x) \in L_2(\Gamma), \quad \mu(t) \in C^2([0,T]), \quad f(x,t) \in L_2(Q) \cap L_\infty(0,T;L_2(\Gamma));$$

$$\varphi_0 > 0, \quad \varphi_0 (c_0 R_0)^{1/2} \|N\|_{L_2(\Gamma)} < \min\left(\frac{2}{3}, \frac{4}{a^2T^2}\right).$$

Then the functions $u_1(x,t)$ and $u_2(x,t)$ coincide almost everywhere in Q, the functions $q_1(t)$ and $q_2(t)$ coincide for almost all t from the segment $[0,T]$.

Proof. The function $w(x,t) = u_1(x,t) - u_2(x,t)$ satisfies the following problem

$$w_{tt} + a^2\Delta w - q_1(t)w = \varphi(t)\Phi_1(t;w)u_2, \quad (x,t) \in Q; \tag{29}$$

$$w(x,0) = w(x,T) = 0, \quad x \in \Omega; \tag{30}$$

$$\left.\frac{\partial w(x,t)}{\partial \nu}\right|_S = 0. \tag{31}$$

Equalities (29) and (31) imply, in particular, the property

$$\left.\frac{\partial \Delta w(x,t)}{\partial \nu}\right|_S = 0. \tag{32}$$

Further, using the procedure for approximating the function $w(x,t)$ by smooth functions while maintaining the property (32), it is easy to show that the equality holds (formally obtained by multiplying equation (29) by the function $\Delta^2 w$ and integrating over Q)

$$\int_Q (\Delta w_t)^2\,dx\,dt + a^2\sum_{i=1}^{n}\int_Q (\Delta w_{x_i})^2\,dx\,dt + \int_Q q(t)(\Delta w)^2\,dx\,dt =$$

$$= \int_Q \varphi(t)\Phi_1(t;w)\left(\sum_{i=1}^{n} u_{2x_i}\Delta w_{x_i}\right)dx\,dt. \qquad (33)$$

We obtain an estimate for the right-hand side of the inequality (33). Using the Cauchy-Bunyakovsky's and Hölder's inequalities, the condition (32) and estimate

$$\int_Q (\Delta w)^2\,dx\,dt \leqslant \frac{T^2}{2}\int_Q (\Delta w_t)^2\,dx\,dt,$$

we obtain the inequality

$$\int_Q (\Delta w_t)^2\,dx\,dt + a^2\sum_{i=1}^{n}\int_Q (\Delta w_{x_i})^2\,dx\,dt \leqslant$$

$$\leqslant \frac{1}{4}a^2T^2\varphi_0(c_0R_0)^{1/2}\|N\|_{L_2(\Gamma)}\int_Q (\Delta w_t)^2\,dx\,dt + \frac{3}{2}a^2\varphi_0(c_0R_0)^{1/2}\|N\|_{L_2(\Gamma)}\sum_{i=1}^{n}\int_Q (\Delta w_{x_i})^2\,dx\,dt.$$

This inequality and the conditions of the theorem imply the identities $\Delta w_t(x,t)\equiv 0$, $\Delta w_{x_i}\equiv 0$ for $(x,t)\in Q$, $i=1,\ldots,n$, and further follows the identity $w(x,t)\equiv 0$ for $(x,t)\in Q$. The last identity means that he functions $u_1(x,t)$ and $u_2(x,t)$ coincide almost everywhere in Q. But then the functions $q_1(t)$ и $q_2(t)$ coincide for almost of all t from the segment $[0,T]$. The theorem is proved. □

5. Comments and appendices

1. Let us show that the set of input data of inverse problems I–III, for which all conditions of the existence and uniqueness theorems are satisfied, is not empty.

Let $u_0(x)$ and $u_1(x)$ be given nonnegative functions in Ω such that, in addition to the conditions of Theorem 2.1, they satisfy the conditions

$$\frac{\partial u_0(x)}{\partial\nu} = \frac{\partial u_1(x)}{\partial\nu} = 0 \quad \textit{for} \quad x\in\Gamma, \qquad \int_\Omega u_0(x)\,dx = \int_\Omega u_1(x)\,dx = 1.$$

Similar functions exist. For example, $u_0(x)=\alpha_0[\rho(x)]^{m_0}$, $u_1(x)=\alpha_1[\rho(x)]^{m_1}$, where $\rho(x)$ is the distance from the point $x\in\Omega$ to the boundary Γ, $m_0\geqslant 3$, $m_1\geqslant 3$. The multipliers α_0 and α_1 are selected so that the required integral equalities hold. Or $u_0(x)$ and $u_1(x)$ can be finite in $\overline{\Omega}$.

Let $N(x)\equiv 1$, $\mu(t)\equiv 1$, $f(x,t)=\widetilde{f}_0(x)$, $\widetilde{f}_0(x)<0$, $x\in\overline{\Omega}$. Then

$$\psi(t) = -\int_\Omega \widetilde{f}_0(x)\,dx = \psi_0 > 0, \quad \Phi(0,u_0)=\Phi(0,u_1)=0.$$

Obviously, the condition $N_2<1$ of Theorem 1 will hold for small numbers T, the number N_5 can also be made arbitrarily small by decreasing the number T. Hence, for the given functions $f(x,t)$, $u_0(x)$, $u_1(x)$, $\mu(t)$ and $N(x)$ for small T all conditions of the Theorem 1 will be satisfied.

Condition $N_6 < 1$ of Theorem 2.2 will be be executed if the functions $u_0(x)$ and $u_1(x)$ or the measure of the region Ω are small, the condition for the number N_7 will be run automatically.

The non-emptiness of the set of input data for which all conditions of Theorem 2.2 are satisfied is also easy to show. Take as $u_0(x)$, $u_1(x)$, $N(x)$ and $\mu(t)$ are identically constant functions, $f(x,t)$ is negative function in $\overline{Q}$. Condition $N_6 < 1$ of Theorem 2.2 will be be executed if the functions $u_0(x)$ and $u_1(x)$ or the measure of the domain Ω are small, the condition for the number N_7 будет will be run automatically.

Conditions of Theorem 3 are satisfied for small numbers T, if the functions $u_0(x)$, $u_1(x)$, $N(x)$ and $\mu(t)$ are identically constant functions, $f(x,t) > 0, (t,x) \in \overline{Q}$ and overdetermination conditions hold.

Obviously, the conditions of the uniqueness theorems (Theorems 4.1–4.3) will obviously be satisfied for small numbers R_0.

2. Inverse problems I − III can also be studied for equations that are more general than (1). Thus, the Laplace operator can be replaced an arbitrary second-order elliptic operator with variable coefficients, into the equation (1) low-order terms with first-order derivatives can be added. The essence of the results obtained is a more general form of the equation (1) will not change, but the number of calculations will increase.

3. If the conditions of existence theorems are satisfied, then for solutions $u(x,t)$ of inverse problems I, II, or III it is easy to establish estimates for quantities defining the sets W_{R_0} or $\widetilde{W}_{R_0}$. The constants in these estimates will be determined by the input data. Using further conditions of the respective theorems of uniqueness, it will be easy to obtain theorems that give both the existence and the uniqueness of solutions to inverse problems I, II, or III.

4. Conditions (3) or (5) in inverse problems I, II, or III can be inhomogeneous. Assuming that there are continuations of the given boundary data into the cylinder Q and using the technique of proving Theorems 2.1–2.2, Theorem 3.1, it will be possible to obtain the solvability of the inverse problems with nonzero boundary data.

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Обратные задачи восстановления младшего коэффициента в эллиптическом уравнении

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Abstract. Изучается разрешимость обратных задач восстановления неотрицательного коэффициента $q(t)$ в эллиптическом уравнении

$$u_{tt} + a^2 \Delta u - q(t)u = f(x,t)$$

($x = (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n$, $t \in (0,T)$, $0 < T < +\infty$, Δ — оператор Лапласа, действующий по переменным x_1, …, x_n). Вместе с естественными для эллиптических уравнений граничными условиями в изучаемых задачах задают также одно из дополнительных условий — либо условие пространственного интегрального переопределения, либо же условие граничного интегрального переопределения. Доказываются теоремы существования и единственности решений.

Ключевые слова: эллиптические уравнения, неизвестный коэффициент, пространтсвенное интегральное переопределение, граничное интегральное переопределение, существование, единственность.