ISSN 1997-1397 (Print)
ISSN 2313-6022 (Online)

Журнал Сибирского федерального университета

Математика и физика

Journal of Siberian Federal University

Mathematics & Physics

2022 15 (3)

ISSN 1997-1997-1397 (Print)

ISSN 2313-6022 (Online)

ЖУРНАЛ СИБИРСКОГО ФЕДЕРАЛЬНОГО УНИВЕРСИТЕТА
Математика и Физика

2022 15 (3)

JOURNAL OF SIBERIAN FEDERAL UNIVERSITY
Mathematics & Physics

Издание индексируется Scopus (Elsevier), Emerging Sources Citation Index (WoS, Clarivate Analytics), Российским индексом научного цитирования (НЭБ), представлено в международных и российских информационных базах: Ulrich's periodicals directiory, ProQuest, EBSCO (США), Google Scholar, MathNet.ru, КиберЛенинке.

Включено в список Высшей аттестационной комиссии «Рецензируемые научные издания, входящие в международные реферативные базы данных и системы цитирования».

Все статьи представлены в открытом доступе http://journal.sfu-kras.ru/en/series/mathematics_physics.

Журнал Сибирского федерального университета. Математика и физика.
Journal of Siberian Federal University. Mathematics & Physics.

Учредитель: Федеральное государственное автономное образовательное учреждение высшего образования "Сибирский федеральный университет"(СФУ)
Главный редактор: А.М. Кытманов. Редакторы: В.Е. Рализняк, А.В. Щуплев.
Компьютерная верстка: Г.В. Хрусталева
№ 3. 26.06.2022. Индекс: 42327. Тираж: 1000 экз. Свободная цена
Адрес редакции и издательства: 660041 г. Красноярск, пр. Свободный, 79, оф. 32-03.
Отпечатано в типографии Издательства БИК СФУ
660041 г. Красноярск, пр. Свободный, 82а.
Свидетельство о регистрации СМИ ПИ № ФС 77-28724 от 27.06.2007 г., выданное Федеральной службой по надзору в сфере массовых коммуникаций, связи и охраны культурного наследия
http://journal.sfu-kras.ru
Подписано в печать 15.06.22. Формат 84×108/16. Усл.печ. л. 12,1.
Уч.-изд. л. 11,9. Бумага тип. Печать офсетная.
Тираж 1000 экз. Заказ 15826
Возрастная маркировка в соответствии с Федеральным законом № 436-ФЗ:16+

CONTENTS

СОДЕРЖАНИЕ

DOI: 10.17516/1997-1397-2022-15-3-267-272
УДК 517.96

Numerical Simulation of Temperature and Thermal Stress Fields in a Carbon Block under External Thermal Effect

Evgeniy N. Vasil'ev*
Institute of Computational Modelling of SB RAS
Krasnoyarsk, Russian Federation

Received 23.11.2021, received in revised form 28.01.2022, accepted 20.03.2022

Abstract. The paper is devoted to modelling thermal and stress-strain state of a carbon block when it is partially immersed in an electrolyte. The temperature field in the block was determined from the solution of a non-stationary three-dimensional heat conduction equation. The calculation of temperature stresses was carried out on the basis of the solution of the Poisson equation for the thermoelastic displacement potential. The temperature fields in the carbon block were obtained at various points in time. The stress-strain field was also obtained. Then the location and magnitude of the maximal temperature stresses were determined. It allows one to assess the fracture of the carbon block.

Keywords: heat conduction equation, Poisson equation, temperature stresses, thermoelastic displacement potential, numerical simulation.

Citation: E.N. Vasil'ev, Numerical Simulation of Temperature and Thermal Stress Fields in a Carbon Block under External Thermal Effect, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 267–272.
DOI: 10.17516/1997-1397-2022-15-3-267-272.

Introduction

The technological process of aluminium production requires regular replacement of carbon blocks (anodes). In the industrial electrolytic cells, when the cold anode is initially immersed in a hot electrolytic solution at a temperature of about 960^oC, a heat wave propagates from the contact boundary into the volume of the anode. An increase in the local temperature causes thermal expansion of the anode material. The difference in the magnitude of the expansion of different zones of the anode leads to the occurrence of thermal stresses. In a zone of the highest temperature gradients significant thermal stresses arise which can exceed the ultimate strength of the material. It leads to the formation of cracks and further fracture of the anode. The phenomena that accompanies the process of immersing a cold carbon anode into the melt is called thermal shock [1, 2]. The state of the carbon block during thermal shock depends on the thermophysical (thermal conductivity, heat capacity) and mechanical (thermal expansion coefficient, shear modulus, Poisson's ratio, tensile strength) properties of graphite as well as on conditions of heat exchange with electrolyte. Numerical simulation allows one to analyse the state of the carbon block taking into account these factors.

*ven@icm.krasn.ru https://orcid.org/0000-0003-0689-2962

The aim of the work is to calculate the temperature field and the stress-strain field of the carbon block of the electrolytic cell. To describe the formation of thermal stresses, the mathematical modelling procedure includes two consecutive stages:

1. Determination of the temperature field in the volume of the carbon block is based on the solution of 3D heat conduction problem.
2. Calculation of thermal stresses is based on the solution of the Poisson equation for the obtained temperature field at various points in time.

1. Determination of the temperature field of the carbon block

The anode block is a parallelepiped made of carbon graphite (Fig. 1). In the electrolysis cell, the anode block is mounted using a steel bracket. The geometrical dimensions of the anode along the x, y, and z axes are $1450 \times 700 \times 600$ mm^3.

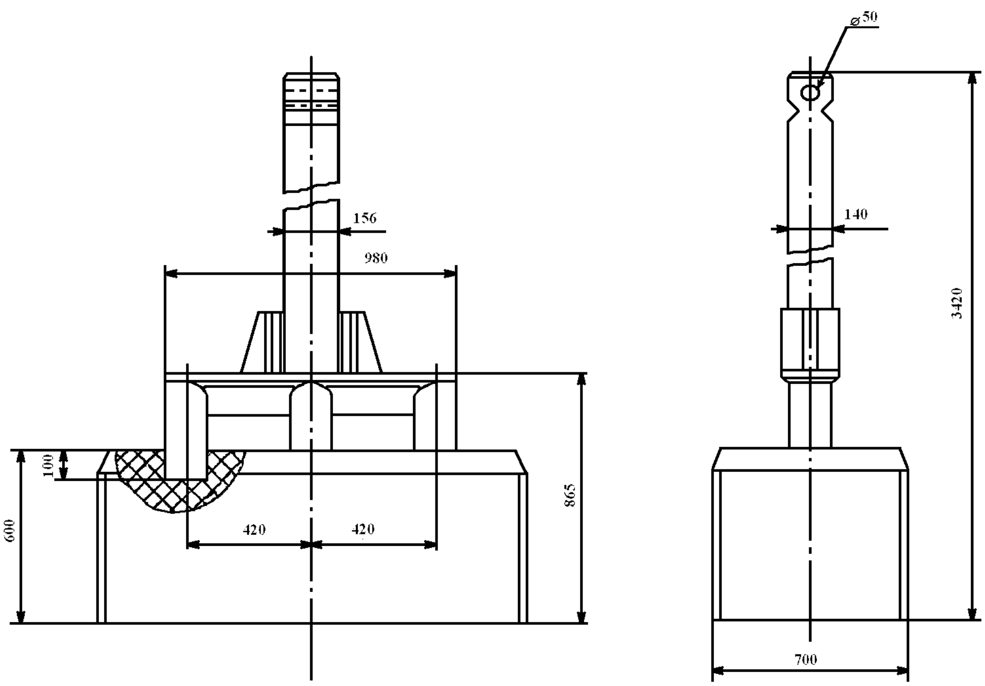


Fig. 1. The anode block of the industrial electrolytic cell

The heat transfer process in a carbon block is described by non-stationary three-dimensional heat conduction equation

$$c\rho\frac{\partial T}{\partial t} = \lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right), \qquad (1)$$

where c, ρ are specific volumetric heat capacity and density of the material; T is temperature; λ is the coefficient of thermal conductivity; t is time; x, y, z are spatial coordinates. The solution of equation (1) was obtained with the use of the method of finite differences with the splitting of the problem in spatial coordinates [3, 4].

Calculations were performed for the anode block shown in Fig. 1. The size of the part of the block immersed in the electrolyte is 120 mm. For graphite, the following thermophysical properties were set: $\lambda = 4.4$ W/(m·K), $c = 942$ J/(kg·K), $\rho = 1560$ kg/m^3 [5, 6].The heat exchange coefficient of the surface of the anode block with air $\beta_A = 10$ W/(m^2·K) and with the electrolyte solution $\beta_E = 18$ W/(m^2·K). A homogeneous spatial grid with the number of nodes $146 \times 71 \times 61$ was used for calculations, and the time step was 5 s.

The results of calculation of the temperature field of the anode for the moment of time $\Delta t = 15$ min are shown in Fig. 2. The temperature field of the lower part of the anode in the middle cross-section of the xz plane is shown in the left figure. Taking into account the symmetry of the problem, one quarter of the temperature field of the lower surface of the anode (plane xy) is shown in the right figure. The temperature values on the isolines are given in degrees Celsius. The most intense heating is observed in the zone where the unit is in contact with the electrolyte. In this area, the highest temperature is in the lower corner and and it is 467^{o}C.

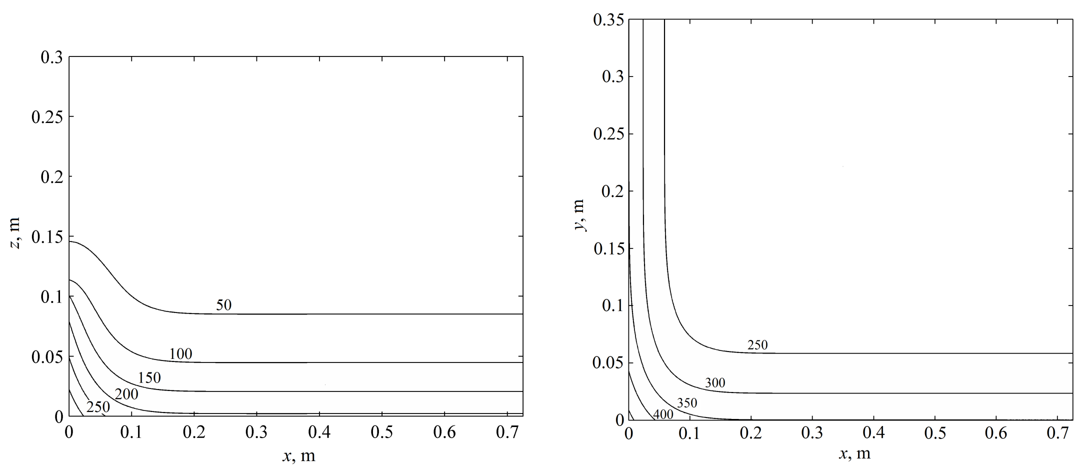


Fig. 2. Distribution of temperature in the middle xz and bottom xy planes of the anode block

The temperature distributions obtained from the solution of equation (1) at various times are the initial data for solving the problem of the stress-strain state of the carbon block.

2. Calculation of the temperature stresses in the carbon block

The temperature stresses are calculated by solving the Poisson equation for the thermoelastic displacement potential [7]

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{\alpha \Theta (1+\mu)}{1-\mu}, \tag{2}$$

where Φ is the thermoelastic potential of displacements; μ, α are Poisson's ratio and coefficient of thermal expansion; $\Theta = (T - T_0)$ is the temperature increment with respect to the temperature of the natural state of the body T_0. Equation (2) is supplemented with the conditions of the absence of externally applied normal and tangential stresses on the carbon block surface: $\sigma_z = 0$, $\tau_{xz} = 0$, $\tau_{yz} = 0$.

The values of the thermoelastic potential Φ were used to determine the stresses at the corresponding points of the difference grid

$$\sigma_x = 2G\left(\frac{\partial^2\Phi}{\partial x^2} + \bigtriangledown^2\Phi\right), \tag{3}$$

$$\tau_{xy} = 2G\frac{\partial^2\Phi}{\partial x\partial y}(xyz), \tag{4}$$

where σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{xz} are normal and tangential elastic stresses; G is the shear modulus of the material at a given point and at a given moment in time, (xyz) is the symbol of cyclic permutation of x, y, z. The number of nodes of the grid in thermal stresses equations (2)–(4) corresponds to the thermal problem.

The distribution of thermal normal stresses of the anode for a time instant of 15 minutes is shown in Fig. 3. The distribution of thermal normal stresses for the middle xz plane is shown in the left figure. The magnitude of the temperature stresses in this plane reaches 3.4 MPa. The highest stresses are observed in zones of the highest temperature gradients. The maximum values of temperature gradients and stresses occur at the corners of the anode, they can be displayed in the vertical diagonal section passing along the bisector of the angle of the anode base. The right figure shows the distribution of thermal stresses in this diagonal plane. Comparison of the distributions in the middle xz and the diagonal vertical planes shows that in the second case the values of maximum stress are more than 1.5 times higher.

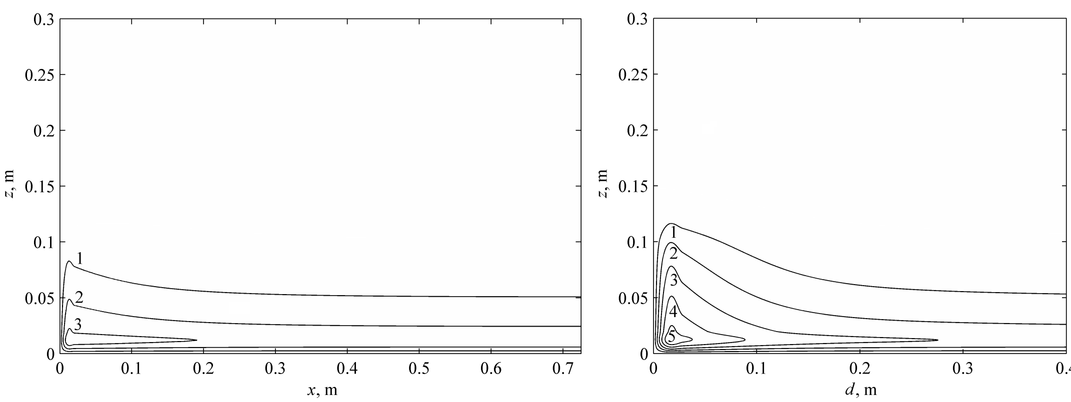


Fig. 3. Distribution of normal temperature stresses in the middle xz and diagonal vertical planes of the anode block

The variation of the maximum normal stresses in the anode block with time is shown in Fig. 4. The calculated maximum values of thermal stresses at various moments of time are marked with circles. The dotted line is obtained with the use of interpolation. The greatest increase in thermal stresses occurs at the initial stage of the process, and then the slope of the curve is significantly decreased. Considering results of calculation of the stress-strain state, it is possible to assess the possibility of fracture of the anode by comparing stresses with the ultimate strength of carbon. The most important from the point of view of cracking of the anode are tensile stresses that arise from the inhomogeneous thermal expansion of the material during heating. The ultimate tensile strength of the anode material is in the range of 5–15 MPa [2, 8, 9]. The scatter in the data of the limiting values of thermal stresses for graphite depends on the manufacturing technology and composition. It follows from the results of calculations that when graphite with a low ultimate strength (less than 8 MPa) is used there is a probability of fracture of the anode block.

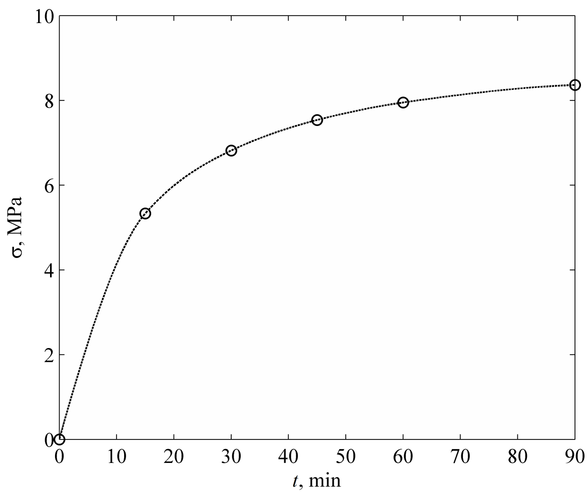


Fig. 4. The time dependence of the maximum normal temperature stresses in the anode block

Conclusion

Numerical modelling of thermal processes occurring when a carbon block is immersed in a hot electrolyte allows one to determine the magnitude and location of the maximal temperature gradients and stresses at various moments of time. Calculations have shown that the maximum values of temperature stresses in the corners of the anode block exceed the lower limit of the tensile strength of graphite. This points up the possible fracture of the anode block.

References

[1] M.W.Meier, W.K.Fischer, R.C.Perruchoud, L.J.Gauckler, Thermal shock of anodes–a solved problem?, Light Metals, 1994, 685–694.

[2] Yu.G.Mikhalev, P.V Polyakov, A.S.Yasinskiy, S.G.Shahrai, A.I.Bezrukikh, A.V.Zavadyak, Anode processes malfunctions causes. An overview, *Journal of Siberian Federal University. Engineering and Technologies*, **10**(2017), no. 5, 593–606 (in Russian). DOI: 10.17516/1999-494X-2017-10-5-593-606

[3] A.A.Samarskii, The theory of difference schemes, Nauka, Moscow, 1989 (in Russian).

[4] E.N.Vasil'ev, Calculation of the Thermal Resistance of a Heat Distributer in the Cooling System of a Heat-Loaded Element, *Technical Physics*, **63**(2018), no. 4, 471–475. DOI: 10.1134/S1063784218040266

[5] J.P.Schneider, B.Coste, Thermomechanical modelling of thermal shock in anodes, Light Metals, 1993, 621–628.

[6] S.N.Akhmedov, V.V.Tikhomirov, B.S.Gromov, R.V.Pak, A.I.Ogurtsov, Specific features of the lining deformation of the cathode devices of aluminum electrolysers, *Tsvetnye metally*, (2004), no. 1, 48–51 (in Russian).

[7] N.I.Bezukhov, V.L.Bazhanov, I.I.Gol'denblatt, N.A.Nikolaenko, A.M.Sinyukov, The Calculations for Strength, Stability, and Oscillations in High Temperature Conditions, Mashinostroenie, Moscow, 1965 (in Russian).

[8] S.E.Vyatkin, A.N.Deev, V.G.Nagornyi, V.S.Ostrovskii, A.M.Sigarev, G.A.Sokker, Nuclear Graphite, Atomizdat, Moscow, 1967, (in Russian).

[9] D.H.Andersen, Z.L.Zhang, Fracture and physical properties of carbon anodes for the aluminum reduction cell, *Engineering Fracture Mechanics*, **78**(2011), 2998–3016.

Вычислительное моделирование полей температур и термических напряжений в угольном блоке при внешних тепловых воздействиях

Евгений Н. Васильев
Институт вычислительного моделирования СО РАН
Красноярск, Российская Федерация

Аннотация. Работа посвящена моделированию теплового режима и напряженно-деформированного состояния угольного блока при его частичном погружении в электролит. Температурное поле в блоке определялось из решения нестационарного трехмерного уравнения теплопроводности. Расчет температурных напряжений проводился на основе решения уравнения Пуассона, записанного для термоупругого потенциала перемещений. В результате моделирования теплового режима получены температурные поля в угольном блоке для разных моментов времени. Расчет напряженно-деформированного состояния определил величину и расположение наибольших температурных напряжений и позволил оценить возможность разрушения угольного блока.

Ключевые слова: уравнение теплопроводности, уравнение Пуассона, термические напряжения, термоупругий потенциал перемещений, вычислительное моделирование.

DOI: 10.17516/1997-1397-2022-15-3-273-280
УДК 536.252

Solution of the Linear Problem of Thermal Convection in Liquid Rotating Layer

Victor K. Andreev*
Institute of Computational Modeling SB RAS
Krasnoyarsk, Russian Federation
Liliya I. Latonova†
Siberian Federal University
Krasnoyarsk, Russian Federation

Received 29.10.2021, received in revised form 19.12.2021, accepted 10.02.2022

Abstract. The initial boundary problem arising in the modeling of viscous fluid creeping rotational motion in a flat layer was solved. A stationary solution was found. The quadrature solution in images was obtained using the Laplace transform method. The time convergence of the the non-stationary problem solution to the established stationary solution was proved under certain conditions on the temperature distribution on the walls.

Keywords: thermal convection, Laplace transform, stationary solution.

Citation: L.I. Latonova, V.K. Andreev, Solution of the Linear Problem of Thermal Convection in Liquid Rotating Layer, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 273–280.
DOI: 10.17516/1997-1397-2022-15-3-273-280.

1. Problem statement

Let us assume that the fields of pressure and temperature velocities are rotationally symmetrical. Then, their values depend only on $r = \sqrt{x^2 + y^2}$, z, and time t in a cylindrical coordinate system. Moreover, we suppose that the only external force acting on the fluid is the centrifugal force. Then [1], the momentum, continuity, and energy equations can be written as

$$\begin{aligned}
&u_t + uu_r + wu_z - 2\omega v - \frac{v^2}{r} = -\frac{1}{\rho}\, p_r + \nu\left(\Delta u - \frac{u}{r^2}\right) - \omega^2\beta r\Theta,\\
&v_t + uv_r + wv_z + 2\omega u + \frac{uv}{r} = \nu\left(\Delta v - \frac{v}{r^2}\right),\\
&w_t + uw_z + ww_z = \frac{1}{\rho}\, p_z + \nu\Delta w,\\
&u_r + \frac{u}{r} + w_z = 0,\\
&\Theta_t + u\Theta_r + w\Theta_z = \chi\Delta\Theta,
\end{aligned} \tag{1.1}$$

where $\Delta = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + \partial^2/\partial z^2$ is the axisymmetric part of Laplace operator.

Equations (1.1) are written in the rotating coordinate system with constant angular velocity ω relatively to the original inertial system. Its rotation axis and the z axis of the cylindrical

*andr@icm.krasn.ru
†llatonova@sfu-kras.ru

coordinate system r, φ, z are coincide. The radial and axial components of the velocity are denoted as u and w, respectively, and v is the deviation of the rotational velocity component from the solid rotation velocity ωr. The quantity p characterizes the pressure deviation from equilibrium pressure: $\rho\omega^2 r^2/2$; and Θ is the temperature deviation from the mean value. The positive constants ρ, ν, χ, β are the physical liquid characteristics: density, kinematic viscosity, thermal diffusivity, and volumetric expansion coefficient.

The solution of a system (1.1) is sought in the form [2]

$$\begin{aligned} &u = rf(z,t), \quad v = rg(z,t), \quad w = w(z,t),\\ &p = \frac{1}{2}K(t)r^2 + \frac{A\rho\beta\omega^2}{2}r^2\left(\ln\frac{r}{a} - \frac{1}{2}\right) + h(z,t),\\ &\Theta = A\ln\frac{r}{a} + T(z,t), \end{aligned} \tag{1.2}$$

where A and a is the constant dimensions of temperature and length correspondingly. The substitution of (1.2) in (1.1) results in the system

$$\begin{aligned} &f_t + wf_z - 2\omega g + f^2 - g^2 = -\frac{1}{\rho}\,K(t) + \nu f_{zz} - \omega^2\beta T,\\ &g_t + wg_z + 2\omega f + 2fg = \nu g_{zz}, \quad 2f + w_z = 0,\\ &T_t + wT_z + Af = \chi T_{zz}, \quad w_t + ww_z = -\frac{1}{\rho}\,h_z + \nu w_{zz}. \end{aligned} \tag{1.3}$$

The solution of (1.2) may be interpreted as the following. A viscous heat-conductive liquid fills the layer between flat walls $z = \pm a$ rotating with angular velocity $\omega = const$ around the z axis. The no-slip condition $u(r,\pm a,t) = 0$, $v(r,\pm a,t) = 0$, $w(r,\pm a,t) = 0$ is satisfied on them. At the initial instant the velocity and temperature distributions are specified consistent with (1.2) formulas. On the rotation axis $r = 0$ sinks or sources of heat are distributed with constant linear density $2\pi Ak$ ($k > 0$ is the constant liquid thermal diffusivity coefficient). The solid walls (planes) bounding the liquid are ideally heat conductive. All the assumptions above lead to the formulation of an initial boundary value problem for the system (1.3)

$$f = -\frac{1}{2}w_{0\,z}(z), \quad g = g_0(z), \quad w = w_0(z), \quad T = T_0(z), \quad |z| \leqslant a, \quad t = 0; \tag{1.4}$$

$$f = g = 0, \quad w = 0, \quad T = T_{1,2}(t), \quad z = \pm a, \quad t > 0; \tag{1.5}$$

with the specified functions $w_0(z)$, $g_0(z)$, $T_0(z)$, $T_{1,2}(t)$. The conditions of thermal insulation of one (or both) walls can be used instead of the last in (1.5), for instance $T(-a,t) = T_1(t)$, $T_z(a,t) = 0$. Note, that for smooth solutions the agreement conditions should be satisfied

$$\begin{aligned} &w_0(\pm a) = 0, \quad w_{0\,z}(\pm a) = 0, \quad g_0(\pm a) = 0,\\ &T_0(\pm a) = T_{1,2}(0) \quad (T_0(-a) = T_1(0), \quad T_{0\,z}(a) = 0). \end{aligned} \tag{1.6}$$

Let us introduce the dimensionless variables by

$$\begin{aligned} &t = \frac{a^2}{\nu}\bar{t}, \quad z = a\bar{z}, \quad f = \omega R^2\overline{f}, \quad g = \omega R\overline{g} \quad w = a\omega R^2\overline{w}, \quad T = RA\overline{T},\\ &K = \rho\omega^2 R\overline{K}, \quad h = \rho\omega^2 a^2 R\overline{h} \quad R = \frac{a^2\omega}{\nu}, \quad P = \frac{\nu}{\chi}, \quad \varepsilon = \beta A, \end{aligned} \tag{1.7}$$

where R, P, ε are the Reynolds, Prandtl, and Boussinesq numbers correspondingly. Since $\partial^2/\partial t = \nu a^{-2}\partial^2/\partial\bar{t}$, $\partial^2/\partial z = a^{-1}\partial^2/\partial\bar{z}$, we obtain the following system by substituting (1.7) into (1.3) and omitting the upper bars

$$\begin{gathered} f_t + R^3 w f_z - 2g + R^3 f^2 - R g^2 = f_{zz} - K(t) - \varepsilon T, \\ g_t + R^3 w g_z + 2R^2 f = g_{zz}, \quad 2f + w_z = 0, \\ T_t + R^2 w T_z + R^2 f = \frac{1}{P} T_{zz}, \quad w_t + R^3 w w_z = -h_z + w_{zz}, \quad |z| < 1, \quad t > 0. \end{gathered} \tag{1.8}$$

The conditions (1.4), (1.5), (1.6) remain unchangeable, it is just needed to take into account that $|z| \leqslant 1$. In addition, $w_0(z) = \omega R^2 \overline{w}_0(\overline{z})$, $g_0(z) = \omega R \overline{g}_0(\overline{z})$, $T_0(z) = RA\overline{T}_0(\overline{z})$, $T_{1,2}(t) = RA\overline{T}_{1,2}(\overline{t})$ in the initial data.

2. Linear initial boundary value problem

Let be $R << 1$; such movements are called creeping. In practice they arise due to the high kinematic viscosity, cross-sectional layer size fineness or small angular velocity ω. Assuming that

$$\begin{gathered} f = f_0 + R f_1 + \cdots, \quad g = g_0 + R g_1 + \cdots, \quad w = w_0 + R w_1 + \cdots, \\ T = T_0 + R T_1 + \cdots, \quad K = K_0 + R K_1 + \cdots, \end{gathered} \tag{2.1}$$

and substituting it into (1.8) we obtain the initial boundary value problem in the zero approximation (the subscript "0" is omitted)

$$\begin{aligned} & f_t - 2g = f_{zz} - K(t) - \varepsilon T, \\ & g_t = g_{zz}, \quad 2f + w_z = 0, \\ & T_t = \frac{1}{P} T_{zz}, \quad w_t = w_{zz} - h_z, \quad |z| < 1, \quad t \geqslant 0; \end{aligned} \tag{2.2}$$

$$\begin{gathered} f(z,0) = -\frac{1}{2} w_{0\,z}(z), \quad g(z,0) = g_0(z), \quad T(z,0) = T_0(z), \\ w(z,0) = w_0(z), \quad |z| \leqslant 1; \end{gathered} \tag{2.3}$$

$$f(\pm 1, t) = 0, \quad g(\pm 1, t) = 0, \quad T(\pm 1, t) = T_{1,2}(t), \quad w(\pm 1, t) = 0, \quad t \geqslant 0. \tag{2.4}$$

Note, that

$$\int_{-1}^{1} f(z,t)\,dz = 0, \tag{2.5}$$

what follows from the third equation in (2.2) and non-slip condition (2.4): $w(\pm 1, t) = 0$. The integral equality (2.5) is correct also for the general problem (1.3), (1.4), (1.5). This additional condition is used to compute the part of radial pressure "gradient", which is the function $K(t)$, see (1.2). Thus, the problem under consideration *is an inverse problem*.

Let us find the stationary solution of system (2.2)–(2.5). It is denoted as $f^s(z)$, $g^s(z)$, $w^s(z)$, $T^s(z)$, K^s, $h^s(z)$ and corresponds to the data $T^s_{1,2} = const$. Simple calculations lead to the next formulas

$$\begin{gathered} g^s(z) \equiv 0, \quad T^s(z) = \frac{1}{2}\left((T_2^s - T_1^s)\, z + T_1^s + T_2^s\right), \\ f^s(z) = \frac{\varepsilon}{12}\left(T_2^s - T_1^s\right)\left(z^3 - z\right), \\ K^s = -\frac{\varepsilon}{2}\left(T_1^s + T_2^s\right), \\ w^s(z) = \frac{\varepsilon}{24}\left(T_1^s - T_2^s\right)\left(z^2 - 1\right)^2, \\ h^s(z) = h_0^s + \frac{\varepsilon}{6}\left(T_1^s - T_2^s\right) z\left(z^2 - 1\right), \quad h_0^s = const. \end{gathered} \tag{2.6}$$

The real fields of velocities $u^s(r,z)$, $v^s(r,z)$, $w^s(z)$, pressure $p^s(r,z)$, and temperature $\Theta^s(r,z)$ are given by (1.2).

The solution of inverse problem (2.2)–(2.5) can be obtained using the partition method in the form of Fourier series. First, the functions $g(z,t)$, $T(z,t)$ are to be found as solutions of the first classical initial boundary value problems for the heat conduction equations [3]. After that $f(z,t)$ and $K(t)$ should be determined taking into account the overloading condition (2.5). The function $w(z,t)$ can be recovered by quadrature from the third equation of the system (2.2), and $h(z,t)$ can be found by the latter from (2.2). This solution procedure is rather cumbersome. Here we use the Laplace transform method to find a solution [4].

Let

$$\widehat{u}(z,s) = \int_0^\infty u(z,t)e^{-st}\, dt$$

be the Laplace transform for the function u. Since

$$\widehat{u}_t(z,s) = s\widehat{u}(z,s) - u_0(z), \quad \widehat{u_{zz}} = \frac{\partial}{\partial z^2}\widehat{u},$$

the problem for $\widehat{f}(z,s)$, $\widehat{g}(z,s)$, $\widehat{T}(z,s)$, $\widehat{K}(s)$ takes the form

$$\begin{gathered} \widehat{f}_{zz} - s\widehat{f} = \varepsilon\widehat{T} - 2\widehat{g} + \widehat{K} - f_0(z) \\ \widehat{g}_{zz} - s\widehat{g} = -g_0(z), \quad \widehat{T}_{zz} - Ps\widehat{T} = -PT_0(z), \quad |z| < 1, \end{gathered} \tag{2.7}$$

where $\widehat{T}_{1,2}(s)$ is the Laplace transform of the specified functions $T_{1,2}(t)$. Moreover, the next conditions are satisfied

$$\begin{gathered} \widehat{f}(\pm 1,s) = 0, \quad \widehat{g}(\pm 1,s) = 0, \quad \widehat{T}(\pm 1,s) = \widehat{T}_{1,2}(s), \\ \int_{-1}^{1} \widehat{f}(z,s)\, dz = 0. \end{gathered} \tag{2.8}$$

Thus, we obtain the boundary value problem (2.7), (2.8) in Laplace images for ODE systems.

Remark 1. The functions $T_{1,2}(t)$ can have a finite number of the discontinuities of the first kind [4].

After simple calculations, we obtain a quadrature representation of the solution to the problem (2.7), (2.8)

$$\begin{gathered}
\widehat{g}(z,s)=\frac{1}{\sqrt{s}\,\mathrm{sh}(2\sqrt{s})}\int_{-1}^{1}g_0(y)\,\mathrm{sh}\left[\sqrt{s}(1-y)\right]\,dy\,\mathrm{sh}\left[\sqrt{s}(z+1)\right]-\\
-\frac{1}{\sqrt{s}}\int_{-1}^{z}g_0(y)\,\mathrm{sh}\left[\sqrt{s}(z-y)\right]\,dy,\\
\widehat{T}(z,s)=\frac{1}{\mathrm{sh}(2\sqrt{Ps})}\Bigg\{\widehat{T}_1(s)\,\mathrm{sh}\left[\sqrt{Ps}(1-z)\right]+\widehat{T}_2(s)\,\mathrm{sh}\left[\sqrt{Ps}(z+1)\right]+\\
+\sqrt{\frac{P}{s}}\int_{-1}^{1}T_0(y)\,\mathrm{sh}\left[\sqrt{Ps}(1-y)\right]\,dy\;\mathrm{sh}\left[\sqrt{Ps}(z+1)\right]\Bigg\}-\\
-\sqrt{\frac{P}{s}}\int_{-1}^{z}T_0(y)\,\mathrm{sh}\left[\sqrt{Ps}(z-y)\right]\,dy,\\
\widehat{f}(z,s)=\frac{\widehat{K}(s)}{s}\left(\frac{\mathrm{ch}\,(\sqrt{s}z)}{\mathrm{ch}\,\sqrt{s}}-1\right)-\frac{1}{\sqrt{s}\,\mathrm{sh}(2\sqrt{s})}\times\\
\times\int_{-1}^{1}F(y,s)\,\mathrm{sh}\left[\sqrt{s}(1-y)\right]\,dy\,\mathrm{sh}\left[\sqrt{s}(z+1)\right]+\frac{1}{\sqrt{s}}\int_{-1}^{z}F(y,s)\,\mathrm{sh}\left[\sqrt{s}(z-y)\right]\,dy,
\end{gathered}\tag{2.9}$$

where

$$F(z,s)=\varepsilon\widehat{T}(z,s)-2\widehat{g}(z,s)-f_0(z).\tag{2.10}$$

Now, from equality (2.8) and representation $\widehat{f}(z,s)$ (2.9) we obtain

$$\begin{gathered}
\widehat{K}(s)=\frac{3}{2s\,\mathrm{cth}\,\sqrt{s}}\left[\frac{(1-\mathrm{ch}\,(2\sqrt{s}))}{\sqrt{s}\,\mathrm{sh}\,(2\sqrt{s})}\int_{-1}^{1}F(y,s)\,\mathrm{sh}\left[\sqrt{s}(1-y)\right]\,dy+\right.\\
\left.+\int_{-1}^{1}\int_{-1}^{z}F(y,s)\,\mathrm{sh}\left[\sqrt{s}(z-y)\right]\,dydz\right]
\end{gathered}\tag{2.11}$$

with $F(z,s)$ defining by (2.9), (2.10).

The functions $\widehat{w}(z,s)$, $\widehat{h}(z,s)$ are determined from (2.2) taking into account differentiation properties of the Laplace transform by the following formulas

$$\begin{gathered}
\widehat{w}(z,s)=-2\int_{-1}^{z}\widehat{f}(y,s)\,dy,\\
\widehat{h}(z,s)=h_0(s)+\widehat{w}_z(z,s)-s\widehat{w}(z,s)+w_0(z)=\\
=h_0(s)-2\widehat{f}(z,s)+2s\int_{-1}^{z}\widehat{f}(y,s)\,dy+w_0(z)
\end{gathered}\tag{2.12}$$

with an arbitrary function $h_0(s)$ and the function $\widehat{f}(z,s)$ determined by (2.9).

Under the assumptions that the Laplace transform $\widehat{T}_{1,2}(s)$, $\widehat{\partial T_{1,2}/\partial t}$ exists and that there is the limit $\lim_{t\to\infty}T_{1,2}(t)=T^s_{1,2}=const$ the following holds because of the property of limit relations for the Laplace transform (see [4])

$$\lim_{s\to 0}s\widehat{T}_{1,2}(s)=\lim_{t\to\infty}T_{1,2}(t)=T^s_{1,2}.\tag{2.13}$$

Let us demonstrate that $\lim_{s\to 0}s\widehat{K}(s)=K^s$, where K^s is given by (2.6), i. e. that $\lim_{t\to\infty}K(t)=K^s$. It is obviously that $s\widehat{g}(z,s)\approx 0$, $s\to 0$. Now we proceed to consider the first approximation of the function $s\widehat{T}(z,s)$ using the Taylor series expansion of hyperbolic functions:

$$\begin{aligned} s\widehat{T}(z,s) &\approx \frac{1}{2\sqrt{Ps}}\left[T_1^s\sqrt{Ps}(1-z)+T_2^s\sqrt{Ps}(z+1)\right]= \\ &= \frac{1}{2}\left[T_1^s+T_2^s+\left(T_2^s-T_1^s\right)z\right]=\widehat{T}^s(z). \end{aligned} \tag{2.14}$$

Taking into account (2.6) and (2.13) we can obtain provided $s\to 0$

$$\begin{aligned} s\widehat{K}(s) &\approx \frac{3}{2\sqrt{s}}\left[-\int_{-1}^{1}\varepsilon T^s\sqrt{s}(1-y)\,dy+\int_{-1}^{1}\int_{-1}^{z}\varepsilon T^s\sqrt{s}(z-y)\,dydz\right]= \\ &= \frac{3\varepsilon}{4}\left[-\int_{-1}^{1}((T_2^s-T_1^s)y+T_1^s+T_2^s)(1-y)\,dy+\right. \\ &\left.+\int_{-1}^{1}\int_{-1}^{z}((T_2^s-T_1^s)y+T_1^s+T_2^s)(z-y)\,dydz\right]= \\ &= \frac{3\varepsilon}{4}\left[\frac{2}{3}(T_2^s-T_1^s)-2(T_1^s+T_2^s)--\frac{2}{3}(T_2^s-T_1^s)+\frac{4}{3}(T_1^s+T_2^s)\right]= \\ &= \frac{3\varepsilon}{4}\left(-\frac{2}{3}(T_1^s+T_2^s)\right)=K^s. \end{aligned} \tag{2.15}$$

Here, the Taylor series expansions of the following functions were taken into account with the retention of the main terms

$$\begin{aligned} \frac{3}{2s\operatorname{cth}\sqrt{s}} &= \frac{3}{2s\left(\frac{1}{\sqrt{s}}+\frac{(\sqrt{s})}{3}+\cdots\right)}=\frac{3}{2\sqrt{s}\,(1+o(s))}\approx\frac{3}{2\sqrt{s}}, \\ \frac{1-\operatorname{ch}(2\sqrt{s})}{\sqrt{s}\operatorname{sh}(2\sqrt{s})} &= \frac{1-1-\frac{4s}{2}-\cdots}{\sqrt{s}\left(2\sqrt{s}+\frac{(2\sqrt{s})^3}{6}+\cdots\right)}\approx -1. \end{aligned} \tag{2.16}$$

Now consider the limit $\lim_{s\to 0} s\widehat{f}(s,z)$. Since

$$\frac{K^s}{s}\left(\frac{\operatorname{ch}(\sqrt{s}z)}{2\operatorname{ch}\sqrt{s}}-1\right)=\frac{K^s}{s}\left(\frac{1+(\sqrt{s}z)^2/2+o(s^2)-1-(\sqrt{s})^2/2-o(s^2)}{1+(\sqrt{s})^2/2+o(s^2)}\right)\approx K^s\left(\frac{z^2}{2}-\frac{1}{2}\right),$$

the following can be derived

$$\begin{aligned} s\widehat{f}(z,s) &\approx K^s\left(\frac{z^2}{2}-\frac{1}{2}\right)-\frac{\varepsilon}{4s}\int_{-1}^{1}\left[\left(T_2^s-T_1^s\right)y+T_1^s+T_2^s\right]\sqrt{s}(1-y)\,dy\sqrt{s}(z+1)+ \\ &+\frac{\varepsilon}{2\sqrt{s}}\int_{-1}^{z}\left[\left(T_2^s-T_1^s\right)y+T_1^s+T_2^s\right]\sqrt{s}(z-y)\,dy= \\ &= \frac{\varepsilon}{2}\left[(T_2^s-T_1^s)\frac{z^3}{6}-(T_2^s-T_1^s)\frac{z}{6}\right]=f^s. \end{aligned} \tag{2.17}$$

By direct substitution it is easy to show that

$$\begin{aligned} s\widehat{w}(z,s) &\approx -2\int_{-1}^{z}f^s dy=-\frac{\varepsilon}{6}\left(T_2^s-T_1^s\right)\left(\frac{y^4}{4}-\frac{y^2}{2}\right)\Bigg|_{-1}^{z}= \\ &= \frac{\varepsilon}{24}\left(T_1^s-T_2^s\right)\left(z^4-2z^2+1\right)=w^s, \\ s\widehat{h}(z,s) &\approx h_0^s-2f^s=h^s, \end{aligned} \tag{2.18}$$

where $h_0^s = \lim_{s\to 0} sh_0(s)$.

We have proven the

Theorem. *Under conditions* (2.13), $f_0(z)$, $g_0(z)$, $T_0(z) \in C[-1,1]$ *the solution of a nonstationary inverse initial boundary value problem* (2.2)–(2.5) *converges to the stationary solution* (2.6) *with* $t \to \infty$.

Note, that initial values of function $K(t)$ can be found directly from the problem (2.2)–(2.5).

The solution formulas (2.9) obtained in the images can be transformed into Fourier series. To show it for the function $g(z,t)$ we will use the first formula for $\widehat{g}(z,s)$ from (2.9). Note, that $\widehat{g}(z,s)$ cannot be translated directly into the original space since the second term does not tend to zero at $s\to\infty$. It can be seen that

$$\widehat{g}(z,s) = \int_{-1}^{1} G(z,y) g_0(y)\, dy, \tag{2.19}$$

where

$$G(z,y,s) = \frac{1}{\sqrt{s}\,\mathrm{sh}(2\sqrt{s})}\begin{cases}\mathrm{sh}\,\sqrt{s}(y+1)\,\mathrm{sh}\,\sqrt{s}(1-z), & -1\leqslant y\leqslant z;\\ \mathrm{sh}\,\sqrt{s}(z+1)\,\mathrm{sh}\,\sqrt{s}(1-y), & z\leqslant y\leqslant 1\end{cases} \tag{2.20}$$

is the Green's function for the operator $d^2/dz^2 - s$ with zero first-type boundary conditions at $z\equiv\pm 1$. It is clear that $G(z,y,s)\to 0$ at $s\to\infty$ for any $z,y\in[-1;1]$.

Now we can use the result from [5], p. 273, formula No. 188, namely that the image of the function $G(z,y,s)$ corresponds to the original

$$\sum_{n=1}^{\infty} \sin n\pi z\, \sin n\pi y e^{-n^2\pi^2 t} = \Gamma(z,y,t), \tag{2.21}$$

therefore $\widehat{g}(z,s)$ corresponds to the Fourier series

$$g(z,t) = \int_{-1}^{1}\Gamma(z,y,t) g_0(y)\,dy = \sum_{n=1}^{\infty}\int_{-1}^{1} g_0(y)\sin n\pi y\,dy \sin n\pi z e^{-n^2\pi^2 t}. \tag{2.22}$$

It is easy to verify that the series (2.21) are the solution to the initial boundary value problem for $g(z,t)$. It is classical provided there is the agreement condition $g_0(-1) = g_0(1) = 0$ and $g_0'(y)\in L_2(-1,1)$

$$\begin{aligned} g_{0n} = \int_{-1}^{1} g_0(y)\sin n\pi y\,dy = -\frac{1}{\pi n}\left[g_0(y)\cos\, n\pi y\Big|_{-1}^{1} - \int_{-1}^{1} g_0'(y)\cos\, n\pi y\,dy\right] = \\ = \frac{1}{\pi n}\int_{-1}^{1} g_0'(y)\cos\, n\pi y\,dy = \frac{1}{n}\beta(n). \end{aligned} \tag{2.23}$$

whence it follows that $|g_{0n}| \leqslant \dfrac{1}{2}\dfrac{1}{n^2} + \dfrac{1}{2}\beta^2(n)$. Then

$$|g(z,t)| \leqslant \sum_{n=1}^{\infty}|g_{0n}| \leqslant \sum_{n=1}^{\infty}\frac{1}{2}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{1}{2}\beta^2(n) < \infty, \tag{2.24}$$

as $\int\limits_{-1}^{1} g_0'(y)\cos\, n\pi y\,dy\to 0$, $n\to\infty$. The convergence to zero velocity for the function $g(z,t)$ is determined from the inequality

$$|g(z,t)| \leqslant e^{-\pi^2 t}\sum_{n=1}^{\infty}|g_{0n}|e^{-\pi^2(n-1)t} \leqslant e^{-\pi^2 t}\sum_{n=1}^{\infty}|g_{0n}| = Ce^{-\pi^2 t}, \tag{2.25}$$

since the series $\sum\limits_{n=1}^{\infty} |g_{0n}|$ converges as noted above.

This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation in the framework of the establishment and development of regional Centers for Mathematics Research and Education (Agreement No. 075-02-2022-876).

References

[1] G.Z.Gershuni, E.M.Zhukhovitsky, Convective stability of an incompressible fluid, Moscow, Nauka, 1972 (in Russian).

[2] V.V.Pukhnachev, Exact Solutions of the Hydrodynamic Equations Derived from Partially Invariant Solutions, *Applied Mechanics and Technical Physics*, **44**(2003), no. 3, 317–323. DOI: 10.1023/A:1023472921305

[3] A.D.Polyanin, Handbook. Linear equations of mathematical physics, Moscow, Fizmatlit, 2001 (in Russian).

[4] M.A.Lavrentyev, B.V.Shabat, Methods of Theory of Complex Variable Functions, Moscow, Nauka, 1973 (in Russian).

[5] G.Doetsch, Introduction to the Theory and Applications of the Laplace Transformation, Springer Verlag, Berlin, 1974.

Решение линейной задачи тепловой конвекции во вращающемся слое жидкости

Виктор К. Андреев
Институт вычислительного моделирования СО РАН
Krasnoyarsk, Российская Федерация
Лилия И. Латонова
Siberian Federal University
Красноярск, Российская Федерация

Аннотация. Решена начально-краевая задача, возникающая при моделировании ползущего вращательного движения вязкой жидкости в плоском слое. Найдено стационарное решение. С помощью метода преобразования Лапласа решение в изображениях получено в квадратурах. Доказано, что при некоторых условиях на распределение температуры на стенках решение нестационарной задачи сходится с ростом времени к найденному стационарному решению.

Ключевые слова: тепловая конвекция, преобразование Лапласа, стационарное решение.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(3), 281–291

DOI: 10.17516/1997-1397-2022-15-3-281-291
УДК 519.21

Central Limit Theorem for Weakly Dependent Random Variables with Values in $D[0,1]$

Olimjon Sh. Sharipov*
National University of Uzbekistan
Tashkent, Uzbekistan
V. I. Romanovskiy Institute of Mathematics
Academy of Sciences of Uzbekistan
Tashkent, Uzbekistan

Anvar F. Norjigitov†
Academy of Sciences of Uzbekistan
Tashkent, Uzbekistan

Received 03.10.2021, received in revised form 04.12.2021, accepted 09.02.2022

Abstract. The main goal of this article is to prove the central limit theorem for sequences of random variables with values in the space $D[0,1]$. We assume that the sequence satisfies the mixing conditions. In the paper the central limit theorems for sequences with strong mixing and ρ_m-mixing are proved.

Keywords: central limit theorem, mixing sequence, $D[0,1]$ space.

Citation: O.Sh. Sharipov, A.F. Norjigitov, Central Limit Theorem for Weakly Dependent Random Variables with Values in $D[0,1]$, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 281–291.
DOI: 10.17516/1997-1397-2022-15-3-281-291.

1. Preliminaries

A central limit theorem for Banach space valued dependent random variables have been studied by many authors (see [6, 11, 15–17] and references therein). It is known that validity of the central limit theorem depends on the geometric structure of Banach space. One of the most difficult space in this sense is $D[0,1]$ (the space of all real-valued functions that are right continuous and have left limits, which is endowed with the Skorohod topology) space. In this paper we will prove the central limit theorem for mixing random variables with values in $D[0,1]$.

Let $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ be a sequence of random variables with values in $D[0,1]$. We say that $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ satisfies a central limit theorem if the distribution of $\frac{1}{\sqrt{n}}\left(X_1(t) + \ldots + X_n(t)\right)$ converges weakly to a Gaussian distribution in $D[0,1]$.

The central limit theorem in $D[0,1]$ is very important from applications point of view. It immediately implies asymptotic normality of empirical and weighted empirical processes. The central limit theorem for the sequence of independent identically distributed (i.i.d) random variables with values in $D[0,1]$ were studied by many authors (see [**?**, 1, 2, 8, 12]) and references therein).

*osharipov@yahoo.com
†anvar2383@mail.ru

The first central limit theorem was proved by Hahn [8]. Later the central limit theorem in $D[0,1]$ was proved by D. Juknevičienė (1985), V. Paulauskas and Ch. Stive (1990), P.H. Bezandry and X. Fernique (1992), M. Bloznelis and V. Paulauskas (1993), X. Fernique (1994). The result of M.G. Hahn [8] can be formulated as follows.

Theorem 1.1 ([8])**.** *Let $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ be a sequence of i.i.d. random variables with values in $D[0,1]$ such that*

$$EX_1(t) = 0, \quad EX_1^2(t) < \infty \quad \textit{for all} \quad t \in [0,1]. \tag{1}$$

Assume that there exist nondecreasing continuous functions G and F on $[0,1]$ and numbers $\alpha > 0.5,\ \beta > 1$ such that for all $0 \leqslant s \leqslant t \leqslant u \leqslant 1$ the following two conditions hold:

$$E\left(X_1(u) - X_1(t)\right)^2 \leqslant \left(G(u) - G(t)\right)^{\alpha},$$

$$E\left(X_1(u) - X_1(t)\right)^2 \left(X_1(t) - X_1(s)\right)^2 \leqslant \left(F(u) - F(s)\right)^{\beta}. \tag{2}$$

Then $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ satisfies the central limit theorem in $D[0,1]$ and the limiting Gaussian process is sample continuous.

As it was already noticed in [2], the condition (2) is connected with the fourth moments of the process $X_1(t)$. This conditions does not allow us to apply Theorem 1.1 to a wide class of weighted empirical processes. In [2] and [3] authors obtained the following results (where $a \wedge b$ denotes $\min(a,b)$):

Theorem 1.2 ([2])**.** *Let $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ be a sequence of i.i.d. random variables with values in $D[0,1]$ satisfying the condition (1) and assume that there exist nondecreasing continuous functions G and F on $[0,1]$ and numbers $\alpha, \beta > 0$ such that for all $0 \leqslant s \leqslant t \leqslant u \leqslant 1$ the following two conditions hold:*

$$E\left(X_1(u) - X_1(t)\right)^2 \leqslant \left(G(u) - G(t)\right)^{1/2} \log^{-4,5-\alpha}\left(1 + \left(G(u) - G(t)\right)^{-1}\right), \tag{3}$$

$$\begin{aligned} &E\left(|X_1(t) - X_1(s)| \wedge 1\right)^2 \left(X_1(u) - X_1(t)\right)^2 \leqslant \\ &\leqslant \left(F(u) - F(s)\right) \log^{-5-\beta}\left(1 + \left(F(u) - F(s)\right)^{-1}\right). \end{aligned} \tag{4}$$

Then $\{X_n(t),\, t \in [0,1],\, n \geqslant 1\}$ satisfies the central limit theorem in $D[0,1]$ and the limiting Gaussian process is sample continuous.

Theorem 1.3 ([2])**.** *The statement of Theorem 1.2 remains true if conditions (3) and (4) are replaced by*

$$E\left(X_1(u) - X_1(t)\right)^2 \leqslant \left(G(u) - G(t)\right)^{1/2} \log^{-2,5-\alpha}\left(1 + \left(G(u) - G(t)\right)^{-1}\right), \tag{5}$$

$$E\left(X_1(t) - X_1(s)\right)^2 \left(X_1(u) - X_1(t)\right)^2 \leqslant \left(F(u) - F(s)\right) \log^{-5-\beta}\left(1 + \left(F(u) - F(s)\right)^{-1}\right). \tag{6}$$

Theorem 1.4 ([3])**.** *Assume $p, q \geqslant 2$. Let f, g be nonnegative functions on $[0, +\infty)$ which are nondecreasing near 0 and let F, G be increasing continuous functions on $[0,1]$. Let $X(t)$ be a random process with mean 0, finite second moment, and sample path in D satisfying*

$$E\left(|X(s) - X(t)| \wedge |X(t) - X(u)|\right)^p \leqslant f\left(F(u) - F(s)\right),$$

$$E\left|X(s)-X(t)\right|^{q} \leqslant g\left(G(t)-G(s)\right),$$

for $0 \leqslant s \leqslant t \leqslant u \leqslant 1,\ u-s$ *small and*

$$\int_0 f^{1/p}(u)\cdot u^{-1-1/p}du<\infty, \qquad \int_0 g^{1/q}(u)\cdot u^{-1-1/(2q)}du<\infty.$$

Then $\{X_n(t),\, t\in[0,1],\, n\geqslant 1\}$ *satisfies the central limit theorem in* $D[0,1]$ *and the limiting Gaussian process is sample continuous.*

2. Main results

The main goal of this article is to prove the central limit theorem for mixing sequences of random variables with values in space $D[0,1]$.

Below, we give the definitions of mixing coefficients for a sequence of random variables with values in a separable Banach space $\mathbf{B}$. In Definition 2 it is assumed that $\mathbf{B}$ is an infinite-dimensional space.

Definition 1. *For a sequence* $\{X_n(t),\, t\in[0,1],\, n\geqslant 1\}$ *the coefficients of* ρ, α*-mixing are defined by the following equalities.*

$$\rho(n)=\sup\left\{\frac{|E(\xi-E\xi)(\eta-E\eta)|}{E^{\frac{1}{2}}(\xi-E\xi)^2E^{\frac{1}{2}}(\eta-E\eta)^2}:\ \xi\in L_2(F_1^k),\ \eta\in L_2(F_{n+k}^{\infty}),\ k\in N\right\},$$

$$\alpha(n)=\sup\left\{|P(AB)-P(A)P(B)|:\ A\in F_1^k,\ B\in F_{k+n}^{\infty},\ k\in N\right\}.$$

where F_a^b *is the* σ*-algebra generated by random processes* $X_a(t),\dots,X_b(t)$ *and* $L_2(F_a^b)$ *is the space of all square integrable random variables measurable with respect to* F_a^b.

Definition 2. *For the sequence* $\{X_n(t),\, t\in[0,1],\, n\geqslant 1\}$ *the coefficients of* $\rho_m(n)$*-mixing and* $\alpha_m(n)$*-mixing are defined by the following equalities*

$$\rho_m(n)=\sup_{\boldsymbol{R}^m}\sup\left\{\frac{|E(\xi-E\xi)(\eta-E\eta)|}{E^{\frac{1}{2}}(\xi-E\xi)^2E^{\frac{1}{2}}(\eta-E\eta)^2}:\ \xi\in L_2(F_1^k(\boldsymbol{R}^m)),\ \eta\in L_2(F_{n+k}^{\infty}(\boldsymbol{R}^m)),\ k\in N\right\},$$

$$\alpha_m(n)=\sup_{\boldsymbol{R}^m}\sup\left\{|\ P(AB)-P(A)P(B)|:\ A\in F_1^k(\boldsymbol{R}^m),\ B\in F_{k+n}^{\infty}(\boldsymbol{R}^m),\ k\in N\right\},$$

where $F_a^b(\boldsymbol{R}^m)$ *is the* σ*-algebra generated by random processes* $\prod_m X_a(t),\dots,\prod_m X_b(t)$ *and* $\prod_m$ *is a projection operator* $\boldsymbol{B}$ *in m-dimensional subspace* $\boldsymbol{R}^m$ *i.e.* $\prod_m:\boldsymbol{B}\to\boldsymbol{R}^m$. *A sequence is called* ρ*-mixing (or* ρ_m-, $\alpha-$, α_m- *mixing) if*

$$\rho(k)\to 0 \quad as \quad k\to\infty, \tag{7}$$

$$\rho_m(k)\to 0 \quad as \quad k\to\infty \quad and \quad m=1,2,\dots, \tag{8}$$

$$\alpha(k)\to 0 \quad as \quad k\to\infty, \tag{9}$$

$$\alpha_m(k)\to 0 \quad as \quad k\to\infty \quad and \quad m=1,2,\dots \tag{10}$$

respectively.

As the example given in Zhurbenko [13] shows, in general (8) does not imply (7), though (7) always implies (8), the same is true with (9) and (10). In (8) and (10) it is actually required that all finite-dimensional projections of the sequence $\{X_n(t), t \in [0,1], n \geqslant 1\}$ satisfy the mixing condition and these conditions are weaker than the conditions (7) and (9).

Set $S_n(t) = \frac{1}{\sqrt{n}}(X_1(t) + \cdots + X_n(t))$ and in what follows $\Rightarrow$ denotes weak convergence.

Now we formulate our theorems.

Theorem 2.1. *Let $\{X_n(t),\ t \in [0,1],\ n \geqslant 1\}$ be a strictly stationary sequence of ρ_m-mixing random variables with values in $D[0,1]$ such that*

$$EX_1(t) = 0, \quad E|X_1(t)|^2 < \infty \quad \textit{for all} \quad t \in [0,1].$$

Assume that there exists a nondecreasing continuous function F on $[0,1]$ such that for all $0 \leqslant s \leqslant t \leqslant 1$ and $\varepsilon > 0$ the following hold:

$$E(X_1(t) - X_1(s))^2 \leqslant (F(t) - F(s)) \log^{-(3+\varepsilon)}\left(1 + (F(t) - F(s))^{-1}\right), \tag{11}$$

$$\lim_{n\to\infty} E(X_1 + \cdots + X_n)^2 = \infty \quad \textit{for all} \quad t \in [0,1],$$

$$\sum_{k=1}^{n} \rho_m\left(2^k\right) < \infty, \quad m = 1, 2, \dots .$$

Then $\{X_n(t),\ t \in [0,1],\ n \geqslant 1\}$ satisfies the central limit theorem i.e.

$$S_n(t) \Rightarrow N(t) \quad \textit{as} \quad n \to \infty$$

and the limiting mean-zero, sample continuous Gaussian process has the covariance function:

$$F(t_1, t_2) = \lim_{n\to\infty} ES_n(t_1)S_n(t_2), \quad t_1, t_2 \in [0,1].$$

Theorem 2.2. *Let $\{X_n(t),\ t \in [0,1],\ n \geqslant 1\}$ be a strictly stationary sequence of ρ_m-mixing random variables with values in $D[0,1]$ such that*

$$EX_1(t) = 0, \quad E|X_1(t)|^{2+\varepsilon} < \infty, \quad \textit{for all} \quad t \in [0,1] \quad \textit{and some} \quad \varepsilon > 0.$$

Assume that there exists a nondecreasing continuous function F on $[0,1]$ such that for all $0 \leqslant s \leqslant t \leqslant 1$ and the following hold:

$$E|X_1(s) - X_1(t)|^{2+\varepsilon} \leqslant (F(s) - F(t)) \log^{-(3+2\varepsilon)}\left(1 + (F(s) - F(t))^{-1}\right), \tag{12}$$

$$\sum_{k=1}^{n} \rho_m^{\frac{2}{2+\varepsilon}}(2^k) < \infty, \quad m = 1, 2, \dots .$$

Then $\{X_n(t),\ t \in [0,1],\ n \geqslant 1\}$ satisfies the central limit theorem i.e.

$$S_n(t) \Rightarrow N(t) \quad \textit{as} \quad n \to \infty$$

and the limiting mean-zero, sample continuous Gaussian process has the covariance function:

$$F(t_1, t_2) = \lim_{n\to\infty} ES_n(t_1)S_n(t_2), \quad t_1, t_2 \in [0,1].$$

Theorem 2.3. *Let $\{X_n(t),\ t \in [0,1],\ n \geqslant 1\}$ be a strictly stationary sequence of α_m-mixing random variables with values in $D[0,1]$ such that*

$$EX_1(t) = 0, \quad E|X_1(t)|^{2+\delta} < \infty, \quad \textit{for all} \quad t \in [0,1] \quad \textit{and some} \quad \delta > 0.$$

Assume that there exists a nondecreasing continuous function F on $[0,1]$ such that for all $0 \leqslant s \leqslant t \leqslant 1$ and $\varepsilon > 0$ the following hold:

$$\left(E|X_1(t) - X_1(s)|^{2+\delta}\right)^{\frac{2+\varepsilon}{2+\delta}} \leqslant (F(t) - F(s)) \log^{-(3+2\varepsilon)}\left(1 + (F(t) - F(s))^{-1}\right), \tag{13}$$

$$\sum_{k=1}^{n} \alpha_m^{\frac{\delta}{2+\delta}}(k) < \infty, \quad m = 1, 2, \ldots .$$

Then $\{X_n(t),\ t \in [0,1],\ n \geqslant 1\}$ satisfies the central limit theorem i.e.

$$S_n(t) \Rightarrow N(t) \quad \textit{as} \quad n \to \infty$$

and the limiting mean-zero, sample continuous Gaussian process has the covariance function:

$$F(t_1, t_2) = \lim_{n\to\infty} ES_n(t_1)S_n(t_2), \quad t_1, t_2 \in [0,1].$$

Theorems 2.1–2.2 improve the results of [11].

3. Preliminary results

The proofs of the theorems are based on the following lemmas.

Lemma 1 ([2]). *Let $X_1(t), X_2(t), \ldots, X_n(t), \ldots$ be random variables with values in $D[0,1]$. Assume that there exist a nondecreasing continuous function F on $[0,1]$ and positive numbers $\gamma_1,\ c_1,\ \varepsilon_1$ such that for all $\lambda > 0$ and $0 \leqslant s \leqslant t \leqslant u \leqslant 1$.*

$$P(|X_n(t) - X_n(s)| \wedge |X_n(u) - X_n(t)| \geqslant \lambda) \leqslant c_1 \lambda^{-2\gamma_1} g_{2\gamma_1+1+\varepsilon_1}(F(u) - F(s)),$$

where $g_p(u) = u|\log u|^{-p},\ p > 0$. If for all $t_1, \ldots, t_k \in [0,1]$, $k = 1, 2, \ldots$

$$(X_n(t_1), \ldots, X_n(t_k)) \Rightarrow (X(t_1), \ldots, X(t_k)) \quad \textit{as} \quad n \to \infty$$

and

$$P(X(1) = \lim_{t\to 1} X(t)) = 1.$$

Then $X_n \Rightarrow X$ as $n \to \infty$.

Lemma 2 ([9]). *Let $\{X_i,\ i \geqslant 1\}$ be a strictly stationary sequence of real valued random variables with ρ-mixing and*

$$EX_1 = 0, \quad EX_1^2 < \infty,$$

$$\lim_{n\to\infty} E(X_1 + \cdots + X_n)^2 = \infty,$$

$$\sum_{k=1}^{n} \rho(2^k) < \infty.$$

Then

$$\frac{1}{\sqrt{n}}\left(X_1+\cdots+X_n\right) \Rightarrow N\left(0, \sigma^2\right) \quad \text{as } n \rightarrow \infty,$$

where $N\left(0, \sigma^2\right)$ *Gaussion random variable with zero-mean and variance*

$$\sigma^2=\lim_{n \rightarrow \infty} \frac{1}{n} E\left(X_1+\cdots+X_n\right)^2>0.$$

Lemma 3 ([14]). *Let* $\{X_i,\, i \geqslant 1\}$ *be a sequence of real-valued random variables with* ρ*-mixing and for some* $q \geqslant 2$

$$E X_1=0, \quad E\left|X_1\right|^q<\infty,$$

$$\sum_{k=1}^n \rho^{\frac{2}{q}}\left(2^k\right)<\infty.$$

Then there exists a constant K *such that the following inequality holds:*

$$E\left|X_1+\cdots+X_n\right|^q \leqslant K\left(n^{\frac{q}{2}} \max_{1 \leqslant i \leqslant n}\left(E\left|X_i\right|^2\right)^{\frac{q}{2}}+n \max_{1 \leqslant i \leqslant n} E\left|X_i\right|^q\right).$$

Lemma 4 ([13]). *Let* $\{X_i,\, i \geqslant 1\}$ *be a stationary sequence of random variables with* α*-mixing and*

$$E X_1=0, \quad E\left|X_1\right|^{2+\delta}<\infty,$$

$$\sum_{k=1}^{\infty} \alpha^{\frac{\delta}{2+\delta}}(k)<\infty,$$

for some $\delta>0$. *Then*

$$\sigma^2=E X_1^2+2 \sum_{j=2}^{\infty} E(X_1 X_j)<\infty \quad \text{when} \quad \sigma^2>0,$$

$$\frac{1}{\sigma \sqrt{n}}\left(X_1+\cdots+X_n\right) \Rightarrow N(0,1) \quad \text{as} \quad n \rightarrow \infty.$$

Lemma 5 ([4]). *Let* $\{X_i,\, i \geqslant 1\}$ *be a strictly stationary sequence of random variables with* α*-mixing and*

$$E X_1=0, \quad E\left|X_1\right|^{2+\delta}<\infty,$$

$$\sum_{k=1}^{\infty} n^{\frac{t}{2}-1} \alpha^{\frac{2+\delta-t}{2+\delta}}(k)<\infty,$$

for some $0<\delta \leqslant \infty$ *and* $2 \leqslant t<2+\delta$. *Then*

$$E\left|\sum_{k=1}^n\left(X_k-\mu\right)\right|^t \leqslant C n^{\frac{t}{2}}\left(E\left|X_1\right|^{2+\delta}\right)^{\frac{t}{2+\delta}}.$$

4. Proof of Theorems

Proof of Theorem 2.1. We will use the method developed in the papers [2,8] and [12]. It follows from Lemma 1 that it suffices to prove

$$P\left(|S_n(t)-S_n(s)|\wedge|S_n(u)-S_n(t)|\geqslant\lambda\right)\leqslant c_1\lambda^{-(2+\varepsilon)}g_{3+\varepsilon}\left(F(u)-F(s)\right),$$

where $\lambda\in(0,1]$, $0\leqslant s\leqslant t\leqslant u\leqslant 1$.

It is easy to see that the following inequality holds for $\lambda\in(0,1]$.

$$P\left(|S_n(t)-S_n(s)|\wedge|S_n(u)-S_n(t)|\geqslant\lambda\right)\leqslant P\left(|S_n(t)-S_n(s)|\,|S_n(u)-S_n(t)|\geqslant\lambda^2\right).$$

We have

$$J=|S_n(t)-S_n(s)|\,|S_n(u)-S_n(t)|=$$
$$=\left(\left|n^{-\frac12}\sum_{k=1}^{n}\left(X_k(t)-X_k(s)\right)\right|\right)\left(\left|n^{-\frac12}\sum_{k=1}^{n}\left(X_k(u)-X_k(t)\right)\right|\right)\leqslant$$
$$\leqslant\frac12\left(n^{-\frac12}\sum_{k=1}^{n}\left(X_k(t)-X_k(s)\right)\right)^2+\frac12\left(n^{-\frac12}\sum_{k=1}^{n}\left(X_k(u)-X_k(t)\right)\right)^2=J_1+J_2.$$

In what follows we denote by C the constants (possibly depending on different parameters) which can be different even in the same chain of inequalities.

We have

$$P\left(J\geqslant\lambda^2\right)\leqslant P\left(J_1\geqslant\frac12\lambda^2\right)+P\left(J_2\geqslant\frac12\lambda^2\right).$$

We evaluate each of the summands individually. Using the Markov inequality and Lemma 3, we obtain

$$\begin{gathered}P\left(J_1\geqslant\lambda^2\right)=P\left(\left(n^{-\frac12}\sum_{k=1}^{n}\left(X_k(t)-X_k(s)\right)\right)^2\geqslant\lambda^2\right)\leqslant\\ \leqslant\lambda^{-2}E\left(n^{-\frac12}\sum_{k=1}^{n}\left(X_k(t)-X_k(s)\right)\right)^2\leqslant\lambda^{-2}CE\left(X_1(t)-X_1(s)\right)^2,\end{gathered}\tag{14}$$

$$P\left(J_2\geqslant\lambda^2\right)\leqslant\lambda^{-2}CE\left(X_1(u)-X_1(t)\right)^2.\tag{15}$$

From (14) and (15) we get

$$P\left(J\geqslant\lambda^2\right)\leqslant\lambda^{-2}CE\left(X_1(t)-X_1(s)\right)^2+\lambda^{-2}CE\left(X_1(u)-X_1(t)\right)^2.$$

From the conditions of Theorem 2.1

$$P\left(|S_n(t)-S_n(s)|\,|S_n(u)-S_n(t)|\geqslant\lambda^2\right)\leqslant$$
$$\leqslant\lambda^{-2}CE\left(X_1(t)-X_1(s)\right)^2+\lambda^{-2}CE\left(X_1(u)-X_1(t)\right)^2\leqslant$$
$$\leqslant\lambda^{-2}C\left(F(t)-F(s)\right)\log^{-(3+\varepsilon)}\left(1+\left(F(t)-F(s)\right)^{-1}\right)+$$
$$+\lambda^{-2}C\left(F(u)-F(t)\right)\log^{-(3+\varepsilon)}\left(1+\left(F(u)-F(t)\right)^{-1}\right)\leqslant$$
$$\leqslant2\lambda^{-2}C\left(F(u)-F(s)\right)\log^{-(3+\varepsilon)}\left(1+\left(F(u)-F(s)\right)^{-1}\right)\leqslant2C\lambda^{-2}g_{3+\varepsilon}\left(F(u)-F(s)\right).$$

Above we used the inequality

$$log^{-1}\left(1+(F(u)-F(s))^{-1}\right) \leqslant 2\left|\log\left(F(u)-F(s)\right)\right|^{-1} \tag{16}$$

for

$$F(u)-F(s) \leqslant 0.25.$$

Now, to complete the proof of the theorem, it remains to prove the convergence of the finite-dimensional distributions $S_n(t)$. The convergence of finite-dimensional distributions follows from Lemma 2 and the Cramer-Wold device [5]. Thus, Theorems 2.1 is proved. □

Proof of Theorem 2.2.

We will prove Theorem 2.2 by the same method as Theorem 2.1. It follows from Lemma 1 that it suffices to prove

$$P\left(|S_n(t)-S_n(s)| \wedge |S_n(u)-S_n(t)| \geqslant \lambda\right) \leqslant c_1\lambda^{-(2+\varepsilon)} g_{3+2\varepsilon}\left(F(u)-F(s)\right),$$

where $\lambda \in (0,1]$, $0 \leqslant s \leqslant t \leqslant u \leqslant 1$.

It is easy to see that the following inequality holds for $\lambda \in (0,1]$.

$$P\Big(|S_n(t)-S_n(s)| \wedge |S_n(u)-S_n(t)| \geqslant \lambda\Big) \leqslant P\Big(|S_n(t)-S_n(s)|^{\frac{2+\varepsilon}{2}}|S_n(u)-S_n(t)|^{\frac{2+\varepsilon}{2}} \geqslant \lambda^{2+\varepsilon}\Big).$$

We have

$$I = |S_n(t)-S_n(s)|^{\frac{2+\varepsilon}{2}}|S_n(u)-S_n(t)|^{\frac{2+\varepsilon}{2}} =$$

$$= \left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k(t)-X_k(s))\right|^{\frac{2+\varepsilon}{2}}\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k(u)-X_k(t))\right|^{\frac{2+\varepsilon}{2}} \leqslant$$

$$\leqslant \frac{1}{2}\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k(t)-X_k(s))\right|^{2+\varepsilon} + \frac{1}{2}\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k(u)-X_k(t))\right|^{2+\varepsilon} = I_1+I_2.$$

We have

$$P\left(I \geqslant \lambda^{2+\varepsilon}\right) \leqslant P\left(I_1 \geqslant \frac{1}{2}\lambda^{2+\varepsilon}\right) + P\left(I_2 \geqslant \frac{1}{2}\lambda^{2+\varepsilon}\right).$$

Using the Markov inequality and Lemma 3, we obtain

$$P\left(I_1 \geqslant \lambda^{2+\varepsilon}\right) = P\left(\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k(t)-X_k(s))\right|^{2+\varepsilon} \geqslant \lambda^{2+\varepsilon}\right) \leqslant$$

$$\leqslant \lambda^{-(2+\varepsilon)}E\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k(t)-X_k(s))\right|^{2+\varepsilon} \leqslant$$

$$\leqslant C\lambda^{-(2+\varepsilon)}n^{-(2+\varepsilon)/2}n^{(2+\varepsilon)/2}\left(E|X_1(t)-X_1(s)|^2\right)^{(2+\varepsilon)/2} +$$

$$+C\lambda^{-(2+\varepsilon)}n^{-(2+\varepsilon)/2}nE|X_1(t)-X_1(s)|^{2+\varepsilon} \leqslant$$

$$\leqslant \lambda^{-(2+\varepsilon)}C\left(E|X_1(t)-X_1(s)|^2\right)^{(2+\varepsilon)/2} + \lambda^{-(2+\varepsilon)}Cn^{-\varepsilon/2}E|X_1(t)-X_1(s)|^{2+\varepsilon} \leqslant$$

$$\leqslant 2C\lambda^{-(2+\varepsilon)}E|X_1(t)-X_1(s)|^{2+\varepsilon}.$$

$$P\left(I_2 \geqslant \lambda^{2+\varepsilon}\right) \leqslant 2C\lambda^{-(2+\varepsilon)}E|X_1(u)-X_1(t)|^{2+\varepsilon}.$$

From the conditions of Theorem 2.2 and using (16) we have

$$P\left(|S_n(t)-S_n(s)|^{\frac{2+\varepsilon}{2}}\,|S_n(u)-S_n(t)|^{\frac{2+\varepsilon}{2}}\geqslant\lambda^{2+\varepsilon}\right)\leqslant$$

$$\leqslant C\lambda^{-(2+\varepsilon)}\left(F(t)-F(s)\right)\log^{-(3+2\varepsilon)}\left(1+\left(F(t)-F(s)\right)^{-1}\right)+$$

$$+C\lambda^{-(2+\varepsilon)}\left(F(u)-F(t)\right)\log^{-(3+2\varepsilon)}\left(1+\left(F(u)-F(t)\right)^{-1}\right)\leqslant$$

$$\leqslant 2C\lambda^{-(2+\varepsilon)}\left(F(u)-F(s)\right)\log^{-(3+2\varepsilon)}\left(1+\left(F(u)-F(s)\right)^{-1}\right)\leqslant 2C\lambda^{-(2+\varepsilon)}g_{3+2\varepsilon}\left(F(u)-F(s)\right).$$

To complete the proof of the theorem, it remains to prove the convergence of the finite-dimensional distributions $S_n(t)$. The convergence of finite-dimensional distributions follows from Lemma 2 and the Cramer-Wold device [5]. Thus, Theorems 2.2 is proved. □

Proof of Theorem 2.3.

To prove Theorem 2.3, we estimate I as in the proof of Theorem 2.2 by I_1 and I_2. Using the Markov inequality and Lemma 5, we have (where $\varepsilon+\varepsilon_1=\delta,\ \varepsilon_1>0$)

$$P\left(I_1\geqslant\lambda^{2+\varepsilon}\right)=P\left(\left|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\left(X_k(t)-X_k(s)\right)\right|^{2+\varepsilon}\geqslant\lambda^{2+\varepsilon}\right)\leqslant$$

$$\leqslant C\alpha\left(k\right)\lambda^{-(2+\varepsilon)}\frac{1}{n^{\frac{2+\varepsilon}{2}}}E\left|\sum_{k=1}^{n}\left(X_k(t)-X_k(s)\right)\right|^{2+\varepsilon}\leqslant$$

$$\leqslant C\lambda^{-(2+\varepsilon)}\left(E\left|X_1(t)-X_1(s)\right|^{2+\varepsilon+\varepsilon_1}\right)^{\frac{2+\varepsilon}{2+\varepsilon+\varepsilon_1}}\leqslant$$

$$\leqslant C\lambda^{-(2+\varepsilon)}\left(E\left|X_1(t)-X_1(s)\right|^{2+\delta}\right)^{\frac{2+\varepsilon}{2+\delta}}.$$

$$P\left(I_2\geqslant\lambda^{2+\varepsilon}\right)=C\lambda^{-(2+\varepsilon)}\left(E\left|X_1(u)-X_1(t)\right|^{2+\delta}\right)^{\frac{2+\varepsilon}{2+\delta}}.$$

$$P\left(I\geqslant\lambda^{2}\right)\leqslant C\lambda^{-(2+\varepsilon)}\left(\left(E\left|X_1(t)-X_1(s)\right|^{2+\delta}\right)^{\frac{2+\varepsilon}{2+\delta}}+\left(E\left|X_1(u)-X_1(t)\right|^{2+\delta}\right)^{\frac{2+\varepsilon}{2+\delta}}\right).$$

From the conditions of Theorem 2.3 and using (16) we have

$$P\left(|S_n(t)-S_n(s)|^{\frac{2+\varepsilon}{2}}\,|S_n(u)-S_n(t)|^{\frac{2+\varepsilon}{2}}\geqslant\lambda^{2+\varepsilon}\right)\leqslant$$

$$\leqslant C\lambda^{-(2+\varepsilon)}\left(F(t)-F(s)\right)\log^{-(3+2\varepsilon)}\left(1+\left(F(t)-F(s)\right)^{-1}\right)+$$

$$+C\lambda^{-(2+\varepsilon)}\left(F(u)-F(t)\right)\log^{-(3+2\varepsilon)}\left(1+\left(F(u)-F(t)\right)^{-1}\right)\leqslant$$

$$\leqslant 2C\lambda^{-(2+\varepsilon)}\left(F(u)-F(s)\right)\log^{-(3+2\varepsilon)}\left(1+\left(F(u)-F(s)\right)^{-1}\right)\leqslant 2C\lambda^{-(2+\varepsilon)}g_{3+2\varepsilon}\left(F(u)-F(s)\right).$$

Again as in the proof of previous theorems, to complete the proof of the theorem, it remains to prove the convergence of the finite-dimensional distributions $S_n(t)$. The convergence of finite-dimensional distributions follows from Lemma 4 and the Cramer–Wold device [5]. Thus, Theorems 2.3 is proved. □

References

[1] P.H.Bezandry, X.Fernique, Sur la propriete de la limite centrale dans $D\,[0,1]$, *Ann. Inst. H. Poincare*, **28**(1992), 31–46.

[2] M.Bloznelis, V.Paulauskas, On the central limit theorem in $D\,[0,1]$, *Stat. Prob. Lett.*, **17**(1993), 105–111.

[3] M.Bloznelis, V.Paulauskas, A note on the central limit theorem for stochastically continuous processes, *Stochastic Processes and their Applications*, **53**(1994), 351–361.

[4] D.Bosq, Linear processes in function spaces. Lecture Notes in Statistics, Vol. 149, New York, Springer-Verlag, 2000.

[5] P.Billingsley, Convergence of Probability Measures, Wiley, New York, 1968.

[6] H.Dehling, O.Sh. Sharipov, M.Wendler, Bootstrap for dependent Hilbert space-valued random variables with application to von Mises statistics, *Journal of Multivariate analysis*, **133**(2015), 200–215.

[7] X.Fernique, Les functions aleatoires càdlàg, la compacite de laurs lois, *Liet. Mat. Rink.*, **34**(1994), 288–306.

[8] M.G.Hahn, Central limit theorems in $D\,[0,1]$, *Z. Wahrsch. Verw. Gebiete.*, **44**(1978), 89–101.

[9] I.A.Ibragimov, A Remark on the Central Limit Theorem for Dependent Random Variables, *Theor. Prob. Appl.*, **20**(1975), 134–140.

[10] D.Juknevičienė, Central limit theorem in the space $D\,[0,1]$, *Lithuanian Math. J.*, **25**(1985), 293–298.

[11] A.F.Norjigitov, O.Sh.Sharipov, The central limit theorem for weakly dependent random variables with values in the space $D\,[0,1]$, *Uzbek Math. J.*, **2**(2010), 103–112 (in Russian).

[12] V.Paulauskas, Ch.Stieve, On the central limit theorem in $D\,[0,1]$ and $D([0,1]\,,H)$, *Lithuanian Math. J.*, **30**(1990), 267–276.

[13] I.G.Zhurbenko, On the conditions for mixing random processes with values in Hilbert spaces, *Dokl. AN USSR.*, **278**(1984), no. 4, 792–796.

[14] Q.M.Shao, Maximal inequalities for partial sums of ρ-mixing sequences, *Ann. Prob.*, **23**(1995), 948–965.

[15] O.Sh.Sharipov, On probabilistic characterization of Banach space by weakly dependent random elements, *Uzbek Math. J.*, **2**(1998), 86–91 (in Russian).

[16] O.Sh.Sharipov, A central limit theorem for weakly dependent random variables with values in co space, In: Modern problems of Probability theory and Mathematical statistics, Tashkent. "FAN", 2000, 193–197 (in Russian).

[17] O.Sh.Sharipov, A.F.Norjigitov, Central limit theorem for weakly dependent random variables and autoregressive processes with values in $C\,[0,1]$, *Uzbek Math. J.*, **2**(2016), 149–157 (in Russian).

Центральная предельная теорема для слабо зависимых случайных величин со значениями в $D\,[0,1]$

Олимжон Ш. Шарипов
Национальный университет Узбекистана
Ташкент, Узбекистан
Математический институт им. В. И. Романовского АНУз
Ташкент, Узбекистан
Анвар Ф. Норжигитов
Математический институт им. В. И. Романовского АНУз
Ташкент, Узбекистан

Аннотация. Основной целью настоящей статьи является доказательство центральной предельной теоремы для последовательностей случайных величин со значениями в пространстве $D\,[0,1]$. Мы предполагаем, что последовательность удовлетворяет условиям перемешивания. В статье доказаны центральные предельные теоремы для последовательностей с сильным перемешиванием и ρ_m-перемешиванием.

Ключевые слова: центральная предельная теорема, последовательность с перемешиванием, пространство $D\,[0,1]$.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(3), 292–307

DOI: 10.17516/1997-1397-2022-15-3-292-307
УДК 519.2

On Approximation of Empirical Kac Processes under General Random Censorship Model

Abdurahim A. Abdushukurov*
Moscow State University named after M. V. Lomonosov, Tashkent Branch
Tashkent, Uzbekistan

Gulnoz S. Saifulloeva†
Navoi State Pedagogical Institute
Navoi, Uzbekistan

Received 07.11.2021, received in revised form 10.12.2021, accepted 20.02.2022

Abstract. A general random censorship model is considered in the paper. Approximation results are proved for empirical Kac processes. This model includes important special cases such as random censorship on the right and competing risks model. The obtained results use strong approximation theory and optimal approximation rates are built. Cumulative hazard processes are also investigated in a similar manner in the general setting. These results are also used for estimating of characteristic functions in random censorship model on the right.

Keywords: censored data, competing risks, empirical estimates, Kac estimate, strong approximation, Gaussian process, characteristic function.

Citation: A.A. Abdushukurov, G.S. Saifulloeva, On Approximation of Empirical Kac Processes under General Random Censorship Model, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 292–307.
DOI: 10.17516/1997-1397-2022-15-3-292-307.

1. Introduction and preliminaries

Following of ([3–5]) we define a general random censorship model in the following way: Let Z be a real random variable (r.v.) with distribution function (d.f.) $H(x) = P(Z \leqslant x)$, $x \in \mathbb{R}$. Let us assume that $A^{(1)}, \dots, A^{(k)}$ are pairwise disjoint random events for a fixed integer $k \geqslant 1$. Let us define the subdistribution functions $H(x;i) = P(Z \leqslant x, A^{(i)}), i \in \Im = \{1, \dots, k\}$. Suppose that when observing Z we are interested in the joint behaviour of the pairs $(Z, A^{(i)})$, $i \in \Im$. Let $\{(Z_j, A_j^{(1)}, \dots, A_j^{(k)}),\ j \geqslant 1\}$ be a sequence of independent replicas of $(Z, A^{(1)}, \dots, A^{(k)})$ defined on some probability space $\{\Omega, A, P\}$. We assume throughout that functions $H(x), H(x;1), \dots, H(x;k)$ are continuous. Let us denote the ordinary empirical d.f. of $Z_1, \dots, Z_n$ by $H_n(x)$ and introduce the empirical sub d.f. $H_n(x;i)$, $i \in \Im$

$$H_n(x;i) = \frac{1}{n}\sum_{j=1}^{n} \delta_j^{(i)} I(Z_j \leqslant x),\ (x;i) \in \overline{\mathbb{R}} \times \Im,$$

where $\overline{\mathbb{R}} = [-\infty; \infty]$, $\delta_j^{(i)} = I(A_j^{(i)})$ is the indicator of event $A_j^{(i)}$ and

*a-abdushukurov@rambler.ru
†sayfulloyevagulnoz@gmail.com

$$H_n(x;1)+\cdots+H_n(x;k)=\frac{1}{n}\sum_{j=1}^{n}I(Z_j\leqslant x)=H_n(x),\ \ x\in\overline{\mathbb{R}},$$

is the ordinary empirical d.f.. Properties of many biometric estimates depend on the limit behaviour of proposed empirical statistics. The following results are straightforward consequences of the Dvoretzky–Kiefer–Wolfowitz exponential inequality with constant $D=2$ [8,12] :

For all $n=1,2,\ldots$ and $\varepsilon>0$

$$P\left(\sup_{|x|<\infty}\bigl|H_n(x)-H(x)\bigr|>\left(\frac{(1+\varepsilon)}{2}\cdot\frac{\log n}{n}\right)^{1/2}\right)\leqslant 2n^{-(1+\varepsilon)},\tag{1}$$

and

$$P\left(\sup_{|x|<\infty}\bigl|H_n(x;i)-H(x;i)\bigr|>2\Bigl(\frac{(1+\varepsilon)}{2}\frac{\log n}{n}\Bigr)^{1/2}\right)\leqslant 4n^{-(1+\varepsilon)}.\tag{2}$$

Vector-valued empirical process $\Bigl\{a_n(t)=\Bigl(a_n^{(0)}(t_0),a_n^{(1)}(t_1),\ldots,a_n^{(k)}(t_k)\Bigr),\ t=(t_0,\ldots,t_k)\in\overline{\mathbb{R}}^{k+1}\Bigr\}$ plays a decisive role, where $a_n^{(0)}(x)=\sqrt{n}(H_n(x)-H(x))$, $a_n^{(i)}(x)=\sqrt{n}(H_n(x;i)-H(x;i))$, $i\in\Im$. The following Burke–Csörgő–Horváth theorem [3,4] is an extended analogue of Komlós–Major–Tusnády's result [9–11].

Theorem A([3,4]). *If the underlying probability space $\{\Omega,\mathcal{A},P\}$ is rich enough then one can define $k+1$ sequences of Gaussian processes $B_n^{(0)}(x),B_n^{(1)}(x),\ldots,B_n^{(k)}(x)$ such that for $a_n(t)$ and $B_n(t)=(B_n^{(0)}(x_0),B_n^{(1)}(x_1),\ldots,B_n^{(k)}(x_k))$, $t=(t_0,\ldots,t_k)$ we have*

$$P\left\{\sup_{t\in\overline{\mathbb{R}}^{k+1}}\bigl\|a_n(t)-B_n(t)\bigr\|^{(k+1)}>n^{-\frac{1}{2}}(M(\log n)+z)\right\}\leqslant K\exp\left(-\lambda z\right),\tag{3}$$

for all real z, where $M=(2k+1)A_1$, $K=(2k+1)A_2$ and $\lambda=A_3/(2k+1)$ with A_1,A_2 and A_3 are absolute constants. Moreover, B_n is $(k+1)$-dimensional vector-valued Gaussian process that has the same covariance structure as the vector $a_n(t)$, namely, $EB_n^{(i)}(x)=0$, $(x,i)\in\overline{\mathbb{R}}\times\overline{\Im}=\Im\cup\{0\}$. We have for any $i,j\in\Im$, $i\neq j$, $x,y\in\overline{\mathbb{R}}$ that

$$\begin{aligned}
&EB_n^{(0)}(x)B_n^{(0)}(y)=\min\bigl\{H(x),H(y)\bigr\}-H(x)\cdot H(y),\\
&EB_n^{(i)}(x)B_n^{(i)}(y)=\min\bigl\{H(x;i),H(y;i)\bigr\}-H(x;i)\cdot H(y;i),\\
&EB_n^{(i)}(x)B_n^{(j)}(y)=-H(x;i)\cdot H(y;j),\\
&EB_n^{(0)}(x)B_n^{(i)}(y)=\min\bigl\{H(x;i),H(y;j)\bigr\}-H(x)\cdot H(y;i).
\end{aligned}\tag{4}$$

If we set $z=\left(\dfrac{(1+\varepsilon)}{\lambda}\log n\right)$ in (3) then

$$P\left\{\sup_{t\in\overline{\mathbb{R}}^{k+1}}\bigl\|a_n(t)-B_n(t)\bigr\|^{(k+1)}>Cn^{-\frac{1}{2}}\log n\right\}\leqslant Kn^{-(1+\varepsilon)},$$

where $C=(2k+1)\left(A_1+\dfrac{(1+\varepsilon)}{A_3}\right)$. Then

$$\Bigl\|a_n(t)-B_n(t)\Bigr\|^{(k+1)}\overset{a.s.}{=}O\Bigl(n^{-\frac{1}{2}}\log n\Bigr).$$

Let us note that in proving Theorem A (Theorem 3.1 in [4]) the sequence of two-parametrical Gaussian processes $\mathbb{Q}^{(0)}(x,n), \mathbb{Q}^{(2)}(x,n), \ldots, \mathbb{Q}^{(k)}(x,n)$ was constructed such that for $a_n(t)$ and $\mathbb{Q}(t;n) = \left(\mathbb{Q}^{(0)}(x;n), \ldots, \mathbb{Q}^{(k)}(x;n)\right), \quad t \in \overline{\mathbb{R}}^{k+1}$ the following approximation was used

$$\left\| a_n(t) - n^{-\frac{1}{2}}\mathbb{Q}(t,n) \right\|^{(k+1)} \overset{a.s.}{=} O\left(n^{-\frac{1}{2}} \log^2 n\right),$$

where $\mathbb{Q}(t,n)$ is the $(k+1)$ dimensional vector-valued Gaussian process and $\mathbb{Q}(t;n) \overset{D}{=} n^{\frac{1}{2}} a_n(t)$. Hence

$$E\mathbb{Q}^{(i)}(x;n) = 0, \quad (x,i) \in \overline{\mathbb{R}} \times \overline{\Im}$$

and we have for any $i,j \in \Im,\ i \neq j,\ x,y \in \overline{\mathbb{R}}$ that

$$\begin{aligned} &E\mathbb{Q}^{(0)}(x;n)\mathbb{Q}^{(0)}(y;m) = \min(n,m)\big\{\min\{H(x),H(y)\} - H(x)H(y)\big\},\\ &E\mathbb{Q}^{(0)}(x;n)\mathbb{Q}^{(i)}(y;m) = \min(n,m)\big\{\min\{H(x;i),H(y;i)\} - H(x)H(y;i)\big\},\\ &E\mathbb{Q}^{(i)}(x;n)\mathbb{Q}^{(i)}(y;m) = \min(n,m)\big\{\min\{H(x;i),H(y;i)\} - H(x;i)H(y;j)\big\},\\ &E\mathbb{Q}^{(i)}(x;n)\mathbb{Q}^{(j)}(y;m) = -\min(n,m)H(x;i)\cdot H(y;j). \end{aligned} \tag{5}$$

Let us observe that $\{\mathbb{Q}^{(i)}, i \in \overline{\Im}\}$ are Kiefer processes and they satisfy the distributional equality

$$\mathbb{Q}^{(i)}(x;n) \overset{D}{=} W^{(i)}(H(x;i);n) - H(x;i)W^{(i)}(1;n), \tag{6}$$

where $\big\{W^{(i)}(y;n), 0 \leqslant y \leqslant 1, n \geqslant 1, i \in \Im\big\}$ are two-parametric Wiener processes with $EW^{(i)}(y;n) = 0$ and

$$EW^{(i)}(y;n)W^{(i)}(u;m) = \min(n,m)\min(y,u), \quad i \in \Im.$$

It is important to note that though Kiefer processes $\big\{\mathbb{Q}^{(i)},\ i \in \Im\big\}$ are dependent processes, corresponding Wiener processes are independent. Indeed, it follows from the proof of Theorem A that

$$\begin{aligned} &\mathbb{Q}^{(1)}(x;n) \overset{D}{=} \widetilde{K}(H(x;1);n),\\ &\mathbb{Q}^{(2)}(x;n) \overset{D}{=} \widetilde{K}(H(x;2) - H(+\infty;1);n) - \widetilde{K}(H(+\infty;1);n),\\ &\ldots\ldots\\ &\mathbb{Q}^{(i)}(x;n) \overset{D}{=} \widetilde{K}(H(x;i) + H(+\infty;1) + \cdots + H(+\infty;i-1);n) -\\ &\qquad - \widetilde{K}(H(+\infty;1) + \cdots + H(+\infty;i-1);n), \quad i \in \Im, \end{aligned}$$

where $H(+\infty;i) = \lim\limits_{x\uparrow+\infty} H(x;i),\ H(+\infty;1) + \cdots + H(+\infty;k) = 1$.

The Kiefer processes $\big\{\widetilde{K}(y;n), 0 \leqslant y \leqslant 1, n \geqslant 1\big\}$ are represented in terms of two-parametrical Wiener processes $\big\{W(y;n), 0 \leqslant y \leqslant 1, n \geqslant 1\big\}$ by distributional equality

$$\big\{\widetilde{K}(y;n), \quad 0 \leqslant y \leqslant 1, n \geqslant 1\big\} \overset{D}{=} \big\{W(y;n) - yW(1;n), 0 \leqslant y \leqslant 1, n \geqslant 1\big\}. \tag{7}$$

Then, taking into account (6) and (7), the Wiener process $\{W^{(i)}, i \in \Im\}$ also admits the following representations for all $(x;i) \in \overline{\mathbb{R}} \times \Im$

$$\begin{aligned} &W^{(1)}(H(x;1);n) \overset{D}{=} W(H(x;1);n),\\ &W^{(2)}(H(x;2);n) \overset{D}{=} W(H(x;2) + H(+\infty;1);n) - W^{(1)}(H(+\infty;1);n), \ldots,\\ &W^{(i)}(H(x;i);n) \overset{D}{=} W(H(x;i) + H(+\infty;i-1);n) - W(H(+\infty;1) + \cdots + H(+\infty;i-1);n). \end{aligned}$$

Now performing direct calculations of covariances of processes $\{W^{(i)},\ i \in \Im\}$, it is easy to show that these processes are independent.

2. Kac processes under general censoring

Following [9] we introduce the modified empirical d.f. of Kac by the following way. Along with sequence $\{Z_j, j \geqslant 1\}$ on a probability space $\{\Omega, \mathcal{A}, P\}$ consider also a sequence $\{\nu_n,\ n \geqslant 1\}$ of r.v.-s that has Poisson distribution with parameter $E\nu_n = n,\ n = 1, 2, \dots$. Let us assume throughout that two sequences $\{Z_j, j \geqslant 1\}$ and $\{\nu_n, n \geqslant 1\}$ are independent. The Kac empirical d.f. is

$$H_n^*(x) = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{j=1}^{\nu_n} I(Z_j \leqslant x) \quad \text{if} \quad \nu_n \geqslant 1 \quad a.s., \\ 0 \quad \text{if} \quad \nu_n = 0 \quad a.s., \end{cases}$$

while the empirical sub-d.f. is

$$H_n^*(x; i) = \begin{cases} \dfrac{1}{n}\displaystyle\sum_{j=1}^{\nu_n} I\big(Z_j \leqslant x, A_j^{(i)}\big), \quad i \in \Im \quad if \quad \nu_n \geqslant 1 \quad a.s., \\ 0, \quad i \in \Im \quad if \quad \nu_n = 0 \quad a.s., \end{cases}.$$

with $H_n^*(x;1) + \dots + H_n^*(x;k) = H_n^*(x)$ for all $x \in \overline{\mathbb{R}}$. Here we suppose that sequence $\{\nu_n,\ n \geqslant 1\}$ is independent of random vectors $\big\{\big(Z_j, \delta_j^{(1)}, \dots, \delta_j^{(k)}\big), j \geqslant 1\big\}$, where $\delta_j^{(i)} = I(A_j^{(i)})$. Let us note that statistics $H_n^*(x;i)$ (and also $H_n^*(x)$) are unbiased estimators of $H(x;i)$, $i \in \Im$ (and also of $H(x)$)

$$\begin{aligned} E(H_n^*(x;i)) &= \frac{1}{n} E\left\{\sum_{m=1}^{\infty} E\left[\sum_{k=1}^{n} \delta_k^{(i)} \cdot I(Z_k \leqslant x)\right], \nu_n = m\right\} = \\ &= \frac{1}{n} E\left\{\sum_{m=1}^{\infty} E\left[\sum_{k=1}^{n} \delta_k^{(i)} \cdot I(Z_k \leqslant x)/\nu_n = m\right] \cdot P(\nu_n = m)\right\} = \\ &= \frac{1}{n}\sum_{m=1}^{\infty} H(x;i) m P(\nu_n = m) = \frac{1}{n} H(x;i) \sum_{m=1}^{\infty} m \cdot \frac{n^m e^{-n}}{m!} = \\ &= H(x;i) e^{-n} \sum_{m=0}^{\infty} \frac{n^m}{m!} = H(x;i), \quad (x;i) \in \overline{\mathbb{R}} \times \Im. \end{aligned}$$

Consequently,

$$E\big[H_n^*(x)\big] = \sum_{i=1}^{k} E\big[H_n^*(x;i)\big] = \sum_{i=1}^{k} H(x;i) = H(x), \quad x \in \overline{\mathbb{R}}.$$

Let us define the empirical Kac processes $a_n^{(i)*}(x) = \sqrt{n}\big(H_n^*(x;i) - H(x;i)\big)$, $i \in \Im$ and $a_n^{(0)*}(x) = \sqrt{n}\big(H_n^*(x) - H(x)\big)$.

Theorem 1. *If the underlying probability space $\{\Omega, \mathcal{A}, P\}$ is rich enough then one can define $k+1$ sequences of Gaussian processes $W_n^{(0)}(x), W_n^{(1)}(x), \dots, W_n^{(k)}(x)$ such that for $a_n^*(t) = \big(a_n^{(0)*}(t_0), a_n^{(1)*}(t_1), \dots, a_n^{(k)*}(t_k)\big)$ and $W_n^*(t) = \big(W_n^{(0)}(t_0), W_n^{(1)}(t_1), \dots, W_n^{(k)}(t_k)\big)$, $t = (t_0, t_1, \dots, t_k)$ we have*

$$P\left\{\sup_{t \in \overline{\mathbb{R}}^{k+1}} \left\| a_n^*(t) - W_n^*(t) \right\|^{(k+1)} > C^* n^{-\frac{1}{2}} \log n \right\} \leqslant K^* n^{-r}, \tag{8}$$

where $r \geqslant 2$ is an arbitrary integer, $C^* = C^*(r)$ depends only on r, and K^* is an absolute constant. Moreover, $W_n^*(t)$ is $(k+1)$-dimensional vector-valued Gaussian process with expectation

$EW^{(i)}(x) = 0$, $(x,i) \in \overline{\mathbb{R}} \times \overline{\Im}$. *We have for any* $i, j \in \Im$, $i \neq j$, $x, y \in \overline{\mathbb{R}}$ *that*

$$\begin{aligned} &EW_n^{(0)}(x)W_n^{(0)}(y) = \min\{H(x), H(y)\}, \\ &EW_n^{(i)}(x)W_n^{(j)}(y) = \min\{H(x;i), H(y;j)\}, \\ &EW_n^{(i)}(x)W_n^{(0)}(y) = \min\{H(x;i), H(y)\}. \end{aligned} \tag{9}$$

The basic relation between $a_n(t)$ and $a_n^*(t)$ is the following easily checked identity

$$a_n^*(x) = \sqrt{\frac{\nu_n}{n}}\, a_{\nu_n}^{(i)}(x) + H(x;i)\frac{(\nu_n - n)}{\sqrt{n}}, \quad i \in \Im. \tag{10}$$

Hence, the approximating sequence have the form

$$W_n^{(i)}(x) = B_{\nu_n}^{(i)}(x) + H(x;i)\frac{W^*(n)}{\sqrt{n}}, \quad i \in \Im,$$

where $B_{\nu_n}^{(i)}(x)$ is a Poisson indexed Brownian bridge type process of Theorem A and $\{W^{(*)}(x),\ x \geqslant 0\}$ is a Wiener process. It is easy to verify that $\{W_n^{(i)}(x), (x;i) \in \overline{\mathbb{R}} \times \overline{\Im}\} \overset{D}{=} \{W^*(H(x;i)), (x,i) \in \overline{\mathbb{R}} \times \overline{\Im}\}$. The proof of Theorem 1 is similar to the proof of Theorem 1 of Stute [6] and, it is omitted.

Since $\lim\limits_{x \uparrow +\infty} H_n^*(x) = H_n^*(+\infty) = \dfrac{\nu_n}{n}$ then using Stirlings formula, we obtain

$$P(\nu_n = n) = P\big(H_n^*(+\infty) = 1\big) = \frac{n^n e^{-n}}{n!} = \frac{1}{\sqrt{2\pi n}}(1 + o(1)), \quad n \to \infty,$$

and

$$P\big(H_n^*(+\infty) > 1\big) = P(\nu_n > n) = \sum_{k=n+1}^{\infty} \frac{n^k e^{-n}}{k!} = o(1), \quad n \to \infty.$$

Thus $H_n^*(x)$ with positive probability is greater than 1. In order to avoid these undesirable property the following modifications of the Kac statistics is proposed

$$\begin{aligned} &\widetilde{H}_n(x) = 1 - \big(1 - H_n^*(x)\big) I\big(H_n^*(x) < 1\big), \quad x \in \overline{\mathbb{R}}, \\ &\widetilde{H}_n(x;i) = 1 - \big(1 - H_n^*(x;i)\big) I\big(H_n^*(x;i) < 1\big), \quad (x;i) \in \overline{\mathbb{R}} \times \Im. \end{aligned} \tag{11}$$

The following inequalities are useful in studying the Kac processes.

Theorem 2. *Let* $\{\nu_n, n \geqslant 1\}$ *be a sequence of Poisson r.v.-s with* $E\nu_n = n$. *Then for any* $\varepsilon > 0$ *such that*

$$\frac{n}{\log n} \geqslant \frac{\varepsilon}{8(1 + \frac{e}{3})^2}, \quad e = \exp(1), \tag{12}$$

we have

$$P\left(\big|\nu_n - n\big| > \frac{1}{2}\left(\frac{\varepsilon}{2} n \log n\right)^{\frac{1}{2}}\right) \leqslant 2n^{-\varepsilon w}, \tag{13}$$

$$P\left(\sup_{|x| < \infty} \big|H_n^*(x;i) - H(x;i)\big| > 2\left(\frac{\varepsilon \log n}{2n}\right)^{\frac{1}{2}}\right) \leqslant 4n^{-4\varepsilon w},\ i \in \Im, \tag{14}$$

$$P\left(\sup_{|x| < \infty} \big|\widetilde{H}_n(x;i) - H(x;i)\big| > 2\left(\frac{\varepsilon \log n}{2n}\right)^{\frac{1}{2}}\right) \leqslant 4n^{-4\varepsilon w},\ i \in \Im, \tag{15}$$

where $w = \left[16\left(1 + \dfrac{e}{3}\right)\right]^{-1}$.

Proof. Let $\gamma_1, \gamma_2, \dots$ be a sequence of Poisson r.v.-s with $E\gamma_k = 1$ for all $k = 1, 2, \dots$. Then $S_n = \nu_n - n = \sum\limits_{k=1}^{n} (\gamma_k - 1) = \sum\limits_{k=1}^{n} \xi_k$ and

$$E\exp(t\xi_k) = e^{-t}\exp(t\gamma_1) = \exp\big(-(t+1)\big)\sum_{k=0}^{\infty}\frac{(e^t)^k}{k!} = \exp\big\{e^t - (t+1)\big\}.$$

Using Taylor expansion for e^t, we obtain

$$E\exp(t\xi_k) = \exp\left\{1 + t + \frac{t^2}{2} + \psi(t) - (t+1)\right\} = \exp\left\{\frac{t^2}{2} + \psi(t)\right\},$$

where $\psi(t) = \dfrac{t^3}{6}\exp(\theta t)$, $\quad 0 < \theta < 1$. Taking into account that $t^3 \leqslant t^2$ for $0 \leqslant t \leqslant 1$, we obtain the estimate for $\psi(t)$: $\psi(t) \leqslant \dfrac{t^3}{6}e \leqslant e\dfrac{t^2}{6}$. Thus, $E\exp(t\xi_k) = \exp\left\{\dfrac{t^2}{2}\left(1 + \dfrac{e}{3}\right)\right\}, \quad 0 \leqslant t \leqslant 1.$

The following result (from [13]) is necessary for further considerations.

Lemma 1 ([13])**.** *Let $\{\xi_n, n \geqslant 1\}$ be a sequence of independent r.v.-s with $E\xi_n = 0$, $n = 1, 2, \dots$. Suppose that $U, \lambda_1, \dots, \lambda_n$ are positive real numbers such that*

$$E\exp(t\xi_k) \leqslant \exp\left(\frac{1}{2}\lambda_k t_k^2\right) \quad for \quad k = 1, 2, \dots, n \quad |t| \leqslant U. \tag{16}$$

Let $\Lambda = \lambda_1 + \dots + \lambda_n$. Then

$$P\Big(\big|\xi_1 + \dots + \xi_k\big| \geqslant z\Big) \leqslant \begin{cases} 2\exp\left(-\dfrac{z^2}{2\Lambda}\right) & if \quad o \leqslant z \leqslant \Lambda U, \\ 2\exp\left(-\dfrac{Uz}{2}\right) & if \quad z > \Lambda U. \end{cases}$$

Let us assume that $\lambda_k = 1 + \dfrac{e}{3}$, $U = 1$, $z = \dfrac{1}{2}\left(\frac{\varepsilon}{2}n\log n\right)^{1/2}$ in Lemma 1 then we obtain (13). Here $0 \leqslant z = \dfrac{1}{2}\left(\frac{\varepsilon}{2}n\log n\right)^{1/2} \leqslant \left(1 + \frac{e}{3}\right)n = \Lambda U$. Consider probability in (14). Using total probability formula, we have

$$P\left(\sup_{|x|<\infty}\big|H_n^*(x;i) - H(x;i)\big| > 2\left(\frac{\varepsilon\log n}{2n}\right)^{\frac{1}{2}}\right) =$$

$$= P\left(\sup_{|x|<\infty}\left|H_n(x;i) - H(x;i) + \frac{1}{n}\sum_{k=n+1}^{\nu_n}\delta_k^{(i)}I(Z_k \leqslant x)\right| > 2\left(\frac{\varepsilon\log n}{2n}\right)^{\frac{1}{2}} \Big/ \nu_n > n\right) \cdot P(\nu_n > n) +$$

$$+ P\left(\sup_{|x|<\infty}\left|H(x;i) - H(x;i) - \frac{1}{n}\sum_{k=\nu_n+1}^{n}\delta_k^{(i)}I\big(Z_k \leqslant x\big)\right| > 2\left(\frac{\varepsilon\log n}{2n}\right)^{\frac{1}{2}} \Big/ \nu_n \leqslant n\right) \cdot P(\nu_n \leqslant n) \leqslant$$

$$\leqslant P\left(\sup_{|x|<\infty}\big|H_n(x;i) - H(x;i)\big| > \left(\frac{\varepsilon\log n}{2n}\right)^{\frac{1}{2}}\right) + P\left(\sup_{|x|<\infty}\left|\frac{1}{n}\sum_{k=\min(n,\nu_n)+1}^{\max(n,\nu_n)}\delta_k^{(i)}I(Z_k \leqslant x)\right| >\right.$$

$$\left. > \left(\frac{\varepsilon\log n}{2n}\right)^{\frac{1}{2}}\right) \leqslant 2n^{-4\varepsilon} + P\left(\left|\frac{\nu_n - n}{n}\right| > \left(\frac{\varepsilon\log n}{2n}\right)^{\frac{1}{2}}\right) \leqslant 2n^{-4\varepsilon} + 2n^{-4w\varepsilon} \leqslant 4n^{-4w\varepsilon}, \quad i \in \Im,$$

where we applied (2) and (13) that proves (14). Let us define $T_n^{(i)} = \inf\big\{x : \widetilde{H}_n(x;i) = 1\big\}$, $i \in \Im$. If $x \geqslant \widetilde{T}_n^{(i)}$ and $\nu_n > n$ then $\widetilde{H}_n(x;i) = 1$ and $H_n^*(x;i) - H\,(x;i) \geqslant H_n^*(x;i) - \widetilde{H}(x;i) \geqslant 0$. Then assuming $\nu_n > n$, we obtain

$$\sup_{|x|<\infty}\left|\widetilde{H}_n(x;i)-H(x;i)\right| = \left\{\max\left[\sup_{x<\widetilde{T}_n^{(i)}}\left|H_n^*(x;i)-H(x;i)\right|, \sup_{x\geqslant\widetilde{T}_n^{(i)}}\left|\widetilde{H}_n(x;i)-H(x;i)\right|\right]\right\} \leqslant$$

$$\leqslant \left\{\max\left[\sup_{x<T_n^{(i)}}\left|H_n^*(x;i)-H(x;i)\right|, \sup_{x\geqslant T_n^{(i)}}\left|H_n^*(x;i)-H(x;i)\right|\right]\right\} =$$

$$= \sup_{|x|<\infty}\left|H_n^*(x;i)-H(x;i)\right|, \quad i\in\Im. \tag{17}$$

With $\nu_n \leqslant n$, it is obvious that $\widetilde{H}_n(x;i) = H_n^*(x;i)$ for all $(x;i)\in\overline{\mathbb{R}}\times\Im$.

Now taking into account the last two relations, total probability formula and (14), we obtain (15). Theorem 2 is proved. □

Let $\widetilde{a}_n(t)=\left(\widetilde{a}_n^{(0)}(t_0),\widetilde{a}_n^{(1)}(t_1),\ldots,\widetilde{a}_n^{(k)}(t_k)\right)$, where $\widetilde{a}_n^{(0)}(x)=\sqrt{n}\left(\widetilde{H}_n(x)-H(x)\right)$, $\widetilde{a}_n^{(i)}(x)= = \sqrt{n}\left(\widetilde{H}_n(x;i)-H(x;i)\right)$, $(x;i)\in\overline{\mathbb{R}}\times\Im$. We will prove an approximation theorem of the vector-valued modified empirical Kac process $\widetilde{a}_n(t)$ by the appropriate Gaussian vector-valued process $W_n^*(t)$, $t\in\overline{\mathbb{R}}^{k+1}$ from Theorem 2.

Theorem 3. *Let $\{T_n, n\geqslant 1\}$ be a numerical sequence satisfying for each n the condition $T_n < T_H = \inf\{x: H(x)=1\}\leqslant\infty$ such that*

$$\min_{i\in\Im}\left\{P(A^{(i)})-H(T_n,i)\right\}\geqslant 1-H(T_n)\geqslant 2\left(\frac{r\log n}{2wn}\right)^{1/2}. \tag{18}$$

If for any $\varepsilon>0$ condition (12) *holds then on the probability space of Theorem 2 one can define $k+1$ sequences of mean zero Gaussian processes $W_n^{(0)}(x), W_n^{(1)}(x),\ldots,W_n^{(k)}(x)$ with the covariance structure* (9) *such that for $\widetilde{a}_n(t)$ and $W_n^*(t)=\left(W_n^{(0)}(t_0),W_n^{(1)}(t_1),\ldots,W_n^{(k)}(t_k)\right)$ we have*

$$P\left\{\sup_{t\in(-\infty;T_n]^{(k+1)}}\|\widetilde{a}_n(t)-W_n^*(t)\|^{(k+1)} > \widetilde{C}n^{\frac{1}{2}}\log n\right\}\leqslant\widetilde{K}n^{-\beta}, \tag{19}$$

where $\widetilde{K}$ is an absolute constant, $\widetilde{C}=\widetilde{C}(\varepsilon)$ and $\beta=\min(r,\varepsilon w)$ for any $\varepsilon>0$.

Proof. It is easy to see that probability in (19) can be estimated by the sum

$$\begin{gathered}-P\left\{\sup_{x\leqslant T_n}\left|\widetilde{a}_n^{(0)}(x)-W_n^{(0)}(x)\right|>\widetilde{C}n^{\frac{1}{2}}\log n\right\}+\\ +\sum_{i=1}^{k}P\left(\sup_{x\leqslant T_n}\left|\widetilde{a}_n^{(i)}(x)-W_n^{(i)}(x)\right|>\widetilde{C}n^{\frac{1}{2}}\log n\right)=q_{1n}+q_{2n}.\end{gathered}\tag{20}$$

Taking into account that for any $x\leqslant T_n$, $H_n^*(x)\leqslant H_n^*(T_n)$, and if $H_n^*(T_n)\leqslant 1$ then $\widetilde{a}_n^{(0)}(x)=\widetilde{a}_n^{(0)*}(x)$. Using formula of total probability, we have

$$\begin{gathered}q_{1n}\leqslant P\left(\sup_{x\leqslant T_n}\left|\widetilde{a}_n^{(0)}(x)-W_n^{(0)}(x)\right|>C^*n^{-\frac{1}{2}}\log n\Big/H_n^*(T_n)\leqslant 1\right)+P\left(H_n^*(T_n)>1\right)\leqslant\\ \leqslant P\left(\sup_{x\leqslant T_n}\left|a_n^{(0)*}(x)-W_n^{(0)}(x)\right|>C^*n^{-\frac{1}{2}}\log n\right)+P(H_n^*(T_n)>1)\leqslant\\ \leqslant Kn^{-r}+P\left(H_n^*(T_n)-H(T_n)>1-H(T_n)\right)\leqslant\\ \leqslant K^*n^{-r}+P\left(\sup_{|x|<\infty}|H_n^*(x)-H(x)|>\left(\frac{r\log n}{2wn}\right)^{\frac{1}{2}}\right)\leqslant Ln^{-r},\end{gathered}\tag{21}$$

where Theorem 1 and the analogue of (14) for $H_n^* - H,\ L = K^* + 4$ are used. Analogously,

$$\begin{aligned} q_{2n} \leqslant & \sum_{i=1}^{k} P\left(\sup_{x \leqslant T_n} \left|\widetilde{a}_n^{(i)}(x) - W_n^{(i)}(x)\right| > C^* n^{\frac{1}{2}} \log n\right) + \sum_{i=1}^{k} P\left(H_n^*(T_n; i) > P\left(A^{(i)}\right)\right) \leqslant \\ & \leqslant \sum_{i=1}^{k} P\left(\sup_{x \leqslant T_n} \left|a_n^{(i)*}(x) - W^{(i)}(x)\right| > C^* n^{-\frac{1}{2}} \log n\right) + \\ & + \sum_{i=1}^{k} P\left(\sup_{|x| < \infty} \left|a_n^{(i)*}(x) - W^{(i)}(x)\right| > C^* n^{-\frac{1}{2}} \log n\right) + \\ & + kP\left(\frac{|\nu_n - n|}{n} > \frac{1}{2}\left(\frac{4r \log n}{2wn}\right)^{\frac{1}{2}}\right) \leqslant kLn^{-r} + 2kn^{-4r}, \end{aligned} \tag{22}$$

where inequalities (13), (15) and Theorem 1 are used. Now (19) follows from (21) and (22). Theorem 3 is proved. □

3. Estimation of exponential-hazard function

In many practical situations when we are interested in the joint behaviour of the pairs $\left\{\left(Z, A^{(i)}\right), i \in \Im\right\}$ the so-called cumulative hazard functions $\left\{S^{(i)}(x) = \exp\left(-\Lambda^{(i)}(x)\right),\ i \in \Im\right\}$ plays a crucial role. Here $\Lambda^{(i)}(x)$ is the i-th hazard function $\left(\int\limits_{-\infty}^{x} = \int\limits_{(-\infty;x]}\right)$

$$\Lambda^{(i)}(x) = \int_{-\infty}^{x} \frac{dH(u;i)}{1 - H(u)}, \quad i \in \Im,$$

where $\Lambda^{(1)}(x) + \cdots + \Lambda^{(k)}(x) = \Lambda(x) = \int\limits_{-\infty}^{x} \frac{dH(u)}{1 - H(u)}$ is the corresponding hazard function of d.f. $H(x)$.

Let us consider two important special cases of the considered generalized censorship model:

1. Let $\{X_1, X_2, \dots\}$ be a sequence of independent r.v.-s with common continuous d.f. F. They are censored on the right by a sequence $\{Y_1, Y_2, \dots\}$ of independent r.v.-s. They are independent of the X-sequence with common continuous d.f. G. One can only observe the sequence of pairs $\left\{(Z_k, \delta_k), k = \overline{1,n}\right\}$, where $Z_j = \min(X_j, Y_j)$ and $\delta_j = \delta_j^{(1)}$ is the indicator of event $A_j = A_j^{(1)} = \{Z_j = X_j\}$. In this case $k = 2,\ 1 - H(x) = (1 - F(x))(1 - G(x))$, $H(x; 1) = \int\limits_{-\infty}^{x} (1 - G(u))dF(u)$. Thus $S^{(1)}(x) = S(x) = 1 - F(x)$. The useful special case is $1 - G(x) = (1 - F(x))^{\beta},\ \beta > 0$ which corresponds to independence of r.v.-s Z_j and $\delta_j, j \geqslant 1$.

2. Let us assume that $k > 1$ and consider independent sequences $\left\{Y_1^{(i)}, Y_2^{(i)}, \dots\right\}$ $(i = 1, \dots, k)$ of independent r.v.-s with common continuous d.f. F. Let $Z_j = \min\left(Y_j^{(1)}, \dots, Y_j^{(k)}\right)$. Let us observe the sequences $\left\{\left(Z_j, \delta_j^{(i)}\right),\ i = \overline{1,k}\right\}_{j=1}^{n}$, where $\delta_j^{(i)}$ is the indicator of the event $A_j^{(i)} =$ $= \left\{Z_j = Y_j^{(i)}\right\}$. This is the competing risks model with $S^{(i)}(x) = 1 - F^{(i)}(x),\ i \in \Im$.

Let us define the natural Kac-type estimator

$$\widetilde{\Lambda}_n^{(i)}(x)=\int_{-\infty}^{x}\frac{d\widetilde{H}(u;i)}{1-\widetilde{H}_n(u)},\quad i\in\Im$$

of $\Lambda^{(i)}(x),\ i\in\Im$. Let $w_n^{(i)}(x)=\sqrt{n}\left(\widetilde{\Lambda}_n^{(i)}(x)-\Lambda^{(i)}(x)\right),\ i\in\Im$, is an Kac-type hazard process, $w_n(t)=\left(w_n^{(1)}(t_1),\ldots,w_n^{(k)}(t_k)\right),\ t=(t_1,\ldots,t_k)$, and $Y_n(t)=\left(Y_n^{(1)}(t_1),\ldots,Y_n^{(k)}(t_k)\right)$ is the corresponding vector process with

$$Y_n^{(i)}(x)=\int_{-\infty}^{x}\frac{W_n^{(0)}(u)dH(u;i)}{(1-H(u))^2}+\frac{W_n^{(i)}(x)}{1-H(x)}-\int_{-\infty}^{x}\frac{W_n^{(i)}(u)dH(u)}{(1-H(u))^2},\quad i\in\Im$$

and $\left\{W_n^{(0)}(x),W_n^{(1)}(x),\ldots,W_n^{(k)}(x)\right\}$ are Wiener processes with the covariance structure (9). Then for $i\in\Im,\quad EY_n^{(i)}=0$ and

$$EY_n^{(i)}(x)Y_n^{(i)}(y)=C(x,y),$$

where $x,y\leqslant T_H=\inf\{x:H(x)=1\}\leqslant\infty$.

Theorem 4. *Let $\{T_n,n\geqslant 1\}$ be a numerical sequence satisfying for each n the condition $T_n<T_H$ such that*

$$\frac{n}{\log n}\geqslant\max\left\{32\varepsilon w^2,\frac{2rb_n^2}{w},\frac{2\varepsilon b_n^2}{w}\right\},\tag{23}$$

where $b_n=(1-H(T_n))^{-1},\ \varepsilon>0,\ r\geqslant 2$. Then

$$P\left(\sup_{t\in(-\infty;T_n]^{(k)}}\|w_n(t)-Y_n(t)\|^{(k)}>r(n)\right)\leqslant k\Phi_1 n^{-\beta},\tag{24}$$

on a probability space of Theorem 2, where $r(n)=\Phi_0 b_n^2 n^{-\frac{1}{2}}\log n$,
$Phi_0=\Phi_0(\varepsilon,r),\ \Phi_1-$ are absolute constants.

Proof. It is sufficient to prove that for each $i\in\Im$

$$P\left(\sup_{x\leqslant T_n}\left(w_n^{(i)}(x)-Y_n^{(i)}(x)\right)>r(n)\right)\leqslant\Phi_1 n^{-\beta}.\tag{25}$$

We have representation for each $i\in\Im$ for difference

$$w_n^{(i)}(x)-Y_n^{(i)}(x)=\int_{-\infty}^{x}\frac{\left(\widetilde{a}_n^{(0)}(u)-W_n^{(0)}(u)\right)dH(u;i)}{(1-H(u))^2}+\frac{\widetilde{a}_n^{(i)}(x)-W_n^{(i)}(x)}{1-H(x)}-$$
$$-\int_{-\infty}^{x}\frac{\left(a_n^{(i)}(u)-W_n^{(i)}(u)\right)dH(u)}{(1-H(u))^2}+n^{-\frac{1}{2}}\int_{-\infty}^{x}\frac{\left(\widetilde{a}_n^{(0)}(u)\right)^2dH(u;i)}{(1-H(u))^2\left(1-\widetilde{H}_n(u)\right)}+$$
$$+n^{-\frac{1}{2}}\int_{-\infty}^{x}\frac{\widetilde{a}_n^{(0)}(u)d\widetilde{a}_n^{(i)}(u)}{(1-H(u))\left(1-\widetilde{H}_n(u)\right)}=\sum_{m=1}^{4}R_{mn}^{(i)}(x).$$

Using (15) and (19), we have for sum $R_{1n}^{(i)}(x)+R_{2n}^{(i)}(x)+R_{3n}^{(i)}(x)$

$$P\left(\sup_{x\leqslant T_n}\left|\sum_{m=1}^{4}R_{mn}^{(i)}(x)\right|>3\widetilde{C}n^{-\frac{1}{2}}\log n+\varepsilon n^{-\frac{1}{2}}b_n^3\log n\right)\leqslant$$
$$\leqslant 3\widetilde{K}n^{-\beta}+2Ln^{-w\varepsilon}\leqslant(3\widetilde{K}+2L)n^{-\beta},\quad i\in\Im.\tag{26}$$

Rewrite $R_{4n}^{(i)}$ in the form

$$\begin{aligned} R_{4n}^{(i)}(x) = n^{-\frac{1}{2}} \int_{-\infty}^{x} \frac{\left(\widetilde{a}_n^{(0)}(u)\right)^2 d(H(u;i) - H(u;i))}{(1-H(u))^2\left(1-\widetilde{H}_n(u)\right)} + \\ + n^{-\frac{1}{2}} \int_{-\infty}^{x} \frac{\widetilde{a}_n^{(0)}(u) d\widetilde{a}_n^{(i)}(u)}{(1-H(u))^2} = \overline{R}_{4n}^{(i)}(x) + \overline{\overline{R}}_{4n}^{(i)}(x). \end{aligned} \tag{27}$$

Then taking into account (15), we obtain for $i \in \Im$

$$P\left(\sup_{x\leqslant T_n} \left|\overline{R}_{4n}^{(i)}(x)\right| > 2\varepsilon n^{-\frac{1}{2}} b_n^3 \log n\right) \leqslant 2Ln^{-w\varepsilon} \leqslant 2Ln^{-\beta}. \tag{28}$$

There exists an absolute constant A such that

$$\begin{aligned} P\left(\sup_{x\leqslant T_n} \left|\overline{R}_{4n}^{(i)}(x)\right| > 3An^{-\frac{1}{2}} b_n^2 \log n\right) \leqslant P\left(H_n^*(T_n) > 1\right) + \\ + P\left(\sup_{x\leqslant T_n} n^{-\frac{1}{2}} \left|\int_{-\infty}^{x} \frac{a_n^{(0)*}(u) da_n^{(i)*}(u)}{(1-H(u))^2}\right| > 3An^{-\frac{1}{2}} b_n^2 \log n\right) \leqslant Ln^{-r} + p_n, \end{aligned} \tag{29}$$

so that for any $x \leqslant T_n, H_n^*(x) \leqslant H_n^*(T_n)$ and if $H_n^*(T_n) \leqslant 1$ then $H_n^*(x;i) \leqslant H_n^*(T_n)$ and hence $\widetilde{a}_n^{(i)}(x) = a_n^{(i)*}(x)$ for $i \in \Im$. It is sufficient to estimate probability p_n. According to proof of Theorem 1 in [6], supposing $a_{\nu_n}^{(0)}(x) = \sqrt{\nu_n}(H_{\nu_n}^*(x) - H(x))$, $a_{\nu_n}^{(i)}(x) = \sqrt{\nu_n}(H_{\nu_n}^*(x;i) - H(x;i))$, $i \in \Im$ and using representation (10), we have $p_n = p_{1n} + p_{2n} + p_{3n} + p_{4n}$, where

$$\begin{aligned} p_{1n} &= P\left(\frac{\nu_n}{n} \sup_{x\leqslant T_n} \left|\int_{-\infty}^{x} \frac{a_{\nu_n}^{(0)}(u) da_{\nu_n}^{(i)}(u)}{(1-H(u))^2}\right| > 3An^{-\frac{1}{2}} b_n^2 \log n\right), \\ p_{2n} &= P\left(\sqrt{\frac{\nu_n}{n}} \cdot \frac{|\nu_n - n|}{n} \sup_{x\leqslant T_n} \left|\int_{-\infty}^{x} \frac{a_{\nu_n}^{(0)}(u) dH(u;i)}{(1-H(u))^2}\right| > \frac{\varepsilon}{2}\left(\frac{3}{2}\right)^{-\frac{1}{2}} n^{-\frac{1}{2}} b_n^2 \log n\right), \\ p_{3n} &= P\left(\sqrt{\frac{\nu_n}{n}} \cdot \frac{|\nu_n - n|}{n} \sup_{x\leqslant T_n} \left|\int_{-\infty}^{x} \frac{H(u) da_{\nu_n}(u)}{(1-H(u))^2}\right| > \frac{\varepsilon}{2}\left(\frac{3}{2}\right)^{-\frac{1}{2}} n^{-\frac{1}{2}} b_n^2 \log n\right), \\ p_{4n} &= P\left(\cdot \frac{|\nu_n - n|^2}{\sqrt{n}} \sup_{x\leqslant T_n} \left\{\int_{-\infty}^{x} \frac{H(u) dH(u;i)}{(1-H(u))^2}\right\} > \frac{\varepsilon}{8} n^{-\frac{1}{2}} b_n^2 \log n\right). \end{aligned}$$

Taking into account Lemma in [5], we have

$$P\left(\sup_{x\leqslant T_n} \left|\int_{-\infty}^{x} \frac{a_n^{(0)}(u) da_n^{(i)}(u)}{(1-H(u))^2}\right| > Ab_n^2 \log n\right) \leqslant Bn^{-\varepsilon}, \tag{30}$$

where $A = A(\varepsilon)$ and B is an absolute constant. Moreover, using (13), we have

$$P\left(\frac{|\nu_n - n|}{n} > \frac{1}{2}\right) \leqslant 2n^{-\frac{2nw}{\log n}}. \tag{31}$$

It follows from (30) and (31) that

$$\begin{aligned} p_{1n} = P\left(\sup_{x\leqslant T_n} \left|\int_{-\infty}^{x} \frac{a_{\nu_n}^{(0)}(u) da_{\nu_n}^{(i)}(u)}{(1-H(u))^2}\right| > 2Ab_n^2 \log \nu_n \frac{\log n}{\log \nu_n},\ \frac{n}{2} \leqslant \nu_n \leqslant \frac{3n}{2}\right) + P\left(\frac{|\nu_n - n|}{n} > \frac{1}{2}\right) \leqslant \\ \leqslant 2n^{-\frac{2nw}{\log n}} + P\left(\sup_{x\leqslant T_n} \left|\int_{-\infty}^{x} \frac{a_{\nu_n}^{(0)}(u) da_{\nu_n}^{(i)}(u)}{(1-H(u))^2}\right| > Ab_n^2 \log \nu_n\right) + 2n^{-\frac{2nw}{\log n}} \leqslant \end{aligned}$$

$$\leqslant e^{-n} + \sum_{m=1}^{\infty} P\left(\sup_{x\leqslant T_n}\left|\int_{-\infty}^{x}\frac{a_m^{(0)}(u)da_m^{(i)}(u)}{(1-H(u))^2}\right| > Ab_n^2\log m\right)P(\nu_n=m) + 2n^{-\frac{2nw}{\log n}} \leqslant$$

$$\leqslant e^{-n} + 2n^{-\frac{2nw}{\log n}} + B\sum_{m=1}^{\infty} m^{-\varepsilon}\cdot\frac{n^m}{m!}e^{-n} \leqslant e^{-n} + 2n^{-\frac{2nw}{\log n}} + \widetilde{B}n^{-\varepsilon}. \qquad (32)$$

Analogously, using (31) and (1), we obtain

$$p_{2n} = P\left(\sqrt{\frac{\nu_n}{n}}\frac{|\nu_n-n|}{n}\sup_{x\leqslant T_n}\left|\int_{-\infty}^{x}\frac{a_{\nu_n}^{(0)}(u)dH(u;i)}{(1-H(u))^2}\right| > \frac{\varepsilon}{2}\left(\frac{3}{2}\right)^{\frac12} n^{-\frac12}b_n^2\log n,\ \frac{n}{2}\leqslant\nu_n\leqslant\frac{3n}{2}\right)+$$

$$+P\left(\frac{|\nu_n-n|}{n} > \frac12\right) \leqslant 2n^{-\frac{2nw}{\log n}} + 2n^{-w\varepsilon} + P\left(\sup_{|x|<\infty}\left|a_{\nu_n}^{(0)}(x)\right| > \left(\frac{\varepsilon}{2}\log\nu_n\right)^{\frac12}\right) \leqslant$$

$$\leqslant 2n^{-\frac{2nw}{\log n}} + 2n^{-w\varepsilon} + e^{-n} + \widetilde{D}n^{-\varepsilon}. \qquad (33)$$

Integrating by parts and using (2), we obtain

$$\begin{gathered} p_{3n} \leqslant 2n^{-\frac{2nw}{\log n}} + 2n^{-w\varepsilon} + P\left(\sup_{|x|<\infty}\left|a_{\nu_n}^{(i)}(x)\right| > (2\varepsilon\log\nu_n)^{\frac12}\right) \leqslant \\ \leqslant 2n^{-\frac{2nw}{\log n}} + 2n^{-w\varepsilon} + e^{-n} + 2Dn^{-\varepsilon}. \end{gathered} \qquad (34)$$

Finally, using (13), we have

$$p_{4n} \leqslant P\left(\frac{|\nu_n-n|}{n^{\frac12}} > \frac12\left(\frac{\varepsilon}{2}\log n\right)^{\frac12}\right) \leqslant 2n^{-w\varepsilon}. \qquad (35)$$

Now combining (26)-(29) and (32)-(35), we obtain (25). Theorem 4 is proved.

Corollary 1. *It follows from* (24) *that for suitable* $r\geqslant 2$ *and* $\varepsilon>0$ *one can obtain an approximation on* $(-\infty;T]^{(k)}$ *with* $b^{-1}=1-H(T)>0$:

$$\sup_{t\in(-\infty;T]^{(k)}}\|w_n(t)-Y_n(t)\|^{(k)} \overset{a.s.}{=} O\left(n^{-\frac12}\log n\right),\quad n\geqslant 2. \qquad (36)$$

Now we consider joint estimation of exponential-hazard functions $\left\{S^{(i)}x) = \exp\left(-\Lambda^{(i)}(x)\right),\ i\in\Im\right\}$. Let us consider hazard function estimate

$$\Lambda_n(x) = \int_{-\infty}^{x}\frac{d\widetilde{H}_n(u)}{1-\widetilde{H}_n(u)}$$

and corresponding hazard process $w^{(0)}(x) = \sqrt{n}\left(\Lambda_n(x)-\Lambda(x)\right)$. In the next Theorem 5 we approximate $w_n^{(0)}(x)$ by sequence of Gaussian processes $Y_n^{(0)}(x) = \dfrac{W_n^{(0)}(x)}{1-H(x)}$.

Theorem 5. *Let* $\{T_n, n\geqslant 1\}$ *be a numerical sequence that satisfies the condition* $T_n<T_H$ *for each* n *such that* (23) *holds. Then on a probability space of Theorem 2 we have*

$$P\left(\sup_{x\leqslant T_n}\left|w_n^{(0)}(x)-Y_n^{(0)}(x)\right| > r_0(n)\right) \leqslant \Psi_1 n^{-\beta}, \qquad (37)$$

where $r_0(n)=\Phi_0 b_n^2 n^{-\frac12}\log n$ *and* $\Phi_0=\Phi_0(\varepsilon,r)$, Ψ_1 *are absolute constants.*

Proof. It is easy to verify that

$$w_n^{(0)}(x) - Y_n^{(0)}(x) = \frac{\left(\widetilde{a}_n^{(0)}(x) - W_n^{(0)}(x)\right)}{1 - H(x)} + n^{-\frac{1}{2}} \int_{-\infty}^{x} \frac{\left(\widetilde{a}_n^{(0)}(u)\right)^2 dH(u)}{(1-H(u))^2 \left(1 - \widetilde{H}_n(u)\right)} + $$
$$+ n^{-\frac{1}{2}} \int_{-\infty}^{x} \frac{\widetilde{a}_n^{(0)}(u) da_n^{(0)}(u)}{(1-H(u))\left(1 - \widetilde{H}_n(u)\right)}.$$

Now further proof of (37) is similar to the proof of Theorem 4 and hence details are omitted. Theorem 5 is proved. □

One can obtain from Theorems 4 and 5 the following theorem on deviations of processes $w_n^{(0)}$ and $w_n^{(i)}$, $i \in \Im$.

Theorem 6. *Let $\{T_n, n \geqslant 1\}$ be a numerical sequence that satisfies for each n the condition $T_n < T_H$ such that (23) holds. Then*

$$P\left(\sup_{x \leqslant T_n} \left|w_n^{(0)}(x)\right| > r_0(n) + 2b_n(\varepsilon \log n)^{\frac{1}{2}}\right) \leqslant \Psi_1 n^{-\beta} + n^{-\varepsilon}, \tag{38}$$

and for $i \in \Im$

$$P\left(\sup_{x \leqslant T_n} \left|w_n^{(i)}(x)\right| > r_0(n) + 6b_n^2(\varepsilon \log n)^{\frac{1}{2}}\right) \leqslant \Psi_1 n^{-\beta} + 3n^{-\varepsilon}. \tag{39}$$

Proof. It is easy to verify that for any $n \geqslant 1$

$$W_n^{(0)}(x) \stackrel{D}{=} W\left(H(x)\right) \quad \text{and} \quad W_n^{(i)}(x) \stackrel{D}{=} W\left(H(x;i)\right), \quad (x;i) \in \overline{\mathbb{R}} \times \Im,$$

where $\{W(y), 0 \leqslant y \leqslant 1\}$ is a standard Wiener process on $[0,1]$. Then probability in (38) is not greater than

$$\begin{gathered} P\left(\sup_{x \leqslant T_n} \left|w_n^{(0)}(x) - Y_n^{(0)}(x)\right| > r_0(n)\right) + P\left(\sup_{x \leqslant T_n} \left|Y_n^{(0)}(x)\right| > 2b_n\left(\varepsilon \log n\right)^{\frac{1}{2}}\right) \leqslant \\ \leqslant \Psi_1 n^{-\beta} + P\left(|W(1)| > 2\left(\varepsilon \log n\right)^{\frac{1}{2}}\right) \leqslant \Psi_1 n^{-\beta} + n^{-\varepsilon}, \end{gathered} \tag{40}$$

where inequality (37) and well-known exponential inequality for Wiener process (see [14], Eq. (29.2)) are used. Analogously, (39) follows from (25) and the second estimate in (40). Theorem 6 is proved. □

To estimate the exponential hazard functions $\left\{S^{(i)}(x) = \exp\left(-\Lambda^{(i)}(x)\right), i \in \Im\right\}$ we use the following exponential of Altshuler-Breslow, product-limit of Kaplan-Meier and relative risk power estimates of Abdushukurov ([1–3]):

$$\begin{aligned} &S_{1n}^{(i)}(x) = \exp\left(-\Lambda_n^{(i)}(x)\right), \\ &S_{2n}^{(i)}(x) = \textstyle\prod_{u \leqslant x}\left(1 - \Delta\Lambda_n^{(i)}(x)\right), \\ &S_{3n}^{(i)}(x) = \left[1 - H_n(x)\right]^{R_n^{(i)}(x)}, \end{aligned} \tag{41}$$

where $R_n^{(i)}(x) = \Lambda_n^{(i)}(x)(\Lambda_n(x))^{-1}$, $i \in \Im$.

It follows from the proof of Theorem 1.4.1 in [3] that for all $(x;i) \in (-\infty, Z_{(n)}) \times \Im$, $Z_{(n)} = \max(Z_1, \ldots, Z_n)$

$$
\begin{aligned}
&0 \leqslant S_{1n}^{(i)}(x) - S_{2n}^{(i)}(x) \leqslant \frac{1}{2n}\int_{-\infty}^{x} \frac{d\widetilde{H}_n(u;i)}{\left(1-\widetilde{H}_n(u)\right)^2} \overset{a.s.}{=} O\left(\frac{1}{n}\right),\\
&0 \leqslant S_{1n}^{(i)}(x) - S_{3n}^{(i)}(x) \leqslant \frac{1}{2n}\int_{-\infty}^{x} \frac{d\widetilde{H}_n(u;i)}{\left(1-\widetilde{H}_n(u)\right)^2} \overset{a.s.}{=} O\left(\frac{1}{n}\right).
\end{aligned} \tag{42}
$$

Hence it is sufficient to consider only estimator $S_{1n}^{(i)}$. Let us introduce vector-processes $\mathbb{Q}_n(t) = \left(\mathbb{Q}_n^{(1)}(t_1), \ldots, \mathbb{Q}_n^{(k)}(t_k)\right)$ and $\mathbb{Q}_n^*(t) = \left(\mathbb{Q}_n^{(1)*}(t_1), \ldots, \mathbb{Q}_n^{(k)*}(t_k)\right)$, where $\mathbb{Q}_n^{(i)}(x) = \sqrt{n}\left(S_{1n}^{(i)}(x) - S^{(i)}(x)\right)$ and $\mathbb{Q}_n^{(i)*}(x) = S^{(i)}(x)Y_n^{(i)}(x),\ i \in \Im$.

In the next theorem vector-valued process $Q_n(t)$ is approximated by Gaussian vector-valued process $\mathbb{Q}_n^*(t),\ t \in \mathbb{R}^k$.

Theorem 7. *Let $\{T_n,\, n \geqslant 1\}$ be a numerical sequence that satisfies for each n the condition $T_n < T_H$ such that inequality* (23) *holds. Then we have on a probability space of Theorem 2*

$$
P\left(\sup_{t\in(-\infty;T_n]^{(k)}} \|\mathbb{Q}_n(t) - \mathbb{Q}_n^*(t)\|^{(k)} > r^*(n)\right) \leqslant kR^* n^{-\beta}, \tag{43}
$$

where $r^*(n) = \left\{r_0(n) + \frac{1}{2}n^{-\frac{1}{2}}\left(r(n) + 6b_n^2\left(\varepsilon \log n\right)^{\frac{1}{2}}\right)^2\right\}$ *and R^* is an absolute constant.*

Proof. Using Taylor expansion for each $i \in \Im$, we obtain

$$
\mathbb{Q}_n^{(i)}(x) = S^{(i)}(x)w_n^{(i)}(x) + \frac{1}{2}n^{-\frac{1}{2}}\exp\left(-\theta_n^{(i)}(x)\right)\left(w_n^{(i)}(x)\right)^2,
$$

where $\theta_n^{(i)}(x) \in \left[\min\left(\Lambda_n^{(i)}(x), \Lambda^{(i)}(x)\right), \max\left(\Lambda_n^{(i)}(x), \Lambda^{(i)}(x)\right)\right]$. Now using (24), (38) and (39), we obtain the required result. Theorem 7 is proved. □

4. Estimation of characteristic function under random right censoring

Let $X_1, X_2, \ldots$ be independent identically distributed r.v.-s with common continuous d.f. F. They are interpreted as an infinite sample of the random lifetime X. Another sequence of independent and identically distributed r.v.-s $Y_1, Y_2, \ldots$ with common continuous d.f. G censors on the right is introduced. This sequence is independent of $\{X_j\}$. Then the observations available at the n-th stage consist of the pairs $\{(Z_j, \delta_j), 1 \leqslant j \leqslant n\} = \mathbb{C}^{(n)}$, where $Z_j = \min(X_j, Y_j)$ and δ_j is the indicator of the event $A_j = \{Z_j = X_j\} = \{X_j \leqslant Y_j\}$. Let

$$
C(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)
$$

be the characteristic function of d.f. F. The problem consists in estimating of d.f. F from censored sample $\mathbb{C}^{(n)}$. In some situations it is more desirable to estimate $C(t)$ rather then F. We consider estimator for $C(t)$ in this model as Fourier–Stieltjes transform of estimator $F_n(x) = 1 - S_{1n}(x) = 1 - \exp\left(-\Lambda_n^{(1)}(x)\right)$:

$$
C_n(t) = \int_{-\infty}^{\infty} e^{itx} dF_n(x), \quad t \in \mathbb{R}.
$$

It follows from (39) that when $n \to \infty$

$$\sup_{x \leqslant T_n} |F_n(x) - F(x)| \overset{a.s.}{=} O\left(b_n^2 \left(\frac{\log n}{n}\right)^{\frac{1}{2}}\right), \tag{44}$$

where $b_n^{-1} = 1 - H(T_n)$. It also follows from (44) that when $n \to \infty$

$$1 - F_n(T_n) \overset{a.s.}{=} O\left(1 - F(T_n)\right), \quad F_n(-T_n) \overset{a.s.}{=} O\left(F(-T_n)\right). \tag{45}$$

It is obvious that $\Delta_n(\tau) \overset{a.s.}{\to} 0$ when $n \to \infty$ for any $\tau < \infty$, where $\Delta_n(\tau) = \sup\limits_{|t| \leqslant \tau} |C_n(t) - C(t)|$. Let us consider quantity $\Delta_n(\tau_n)$ for some special numerical sequence τ_n that tends to $+\infty$ when $n \to \infty$.

In the following theorem we prove result of uniform convergence for the empirical characteristic function.

Theorem 8. *Let $\{\tau_n, n \geqslant 1\}$ be a numerical sequence that tends to $+\infty$ slowly when $n \to \infty$. Then, $\Delta_n(\tau_n) \overset{a.s.}{\to} 0$ when $n \to \infty$.*

Proof. Let us choose a sequence $\{\tau_n, n \geqslant 1\}$ such that when $n \to \infty$

$$\gamma_n = \max\left\{1 - F(T_n), F(-T_n), b_n^2 \tau_n T_n \left(\frac{\log n}{n}\right)^{\frac{1}{2}}\right\} \to 0, \tag{46}$$

where $\{T_n, \quad n \geqslant 1\}$ is a sequence that satisfies condition (23). Introducing the truncated integrals

$$b_n(t) = \int_{|x| \leqslant T_n} e^{itx} dF_n(x), \quad \widetilde{b}_n(t) = \int_| x| \leqslant T_n e^{itx} dF(x)$$

and introducing $d_n(t) = b_n(t) - \widetilde{b}_n(t)$, we have that

$$\Delta_n(\tau_n) \leqslant \sup_{|t| \leqslant \tau_n} |d_n(t)| + \sup_{|t| \leqslant \tau_n} |b_n(t) - C_n(t)| + \sup_{|t| \leqslant \tau_n} \left|\widetilde{b}_n(t) - C(t)\right|. \tag{47}$$

Integrating by parts, we obtain

$$\begin{gathered}\sup_{|t| \leqslant \tau_n} |d_n(t)| \leqslant \sup_{|t| \leqslant \tau_n} \left| \int_{|t| \leqslant T_n} e^{itx} d\left(F_n(x) - F(x)\right)\right| \leqslant \\ \leqslant \sup_{|t| \leqslant \tau_n} \left[\left|e^{itx}\right| |F_n(x) - F(x)|_{-T_n}^{T_n}\right] + \sup_{|t| \leqslant \tau_n} \left| it \int_{|x| \leqslant T_n} e^{itx} d\left(F_n(x) - F(x)\right)\right| dx \leqslant \\ \leqslant 2(1 + 2\tau_n T_n) \sup_{|x| \leqslant T_n} |F_n(x) - F(x)|. \end{gathered} \tag{48}$$

On the other hand,

$$\sup_{|t| \leqslant \tau_n} |b_n(t) - C_n(t)| \leqslant \sup_{|t| \leqslant \tau_n} \int_{|x| > T_n} \left|e^{itx}\right| dF_n(x) \leqslant 1 - F_n(T_n) + F_n(-T_n) \tag{49}$$

and

$$\sup_{|t| \leqslant \tau_n} \left|\widetilde{b}_n(t) - C(t)\right| \leqslant \sup_{|t| \leqslant \tau_n} \int_{|x| > T_n} \left|e^{itx}\right| dF(x) \leqslant 1 - F(T_n) + F(-T_n). \tag{50}$$

Now adding (44)–(50), we have that $\Delta_n(\tau_n) \overset{a.s.}{=} O(\gamma_n), \quad n \to \infty$. Theorem 8 is proved. □

References

[1] A.A.Abdushukurov, Nonparametric estimation of the distribution function based on relative risk function, *Commun. Statis. Th. Meth.*, **27**(1998), no. 8, 1991–2012.
DOI: 10.1080/03610929808832205

[2] A.A.Abdushukurov, On nonparametric estimation of reliability indices by censored samples, *Theory Probab. Appl.*, **43**(1999), no. 1, 3–11.

[3] A.A.Abdushukurov, Statistics of incomplete observations. University Press, Tashkent, 2009 (in Russian).

[4] M.D.Burke, Csörgő, L.Horváth, Strong approximations of some biometric estimates under random censorship, *Z.Wahrschein. Verw. Gebiete,* **56**(1981), 87–112.

[5] M.D.Burke, S.Csörgő, L.Horváth, A correction to and improvement of "Strong approximations of some biometric estimates under random censorship", *Probab. Th. Ret. Fields.*, **79**(1988), 51–57.

[6] S.Csörgő, Strong approximation of empirical Kac processes, *Carleton Math. Lect. Note.*, **26**(1980), 71–86.

[7] S.Csörgő, L.Horvath, On random censorship from the right, *Acta. Sci. Math.*, **44**(1982), 23–34.

[8] A.Dvoretzky, J.Kiefer, J.Wolfowitz, Asymptotic minimax character of the sample distribution function and of the multinomial estimator, *Ann. Math. Statist.*, **27**(1956), 642–669.

[9] M.Kac, On denations between theoretical and empirical distributions, *Proc. Nat. Acad. Sci. USA*, **35**(1949), 252–257.

[10] D.M.Mason, Classical empirical process theory and wighted approximation, Comunicaciones del CIMAT, N.I-15-03/09-11-2015/ (PE/CIMAT).

[11] D.M.Mason, Selected Definition and Results from Modern Empirical Theory, Comunicaciones del CIMAT, N.I-17-01, 16.03.2017 (PE),.

[12] P.Massart, The tight constant in the Dvoretzky-Kiefer-Wolfowits inequality, *Ann. Probab.*, **18**(1990), no. 3, 1269–1283.

[13] W.W.Petrov, Limit theorems for sum of random variables, Moscow, Nauka, 1987 (in Russian).

[14] A.W.Skorokhod, Random processes of independent increments, Moscow, Nauka, 1964.

Об аппроксимации эмпирических процессов Каца в общей модели случайного цензурирования

Абдурахим А. Абдушукуров
Филиал Московского государственного университета в Ташкенте
Ташкент, Узбекистан
Гульназ С. Сайфуллоева
Навоийский государственный педагогический институт
Навои, Узбекистан

Аннотация. В статье рассматривается общая модель случайного цензурирования и доказываются результаты аппроксимации для эмпирических процессов Каца. Эта модель включает в себя такие важные специальные случаи, как случайное цензурирование справа и модель конкурирующих рисков. Наши результаты включают в себя теорию сильной аппроксимации, и нами построены оптимальные скорости аппроксимации. Также исследованы кумулятивные процессы риска. Эти результаты использованы для оценивания характеристической функции в модели случайного цензурирования справа.

Ключевые слова: цензурированные данные, конкурирующие риски, эмпирические оценки, оценка Каца, сильная аппроксимация, гауссовские процессы, характеристическая функция.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(3), 308–318

DOI: 10.17516/1997-1397-2022-15-3-308-318
УДК 515:17+517:545

Variational Formulas of the Monodromy Group for a Third-Order Equation on a Compact Riemann Surface

Alexander V. Chueshev*
Victor V. Chueshev
Kemerovo State University
Kemerovo, Russian Federation

Received 10.09.2021, received in revised form 10.11.2021, accepted 20.12.2021

Abstract. In the present article, we deduce explicit variational formulas for a solution vector and the elements of its monodromy group for a third-order ordinary differential equation on a compact Riemann surface of genus $g \geqslant 2$ in the spaces of quadratic and cubic holomorphic differentials.

Keywords: Riemann surface, third-order equation on a Riemann surface, variational formula, holomorphic differential.

Citation: A.V. Chueshev, V.V. Chueshev, Variational Formulas of the Monodromy Group for a Third-Order Equation on a Compact Riemann Surface, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 308–318. DOI: 10.17516/1997-1397-2022-15-3-308-318.

Introduction

In the present article, we deduce explicit variational formulas for a solution vector and for the elements of its monodromy group to a third-order ordinary differential equation on a compact Riemann surface of genus $g \geqslant 2$ with respect to a variation in the spaces of quadratic and cubic holomorphic differentials. These theorems are a continuation of results by D. Hejhal, V. V. Chueshev, and M. I. Tulina.

In [1–3], D. Hejhal began to study the dependence of a solution vector and the generators of the monodromy group of the equation on small variations in the space of holomorphic differentials.

Variational formulas found applications in the theory of Teichmüller spaces in connection with the uniformization of compact Riemann surfaces (see [3–4]).

The coefficients of a third-order differential equation on a compact Riemann surface must be the quadratic and cubic differentials at the corresponding derivatives (see [5]).

In the previous papers [4, 6, 7], a compact method was proposed for deducing the variational formulas for the vector solution and the elements of its monodromy group with the use of matrix-vector notation.

In the present article, we obtain formulas for the first variation with respect to a basis in spaces of holomorphic cubic differentials for a solution vector and the monodromy group on a compact Riemann surface for a third-order linear ordinary differential equation with any holomorphic coefficients. Moreover, we find explicit variational formulas for a variation in spaces

*chueshev@ngs.ru https://orcid.org/0000-0002-7185-7736

of holomorphic quadratic differentials for a solution vector as well as the formula for the first variation of the solution vector for a variation with respect to a basis of quadratic holomorphic differentials on a compact Riemannian surface of genus $g \geqslant 2$.

1. Preliminaries

Let F be a compact Riemann surface of genus $g \geqslant 2$, D be an open disk on the plane $\mathbb{C}$. Denote by Γ a Fuchsian group of the first kind uniformizing F in the disk D, i.e., F is conformally equivalent to D/Γ.

Consider an linear ordinary differential equation

$$\frac{d^n v}{dt^n} + q_2(t)\frac{d^{n-2} v}{dt^{n-2}} + q_3(t)\frac{d^{n-3} v}{dt^{n-3}} + \cdots + q_n(t)v = 0,\ t \in D, \tag{1}$$

where $q_j(t)$ is a meromorphic function on $D, j = 2, \ldots, n$. Equation (1) has Fuchsian type on F if it has only regular Fuchsian points and is preserved after the change of variables

$$\omega = v(s)L'(t)^{\frac{n-1}{2}},\ (t, v) \to (s, \omega),\ s = L(t),\ L \in \Gamma. \tag{2}$$

A solution vector is a column-vector consisting of a basis in the space of holomorphic solutions to an equation with holomorphic coefficients. Holomorphic differentials of order q have the form $\Phi(z)dz^q$ and are invariant under a change of coordinates on the surface, i.e.,

$$\Phi(Lz)L'(z)^q = \Phi(z),\ z \in D,\ L \in \Gamma.$$

Denote by $\Omega^q(F)$ the vector space of holomorphic q-differentials on D/Γ, where $q \in \mathbf{N}$ (see [5]).

Lemma 1 ([2,3])**.** *Suppose that a column vector $U(t)$ consists of n linearly independent solutions to equation (1) on $F = D/\Gamma$. Then the equality*

$$U(Lt) = \chi(L)U(t)\xi_L(t)^{n-1},\ L \in \Gamma,\ \xi_L(t) = \sqrt{L'(t)}, \tag{3}$$

uniquely determines the monodromy homomorphism $\chi : \Gamma \to GL(n, \mathbb{C})$ defined by the mapping $L \to \chi(L),\ L \in \Gamma$.

The monodromy group of equation (1) is the image $\chi(\Gamma)$ of the group Γ. This is a matrix group describing the multivaluedness of a solution vector.

Note that for $n = 2$ the variation is possible only with respect to one coefficient of the equation

$$u^{(2)}(z) + (Q_0(z) - \mu r(z))u(z) = 0.$$

For $n = 3$, for the equation

$$u^{(3)}(z) + (Q_0(z) - \lambda q(z))u^{(1)}(z) + (R_0(z) - \mu r(z))u(z) = 0 \tag{4}$$

we have already three substantially different variations: (1) with respect to r, i.e., with respect to μ, in the space of cubic differentials; (2) with respect to q, i.e., with respect to λ, in the space of quadratic differentials; (3) with respect to r and q, i.e., with respect to λ and μ.

Let $U(z) = (u(z), v(z), w(z))^T$ be the solution vector to the Cauchy problem at a point $z_0 \in D$,

$$\begin{pmatrix} u(z_0) \\ v(z_0) \\ w(z_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\quad \begin{pmatrix} u'(z_0) \\ v'(z_0) \\ w'(z_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\quad \begin{pmatrix} u''(z_0) \\ v''(z_0) \\ w''(z_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{5}$$

for the unperturbed equation, i.e., for $\lambda = 0$ and $\mu = 0$.

Put

$$W(x) = \begin{pmatrix} u & \upsilon & w \\ u' & \upsilon' & w' \\ u'' & \upsilon'' & w'' \end{pmatrix}, \quad W(z_0) = E,$$

$$W_1(x) = \begin{vmatrix} 0 & \upsilon & w \\ 0 & \upsilon' & w' \\ f & \upsilon'' & w'' \end{vmatrix} = f(-1)^4 \begin{vmatrix} \upsilon & w \\ \upsilon' & w' \end{vmatrix} = [f = ru] = ru \begin{vmatrix} \upsilon & w \\ \upsilon' & w' \end{vmatrix},$$

$$W_2(x) = \begin{vmatrix} u & 0 & w \\ u' & 0 & w' \\ u'' & f & w'' \end{vmatrix} = f(-1)^5 \begin{vmatrix} u & w \\ u' & w' \end{vmatrix} = [f = r\upsilon] = r\upsilon \begin{vmatrix} u & w \\ u' & w' \end{vmatrix},$$

$$W_3(x) = \begin{vmatrix} u & \upsilon & 0 \\ u' & \upsilon' & 0 \\ u'' & \upsilon'' & f \end{vmatrix} = f(-1)^6 \begin{vmatrix} u & \upsilon \\ u' & \upsilon' \end{vmatrix} = [f = rw] = rw \begin{vmatrix} u & \upsilon \\ u' & \upsilon' \end{vmatrix}.$$

Then

$$V(z) = \begin{pmatrix} W_1(z) & 0 & 0 \\ 0 & W_2(z) & 0 \\ 0 & 0 & W_3(z) \end{pmatrix},$$

is a solution to the Lagrange adjoint unperturbed third-order equation on D/Γ. It is known from [3] that it satisfies the equality

$$V(Lz) = \xi_L(z)^2 V(z)\chi(L)^{-1}, \quad L \in \Gamma, \quad \xi_L(z) = \sqrt{L'(z)}, \ z \in D.$$

2. Expansion of the solution vector in a series under variation in the space of quadratic differentials

Consider the perturbed vector equation

$$U^{(3)}(z) + \big(Q_0(z) - \lambda q(z)\big)U^{(1)}(z) + R_0(z)U(z) = 0, \tag{6}$$

where $\lambda \in \mathbb{C}$, $|\lambda| < \varepsilon$, ε is a sufficiently small number, and $q(z)dz^2$ is a nonzero holomorphic differential on D/Γ.

Denote by

$$U(z;\lambda;0) = \begin{pmatrix} u(z;\lambda;0) & 0 & 0 \\ 0 & \upsilon(z;\lambda;0) & 0 \\ 0 & 0 & \omega(z;\lambda;0) \end{pmatrix} = \begin{pmatrix} u(z;\lambda;0) \\ \upsilon(z;\lambda;0) \\ \omega(z;\lambda;0) \end{pmatrix}$$

the solution vector to the Cauchy problem (5) at a point z_0 for the perturbed equation (6). By Poincaré's small parameter method and the Cauchy–Kovalevskaya theorem, expand the solution vector in the Taylor series

$$U(z;\lambda;0) = U(z) + \lambda U_{10}(z) + \lambda^2 U_{20}(z) + \ldots + \lambda^m U_{m0}(z) + \ldots,$$

convergent for $|\lambda| < \epsilon, z \in D$ (see [2; 3]).

Inserting this series in (6), we obtain the infinite system of differential equations in vector-matrix form
$U^{(3)}(z) + Q_0(z)U^{(1)}(z) + R_0U(z) = 0,$
$U^{(3)}_{10}(z) + Q_0(z)U^{(1)}_{10}(z) + R_0U_{10}(z) = q(z)U^{(1)}(z),$
$U^{(3)}_{20}(z) + Q_0(z)U^{(1)}_{20}(z) + R_0U_{20}(z) = q(z)U^{(1)}_{10}(z),$
...
$U^{(3)}_{n0}(z) + Q_0(z)U^{(1)}_{n0}(z) + R_0U_{n0}(z) = q(z)U^{(1)}_{n-1,0}(z),$
...

Theorem 1. *The solution vector*

$$U^{(3)}(z) + \big(Q_0(z) - \lambda q(z)\big)U^{(1)}(z) + R_0(z)U(z) = 0$$

with condition (5) on a compact Riemann surface F of genus $g \geqslant 2$ satisfies the explicit variational formula

$$U(z;\lambda;0) = \Big[E + \lambda A_0(z) + \lambda^2 A_1(z) + \ldots + \lambda^n A_{n-1}(z) + \ldots\Big]U(z),$$

where $z \in D, |\lambda| < \varepsilon$,

$$A(x) = q(x)U^{(1)}(x)V(x), D(x) = q(x)U(x)V(x), A_0(z) = \int_{z_0}^{z} A(x)dx,$$

$$A_n(z) = \int_{z_0}^{z} \Big[A(x)D^n(x) + A_0(x)A(x)D^{n-1}(x) + A_1(x)A(x)D^{n-2}(x)$$
$$+ \ldots + A_{n-2}(x)A(x)D(x) + A_{n-1}(x)A(x)\Big]dx,$$

and E is the identity matrix of order 3.

Proof. Find the solution to the second equation of the system by Lagrange's method of variation of constants:

$$U_{10}(z) = \int_{z_0}^{z} q(x)U^{(1)}(x)V(x)dxU(z).$$

If $n = 1$ then $U_{10}(z) = A_0(z)U(z)$.

For $n > 1$, denote by $U_{n0}(z) = A_{n-1}(z)U(z)$, where

$$A_{n-1}(z) = \int_{z_0}^{z} q(x)U^{(1)}_{n-1,0}(x)V(x)dx.$$

For $n = 2$, we have $U_{20}(z) = A_1(z)U(z)$. On the other hand,

$$U_{20}(z) = A_1(z)U(z) = \int_{z_0}^{z} q(x)U^{(1)}_{10}(x)V(x)dxU(z) = \int_{z_0}^{z} q(x)[A_0(x)U(x)]'_x V(x)dxU(z).$$

It follows that

$$A_1(z) = \int_{z_0}^{z} q(x)[A_0(x)U(x)]'_x V(x)dx =$$

$$= \int_{z_0}^{z} q(x)\Big[A'_0(x)U(x) + A_0(x)U^{(1)}(x)\Big]V(x)dx =$$

$$= \int_{z_0}^{z} q(x)[A(x)U(x) + A_0(x)U^{(1)}(x)]V(x)dx =$$

$$= \int_{z_0}^{z} A(x)q(x)U(x)V(x)dx + \int_{z_0}^{z} A_0(x)q(x)U^{(1)}(x)V(x)dx =$$

$$= \int_{z_0}^{z} A(x)D(x)dx + \int_{z_0}^{z} A_0(x)A(x)dx.$$

Thus,

$$U_{20}(z) = \Big(\int_{z_0}^{z} A(x)D(x)dx + \int_{z_0}^{z} A_0(x)A(x)dx \Big) U(z).$$

For $n = 3$, we have the equality $U_{30}(z) = A_2(z)U(z)$. On the other hand,

$$U_{30}(z) = A_2(z)U(z) = \int_{z_0}^{z} q(x)U_{20}^{(1)}(x)V(x)dxU(z) = \int_{z_0}^{z} q(x)\Big[A_1(x)U(x)\Big]_x' V(x)dxU(z),$$

where

$$A_2(z) = \int_{z_0}^{z} q(x)\Big[A_1(x)U(x)\Big]_x' V(x)dx = \int_{z_0}^{z} q(x)\Big[A_1'(x)U(x) + A_1(x)U^{(1)}(x)\Big]V(x)dx =$$

$$= \int_{z_0}^{z} q(x)\Big[A(x)D(x)U(x)V(x)dx\Big] + \int_{z_0}^{z} q(x)\Big[A_0(x)A(x)U(x)V(x)dx\Big] +$$

$$+ \int_{z_0}^{z} q(x)\Big[A_1(x)U^{(1)}(x)V(x)dx\Big] = \int_{z_0}^{z} \Big[A(x)D^2(x) + A_0(x)A(x)D(x) + A_1(x)A(x)\Big]dx.$$

Therefore,

$$A_2(z) = \int_{z_0}^{z} \Big[A(x)D^2(x) + A_0(x)A(x)D(x) + A_1(x)A(x)\Big]dx$$

and

$$U_{30}(z) = \left(\int_{z_0}^{z} \Big[A(x)D^2(x) + +A_0(x)A(x)D(x) + A_1(x)A(x)\Big]dx \right) U(z).$$

By the induction assumption, for $n = m$ we have the equality

$$A_m(z) = \int_{z_0}^{z} \Big[A(x)D^m(x) + A_0(x)A(x)D^{m-1}(x) + A_1(x)A(x)D^{m-2}(x) + \ldots + A_{m-1}(x)A(x)\Big]dx.$$

Prove this assertion for the case $n = m + 1$. We have

$$U_{m+1,0}(z) = A_m(z)U(z) = \int_{z_0}^{z} q(x)U_{m0}^{(1)}(x)V(x)dxU(z),$$

where

$$A_m(z) = \int_{z_0}^{z} q(x)\Big[A_{m-1}(x)U(x)\Big]_x' V(x)dx =$$

$$= \int_{z_0}^{z} q(x)\Big[A_{m-1}'(x)U(x) + A_{m-1}(x)U^{(1)}(x)\Big]V(x)dx =$$

$$= \int_{z_0}^{z} q(x)\Big[A(x)D^{m-1}(x) + A_0(x)A(x)D^{m-2}(x) +$$

$$+ A_1(x)A(x)D^{m-3}(x) + \cdots + A_{m-2}(x)A(x)\Big]U(x)V(x)dx + \int_{z_0}^{z} q(x)A_{m-1}(x)U^{(1)}(x)V(x)dx =$$

$$= \int_{z_0}^{z} q(x)A(x)D^{m-1}(x)U(x)V(x)dx + \int_{z_0}^{z} q(x)A_0(x)A(x)D^{m-2}(x)U(x)V(x)dx +$$

$$+ \int_{z_0}^{z} q(x)A_1(x)A(x)D^{m-3}(x)U(x)V(x)dx +$$

$$+ \cdots + \int_{z_0}^{z} q(x)A_{m-2}(x)A(x)U(x)V(x)dx + \int_{z_0}^{z} A_{m-1}(x)A(x)dx =$$

$$= \int_{z_0}^{z} \Big[A(x)D^m(x) + A_0(x)A(x)D^{m-1}(x) +$$

$$+ A_1(x)A(x)D^{m-2}(x) + \cdots + A_{m-2}(x)A(x)D(x) + A_{m-1}(x)A(x)\Big]dx.$$

Consequently, by induction, we have proved the formula for the matrix A_n for any n.

Let us now introduce the explicit variational formula with respect to λ for the solution vector:

$$U(z;\lambda;0) = U(z) + \lambda U_{10}(z) + \lambda^2 U_{20}(z) + \ldots + \lambda^n U_{n0}(z) + \ldots$$

$$= EU(z) + \lambda A_0(z)U(z) + \lambda^2 A_1(z)U(z) + \ldots + \lambda^n A_{n-1}(z)U(z) + \ldots$$

$$= \Big[E + \lambda A_0(z) + \lambda^2 A_1(z) + \ldots + \lambda^n A_{n-1}(z) + \ldots\Big]U(z).$$

Thus, the theorem is proved.

Remark 1. This theorem gives an explicit vatiational formula for the solution vector, i.e., all the variational terms of any order or the whole Taylor series in λ under variation with respect to one holomorphic differential in $\Omega^2(F)$.

Proposition 1. *Let $q_1(z)dz^2, \ldots, q_{3g-3}(z)dz^2$ be a basis of quadratic holomorphic differentials on $F = D/\Gamma$ of genus $g \geqslant 2$. Then the perturbed equation*

$$U^{(3)}(z) + (Q_0(z) - \sum_{j=1}^{3g-3} \lambda_j q_j(z))U^{(1)}(z) + R_0(z)U(z) = 0$$

with condition (5) satisfies the formula for the first variation of the solution vector

$$U(z;\lambda_1,\ldots,\lambda_{3g-3};0) = \Big[E + \sum_{j=1}^{3g-3} \lambda_j A_{0;e_j}(z)\Big]U(z) + o(\lambda_1,\ldots,\lambda_{3g-3}),$$

where $|\lambda_j| \to 0, j = 1,\ldots,3g-3, z \in D,$ $A_{0;e_j}(z) = \int\limits_{z_0}^{z} q_j(x)U_x^{(1)}V(x)dx.$

Proof. Since the coefficient at the first derivative depends holomorphically on $\lambda = (\lambda_1, \ldots, \lambda_{3g-3})$, the solution vector to this equation is representable as

$$U(z;\lambda_1,\ldots,\lambda_{3g-3};0) = U(z) + \sum_{j=1}^{3g-3} \lambda_j U_{10;e_j}(z) + o(\lambda),$$

$|\lambda_j| \to 0$, $j = 1, \ldots, 3g-3$. Here e_j is the vector whose jth coordinate is equal to 1 and all the remaining coordinates are zero. Now, put $d = 3g - 3$.

Inserting this expression in the equation, we obtain the vector equality

$$U^{(3)}(z)+\sum_{j=1}^{3g-3}\lambda_j U^{(3)}_{10;e_j}(z)+o(\lambda)+\Big(Q_0(z)-\lambda_1 q_1(z)-\ \dots\ -\lambda_d q_d(z)\Big)\times$$

$$\times\Big(U^{(1)}(z)+\sum_{j=1}^{3g-3}\lambda_j U^{(1)}_{10;e_j}(z)+o(\lambda)\Big)+R_0(z)\Big(U(z)+\sum_{j=1}^{3g-3}\lambda_j U_{10;e_j}(z)+o(\lambda)\Big)=0.$$

Note that here the following conditions are fulfilled:

$$U_{10;e_j}(z_0)=U^{(1)}_{10;e_j}(z_0)=U^{(2)}_{10;e_j}(z_0)=0,\ j=1,\dots,d.$$

Hence we obtain a system of vector linear differential equations of the form

$$U^{(3)}(z)+Q_0(z)U^{(1)}(z)+R_0(z)U(z)=0,$$

$$U^{(3)}_{10;e_j}(z)+Q_0(z)U^{(1)}_{10;e_j}(z)+R_0(z)U_{10;e_j}(z)=q_j(z)U^{(1)}(z),\ j=1,\dots,d.$$

For each j, $j=1,\dots,d$, solve the equation by Lagrange's method of variation of constants:

$$U_{10;e_j}(z)=\Big[\int_{z_0}^{z}q_j(x)U^{(1)}(x)V(x)dx\Big]U(z).$$

Put $A_j(z)=q_j(z)U^{(1)}(z)V(z)$ and

$$A_{0;e_j}(z)=\int_{z_0}^{z}A_j(x)dx,\ j=1,\dots,d.$$

This gives the equality $U_{10;e_j}(z)=A_{0;e_j}(z)U(z)$, $j=1,\dots,d$. Therefore, we have the formula of the first variation of the solution vector:

$$U(z;\lambda_1,\dots,\lambda_{3g-3};0)=U(z)+\lambda_1A_{0;e_1}(z)U(z)+\dots+\lambda_dA_{0;e_d}(z)U(z)+o(\lambda_1,\dots,\lambda_{3g-3})=$$

$$=\big[E+\lambda_1A_{0;e_1}(z)+\dots+\lambda_dA_{0;e_d}(z)\big]U(z)+o(\lambda_1,\dots,\lambda_{3g-3}),$$

$\lambda_1\to 0,\dots,\lambda_{3g-3}\to 0$ under variation with respect to a basis of quadratic holomorphic differentials on a compact Riemann surface of genus $g>1$.

3. Elements of the monodromy group under a variation with respect to a basis of cubic differentials

Consider the perturbed differential vector equation

$$U^{(3)}(z)+Q_0(z)U^{(1)}(z)+(R_0(z)-\sum_{j=1}^{m}\mu_jr_j)U(z)=0. \tag{9}$$

on the surface $F=D/\Gamma$, where $r_1,\dots,r_m$ is a basis of cubic holomorphic differentials in the space $\Omega^3(F)$, $m=5g-5$, $\mu=(\mu_1,\dots,\mu_m)$. As above, denote by $U(z;0;\mu)=$ $=(u(z;0;\mu),v(z;0;\mu),w(z;0;\mu))^T$ three linearly independent solutions to the Cauchy problem at a point z_0 defined by the conditions

$$U(z_0;0,\mu)=(1,0,0)^T;\ \ U^{(1)}(z_0;0,\mu)=(0,1,0)^T;\ \ U^{(2)}(z_0;0,\mu)=(0,0,1)^T, \tag{10}$$

for every μ. By the Poincaré's small parameter method and the Cauchy–Kovalevskaya theorem, we have the solution to (9) in vector form

$$U(z;0;\mu_1,\ldots,\mu_m) = U(z) + \sum_{k=1}^{m} \mu_k U_{01;\widehat{e}_k}(z) + o(\mu_1,\ldots,\mu_m),$$

where $\mu_1,\cdots,\mu_m \to 0$.

Inserting the last expression in (9), we obtain the vector equalities

$$U^{(3)}(z) + \mu_1 U^{(3)}_{01;\widehat{e}_1}(z) + \cdots + \mu_m U^{(3)}_{01;\widehat{e}_m}(z) + o(\mu) +$$

$$+Q_0(z)(U^{(1)}(z) + \mu_1 U^{(1)}_{01;\widehat{e}_1}(z) + \cdots + \mu_m U^{(1)}_{01;\widehat{e}_m}(z) + o(\mu)) +$$

$$+(R_0(z) - \sum_{j=1}^{m} \mu_j r_j(z))(U(z) + \mu_1 U_{01;\widehat{e}_1}(z) + \cdots + \mu_m U_{01;\widehat{e}_m}(z) + o(\mu)) = 0.$$

Note that the following conditions are satisfied:

$$U_{01;\widehat{e}_k}(z_0) = U^{(1)}_{01;\widehat{e}_k}(z_0) = U^{(2)}_{01;\widehat{e}_k}(z_0) = 0,\ k = 1,\ldots,m.$$

From this we obtain the system of vector linear differential equations

$$U^{(3)}(z) + Q_0(z)U^{(1)}(z) + R_0(z)U(z) = 0;$$

$$U^{(3)}_{01;\widehat{e}_k}(z) + Q_0(z)U^{(1)}_{01;\widehat{e}_k}(z) + R_0(z)U_{01;\widehat{e}_k}(z) = r_k(z)U(z),\ k = 1,\ldots,m.$$

For each $k, k = 1,\ldots,m,$, solve the second equation by Lagrange's method of variation of constants

$$U_{01;\widehat{e}_k}(z) = \int_{z_0}^{z} r_k(t)U(t)V(t)dtU(z).$$

Introduce the notations

$$B_k(z) = r_k(z)U(z)V(z), B_{0;\widehat{e}_k}(z) = \int_{z_0}^{z} B_k(t)dt,\ k = 1,\ldots,m.$$

Hence, we obtain the equalities

$$U_{01;\widehat{e}_k}(z) = B_{0;\widehat{e}_k}U(z),\ k = 1,\ldots,m.$$

Thus,

$$U(z;0;\mu_1,\ldots,\mu_m) = U(z) + \sum_{k=1}^{m} \mu_k B_{0;\widehat{e}_k}(z)U(z) + o(\mu_1,\ldots,\mu_m)$$

$$= \left[E + \sum_{k=1}^{m} \mu_k B_{0;\widehat{e}_k}(z)\right]U(z) + o(\mu_1,\ldots,\mu_m),$$

where $\mu_1,\cdots,\mu_m \to 0$.

For deducing the variational formulas for the elements of the monodromy group, we must express $U_{01;\widehat{e}_k}(Lz)$ through $U(z)$ and the coefficients of the equation. We infer

$$U_{01;\widehat{e}_k}(Lz) = \left[\int_{z_0}^{Lz} B_k(x)dx\right]U(Lz) = \left[\int_{z_0}^{Lz_0} B_k(x)dx + \int_{Lz_0}^{Lz} B_k(x)dx\right]U(Lz) =$$

$$= B_{0;\widehat{e}_k}(Lz_0)\chi(L)U(z)\xi_L(z)^2 + \xi_L(z)^2\chi(L)U_{01;\widehat{e}_k}(z)$$

since

$$\int_{Lz_0}^{Lz} B_k(x)\chi(L)dx\xi_L(z)^2U(z) =< x = Lt >= \int_{z_0}^{z} B_k(Lt)\chi(L)dLt\xi_L(z)^2U(z) =$$

$$= \int_{z_0}^{z} r_k(Lt)U(Lt)V(Lt)\chi(L)L'(t)dt\xi_L(z)^2U(z) =$$

$$= \int_{z_0}^{z} r_k(t)L'(t)^{-3}L'(t)\chi(L)U(t)L'(t)V(t)\chi(L^{-1})\chi(L)L'(t)dt\xi_L(z)^2U(z) =$$

$$= \chi(L)\int_{z_0}^{z} r_k(t)U(t)V(t)dt\xi_L(z)^2U(z) =$$

$$= \chi(L)B_{0;\widehat{e}_k}(z)\xi_L(z)^2U(z) = \chi(L)\xi_L(z)^2U_{01;\widehat{e}_k}(z).$$

Using the above-proven equality for $U_{01;\widehat{e}_k}(Lz)$, deduce the first-order variational formula for the elements of the monodromy group:

$$\xi_L(z)^2\chi(L;0;\mu)U(z;0;\mu) = U(Lz;0;\mu) = U(Lz) + \sum_{k=1}^{m}\mu_kU_{01;\widehat{e}_k}(Lz) + o(\mu) =$$

$$= \chi(L)U(z)\xi_L(z)^2 + \sum_{k=1}^{m}\mu_k[B_{0;\widehat{e}_k}(Lz_0)\chi(L)U(z)\xi_L(z)^2 + \chi(L)\sum_{k=1}^{m}\mu_kU_{01;\widehat{e}_k}(z)\xi_L(z)^2 + o(\mu) =$$

$$= \chi(L)\Big[U(z;0;\mu)\xi_L(z)^2 - o(\mu)\xi_L(z)^2\Big] + \sum_{k=1}^{m}\mu_kB_{0;\widehat{e}_k}(Lz_0)\chi(L)U(z)\xi_L(z)^2 + o(\mu) =$$

$$= \chi(L)U(z;0;\mu)\xi_L(z)^2 - \chi(L)o(\mu)\xi_L(z)^2 +$$

$$+ \sum_{k=1}^{m}\mu_kB_{0;\widehat{e}_k}(Lz_0)\chi(L)[U(z;0;\mu)\xi_L(z)^2 - o(1)\xi_L(z)^2] + o(\mu) =$$

$$= \chi(L)\Big[U(z;0,\mu) - o(\mu)\Big]\xi_L(z)^2 + \sum_{k=1}^{m}\mu_kB_{0;\widehat{e}_k}(Lz_0)\chi(L)U(z;0,\mu)\xi_L(z)^2 -$$

$$- \sum_{k=1}^{m}\mu_kB_{0;\widehat{e}_k}(Lz_0)\chi(L)o(1)\xi_L(z)^2 + o(\mu) =$$

$$= [\chi(L) + \sum_{k=1}^{m}\mu_kB_{0;\widehat{e}_k}(Lz_0)\chi(L)]U(z;0,\mu)\xi_L(z)^2 - \chi(L)o(\mu)U^{-1}(z;0,\mu)U(z;0,\mu)\xi_L(z)^2 -$$

$$- \sum_{k=1}^{m}\mu_kB_{0;\widehat{e}_k}(Lz_0)\chi(L)o(1)U^{-1}(z;0,\mu)U(z;0,\mu)\xi_L(z)^2 + o(\mu).$$

Hence we obtain a formula for the first variation of the elements of the monodromy group:

$$\chi(L;0;\mu) = \left[E + \sum_{k=1}^{m}\mu_kB_{0;\widehat{e}_k}(Lz_0)\right]\chi(L) - o(\mu) - o(\mu),\ \ \mu \to 0.$$

Thus, we have proved the following theorem:

Theorem 2. *The following variational formulas hold for the solution vector and the elements of the monodromy group of equation* (9) *perturbed with respect to the basis of holomorphic cubic differentials* $r_j,\ j = 1, \dots, m = 5g-5$, *with normalization* (10)*:*

$$U(z;0;\mu_1,\dots,\mu_m) = \big[E + \mu_1 B_{0;\widehat{e}_1}(z) + \dots + \mu_m B_{0;\widehat{e}_m}(z)\big]U(z) + o(\mu_1,\dots,\mu_m)$$

and

$$\chi(L;0;\mu) = \left[E + \sum_{k=1}^{m} \mu_k B_{0;\widehat{e}_k}(Lz_0)\right]\chi(L) + o(\mu),$$

$\mu_1, \cdots, \mu_m \to 0$, *where*

$$B_k(z) = r_k(z)U(z)V(z), B_{0;\widehat{e}_k}(z) = \int_{z_0}^{z} B_k(t)dt,\ k = 1,\dots,m.$$

Remark 2. These variational formulas show how the generators of the monodromy group $\chi(A_1),\dots,\chi(A_g),\chi(B_1),\dots,\chi(B_g)$ and the solution vector to the third-order equation depend of the parameters $(\mu_1,\dots,\mu_m)$ under a variation with respect to a basis of cubic holomorphic differentials on F.

Now, consider the equation perturbed simultaneously with respect to $\lambda = (\lambda_1,\dots,\lambda_d)$ and to $\mu = (\mu_1,\dots,\mu_m)$,

$$U^{(3)}(z) + \left(Q_0(z) - \sum_{j=1}^{3g-3} \lambda_j q_j(z)\right)U^{(1)}(z) + \left(R_0(z) - \sum_{j=1}^{5g-5} \mu_j r_j(z)\right)U(z) = 0 \tag{11}$$

and the Cauchy problem at a point z_0 defined by the condition

$$U(z_0;\lambda;\mu) = (1,0,0)^T;\quad U^{(1)}(z_0;\lambda;\mu) = (0,1,0)^T;$$
$$U^{(2)}(z_0;\lambda;\mu) = (0,0,1)^T, \tag{12}$$

for any μ and λ.

Corollary 1. *The solution vector to equation* (11) *with the Cauchy problem* (12) *satisfies the formulas of the first variation*

$$U(z;\lambda;\mu) = \left[E + \sum_{j=1}^{3g-3} \lambda_j A_{0;e_j}(z) + \sum_{j=1}^{5g-5} \mu_j B_{0;\widehat{e}_j}(z)\right]U(z) + o(\lambda_1,\dots,\lambda_{3g-3}) + o(\mu_1,\dots,\mu_{5g-5}),$$

$\lambda_1,\dots,\lambda_{3g-3} \to 0,\ \mu_1,\dots,\mu_{5g-5} \to 0$, *where*

$$A_{0;e_j}(z) = \int_{z_0}^{z} q_j(x)U_x^{(1)}V(x)dx,\ j = 1,\dots,3g-3,$$

$$B_{0;\widehat{e}_k}(z) = \int_{z_0}^{z} r_k(x)U(x)V(x)dx,\ k = 1,\dots,5g-5.$$

Remark 3. The equality $U(Lz)(L'(z))^{-1} = \chi(L)U(z),\ L \in \Gamma$, means that the solution vector $U(z)$ for the Cauchy problem at z_0 is the form of vector third-order Prym 1-differentials on $F = D/\Gamma$ with respect to the matrix character χ of the group Γ with values in $GL(3,\mathbb{C})$, or, more exactly, $U(z)$ is a holomorphic section of the vector bundle $\chi \otimes K^{-1}$, where K is the canonical bundle on $F = D/\Gamma$ [5].

The research was supported by the Russian Foundation for Basic Research (Grants 15-01-07906, 18-01-00420) and a grant of the Government of the Russian Federation at Siberian Federal University (Contract no. 14.Y26.31.0006).

References

[1] D.A.Hejhal, Monodromy groups for higher-order differentials equation, *Bull. Amer. Math. Soc.*, **81**(1975), no. 3, 590–592.

[2] D.A.Hejhal, Monodromy groups and linearly polymorphic functions, *Acta Math.*, **135**(1975), no. 1-2, 1–55.

[3] D.A.Hejhal, The variational theory of linearly polymorphic functions, *J. d'Analyse Math.*, **30**(1976), 215–264.

[4] V.V.Chueshev, An Explicit Variational Formula for the Monodromy Group, *Siberian Adv. Math.*, **15**(2005), no. 2, 1–32; in Russian: *Mat. Tr.*, **7**(2004), no. 2, 126–158.

[5] H.M.Farkas, I.Kra, Riemann surfaces, Grad. Text's Math., vol. 71, Springer, New-York, 1992.

[6] M.I.Tulina, Explicit variational formulas for third-order equations on Riemann surfaces, *Journal of Siberian Federal University. Mathematics & Physics*, **6**(2013), no. 3, 365–375.

[7] A.V.Chueshev, V.V.Chueshev, The residue theorem and an analog of P. Appell's formula for finite Riemann surface, *Science Evolution*, **1**(2016), no. 1, 40–45.
DOI: 10.21603/2500-1418-2016-1-1-40-45

Вариационные формулы группы монодромии для уравнения третьего порядка на компактной римановой поверхности

Александр В. Чуешев
Виктор В. Чуешев
Кемеровский государственный университет
Кемерово, Российская Федерация

Аннотация. В данной статье выводятся явные вариационные формулы для вектор-решения и для элементов его группы монодромии обыкновенного дифференциального уравнения третьего порядка на компактной римановой поверхности рода $g \geqslant 2$ относительно вариации в пространствах квадратичных и кубических голоморфных дифференциалов.

Ключевые слова: римановы поверхности, уравнение третьего порядка на римановой поверхности, вариационные формулы, голоморфные дифференциалы.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(3), 319–328

DOI: 10.17516/1997-1397-2022-15-3-319-328
УДК 517.9

A Further Generalization of the Reverse Minkowski Type Inequality via Hölder and Jensen Inequalities

Bouharket Benaissa*
Laboratory of Informatic anf Mathematics
Faculty of Material Science
University of Tiaret
Algeria, Algeria

Received 10.11.2021, received in revised form 25.01.2022, accepted 09.03.2022

Abstract. The main objective of this article is to establish new generalizations of the reverse Minkowski's integral inequalities by introducing weighted functions and two integrability parameters. Two new theorems will be proved using Jensen's integral inequality and Hölder's two-parameter inequality, some reverse Minkowski type Integral inequalities are also obtained.

Keywords: convex function, Hölder inequality, Minkowski inequality, Jensen inequality.

Citation: B.Benaissa, A further generalization of the reverse Minkowski Type Inequality via Hölder and Jensen Inequalities, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 319–328.
DOI: 10.17516/1997-1397-2022-15-3-319-328.

Introduction

In recent years, many researchers have paid great attention to generalizations, extensions, and variations of Minkowski's inverse inequalities (see [1–7]). On the other hand, the convex functions have a very useful structure in terms of properties and play an important role in inequality theory, this class of functions has many applications in different branches of mathematics (functional analysis, numerical computation, probability theory, etc.). Many inequalities and results are obtained by the Jensen inequality, and many articles relating to different versions of this inequality have been found in the literature.

In this work, we will establish two results on the reverse Minkowski type integral inequalities, the first one involving Hölder inequality with two parameters, Also, we will investigate a second result via the Jensen integral inequality (convex function). Special cases will be given as generalizations to some known results.

1. Model inequalities

The following inequality is well known in the literature as Minkowski's inequality, it states that, for$p \geqslant 1$, if

$$0 < \int_a^b f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_a^b g^p(x)dx < \infty,$$

*bouharket.benaissa@univ-tiaret.dz https://orcid.org/0000-0002-1195-6169

then

$$\left(\int_a^b (f(x)+g(x))^p dx\right)^{\frac{1}{p}} \leqslant \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)dx\right)^{\frac{1}{p}}.$$

In this section, we give some recent results about the reverse Minkowski's inequality. Sulaiman [2] presented the following result related to the reverse Minkowski's inequality: for any $f, g > 0$, if $p \geqslant 1$ and

$$1 < m \leqslant \frac{f(x)}{g(x)} \leqslant M$$

for all $x \in [a, b]$, then

$$\begin{aligned} \frac{M+1}{M-1}\left(\int_a^b (f(x)-g(x))^p dx\right)^{\frac{1}{p}} &\leqslant \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)dx\right)^{\frac{1}{p}} \leqslant \\ &\leqslant \frac{m+1}{m-1}\left(\int_a^b (f(x)-g(x))^p dx\right)^{\frac{1}{p}}. \end{aligned} \tag{1}$$

Banyat Sroysang in [3] proved a significant extension of the above inequality as follows: for any $f, g > 0$, if $p \geqslant 1$ and

$$0 < c < m \leqslant \frac{f(x)}{g(x)} \leqslant M$$

for all $x \in [a, b]$, then

$$\begin{aligned} \frac{M+1}{M-c}\left(\int_a^b (f(x)-cg(x))^p dx\right)^{\frac{1}{p}} &\leqslant \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)dx\right)^{\frac{1}{p}} \leqslant \\ &\leqslant \frac{m+1}{m-c}\left(\int_a^b (f(x)-cg(x))^p dx\right)^{\frac{1}{p}}. \end{aligned} \tag{2}$$

Benaissa in [1] gave a new result to the inverse Minkowski inequality according to the following formula: For any $f, g > 0$, $\alpha > 0$, if $p \geqslant 1$ and

$$0 < c < m \leqslant \frac{\alpha f(x)}{g(x)} \leqslant M$$

for all $x \in [a, b]$, then

$$\begin{aligned} \frac{M+\alpha}{\alpha(M-c)}\left(\int_a^b (\alpha f(x)-cg(x))^p dx\right)^{\frac{1}{p}} &\leqslant \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)dx\right)^{\frac{1}{p}} \leqslant \\ &\leqslant \frac{m+\alpha}{\alpha(m-c)}\left(\int_a^b (\alpha f(x)-cg(x))^p dx\right)^{\frac{1}{p}}. \end{aligned} \tag{3}$$

2. Main results

Motivated by the above Theorems, we give a further improvement of the reverse Minkowski Type inequality by introducing weight function and two parameters $p, q > 0$. Throughout this section, the functions f, g are measurable and non-negative on interval (a, b), and w is weight function (measurable and positive) on (a, b). In order to demonstrate our main results, we need the following Lemma:

Lemma 1. *Let $0 < p \leqslant q < \infty$ and f, w be non-negative measurable functions on (a, b) and suppose that $0 < \int\limits_a^b f^q(t)w(t)dt < \infty$, then*

$$\int_a^b f^p(t)w(t)dt \leqslant \left(\int_a^b w(t)dt\right)^{\frac{q-p}{q}} \left(\int_a^b f^q(t)w(t)dt\right)^{\frac{p}{q}}. \tag{4}$$

The inequality (4) hold for $-\infty < q \leqslant p < 0$ and inverted for $0 < q \leqslant p < \infty$.

Proof. Using Hölder inequality for using the parameter $\dfrac{q}{p} \geqslant 1$, we have

$$\begin{aligned} \int_a^b f^p(t)w(t)dt &= \int_a^b \left(w^{\frac{q-p}{q}}(t)\right)\left(f^p(t)w^{\frac{p}{q}}(t)\right)dt \leqslant \\ &\leqslant \left(\int_a^b w(t)dt\right)^{\frac{q-p}{q}} \left(\int_a^b f^q(t)w(t)dt\right)^{\frac{p}{q}}. \end{aligned}$$

Jensen's integral inequality

Let f be an integrable function defined on (a, b) and let $\phi : (a, b) \longrightarrow \mathbb{R}$ be a convex function. If $\phi \circ f \in L(a, b)$, then

$$\phi\left(\frac{1}{b-a}\int_a^b f(t)dt\right) \leqslant \frac{1}{b-a}\left(\int_a^b \phi(f(t))dt\right), \tag{5}$$

the above inequality (5) is inverted if ϕ is a concave function.
Taking $\phi(t) = t^\lambda$, thus the formula (5) can be rewritten in the following forms.

- If $1 \leqslant \lambda$, then
$$\int_a^b f^\lambda(t)dt \geqslant (b-a)^{1-\lambda}\left(\int_a^b f(t)dt\right)^\lambda, \tag{6}$$
- if $0 < \lambda < 1$, then
$$\int_a^b f^\lambda(t)dt \leqslant (b-a)^{1-\lambda}\left(\int_a^b f(t)dt\right)^\lambda. \tag{7}$$

Let $-\infty \leqslant a < b \leqslant +\infty$, for $p > 0$ we suppose that

$$0 < \int_a^b f^p(x)w(x)dx < \infty \quad and \quad 0 < \int_a^b g^p(x)w(x)dx < \infty\,,$$

and we denote by $L_p^w(a, b)$ the space of all Lebesgue measurable functions f on (a, b) for which

$$\|f\|_{L_p^w(a,b)} = \left(\int_a^b f^p(x)w(x)dx\right)^{\frac{1}{p}}.$$

Using the above lemmas, we give and prove the following theorems.

Theorem 1. *Let $f,\ g>0,\ 0<p\leqslant q,\ \alpha>0,\ w$ be a weight function and*

$$0<c<m\leqslant\frac{\alpha f(x)}{g(x)}\leqslant M\quad for\ all\ x\in[a,b], \tag{8}$$

then

$$\begin{aligned}\frac{M+\alpha}{\alpha(M-c)}\,(w(x))^{\frac{p-q}{pq}}\left(\int_a^b(\alpha f(x)-cg(x))^p w(x)dx\right)^{\frac{1}{p}}&\leqslant\\ \leqslant\left(\int_a^b f^q(x)w(x)dx\right)^{\frac{1}{q}}+\left(\int_a^b g^q(x)w(x)dx\right)^{\frac{1}{q}}&\leqslant\\ \leqslant\frac{m+\alpha}{\alpha(m-c)}\left(\int_a^b(\alpha f(x)-cg(x))^q w(x)dx\right)^{\frac{1}{q}}&.\end{aligned} \tag{9}$$

Proof. From the hypothesi (8) we get

$$0<\frac{1}{c}-\frac{1}{m}\leqslant\frac{1}{c}-\frac{g(x)}{\alpha f(x)}\leqslant\frac{1}{c}-\frac{1}{M}$$

then

$$\frac{M}{M-c}\leqslant\frac{\alpha f(x)}{\alpha f(x)-c\,g(x)}\leqslant\frac{m}{m-c}, \tag{10}$$

let $0<p\leqslant q$, from the inequality (10) we have

$$\left[\frac{M}{\alpha(M-c)}(\alpha f(x)-c\,g(x))\right]^p w(x)\leqslant f^p(x)w(x),$$

and

$$f^q(x)w(x)\leqslant\left[\frac{m}{\alpha(m-c)}(\alpha f(x)-c\,g(x))\right]^q w(x).$$

Integrating the above inequalities on $[a,b]$, we get

$$\frac{M}{\alpha(M-c)}\left(\int_a^b(\alpha f(x)-c\,g(x))^p w(x)dx\right)^{\frac{1}{p}}\leqslant\left(\int_a^b f^p(x)w(x)dx\right)^{\frac{1}{p}}, \tag{11}$$

and

$$\left(\int_a^b f^q(x)w(x)dx\right)^{\frac{1}{q}}\leqslant\frac{m}{\alpha(m-c)}\left(\int_a^b(\alpha f(x)-c\,g(x))^q w(x)dx\right)^{\frac{1}{q}}, \tag{12}$$

from the inequalities (11) and (4), we get

$$\begin{aligned}\frac{M}{\alpha(M-c)}\left(\int_a^b(\alpha f(x)-c\,g(x))^p w(x)dx\right)^{\frac{1}{p}}&\leqslant\\ \leqslant\left(\int_a^b w(t)dt\right)^{\frac{q-p}{pq}}\left(\int_a^b f^q(t)w(t)dt\right)^{\frac{1}{q}}&,\end{aligned}$$

this is same us

$$\frac{M}{\alpha(M-c)}\left(\int_a^b w(t)dt\right)^{\frac{p-q}{pq}}\left(\int_a^b(\alpha f(x)-c\,g(x))^p w(x)dx\right)^{\frac{1}{p}}\leqslant\left(\int_a^b f^q(x)w(x)dx\right)^{\frac{1}{q}}. \tag{13}$$

From the hypothesi (8), we deduce that

$$0 < m - c \leqslant \frac{\alpha f(x) - c\,g(x)}{g(x)} \leqslant M - c,$$

thus

$$\frac{\alpha f(x) - c\,g(x)}{M - c} \leqslant g(x) \leqslant \frac{\alpha f(x) - c\,g(x)}{m - c}, \tag{14}$$

let $0 < p \leqslant q$, from the inequality (14), we obtain

$$\left[\frac{1}{M - c}(\alpha f(x) - c\,g(x))\right]^p w(x) \leqslant g^p(x)w(x),$$

and

$$g^q(x)w(x) \leqslant \left[\frac{1}{m - c}(\alpha f(x) - c\,g(x))\right]^q w(x),$$

integrating on $[a, b]$, we get

$$\left(\int_a^b g^q(x)w(x)dx\right)^{\frac{1}{q}} \leqslant \frac{1}{m - c}\left(\int_a^b (\alpha f(x) - c\,g(x))^q w(x)dx\right)^{\frac{1}{q}}, \tag{15}$$

and

$$\frac{1}{M - c}\left(\int_a^b (\alpha f(x) - c\,g(x))^p w(x)dx\right)^{\frac{1}{p}} \leqslant \left(\int_a^b g^p(x)w(x)dx\right)^{\frac{1}{p}}, \tag{16}$$

using the inequality (16) and (4), we get

$$\frac{1}{M - c}\left(\int_a^b w(t)dt\right)^{\frac{p-q}{pq}}\left(\int_a^b (\alpha f(x) - c\,g(x))^p w(x)dx\right)^{\frac{1}{p}} \leqslant \left(\int_a^b g^q(x)w(x)dx\right)^{\frac{1}{q}}. \tag{17}$$

By the inequalities (12), (15) and (13), (17) we result the inequality (9). □

Now we present a new result involving Jensen integral inequality.

Theorem 2. *Let f, $g > 0$, $\alpha > 0$, w be a weight function and*

$$0 < c < m \leqslant \frac{\alpha f(x)}{g(x)} \leqslant M \quad for\ all\ x \in [a, b], \tag{18}$$

then, for $1 < p \leqslant q$

$$\begin{aligned} &\frac{M + \alpha}{\alpha(M - c)}(b - a)^{\frac{1-p}{q}}\left(\int_a^b (\alpha f(x) - c\,g(x))^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \leqslant \\ &\leqslant \left(\int_a^b f^q(x)w(x)dx\right)^{\frac{1}{q}} + \left(\int_a^b g^q(x)w(x)dx\right)^{\frac{1}{q}} \leqslant \\ &\leqslant \frac{m + \alpha}{\alpha(m - c)}\left(\int_a^b (\alpha f(x) - cg(x))^q w(x)dx\right)^{\frac{1}{q}}, \end{aligned} \tag{19}$$

for $0 < q \leqslant p \leqslant 1$

$$\begin{aligned} &\frac{M+\alpha}{\alpha(M-c)}\left(\int_a^b(\alpha f(x)-c\,g(x))^{\frac{q}{p}}w(x)dx\right)^{\frac{p}{q}} \leqslant \\ &\leqslant \left(\int_a^b f^q(x)w(x)dx\right)^{\frac{1}{q}} + \left(\int_a^b g^q(x)w(x)dx\right)^{\frac{1}{q}} \leqslant \\ &\leqslant \frac{m+\alpha}{\alpha(m-c)}(b-a)^{\frac{1-p}{q}}\left(\int_a^b(\alpha f(x)-cg(x))^q w(x)dx\right)^{\frac{1}{q}}. \end{aligned} \tag{20}$$

Proof. Firstly let $1 \leqslant p \leqslant q$, from the inequality (10), we have

$$\left[\frac{M}{\alpha(M-c)}(\alpha f(x)-c\,g(x))\right]^{\frac{q}{p}} w(x) \leqslant f^{\frac{q}{p}}(x)w(x),$$

and

$$f^q(x)w(x) \leqslant \left[\frac{m}{\alpha(m-c)}(\alpha f(x)-c\,g(x))\right]^q w(x),$$

Integrating the above inequalities on $[a,b]$, we get

$$\left(\frac{M}{\alpha(M-c)}\right)^{\frac{q}{p}}\int_a^b(\alpha f(x)-c\,g(x))^{\frac{q}{p}}w(x)dx \leqslant \int_a^b f^{\frac{q}{p}}(x)w(x)dx, \tag{21}$$

and

$$\left(\int_a^b f^q(x)w(x)dx\right)^{\frac{1}{q}} \leqslant \frac{m}{\alpha(m-c)}\left(\int_a^b(\alpha f(x)-c\,g(x))^q w(x)dx\right)^{\frac{1}{q}}, \tag{22}$$

apply the Jensen inequality (7) for $\lambda = \dfrac{1}{p}$, hence from the inequality (21), we get

$$\begin{aligned} \left(\frac{M}{\alpha(M-c)}\right)^{\frac{q}{p}}\int_a^b(\alpha f(x)-c\,g(x))^{\frac{q}{p}}w(x)dx &\leqslant \int_a^b f^{\frac{q}{p}}(x)w(x)dx \leqslant \\ &\leqslant (b-a)^{1-\frac{1}{p}}\left(\int_a^b f^q(x)w(x)dx\right)^{\frac{1}{p}}, \end{aligned}$$

this give us

$$\frac{M}{\alpha(M-c)}(b-a)^{\frac{1-p}{q}}\left(\int_a^b(\alpha f(x)-c\,g(x))^{\frac{q}{p}}w(x)dx\right)^{\frac{p}{q}} \leqslant \left(\int_a^b f^q(x)w(x)dx\right)^{\frac{1}{q}}. \tag{23}$$

In another case, from the inequality (14) we result

$$g^q(x)w(x) \leqslant \left[\frac{1}{m-c}(\alpha f(x)-c\,g(x))\right]^q w(x),$$

and

$$\left[\frac{1}{M-c}(\alpha f(x)-c\,g(x))\right]^{\frac{q}{p}} w(x) \leqslant g^{\frac{q}{p}}(x)w(x),$$

integrating on $[a,b]$, we get

$$\left(\int_a^b g^q(x)w(x)dx\right)^{\frac{1}{q}} \leqslant \frac{1}{m-c}\left(\int_a^b (\alpha f(x)-c\,g(x))^q w(x)dx\right)^{\frac{1}{q}}, \tag{24}$$

and

$$\left(\frac{1}{M-c}\right)^{\frac{q}{p}}\int_a^b (\alpha f(x)-c\,g(x))^{\frac{q}{p}} w(x)dx \leqslant \int_a^b g^{\frac{q}{p}}(x)w(x)dx, \tag{25}$$

use Jensen integral inequality (7) and (25), we obtain

$$\frac{1}{M-c}(b-a)^{\frac{1-p}{q}}\left(\int_a^b (\alpha f(x)-c\,g(x))^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}} \leqslant \left(\int_a^b g^q(x)w(x)dx\right)^{\frac{1}{q}}. \tag{26}$$

By the inequalities (22), (24) and (23), (26) we result the inequality (19).

Secondly let $1 \leqslant p \leqslant q$, from the inequality (10) we deduce that

$$\left[\frac{M}{\alpha(M-c)}(\alpha f(x)-c\,g(x))\right]^q w(x) \leqslant f^q(x)w(x),$$

and

$$f^{\frac{q}{p}}(x)w(x) \leqslant \left[\frac{m}{\alpha(m-c)}(\alpha f(x)-c\,g(x))\right]^{\frac{q}{p}} w(x),$$

Integrating the above inequalities on $[a,b]$, we get

$$\frac{M}{\alpha(M-c)}\left(\int_a^b (\alpha f(x)-c\,g(x))^q w(x)dx\right)^{\frac{1}{q}} \leqslant \int_a^b f^q(x)w(x)dx, \tag{27}$$

and

$$\int_a^b f^{\frac{q}{p}}(x)w(x)dx \leqslant \left(\frac{m}{\alpha(m-c)}\right)^{\frac{q}{p}}\int_a^b (\alpha f(x)-c\,g(x))^{\frac{q}{p}} w(x)dx, \tag{28}$$

apply the Jensen inequality (6) for $\lambda = \dfrac{1}{p}$, hence from the inequality (27), we get

$$\begin{aligned}(b-a)^{1-\frac{1}{p}}\left(\int_a^b f^q(x)w(x)dx\right)^{\frac{1}{p}} &\leqslant \int_a^b f^{\frac{q}{p}}(x)w(x)dx \leqslant \\ &\leqslant \left(\frac{m}{\alpha(m-c)}\right)^{\frac{q}{p}}\int_a^b (\alpha f(x)-c\,g(x))^{\frac{q}{p}} w(x)dx,\end{aligned}$$

this give us

$$\left(\int_a^b f^q(x)w(x)dx\right)^{\frac{1}{q}} \leqslant \frac{m}{\alpha(m-c)}(b-a)^{\frac{1-p}{q}}\left(\int_a^b (\alpha f(x)-c\,g(x))^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}}. \tag{29}$$

In another case, from the inequality (14), we deduce that

$$\left[\frac{1}{M-c}(\alpha f(x)-c\,g(x))\right]^q w(x) \leqslant g^q(x)w(x),$$

and

$$g^{\frac{q}{p}}(x)w(x) \leqslant \left[\frac{1}{M-c}(\alpha f(x) - c\, g(x))\right]^{\frac{q}{p}} w(x),$$

integrating on $[a,b]$, we get

$$\frac{1}{M-c}\left(\int_a^b (\alpha f(x) - c\, g(x))^q w(x)dx\right)^{\frac{1}{q}} \leqslant \left(\int_a^b g^q(x)w(x)dx\right)^{\frac{1}{q}}, \tag{30}$$

and

$$\int_a^b g^{\frac{q}{p}}(x)w(x)dx \leqslant \left(\frac{1}{m-c}\right)^{\frac{q}{p}} \int_a^b (\alpha f(x) - c\, g(x))^{\frac{q}{p}} w(x)dx, \tag{31}$$

use the Jensen inequality (6) and (31), we obtain

$$\left(\int_a^b g^q(x)w(x)dx\right)^{\frac{1}{q}} \leqslant \frac{1}{m-c}(b-a)^{\frac{1-p}{q}} \left(\int_a^b (\alpha f(x) - c\, g(x))^{\frac{q}{p}} w(x)dx\right)^{\frac{p}{q}}. \tag{32}$$

By the inequalities (28), (30) and (29), (32) we result the inequality (20). □

3. Application

We now give some new results of the above Theorems.

3.1. Reverse Minkowski weight type inequality

Put $p = q$ in the Theorem 1 and $p = 1$ in the Theorem 2, we get the following corollary.

Corollary 1. *Let f, $g > 0$, $q > 0$, $\alpha > 0$, w be a weight function and*

$$0 < c < m \leqslant \frac{\alpha f(x)}{g(x)} \leqslant M \quad \textit{for all } x \in [a,b],$$

then

$$\begin{aligned} &\frac{M+\alpha}{\alpha(M-c)}\left(\int_a^b (\alpha f(x) - cg(x))^q w(x)dx\right)^{\frac{1}{q}} \leqslant \\ &\leqslant \left(\int_a^b f^q(x)w(x)dx\right)^{\frac{1}{q}} + \left(\int_a^b g^q(x)w(x)dx\right)^{\frac{1}{q}} \leqslant \\ &\leqslant \frac{m+\alpha}{\alpha(m-c)}\left(\int_a^b (\alpha f(x) - cg(x))^q w(x)dx\right)^{\frac{1}{q}}. \end{aligned} \tag{33}$$

3.2. Reverse Minkowski type inequality

Using $w \equiv 1$ in Theorem 1 and Theorem 2, we get the following corollaries.

Corollary 2. *Let f, $g > 0$, $0 < p \leqslant q$, $\alpha > 0$ and*

$$0 < c < m \leqslant \frac{\alpha f(x)}{g(x)} \leqslant M \quad \textit{for all } x \in [a,b],$$

then

$$\begin{aligned}
&\frac{M+\alpha}{\alpha(M-c)}(b-a)^{\frac{p-q}{pq}}\left(\int_a^b(\alpha f(x)-cg(x))^p dx\right)^{\frac{1}{p}} \leqslant \\
&\leqslant \left(\int_a^b f^q(x)dx\right)^{\frac{1}{q}}+\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \leqslant \\
&\leqslant \frac{m+\alpha}{\alpha(m-c)}\left(\int_a^b(\alpha f(x)-cg(x))^q dx\right)^{\frac{1}{q}}.
\end{aligned} \tag{34}$$

Corollary 3. *Let $f,\, g>0,\ \alpha>0$ and*

$$0<c<m\leqslant \frac{\alpha f(x)}{g(x)}\leqslant M \quad for\ all\ x\in[a,b],$$

then, for $1<p\leqslant q$

$$\begin{aligned}
&\frac{M+\alpha}{\alpha(M-c)}(b-a)^{\frac{1-p}{q}}\left(\int_a^b(\alpha f(x)-c\,g(x))^{\frac{q}{p}} dx\right)^{\frac{p}{q}} \leqslant \\
&\leqslant \left(\int_a^b f^q(x)dx\right)^{\frac{1}{q}}+\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \leqslant \\
&\leqslant \frac{m+\alpha}{\alpha(m-c)}\left(\int_a^b(\alpha f(x)-cg(x))^q dx\right)^{\frac{1}{q}}.
\end{aligned} \tag{35}$$

for $0<q\leqslant p\leqslant 1$

$$\begin{aligned}
&\frac{M+\alpha}{\alpha(M-c)}\left(\int_a^b(\alpha f(x)-c\,g(x))^{\frac{q}{p}} dx\right)^{\frac{p}{q}} \leqslant \\
&\leqslant \left(\int_a^b f^q(x)dx\right)^{\frac{1}{q}}+\left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \leqslant \\
&\leqslant \frac{m+\alpha}{\alpha(m-c)}(b-a)^{\frac{1-p}{q}}\left(\int_a^b(\alpha f(x)-cg(x))^q dx\right)^{\frac{1}{q}}.
\end{aligned} \tag{36}$$

The inequalities (34), (35) and (36) are new generalizations of the revers Minkowski inequality with two parameters.

Conclusion

By using Hölder's inequality, Jensen's integral inequality and by introducing two parameters of integrability, new generalizations of the inverse of Minkowski's integral inequality have been established and demonstrated. Two results are given in the application section, the reverse Minkowski weight type inequality and we deduce a particular case the reverse Minkowski type inequality, this is a new generalization of the classic reverse Minkowski inequality known in the literature.

References

[1] B.Benaissa, More on reverses of Minkowski's inequalities and Hardy's integral inequalities, *Asian-Eur. J. Math.*, **13**(2020), no. 3, 1–7. DOI: 10.1142/S1793557120500643

[2] W.T.Sulaiman, Reverses of Minkowski's, Hölder's, and Hardy's integral inequalities, *Int. J. Mod. Math. Sci.*, **1**(2012), no. 1, 14–24.

[3] B.Sroysang, More on Reverses of Minkowski's Integral Inequality, *Mathematica Aeterna*, **3**(2013), no. 7, 597–600.

[4] S.Bobkov, M.Madiman, Reverse Brunn–Minkowski and reverse entropy power inequalities for convex measures, *J. Func. Anal.*, **262**(2012), 3309–3339.

[5] G.Rahman et al, The Minkowski inequalities via generalized proportional fractional integral operators, *Adv. Difference Equ.*, **2019**(2019), no. 287. DOI: 10.1186/s13662-019-2229-7

[6] S.Rashid, F.Jarad, Yu-M.Chu, A Note on Reverse Minkowski Inequality via Generalized Proportional Fractional Integral Operator with respect to Another Function, *Math. Probl. Eng.*, **2020**(2020), 1–12. DOI: 10.1155/2020/7630260

[7] S.Rashid et al, New generalized reverse Minkowski and related integral inequalities involving generalized fractional conformable integrals, *J. Inequal. Appl.*, **2020**(2020), no. 177. DOI: 10.1186/s13660-020-02445-2

Дальнейшее обобщение обратного неравенства типа Минковского с помощью неравенств Гельдера и Йенсена

Буаркет Бенаисса
Лаборатория информатики и математики
Факультет материаловедения
Тиаретский университет
Алжир, Алжир

Аннотация. Основная цель этой статьи — установить новые обобщения обратных интегральных неравенств Минковского путем введения весовых функций и двух параметров интегрируемости. Будут доказаны две новые теоремы с использованием интегрального неравенства Йенсена и двухпараметрического неравенства Гельдера, а также получены некоторые обратные интегральные неравенства типа Минковского.

Ключевые слова: выпуклая функция, неравенство Гельдера, неравенство Минковского, неравенство Йенсена.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(3), 329–342

DOI: 10.17516/1997-1397-2022-15-3-329-342
УДК 517.55

On Properties of the Second Type Matrix Ball $B_{m,n}^{(2)}$ from Space $\mathbb{C}^n[m \times m]$

Uktam S. Rakhmonov*
Tashkent State Technical University
Tashkent, Uzbekistan
Jonibek Sh. Abdullayev†
National University of Uzbekistan
Tashkent, Uzbekistan

Received 02.09.2021, received in revised form 28.01.2022, accepted 20.03.2022

Abstract. The automorphisms of the matrix ball associated with the classical domains of the second type are described in this paper. The properties of the second type matrix ball $B_{m,n}^{(2)}$ are studied.

Keywords: classical domain, matrix ball, automorphism of the matrix ball, volume, Shilov's boundary.

Citation: U.S. Rakhmonov, J.Sh. Abdullayev, On Properties of the Second Type Matrix Ball $B_{m,n}^{(2)}$ from Space $\mathbb{C}^n[m \times m]$, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 329–342.
DOI: 10.17516/1997-1397-2022-15-3-329-342.

1. Introduction, preliminaries and problem statement

The theory of functions of several complex variables, or multidimensional complex analysis, currently is rather rigorously developed (see [1–4]). At the same time, many questions of classical complex analysis still do not have unambiguous multidimensional analogues. The matrix approach to the presentation of the theory of multidimensional complex analysis was widely used (see [5–8]).

In 1935 E.Cartan proved that there are only six possible types of classical domains, including irreducible, homogeneous, bounded, symmetric domains, four of them K_1, K_2, K_3 and K_4 have the form

$$K_1 = \left\{ Z \in \mathbb{C}\left[m \times k\right] : I^{(m)} - ZZ^* > 0 \right\},$$

$$K_2 = \left\{ Z \in \mathbb{C}\left[m \times m\right] : I^{(m)} - Z\overline{Z} > 0, \ \ \forall Z' = Z \right\},$$

$$K_3 = \left\{ Z \in \mathbb{C}\left[m \times m\right] : I^{(m)} + Z\overline{Z} > 0, \quad \forall Z' = -Z \right\},$$

$$K_4 = \left\{ z \in \mathbb{C}^n : \left|zz'\right|^2 + 1 - 2\overline{z}z' > 0, \quad |zz'| < 1 \right\}.$$

Here $I^{(m)}$ is the identity matrix of order m, Z^* is the complex conjugate of transposed matrix Z' ($H > 0$ means that hermitian matrix H is positive definite).

The dimensions of these domains are equal to mk, $m(m+1)/2$, $m(m-1)/2$, n, respectively.

*uktam_rakhmonov@mail.ru https://orcid.org/0000-0001-8926-9505
†jonibek-abdullayev@mail.ru https://orcid.org/0000-0001-8950-2135

All these domains are biholomorphically non-equivalent, therefore, complex analysis is constructed differently for each of them.

It should be noted[‡] that domain K_4 is reducible for $n = 2$ (see [6]). In contrast, the other domains of all four types are irreducible, but the same domains can be found. Switching the places of m and k does not change domains in K_1. Further, the unit circle of the complex plane is obtained when $m = k = 1$ in K_1, $m = 1$ in K_2, $m = 2$ in K_3 and $n = 1$ in K_4. When $m = 3$, $k = 1$ in domain K_1 then K_1 coincides with domain K_3 including $m = 3$. When $m = k = 2$ in domain K_1 then K_1 coincides with domain K_4 including $n = 4$. When $m = 2$ in domain K_2 then K_2 coincides with domain K_4 including $n = 3$. Thus, we obtain different irreducible domains if we demand $m \geqslant k$ in K_1, $m \geqslant 2$ in K_2, $m \geqslant 4$ in K_3 and $n \geqslant 5$ in K_4. So, the number $\psi(n)$ of classes of irreducible bounded symmetric domains of an n-dimensional complex space is equal to the total number of representations of n in one of the following forms

$$K_1 : n = mk \ \ (m \geqslant k),$$

$$K_2 : n = \frac{1}{2}m(m + 1) \ \ (m \geqslant 2),$$

$$K_3 : n = \frac{1}{2}m(m - 1) \ \ (m \geqslant 4),$$

$$K_4 : n = m \quad (n \geqslant 5),$$

$$K_5, K_6 : n = 16, \ n = 27.$$

All irreducible domains obtained in this way are topologically (but not analytically) equivalent to the n-dimensional complex space.

Let us consider the space of m^2 complex variables denoted by $\mathbb{C}^{m^2}$. Points Z of this space can be represented conveniently as a square $[m \times m]$ matrices, i.e., in the form $Z = (z_{ij})_{i,j=1}^{m}$. With this representation of points the space $\mathbb{C}^{m^2}$ is denoted by $\mathbb{C}[m \times m]$. The direct product $\underbrace{\mathbb{C}[m \times m] \times \cdots \times \mathbb{C}[m \times m]}_{n}$ of n copies of $[m \times m]$ matrix spaces is denoted by $\mathbb{C}^n[m \times m]$.

Let $Z = (Z_1, Z_2, \ldots, Z_n)$ be a vector composed of square matrices Z_j of order m considered over the field of complex numbers $\mathbb{C}$. We can assume that Z is an element of the set $\mathbb{C}^n[m \times m] \cong \mathbb{C}^{nm^2}$.

The matrix «scalar product» is defined as $(Z, W \in \mathbb{C}^n[m \times m])$

$$\langle Z, W\rangle = Z_1W_1^* + Z_2W_2^* + \cdots + Z_nW_n^*.$$

It is known that matrix balls $B^{(1)}_{m,n}$, $B^{(2)}_{m,n}$ and $B^{(3)}_{m,n}$ of the first, second, and third types have the following forms, respectively (see [9–11]):

$$B^{(1)}_{m,n} = \{(Z_1, \ldots, Z_n) = Z \in \mathbb{C}^n\, [m \times m] : I - \langle Z, Z\rangle > 0\}\,,$$

$$B^{(2)}_{m,n} = \{Z \in \mathbb{C}^n\, [m \times m] : I - \langle Z, Z\rangle > 0 \ \ \forall\, Z'_\nu = Z_\nu, \ \nu = 1, \ldots, n\}$$

and

$$B^{(3)}_{m,n} = \left\{(Z \in \mathbb{C}^n\, [m \times m] : I + \langle Z, Z\rangle > 0 \ \ \forall\, Z^{'}_\nu = -Z_\nu, \ \nu = 1, \ldots, n\right\}.$$

[‡]When $n = 2$, a homogeneous bounded domain is equivalent to the domain $K = \left\{\zeta \in \mathbb{C}^2 : \max\left(|\zeta_1| < 1, |\zeta_2| < 1\right)\right\}$, after the change of variables: $z_1 = \frac{\zeta_1 + \zeta_2}{2}$, $z_2 = \frac{i(\zeta_1 - \zeta_2)}{2}$.

The skeletons (Shilov boundaries) of the matrix balls $B^{(k)}_{m,n}$ are denoted by $X^{(k)}_{m,n}$, $k = 1, 2, 3$, i.e.,

$$X^{(1)}_{m,n} = \{Z \in \mathbb{C}^n\,[m \times m] : \langle Z, Z\rangle = I\},$$

$$X^{(2)}_{m,n} = \{Z \in \mathbb{C}^n\,[m \times m] : \langle Z, Z\rangle = I,\ Z'_v = Z_\nu,\ \nu = 1, 2, \ldots, n\},$$

$$X^{(3)}_{m,n} = \left\{Z \in \mathbb{C}^n\,[m \times m] : I + \langle Z, Z\rangle = 0,\ Z^{'}_\nu = -Z_\nu,\ \nu = 1, 2, \ldots, n\right\}.$$

Note that $B^{(1)}_{1,1}$, $B^{(2)}_{1,1}$ and $B^{(3)}_{2,1}$ are unit disks, and $X^{(1)}_{1,1}$, $X^{(2)}_{1,1}$, $X^{(3)}_{2,1}$ are unit circles in the complex plane $\mathbb{C}$.

If $n = 1$, $m > 1$ then domains $B^{(k)}_{m,1}$, $k = 1, 2, 3$ are the classical domains of the first, second and the third type (according to the classification of E. Cartan (see [5])). The skeletons $X^{(1)}_{m,1}$, $X^{(2)}_{m,1}$, and $X^{(3)}_{m,1}$ are unitary, symmetric unitary and skew-symmetric unitary matrices, respectively.

The first type of matrix ball was considered by A. G. Sergeev (see [11,26]), G. Khudayberganov (see [12,13]) and S. Kosbergenov (see [14,15]). The volume of a matrix ball of the first type and its skeleton is studied in [16]. Holomorphic automorphisms for a matrix ball of the first type are described in [17]. The integral formulas for the matrix ball of the second type were studied by G. Khudayberganov and Z. Matyakubov [18,19] and the third type of the matrix ball was studied by G. Khudayberganov, U. Rakhmonov, and the integral formulas were found [20,21]. We recall that a bounded domain $D \subset \mathbb{C}^n$ is called classical if the complete group of its holomorphic automorphisms is a classical Lie group and transitive on it. The biholomorphic equivalence of bounded domains in $\mathbb{C}^n$ to their indicatrices for the Carathéodory and Kobayashi metrics was studied [32]. From this, in particular, a description of that domains can be obtained when indicatrices are classical domains. It was proved that first, second and third type matrix balls in space $\mathbb{C}^n[m \times m]$ are equivalent biholomorphically to Siegel domains of the second type [27–29]. However, the question of whether matrix balls $B^{(1)}_{m,n}$, $B^{(2)}_{m,n}$ and $B^{(3)}_{m,n}$ are the classical domains still remains open.

The problem of the holomorphic extendability of a function to a matrix ball, given on a piece of its skeleton was discussed [26]. For this purpose complete orthonormal systems in the matrix ball were used. The total volumes of a matrix ball of the third type and a generalized Lie ball were calculated [22]. The full volumes of these domains are necessary for finding the kernels of the integral formulas for these domains (the Bergman, Cauchy–Szegő kernels, Poisson kernels, etc. [14, 19, 23, 30]). In addition, they are used for the integral representation of a holomorphic function on these domains, in the mean value theorem and in other important concepts. Volumes of classical supermanifolds such as supersphere, complex projective superspace, and the Stifel and Grassmann supermanifolds were calculated with respect to natural metrics of symplectic structures. It was shown that formulas for volumes of these supermanifolds can be obtained by analytic continuation of the parameters from the formulas for the volumes of the corresponding ordinary varieties (see [24]).

In this paper we describe automorphisms of the matrix ball associated with classical domains of the second type, and also study the properties of the second type matrix ball. An automorphism of the second type matrix ball and the characteristic shape of this ball were studied [10]. Writing automorphism in this form causes inconvenience in applying it to practical issues. Therefore, we consider automorphisms of a matrix ball of the second type which are convenient for calculations. In addition, the total volume of the skeleton of this ball is calculated.

2. Automorphisms for a matrix ball of the second type

Let $B_{m,n}^{(2)}$ be a matrix ball of the second type and $X_{m,n}^{(2)}$ is its skeleton. The following lemma describes some properties of a matrix ball of the second type [18].

Lemma 1. *A matrix ball $B_{m,n}^{(2)}$ has the following properties:*
1) $B_{m,n}^{(2)}$ is a bounded domain;
2) $B_{m,n}^{(2)}$ is a full circular domain;
3) $B_{m,n}^{(2)}$ and its skeleton $X_{m,n}^{(2)}$ are invariants under unitary transformations.

It is known that automorphism $B_{m,1}^{(2)}$ which maps the point $P \in B_{m,1}^{(2)}$ to the point 0 has the form [8]

$$W = R(Z - P)(I - \bar{P}Z)^{-1}\bar{R}^{-1},$$

where R is $[m \times m]$ matrix

$$\bar{R}(I - \bar{P}P')R' = I.$$

Our goal is to find automorphisms for a matrix ball of the second type. Let us consider the desired automorphism in the form

$$W_k = \left(A_{00} + \sum_{j=1}^{n} Z_j A_{j0}\right)^{-1}\left(A_{0k} + \sum_{j=1}^{n} Z_j A_{jk}\right), \;\; k = 1, \dots, n. \tag{1}$$

We need to find the coefficients A_{ij} so that map (1) is an automorphism of the matrix ball of the second type.

Let us introduce the following notation of block square matrices of order $n + 1$

$$A = \begin{pmatrix} A_{00} & A_{01} & \dots & A_{0n} \\ A_{10} & A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots & \dots \\ A_{n0} & A_{n1} & \dots & A_{nn} \end{pmatrix}, \;\; H = \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix},$$

where A_{ij} are square matrices of order m.

The following statement holds.

Theorem 1. *Mapping (1) is an automorphism of the matrix ball $B_{m,n}^{(2)}$ if and only if coefficients A_{ij}, $i, j = 0, 1, 2, \dots, n$ satisfy the following relations:*

$$AHA^* = H, \;\; A_{sk}A'_{j0} = A_{j0}A'_{jk}, \;\; s = 0, \dots, n; \; j, k = 0, \dots, n. \tag{2}$$

Proof. This theorem is proved in several stages, according to the properties of a matrix ball of the second type.

$\mathbf{1}^0$. Let us consider a linear transformation

$$\omega_0 = \sum_{j=0}^{n} \zeta_j A_{j0}, \;\; \omega_k = \sum_{j=0}^{n} \zeta_j A_{jk}, \;\; k = 1, \dots, n, \tag{3}$$

where matrix A satisfies relations (2). Then we have

$$AHA^* = \begin{pmatrix} A_{00} & A_{01} & \dots & A_{0n} \\ A_{10} & A_{11} & \dots & A_{1n} \\ \dots & \dots & \dots & \dots \\ A_{n0} & A_{n1} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix} \times$$

$$\times \begin{pmatrix} A_{00}^* & A_{10}^* & \dots & A_{n0}^* \\ A_{01}^* & A_{11}^* & \dots & A_{n1}^* \\ \dots & \dots & \dots & \dots \\ A_{0n}^* & A_{1n}^* & \dots & A_{nn}^* \end{pmatrix} = \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix},$$

$$\begin{pmatrix} A_{00} & -A_{01} & \dots & -A_{0n} \\ A_{10} & -A_{11} & \dots & -A_{1n} \\ \dots & \dots & \dots & \dots \\ A_{n0} & -A_{n1} & \dots & -A_{nn} \end{pmatrix} \begin{pmatrix} A_{00}^* & A_{10}^* & \dots & A_{n0}^* \\ A_{01}^* & A_{11}^* & \dots & A_{n1}^* \\ \dots & \dots & \dots & \dots \\ A_{0n}^* & A_{1n}^* & \dots & A_{nn}^* \end{pmatrix} = \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix}$$

$$\begin{pmatrix} A_{00}A_{00}^* - \dots - A_{0n}A_{0n}^* & A_{00}A_{10}^* - \dots - A_{0n}A_{1n}^* & \dots & A_{00}A_{n0}^* - \dots - A_{0n}A_{nn}^* \\ A_{10}A_{00}^* - \dots - A_{1n}A_{0n}^* & A_{10}A_{10}^* - \dots - A_{1n}A_{1n}^* & \dots & A_{10}A_{n0}^* - \dots - A_{1n}A_{nn}^* \\ \dots & \dots & \dots & \dots \\ A_{n0}A_{00}^* - \dots - A_{nn}A_{0n}^* & A_{n0}A_{10}^* - \dots - A_{nn}A_{1n}^* & \dots & A_{n0}A_{n0}^* - \dots - A_{nn}A_{nn}^* \end{pmatrix} =$$

$$= \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} A_{00}A_{00}^* - \sum\limits_{s=1}^{n} A_{0s}A_{0s}^* = I^{(m)}, \\ A_{j0}A_{k0}^* = \sum\limits_{s=1}^{n} A_{js}A_{ks}^*, \ \ j \neq k, \\ A_{j0}A_{j0}^* - \sum\limits_{s=1}^{n} A_{js}A_{js}^* = -I^{(m)}, \ \ j \geqslant 1. \end{cases} \tag{4}$$

2^0. Let matrix row $\zeta = (\zeta_0, \zeta_1, \dots, \zeta_n)$ covers all matrices consisting of m rows and $(n+1)m$ columns such that $\zeta H \zeta^* > 0$. Then

$$\zeta H \zeta^* = \begin{pmatrix} \zeta_0 & \zeta_1 & \dots & \zeta_n \end{pmatrix} \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix} \begin{pmatrix} \zeta_0^* \\ \zeta_1^* \\ \dots \\ \zeta_n^* \end{pmatrix} =$$

$$= \begin{pmatrix} \zeta_0 & -\zeta_1 & \dots & -\zeta_n \end{pmatrix} \begin{pmatrix} \zeta_0^* \\ \zeta_1^* \\ \dots \\ \zeta_n^* \end{pmatrix} = \zeta_0\zeta_0^* - \zeta_1\zeta_1^* - \dots - \zeta_n\zeta_n^* > 0 \Rightarrow$$

$$\Rightarrow \zeta_0\zeta_0^* > \zeta_1\zeta_1^* + \dots + \zeta_n\zeta_n^* \geqslant 0.$$

Providing $\zeta H \zeta^* > 0$, matrix ζ_0 is not degenerate since otherwise there would be a non-zero m-dimensional vector x such that $x\zeta_0 = 0$.

We have a contradiction since

$$0 = x\zeta_0\zeta_0^* x^* > x(\zeta_1\zeta_1^* + \dots + \zeta_n\zeta_n^*)x^* \geqslant 0.$$

3^0. Now we consider the following matrices

$$Z_k = \zeta_0^{-1}\zeta_k, \ \ k = 1, \dots, n.$$

We obtain the following inequality from condition $\zeta H\zeta^* > 0$

$$\zeta H\zeta^* = \begin{pmatrix} \zeta_0 & \zeta_1 & \dots & \zeta_n \end{pmatrix} \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix} \begin{pmatrix} \zeta_0^* \\ \zeta_1^* \\ \dots \\ \zeta_n^* \end{pmatrix} =$$

$$= \begin{pmatrix} \zeta_0 & -\zeta_1 & \dots & -\zeta_n \end{pmatrix} \begin{pmatrix} \zeta_0^* \\ \zeta_1^* \\ \dots \\ \zeta_n^* \end{pmatrix} = \zeta_0\zeta_0^* - \zeta_1\zeta_1^* - \dots - \zeta_n\zeta_n^* =$$

$$= \zeta_0\big(I - \zeta_0^{-1}\zeta_1\zeta_1^*(\zeta_0^*)^{-1} - \dots - \zeta_0^{-1}\zeta_n\zeta_n^*(\zeta_0^*)^{-1}\big)\zeta_0^* =$$

$$= \zeta_0\left(I^{(m)} - Z_1Z_1^* - \dots - Z_nZ_n^*\right)\zeta_0^* = \zeta_0\big(I - \langle Z, Z\rangle\big)\zeta_0^* > 0$$

$$\Rightarrow I^{(m)} - \langle Z, Z\rangle > 0, \text{ i.e., } Z \in B_{m,n}^{(2)}.$$

4^0. Using (3) we consider the vector

$$\omega = (\omega_0, \omega_1, \dots, \omega_n) = \zeta A$$

and multiply the block matrix by the right of the above-mentioned formula

$$\widetilde{A} = \begin{pmatrix} A_{00}^* & -A_{10}^* & \dots & -A_{n0}^* \\ -A_{01}^* & A_{11}^* & \dots & A_{n1}^* \\ \dots & \dots & \dots & \dots \\ -A_{0n}^* & A_{1n}^* & \dots & A_{nn}^* \end{pmatrix}.$$

Note that the product of block matrices is carried out according to the usual rules for the product of matrices. Since (4) is equivalent to the condition $A\widetilde{A} = I^{(m(n+1))}$, then we have

$$\omega\widetilde{A} = \zeta,$$

i.e., map (3) is invertible (under condition (2)) and the matrix defines the inverse map.

Hence,

$$\omega H\omega^* = \zeta AHA^*\zeta^* = \zeta H\zeta^* > 0. \tag{5}$$

5^0. Now we prove that map W_k is an automorphism. Obviously,

$$\omega H\omega^* = \begin{pmatrix} \omega_0 & \omega_1 & \dots & \omega_n \end{pmatrix} \begin{pmatrix} I^{(m)} & 0 & \dots & 0 \\ 0 & -I^{(m)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -I^{(m)} \end{pmatrix} \begin{pmatrix} \omega_0^* \\ \omega_1^* \\ \dots \\ \omega_n^* \end{pmatrix} =$$

$$= \begin{pmatrix} \omega_0 & -\omega_1 & \dots & -\omega_n \end{pmatrix} \begin{pmatrix} \omega_0^* \\ \omega_1^* \\ \dots \\ \omega_n^* \end{pmatrix} = \omega_0\omega_0^* - \omega_1\omega_1^* - \dots - \omega_n\omega_n^* =$$

$$= \omega_0\big(I - \omega_0^{-1}\omega_1\omega_1^*(\omega_0^*)^{-1} - \dots - \omega_0^{-1}\omega_n\omega_n^*(\omega_0^*)^{-1}\big)\omega_0^* =$$

$$= \omega_0\left(I^{(m)} - W_1W_1^* - \dots - W_nW_n^*\right)\omega_0^* = \omega_0\big(I - \langle W, W\rangle\big)\omega_0^* > 0 \Rightarrow I - \langle W, W\rangle > 0.$$

Then transformation (3) generates a linear-fractional transformation

$$W_k = \omega_0^{-1}\omega_k = \left(\sum_{j=0}^{n}\zeta_j A_{j0}\right)^{-1}\left(\sum_{j=0}^{n}\zeta_j A_{jk}\right) = \left(\zeta_0 A_{00} + \sum_{j=1}^{n}\zeta_j A_{j0}\right)^{-1}\left(\zeta_0 A_{0k} + \sum_{j=1}^{n}\zeta_j A_{jk}\right) =$$
$$= \left(A_{00} + \sum_{j=1}^{n}\zeta_0^{-1}\zeta_j A_{j0}\right)^{-1}\zeta_0^{-1}\zeta_0\left(A_{0k} + \sum_{j=1}^{n}\zeta_0^{-1}\zeta_j A_{jk}\right) =$$
$$= \left(A_{00} + \sum_{j=1}^{n} Z_j A_{j0}\right)^{-1}\left(A_{0k} + \sum_{j=1}^{n} Z_j A_{jk}\right), \quad k = 1, \ldots, n.$$

6^0. Let us show that matrices $W_k, k = 1, \ldots, n$ are symmetric matrices. Let $W_k = \omega_0^{-1}\omega_k$ then $W_k' = \omega_k'(\omega_0')^{-1}$ and

$$W_k - W_k' = \omega_0^{-1}\omega_k - \omega_k'(\omega_0')^{-1} = \omega_0^{-1}(\omega_k\omega_0' - \omega_0\omega_k')(\omega_0')^{-1};$$

$$\omega_k\omega_0' - \omega_0\omega_k' = \sum_{j=0}^{n}\zeta_j A_{jk}\sum_{j=0}^{n} A_{j0}'\zeta_j' - \sum_{j=0}^{n}\zeta_j A_{j0}\sum_{j=0}^{n} A_{jk}'\zeta_j' =$$
$$= (\zeta_0 A_{0k} + \zeta_1 A_{1k} + \cdots + \zeta_n A_{nk})(A_{00}'\zeta_0' + A_{10}\zeta_1' + \cdots + A_{n0}'\zeta_n') -$$
$$-(\zeta_0 A_{00} + \zeta_1 A_{10} + \cdots + \zeta_n A_{n0})(A_{0k}'\zeta_0' + A_{1k}\zeta_1' + \cdots + A_{nk}'\zeta_n') =$$
$$= \zeta_0(A_{0k}A_{00}' - A_{00}A_{0k}')\zeta_0' + \zeta_0(A_{0k}A_{10}' - A_{10}A_{1k}')\zeta_1' + \cdots +$$
$$+\zeta_0(A_{0k}A_{n0}' - A_{00}A_{0k}')\zeta_n' + \zeta_1(A_{1k}A_{00}' - A_{10}A_{0k}')\zeta_0' +$$
$$+\zeta_1(A_{1k}A_{10}' - A_{10}A_{1k}')\zeta_1' + \cdots + \zeta_1(A_{1k}A_{n0}' - A_{10}A_{nk}')\zeta_n' + \cdots +$$
$$+\zeta_n(A_{nk}A_{00}' - A_{n0}A_{0k}')\zeta_0' + \zeta_n(A_{nk}A_{10}' - A_{n0}A_{1k}')\zeta_1' + \cdots +$$
$$+\zeta_n(A_{nk}A_{n0}' - A_{n0}A_{nk}')\zeta_n' = 0.$$

The last equality is valid by virtue of (2).

Theorem 1 is proved. □

Further, using relation $A\widetilde{A} = I^{(m(n+1))}$, we obtain $\widetilde{A}A = I^{(m(n+1))}$. It means that

$$\widetilde{A}A = \begin{pmatrix} A_{00}^* & -A_{10}^* & \ldots & -A_{n0}^* \\ -A_{01}^* & A_{11}^* & \ldots & A_{n1}^* \\ \ldots & \ldots & \ldots & \ldots \\ -A_{0n}^* & A_{1n}^* & \ldots & A_{nn}^* \end{pmatrix}\begin{pmatrix} A_{00} & A_{01} & \ldots & A_{0n} \\ A_{10} & A_{11} & \ldots & A_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ A_{n0} & A_{n1} & \ldots & A_{nn} \end{pmatrix} = I^{(m(n+1))},$$

$$\begin{pmatrix} A_{00}^*A_{00} - \cdots - A_{n0}^*A_{n0} & A_{00}^*A_{01} - \cdots - A_{n0}^*A_{n1} & \ldots & A_{00}^*A_{0n} - \cdots - A_{n0}^*A_{nn} \\ -A_{01}^*A_{00} + \cdots + A_{n1}^*A_{n0} & -A_{01}^*A_{01} + \cdots + A_{n1}^*A_{n1} & \ldots & -A_{01}^*A_{0n} + \cdots + A_{n1}^*A_{nn} \\ \ldots & \ldots & \ldots & \ldots \\ -A_{0n}^*A_{00} + \cdots + A_{nn}^*A_{n0} & -A_{0n}^*A_{01} + \cdots + A_{nn}^*A_{n1} & \ldots & -A_{0n}^*A_{0n} + \cdots + A_{nn}^*A_{nn} \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} A_{00}^*A_{00} - \sum\limits_{j=1}^{n} A_{j0}^*A_{j0} = I^{(m)}, \\ A_{0k}^*A_{0j} = \sum\limits_{s=1}^{n} A_{sk}^*A_{sj}, \quad j \neq k, \\ A_{0k}^*A_{0k} - \sum\limits_{j=1}^{n} A_{jk}^*A_{jk} = -I^{(m)} \end{cases}. \tag{6}$$

Now let the point $P = (P_1, \dots, P_n) \in B^{(2)}_{m,n}$. Let us consider the mapping

$$W_k = R^{-1}(I^{(m)} - \langle Z, P\rangle)^{-1}\sum_{s=1}^{n}(Z_s - P_s)G_{sk},\ k = 0, 1, \dots, n \tag{7}$$

that transfers the point P to 0, where R, G_{sk} are arbitrary matrices.

Theorem 2. *For a mapping of form (7) to be an automorphism of a matrix ball of the second type it is necessary and sufficient that matrices R and G satisfy the following relations*

$$R^*(I^{(m)} - \langle P, P\rangle)R = I^{(m)}, \quad G^*(I^{(mn)} - P^*P)G = I^{(mn)}, \tag{8}$$

where G is a block matrix.

Proof. Necessity. Let mapping of form (7) be an automorphism of the matrix ball $B^{(2)}_{m,n}$ that maps the point P to 0. We have that

$$\begin{gathered} A_{00} = R, \quad A_{j0} = -P_j^* R, \quad j = 1, \dots, n, \\ A_{jk} = G_{jk}, \quad j, k = 1, \dots, n, \\ A_{0k} = -\sum_{s=1}^{n} P_s G_{jk}, \quad k = 1, \dots, n, \end{gathered} \tag{9}$$

$$\begin{aligned} (1) \Rightarrow W_k &= \left(A_{00} + \sum_{j=1}^{n} Z_j A_{j0}\right)^{-1}\left(A_{0k} + \sum_{j=1}^{n} Z_j A_{jk}\right) = \\ &= \left(R - \sum_{j=1}^{n} Z_j P_j^* R\right)^{-1}\left(-\sum_{s=1}^{n} P_s G_{sk} + \sum_{j=1}^{n} Z_j G_{jk}\right) = \\ &= R^{-1}\left(I - \langle Z, P\rangle\right)^{-1}\sum_{s=1}^{n}\left(Z_s - P_s\right)G_{sk}. \end{aligned}$$

Taking into account (6) and (9), we obtain (8)

$$\begin{gathered} R^*R - \sum_{j=1}^{n} R^* P_j P_j^* R = I^{(m)} \;\Rightarrow\; R^*(I^{(m)} - \langle P, P\rangle)R = I^{(m)}, \\ \sum_{s=1}^{n} G^*_{sk} P^*_s \sum_{s=1}^{n} P_s G_{sk} - \sum_{j=1}^{n} G^*_{jk} G_{jk} = -I^{(m)}, \\ \sum_{s=1}^{n} G^*_{sk} P^*_s \sum_{s=1}^{n} P_s G_{sj} = \sum_{s=1}^{n} G^*_{sk} G_{sk}, j \neq k, \\ G^*(I^{(mn)} - P^*P)G = I^{(mn)}, \end{gathered}$$

$$G = \begin{pmatrix} G_{11} & G_{12} & \dots & G_{1n} \\ G_{21} & G_{22} & \dots & G_{2n} \\ \dots & \dots & \dots & \dots \\ G_{n1} & G_{n2} & \dots & G_{nn} \end{pmatrix}, \quad P^*P = \begin{pmatrix} P_1^*P_1 & P_1^*P_2 & \dots & P_1^*P_n \\ P_2^*P_1 & P_2^*P_2 & \dots & P_2^*P_n \\ \dots & \dots & \dots & \dots \\ P_n^*P_1 & P_n^*P_2 & \dots & P_n^*P_n \end{pmatrix}.$$

Sufficiency. Sufficiency of the theorem follows from the existence of matrices R, G_{sk} that satisfy (8). Substituting (9) into (6), we obtain (7).

Theorem 2 is proved. □

3. Volumes of a matrix ball of the second type and its skeleton

The volume of a matrix ball of the second type is calculated with the use of the following theorem [22].

Theorem 3. *Let $m \geqslant 2$ and $Z_\nu[m \times m]$ be a symmetric matrix. Let us consider the integral*

$$J(\lambda) = \int\limits_{I - \langle Z, Z\rangle > 0} [det(I - \langle Z, Z\rangle)]^\lambda \dot{Z},$$

where $\dot{Z} = \prod\limits_{i=1}^{m} \prod\limits_{j=1}^{mn} dx_{ij} dy_{ij}, x_{ij} + iy_{ij} = z_{ij}$. Then

$$J(\lambda) = \frac{\pi^{\frac{m(m+1)}{2}n}}{(\lambda+1)\dots(\lambda+mn)} \cdot \frac{\Gamma(2\lambda+3)\Gamma(2\lambda+5)\dots\Gamma(2\lambda+2mn-1)}{\Gamma(2\lambda+mn+2)\Gamma(2\lambda+mn+3)\dots\Gamma(2\lambda+2mn)}.$$

In particular, when $\lambda = 0$ the volume of a matrix ball of the second type is

$$V(B^{(2)}_{m,n}) = \frac{\pi^{\frac{m(m+1)}{2}n}}{m!} \cdot \frac{2!4!\dots(2mn-3)!}{(mn+1)!(mn+2)!\dots(2mn-1)!}. \tag{10}$$

In particular, when $n = 1$ we obtain from (10) the well-known formula for the volume of classical domain of the second type (see [8]).

Now Let us calculate the volume of the skeleton $X^{(2)}_{m,n}$ of the matrix ball of the second type $B^{(2)}_{m,n}$.

Theorem 4. *The volume of the skeleton of a matrix ball of the second type is calculated as follows*

$$V\left(X^{(2)}_{m,n}\right) = (2\pi)^{\frac{nm(m+1)}{2}} \left(\frac{D(l_1,\dots,l_m)}{1!2!\dots(m-1)! \prod\limits_{1\leqslant s\leqslant j\leqslant m} (l_s+l_j+2)} \right)^n,$$

where

$$D(l_1,l_2,\dots,l_m) = \prod_{1\leqslant s<j\leqslant m} (l_s - l_j), 1 \leqslant l_k \leqslant m$$

and $l_1 + l_2 + \dots + l_m = \dfrac{m(m+1)}{2}$.

Proof. Let $U = (U_1,\dots,U_n) \in X^{(2)}_{m,n}$ and each matrix $U_k,\ k = 1,\dots,n$ is a symmetric matrix. It is known ([8, 25]) that for any symmetric matrix $Z_\nu \in \mathbb{C}[m\times m]$ there exists a unitary matrix $U_\nu \in U(m)$ ($U(m)$ are set classes of the unitary matrices group) and real numbers $\lambda_1^{(\nu)} \geqslant \lambda_2^{(\nu)} \geqslant \dots \geqslant \lambda_m^{(\nu)} \geqslant 0$ such that

$$Z_\nu = U_\nu diag\left(\lambda_1^{(\nu)},\dots,\lambda_m^{(\nu)}\right) U'_\nu = U_\nu \Lambda_\nu U'_\nu, \Lambda_\nu = \begin{pmatrix} \lambda_1^\nu & 0 & \dots & 0 \\ 0 & \lambda_2^\nu & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_m^\nu \end{pmatrix}, \quad \nu = 1,\dots,n. \tag{12}$$

By differentiating (12), we obtain

$$dZ_\nu = dU_\nu \Lambda_\nu U'_\nu + U_\nu d\Lambda_\nu U'{}_\nu + U_\nu \Lambda_\nu dU'{}_\nu .$$

Introducing $\delta U_\nu = U_\nu^* dU_\nu$, we have

$$U_\nu^* dZ_\nu \bar{U}_\nu = \delta U_\nu \Lambda_\nu + d\Lambda_\nu + \Lambda_\nu \delta U'{}_\nu .$$

Next we have

$$Sp\,(dZ_\nu \cdot dZ_\nu^*) = Sp\,(U_\nu^* dZ_\nu \cdot U_\nu U_\nu^* \cdot dZ_\nu^* \cdot U_\nu) =$$
$$= Sp\left\{(\delta U_\nu \cdot \Lambda_\nu + d\Lambda_\nu + \Lambda_\nu \delta U'{}_\nu)\left(\Lambda_\nu \delta U_\nu^* + d\Lambda_\nu + \delta \bar{U}_\nu \cdot \Lambda_\nu\right)\right\} =$$
$$= Sp\,(d\Lambda_\nu \cdot d\Lambda_\nu) + Sp\left\{(\delta U_\nu \cdot \Lambda_\nu + \Lambda_\nu \delta U'{}_\nu)\left(\Lambda_\nu \delta U_\nu^* + \delta \bar{U}_\nu \cdot \Lambda_\nu\right)\right\}.$$

Let us set

$$\delta U_\nu \cdot \Lambda_\nu + \Lambda_\nu \delta U'{}_\nu = \left(dg_{jk}^{(\nu)}\right),\ \left(dg_{jk}^{(\nu)} = dg_{kj}^{(\nu)}\right),$$

then

$$Sp\,(dZ_\nu \cdot dZ_\nu^*) = \sum_{j=1}^{n} d\left(\lambda_j^{(\nu)}\right)^2 + \sum_{j=1}^{n} \left|dg_{jj}^{(\nu)}\right|^2 + 2\sum_{j<k}^{n} \left|dg_{jk}^{(\nu)}\right|^2,$$

where

$$dg_{jk}^{(\nu)} = \lambda_k^{(\nu)} \delta u_{jk}^{(\nu)} + \lambda_j^{(\nu)} u_{kj}^{(\nu)}, \quad j<k,$$
$$dg_{jj}^{(\nu)} = 2i\lambda_j^{(\nu)} \delta u_{jj}^{(\nu)}.$$

Now to define the volume element $\left\{\dot{U}_\nu\right\}$ of the set $U\,(m)$ we set $\delta u_{jk}^{(\gamma)} = \delta u'{}_{jk} + i\delta u''{}_{jk}$. Then we have

$$\dot{U}_\nu = 2^{\frac{m(m-1)}{2}} \prod_{j=1}^{n} \delta u''{}_{jj} \prod_{j<k} \delta u'{}_{jk} \cdot \delta u''{}_{jk}.$$

Thus

$$\dot{Z}_\nu = 2^m \prod_{j<k} \left|\left(\lambda_j^{(\nu)}\right)^2 - \left(\lambda_k^{(\nu)}\right)^2\right| \lambda_1^{(\nu)} \dots \lambda_m^{(\nu)} d\lambda_1^{(\nu)} \dots d\lambda_m^{(\nu)} \dot{U}_\nu, (\lambda_j^{(\nu)} \neq \lambda_k^{(\nu)}, \nu = 1, 2, \dots, n). \tag{13}$$

For any $Z = (Z_1, \dots, Z_n) \in X_{m,n}^{(2)}$ we have $\det\left(I^{(m)} - \langle Z, Z\rangle\right) = 0$. On the other hand, the correspondence Z_ν and $U\,(m) \times \Lambda_\nu$ is one-to-one correspondence for all matrices

$$Z = (U_1 \Lambda_1 U'{}_1, \dots, U_n \Lambda_n U'{}_n) \in X_{m,n}^{(2)}. \tag{14}$$

Then it follows from (12) and (14) that

$$\left(\lambda_1^{(\nu)}\right)^2 + \left(\lambda_2^{(\nu)}\right)^2 + \dots + \left(\lambda_m^{(\nu)}\right)^2 = 1, \quad \nu = 1, 2, \dots, n. \tag{15}$$

Then, using Fubini's theorem for calculation of the volume of the skeleton $X_{m,n}^{(2)}$, we obtain

$$V\left(X_{m,n}^{(2)}\right) = \int\limits_{X_{m,n}^{(2)}} \dot{Z} =$$

$$= 2^{mn} \int\limits_{\{U_1\}\times\{\Lambda_1\}} \prod_{j<k} \left|\left(\lambda_j^{(1)}\right)^2 - \left(\lambda_k^{(1)}\right)^2\right| \lambda_1^{(1)} \dots \lambda_m^{(1)} d\lambda_1^{(1)} \dots d\lambda_m^{(1)} \dot{U}_1 \times$$

$$\times\cdots\times\int\limits_{\{U_n\}\times\{\Lambda_n\}}\prod_{j<k}\left|\left(\lambda_j^{(n)}\right)^2-\left(\lambda_k^{(n)}\right)^2\right|\lambda_1^{(n)}\ldots\lambda_m^{(n)}d\lambda_1^{(n)}\ldots d\lambda_m^{(n)}\dot{U}_n=$$

$$=\int\limits_{\{U_1\}}\dot{U}_1\int\limits_{\left(\lambda_1^{(1)}\right)^2+\left(\lambda_2^{(1)}\right)^2+\cdots+\left(\lambda_m^{(1)}\right)^2<1}2^m\prod_{j<k}\left|\left(\lambda_j^{(1)}\right)^2-\left(\lambda_k^{(1)}\right)^2\right|\lambda_1^{(1)}\ldots\lambda_m^{(1)}d\lambda_1^{(1)}\ldots d\lambda_m^{(1)}\times$$

$$\times\cdots\times\int\limits_{\{U_n\}}\dot{U}_n\int\limits_{\left(\lambda_1^{(n)}\right)^2+\left(\lambda_2^{(n)}\right)^2+\cdots+\left(\lambda_m^{(n)}\right)^2<1}2^m\prod_{j<k}\left|\left(\lambda_j^{(n)}\right)^2-\left(\lambda_k^{(n)}\right)^2\right|\lambda_1^{(n)}\ldots\lambda_m^{(n)}d\lambda_1^{(n)}\ldots d\lambda_m^{(n)}.$$

It is known (Theorem 3.1.1 in [8]) that volume of the manifold $\{U_\nu(m)\}$ of unitary matrices is calculated by the following formula

$$V\left(U_\nu(m)\right)=\frac{(2\pi)^{\frac{m(m+1)}{2}}}{1!2!\ldots(m-1)!}.$$

Providing $\lambda_1^{(\nu)}>\lambda_2^{(\nu)}>\cdots>\lambda_m^{(\nu)}>0$ for all ν-th integral, we have

$$I_\nu=V\left(U_\nu\right)\int\limits_{\left(\lambda_1^{(\nu)}\right)^2+\left(\lambda_2^{(\nu)}\right)^2+\cdots+\left(\lambda_m^{(\nu)}\right)^2<1}2^m\prod_{j<k}\left|\left(\lambda_j^{(\nu)}\right)^2-\left(\lambda_k^{(\nu)}\right)^2\right|\lambda_1^{(\nu)}\ldots\lambda_m^{(\nu)}d\lambda_1^{(\nu)}\ldots d\lambda_m^{(\nu)}=$$

$$=2^mV\left(U_\nu\right)\int\limits_{\left(\lambda_1^{(\nu)}\right)^2+\left(\lambda_2^{(\nu)}\right)^2+\cdots+\left(\lambda_m^{(\nu)}\right)^2<1}\prod_{j<k}\left|\left(\lambda_j^{(\nu)}\right)^2-\left(\lambda_k^{(\nu)}\right)^2\right|\lambda_1^{(\nu)}\ldots\lambda_m^{(\nu)}d\lambda_1^{(\nu)}\ldots d\lambda_m^{(\nu)}=$$

$$=2^mV\left(U_\nu\right)\int\cdots\int\limits_{0<\lambda_m^{(\nu)}<\cdots<\lambda_2^{(\nu)}<\lambda_1^{(\nu)}<1}\prod_{j<k}\left|\left(\lambda_j^{(\nu)}\right)^2-\left(\lambda_k^{(\nu)}\right)^2\right|\lambda_1^{(\nu)}\ldots\lambda_m^{(\nu)}d\lambda_1^{(\nu)}\ldots d\lambda_m^{(\nu)}=$$

$$=(-1)^{\frac{m(m-1)}{2}}V\left(U_\nu\right)\int\cdots\int\limits_{0<\lambda_m^{(\nu)}<\cdots<\lambda_2^{(\nu)}<\lambda_1^{(\nu)}<1}\det\left|\left(\lambda_s^{(\nu)}\right)^{2l_j}\right|_{s,j=1}^m\cdot\lambda_1^{(\nu)}\ldots\lambda_m^{(\nu)}d\lambda_1^{(\nu)}\ldots d\lambda_m^{(\nu)}=$$

$$=\frac{V\left(U_\nu\right)D\left(l_1,\ldots,l_m\right)}{\prod\limits_{1\leqslant s\leqslant j\leqslant m}\left(l_s+l_j+2\right)},$$

where the following conditions are satisfied

$$D(l_1,l_2,\ldots,l_m)=\prod_{1\leqslant s<j\leqslant m}(l_s-l_j),\quad 1\leqslant l_k\leqslant m$$

and $l_1+l_2+\cdots+l_m=\dfrac{m\left(m+1\right)}{2}$.

Here lemma from [8] (page 135) was used. Hence, we obtain relation from the statement of the theorem:

$$V\left(X_{m,n}^{(2)}\right)=(2\pi)^{\frac{nm(m+1)}{2}}\left(\frac{D\left(l_1,\ldots,l_m\right)}{1!2!\ldots(m-1)!\prod\limits_{1\leqslant s\leqslant j\leqslant m}\left(l_s+l_j+2\right)}\right)^n.$$

Theorem 4 is proved. □

Note that when $n = 1$ we obtain the formula for calculating the volume of skeleton of classical domain of the second type (see [8]).

The authors would like to thank professor G. Khudayberganov for his helpful advice in writing this paper.

References

[1] V.S.Vladimirov, Methods of functions of several complex variables, Nauka, Moscow, 1964.

[2] B.V.Shabat, Introduction to Complex Analysis Part II (Functions of Several Variables), Moscow, Nauka, Physical and mathematical literature, 1985.

[3] W.Rudin, Function Theory in the Unit Ball of $\mathbb{C}^n$, New York, Berlin, Heidelberg, Springer–Verlag, 1980.

[4] S.G.Krantz. Harmonic and complex analysis in several variables, Springer Monographs in Mathematics, Gewerbestrasse 11, 6330 Cham, Switzerland, 2017.

[5] E.Cartan, Sur les domaines bornes homogenes de l'espace de n variables complexes, *Abh. Math. Sem. Univ. Hamburg*, **11**(1935), 116–162.

[6] C.L.Siegel, Automorphic functions of several complex variables, Moscow, Inostrannaya Literatura, 1954.

[7] I.I.Pjateckii–Sapiro, Geometry of classical domains and the theory of automorphic functions, Moscow, Fizmatgiz, 1961.

[8] Hua Luogeng, Harmonic analysis of functions of several complex variables, in classical domains, Moscow, IL, 1963.

[9] A.G.Sergeev, On matrix and Reinhardt domains, Preprint, Inst. Mittag–Leffler, Stockholm, 1988.

[10] G.Khudayberganov, Spectral decomposition for functions from several matrices, TashSU, Preprint, Tashkent, 1993 (in Russian).

[11] G.Khudayberganov, B.B.Khidirov, U.S.Rakhmonov, Automorphisms of matrix balls, *Acta NUUz*, 2010, no. 3, 205–210 (in Russian).

[12] G.Khudayberganov, A.M.Kytmanov, B.A.Shaimkulov, Complex analysis in matrix domains, Monograph. Krasnoyarsk, Siberian Federal University, 2011 (in Russian).

[13] G.Khudayberganov, A.M.Khalknazarov, J.Sh.Abdullayev, Laplace and Hua Luogeng operators, *Russian Mathematics (Izvestiya VUZ. Matematika)*, **64**(2020), no. 3, 66–71. DOI: 10.26907/0021-3446-2020-3-74-79

[14] S.Kosbergenov, Holomorphic automorphisms and the Bergman integral for a matrix ball, *Dokl. AN RUz.*, 1998, no. 1, 7–10 (In Russian).

[15] S.Kosbergenov, On a multidimensional boundary Morera theorem for the matrix ball, *Izv. Vyssh. Uchebn. Zaved. Mat.*, (2001), no. 4, 28–32.

[16] A.Khalknazarov, The volume of the matrix ball in the space of matrices, *Uzbek Mathematical Journal*, (2012), no. 3, 135–139 (in Russian).

[17] S.Kosbergenov, On the Carleman formula for a matrix ball, *Russian Math. (Iz. VUZ)*, **43**(1999), no. 1, 72–75. DOI: 10.26907/0021-3446-2020-3-74-79

[18] G.Khudayberganov, U.S.Rakhmonov, Z.Q.Matyakubov, Integral formulas for some matrix domains, *Contemporary Mathematic, AMS*, Volume **662**(2016), 89–95.
DOI: 10.1090/conm/662

[19] G.Khudayberganov, U.S.Rakhmonov, The Bergman and Cauchy-Szegő kernels for matrix ball of the second type, *Journal of Siberian Federal University. Mathematics* & *Physics*, **7**(2014), no. 3, 305–310.

[20] G.Khudayberganov, B.P.Otemuratov, U.S.Rakhmonov, Boundary Morera theorem for the matrix ball of the third type, *Journal of Siberian Federal University. Mathematics & Physics*, **11**(2018), no. 1, 40–45. DOI: 10.17516/1997-1397-2018-11-1-40-45

[21] G.Khudayberganov, U.S.Rakhmonov, Carleman Formula for Matrix Ball of the Third Type, Algebra, In: Complex Analysis, and Pluripotential Theory. USUZCAMP 2017, Springer Proceedings in Mathematics & Statistics, vol. 264, Springer, Cham., 2018, 101–108.
DOI: 10.1007/978-3-030-01144-4_9

[22] U.S.Rakhmonov, J.Sh.Abdullayev, On volumes of matrix ball of third type and generalized Lie balls, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki*, **29**(2019), no. 4, 548–557. DOI: 10.20537/vm190406

[23] G.Khudayberganov, J.Sh.Abdullayev, Relationship between the Kernels Bergman and Cauchy-Szegő in the domains $\tau^{+}(n-1)$ and $\Re_{IV}^{n}$, *Journal of Siberian Federal University. Mathematics* & *Physics*, **13**(2020), no. 5, 559–567.
DOI: 10.17516/1997-1397-2020-13-5-559-567

[24] Th.Th.Voronov, On volumes of classical supermanifolds, *Sbornik: Mathematics*, **207**(2016), no. 11, 1512–1536. DOI:10.4213/sm8705

[25] Hua Luogeng, On the theory of automorphic functions of a matrix variable I-geometrical basis, *American Journal of Mathematics*, **66**(1944), no. 3, 470–488.

[26] G.Khudayberganov, J.Sh.Abdullayev, Holomorphic continuation into a matrix ball of functions defined on a piece of its skeleton, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki,* **31**(2021), no. 2, 296–310.

[27] B.T.Kurbanov, Morera's theorem for some unbounded domains, Proceedings of the international conference "Mathematical models and methods of their research", Krasnoyarsk, Vol. 2, 2001, 49–51.

[28] G.Khudayberganov, Z.K.Matyakubov, Integral formulas in Siegel domains, *Uzbek mathematical journal*, 2013, no. 3, 116–128 (in Russian).

[29] G.Khudayberganov, Z.K.Matyakubov, Bergman's formula for an unbounded matrix domain, *Vestnik NUUz*, 2013, no. 4.1, 36–39 (in Russian).

[30] J.Sh.Abdullayev, An analogue of Bremermann's theorem on finding the Bergman kernel for the Cartesian product of the classical domains $\Re_I(m,k)$ and $\Re_{II}(n)$, *Bul. Acad. Ştiinţe Repub. Mold. Mat.*, 2020, no. 3, 88–96

[31] G.Khudayberganov, J.Sh.Abdullayev, The boundary Morera theorem for domain $\tau^+(n-1)$, *Ufimsk. Mat. Zh.*, **13**(2021), no. 3, 196–210.

[32] Dzh.K.Tishabaev, Invariant metrics and indicatrices of bounded domains in $\mathbb{C}^n$, *Siberian Math. J.,* **30**(1989), no. 1, 166–168.

О свойствах матричного шара второго типа $B^{(2)}_{m,n}$ из пространства $\mathbb{C}^n[m \times m]$

Уктам С. Рахмонов
Ташкентский государственный технический университет
Ташкент, Узбекистан
Джанибек Ш. Абдуллаев
Национальный университет Узбекистана
Ташкент, Узбекистан

Аннотация. В этой работе дано описание автоморфизмов матричного шара $B^{(2)}_{m,n}$, ассоциированных с классическими областями второго типа, также изучены некоторые свойства матричного шара второго типа.

Ключевые слова: классическая область, матричный шар, автоморфизм матричного шара, объем, границы Шилова.

DOI: 10.17516/1997-1397-2022-15-3-343-355
УДК 517

Coupled Fixed Point Theorems Via Mixed Monotone Property in A_b-metric Spaces & Applications to Integral Equations

K. Ravibabu*
Department of Mathematics, G.M.R.I.T
Rajam, Srikakulam, India
G. N. V. Kishore†
Department of Engineering Mathematics & Humanities
Sagi Rama Krishnam Raju Engineering College
Chinamiram, Bhimavaram, Andhra Pradesh, India
Ch. Srinivasa Rao‡
Department of Mathematics, Andhra University
Visakhapatnam, India
Ch. Raghavendra Naidu§
Department of Mathematics, Govt. Degree College
Palakonda, Srikakulam, India

Received 01.10.2021, received in revised form 29.11.2021, accepted 20.02.2022

Abstract. In this paper, we establish some results on the existence and uniqueness of coupled common fixed point theorems in partially ordered A_b-metric spaces. Examples have been provided to justify the relevance of the results obtained through the analysis of extant theorem. Further, we also find application to integral equations via fixed point theorems in A_b-metric spaces.

Keywords: Coupled fixed point, Mixed weakly monotone property, A_b-metric space, Integral equation.

Citation: K. Ravibabu, G.N.V. Kishore, Ch. Srinivasa Rao, Ch. Raghavendra Naidu, Coupled Fixed Point Theorems Via Mixed Monotone Property in A_b-metric Spaces & Applications to Integral Equations, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 343–355. DOI: 10.17516/1997-1397-2022-15-3-343-355.

1. Introduction and preliminaries

The study of fixed point theory comes from wider area of non-linear function analysis. However, its study began almost a century ago in the field of algebraic topology. Fixed point theorems find applications in proving the existence and uniqueness of the solutions of certain differential and integral equations that arise in physical, engineering and other optimization problems. In the study of fixed point theory, some of the generalizations of metric space are 2-metric space, D-metric space, D^*-metric space, G-metric space, S-metric space, Rectangular metric or metric-like space, Partial metric space, Cone metric space. In 1989, I. A. Bakhtin [2] introduced the concept

*ravibabu.k@gmrit.edu.in
†gnvkishore@srkrec.ac.in, kishore.apr2@gmail.com https://orcid.org/0000-0001-7260-4071
‡drcsr41@yahoo.com
§ch.rvnaidu@gmail.com

of b-metric space. Consequent upon the introduction of b-metric space, many generalizations of metric spaces came into existence. In 2015, M. Abbas et al. [1] introduced the concept of n-tuple metric space and studied its topological properties. M. Ughade et al. [15] introduced the notion of A_b-metric spaces as a generalized form of n-tuple metric space. Subsequently N. Mlaiki et al. [11] obtained unique coupled common fixed point theorems in partially ordered A_b-metric spaces.

In this paper, we use the notion of a mixed weakly monotone pair of maps to state a coupled common fixed point theorem on partially ordered A_b-metric spaces. We prove some unique coupled common fixed point theorems in partially ordered A_b-metric space and also provide example to support our results.

First we recall some notions, lemmas and examples which will be useful to prove our results.

Definition 1.1 (M. Abbas et al. [1]). *Let $\Im$ be a non empty set and $n(\geqslant 2)$ be a positive integer. A function $A : \Im^n \to [0,\infty)$ is called an A-metric on $\Im$, if for any ζ_i, $a \in \Im$. $i = 1,2,\dots,n$, the following conditions hold.*

(i) $A(\zeta_1,\zeta_2,\dots,\zeta_{n-1},\zeta_n) \geqslant 0$,

(ii) $A(\zeta_1,\zeta_2,\dots,\zeta_{n-1},\zeta_n) = 0$ *if and only if* $\zeta_1 = \zeta_2 = \dots = \zeta_{n-1} = \zeta_n$,

(iii) $$A(\zeta_1,\zeta_2,\dots\zeta_{n-1},\zeta_n) \leqslant [A(\zeta_1,\zeta_1,\dots,{\zeta_1}_{(n-1)},a) + A(\zeta_2,\zeta_2,\dots,{\zeta_2}_{(n-1)},a) + \\ + \dots + A(\zeta_{n-1},\zeta_{n-1},\dots,{\zeta_{n-1}}_{(n-1)},a) + A(\zeta_n,\zeta_n,\dots,{\zeta_n}_{(n-1)},a)].$$

The pair $(\Im, A)$ is called an A-metric space.

Definition 1.2 (T. G. Bhaskar et al. [6]). *Let X be a non empty set. A b-metric on X is a function $d : X^2 \to [0,\infty)$ such that the following conditions hold for all $x,y,z \in X$.*

(i) $d(x,y) = 0 \iff x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) there exists $s \geqslant 1$, *such that* $d(x,z) \leqslant s[d(x,y) + d(y,z)]$.

The pair (X,d) is called a b-metric space.

Definition 1.3 (M. Ughade et al. [14]). *Let $\Im$ be a non empty set and $n \geqslant 2$. Suppose $b \geqslant 1$ is a real number. A function $A_b : \Im^n \to [0,\infty)$ is called an A_b -metric on $\Im$, if for any $\zeta_i, a \in \Im$, $i = 1,2,\dots,n$, the following conditions hold.*

(i) $A_b(\zeta_1,\zeta_2,\dots,\zeta_{n-1},\zeta_n) \geqslant 0$,

(ii) $A_b(\zeta_1,\zeta_2,\dots,\zeta_{n-1},\zeta_n) = 0$ *if and only if* $\zeta_1 = \zeta_2 = \dots = \zeta_{n-1} = \zeta_n$,

(iii) $$A_b(\zeta_1,\zeta_2,\dots,\zeta_{n-1},\zeta_n) \leqslant b[A_b(\zeta_1,\zeta_1,\dots,{\zeta_1}_{(n-1)},a) + A_b(x_2,x_2,\dots,{x_2}_{(n-1)},a) + \dots \\ + A_b(\zeta_{n-1},\zeta_{n-1},\dots,{\zeta_{n-1}}_{(n-1)},a) + A_b(\zeta_n,\zeta_n,\dots,{\zeta_n}_{(n-1)},a)].$$

The pair $(\Im, A_b)$ is called an A_b-metric space.

Note: In practice we write A for A_b when there is no confusion.

Example 1.4 (M. Ughade et al. [14]). *Let $\Im = [1,\infty)$ and $n \geqslant 2$. Define $A_b : \Im^n \to [1,\infty)$ by $A_b(\zeta_1,\zeta_2,\dots,\zeta_{n-1},\zeta_n) = \sum_{i=1}^{n}\sum_{i<j}|\zeta_i - \zeta_j|^2$, for all $\zeta_i \in \Im$, $i = 1,2,\dots,n$. Then $(\Im, A_b)$ is an A_b-metric space with b=2.*

Lemma 1.5 (M. Ughade et al. [14]). *Let $(\Im, A)$ be A_b metric space, so that $A : \Im^n \to [0,\infty)$ for some $n \geqslant 2$. Then $A(\underbrace{\zeta,\zeta,\dots,\zeta}_{(n-1)times},y) \leqslant bA(\underbrace{y,y,\dots,y}_{(n-1)times},\zeta)$, for all $\zeta, y \in \Im$.*

Lemma 1.6 (M. Ughade et al. [14]). *Let $(\Im, A)$ be A_b metric space, so that $A : \Im^n \to [0,\infty)$ for some $n \geqslant 2$. Then $A(\underbrace{\zeta,\zeta,\dots,\zeta}_{(n-1)times},z) \leqslant (n-1)b\ A(\underbrace{\zeta,\zeta,\dots,\zeta}_{(n-1)times},y) + b^2\ A(\underbrace{y,y,\dots,y}_{(n-1)times},z)$, for all $\zeta, y, z \in \Im$.*

Lemma 1.7 (M. Ughade et al. [14]). *Let (X,A) be A_b metric space. Then (X^2, D_A) is A_b-metric space on $X \times X$ with D_A defined by*
$D_A((x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)) = A(x_1,x_2,\dots,x_n) + A(y_1,y_2,\dots,y_n)$ for all $x_i, y_i \in X$, $i,j = 1,2,\dots,n$.

Definition 1.8. *Let (X,A) be A_b-metric space. A sequence $\{x_n\}$ in X is said to converge to a point $x \in X$, if $A(\underbrace{x_n,x_n,\dots,x_n}_{(n-1)times},x) \to 0$ as $n \to \infty$. That is, to each $\varepsilon \geqslant 0$ there exist $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have $A(\underbrace{x_n,x_n,\dots,x_n}_{(n-1)times},x) \leqslant \varepsilon$ and we write $\lim\limits_{n\to\infty} x_n = x$.*

Note: x is called the limit of the sequence $\{x_n\}$.

Lemma 1.9 (N. Mlaiki et al. [11]). *Let (X,A) be A_b-metric space. If the sequence $\{x_n\}$ in X converges to a point x, then the limit x is unique.*

Definition 1.10. *Let (X,A) be A_b-metric space. A sequence $\{x_n\}$ in X is called a Cauchy sequence, if $A(\underbrace{x_n,x_n,\dots,x_n}_{(n-1)times},x_m) \to 0$ as $n,m \to \infty$. That is, to each $\varepsilon \geqslant 0$, there exists $N \in \mathbb{N}$ such that for all $n,m \geqslant N$, we have $A(\underbrace{x_n,x_n,\dots,x_n}_{(n-1)times},x_m) \leqslant \varepsilon$.*

Lemma 1.11 (N. Mlaiki et al. [11]). *Every convergent sequence in a A_b-metric space is a Cauchy sequence.*

Definition 1.12. *A A_b-metric space (X,A) is said to be complete, if every Cauchy sequence in X is convergent.*

Definition 1.13 (M. E. Gordji et al. [7]). *Let $(X, \leqslant)$ be a partially ordered set and $f,g : X \times X \to X$ be mappings. We say that (f,g) has the mixed weakly monotone property on X, if for any $x,y \in X$,*
$x \leqslant f(x,y),\ y \geqslant f(y,x) \implies f(x,y) \leqslant g((f(x,y),f(y,x)),\ f(y,x) \geqslant g((f(y,x),f(x,y))$
and
$x \leqslant g(x,y),\ y \geqslant g(y,x) \implies g(x,y) \leqslant f((g(x,y),g(y,x)),\ g(y,x) \geqslant f(g(y,x),g(x,y))$.

Definition 1.14. *Let X be a non-empty set and $f,g : X \times X \to X$ be maps on $X \times X$.*
(i) A point $(x,y) \in X \times X$ is called a coupled fixed pint of f, if $x = f(x,y)$ and $y = f(y,x)$
(ii) A point $(x,y) \in X \times X$ is said to be a common coupled fixed pint of f and g, if $x = f(x,y) = g(x,y)$ and $y = f(y,x) = g(y,x)$.

Note: (x,y) is said to be a Coupled coincidence point of f and g, if $f(x,y) = g(x,y)$ and $f(y,x) = g(y,x)$.

We observe that a common coupled fixed pint of f and g is necessarily a Coupled coincidence point of f and g.

2. Main results

Now we prove our first main result.

Theorem 2.1. *Let $(X, \leqslant, A)$ be a partially ordered, complete A_b-metric space and let $f, g : X \times X \to X$ be the mappings such that*

(i) the pair (f, g) has mixed weakly monotone property on X and there exists $x_0, y_0 \in X$ such that $x_0 \leqslant f(x_0, y_0)$, $f(y_0, x_0) \leqslant y_0$ or $x_0 \leqslant g(x_0, y_0)$, $g(y_0, x_0) \leqslant y_0$,

(ii) there is an α such that $\alpha b^2((n-1)b+1) < 1$ and

$$A(f(x,y), f(x,y), \ldots, f(x,y), g(u,v)) + A(f(y,x), f(y,x), \ldots, f(y,x), g(v,u)) \leqslant \alpha\, M,$$

where

$$\begin{aligned}M = \max\Bigg\{ &\Bigg[\big(1 + D((x,y),(x,y),\ldots,(x,y),(f(x,y),f(y,x)))\big)\times \\ &\times \frac{(D((u,v),(u,v),\ldots,(u,v),(g(u,v),g(v,u))))}{(1 + D((x,y),(x,y),\ldots,(x,y),(u,v)))}\Bigg], D\big((x,y),(x,y),\ldots,(x,y),(u,v)\big), \\ &\big(D((x,y),(x,y),\ldots (x,y),(f(x,y),f(y,x))) + D((u,v),(u,v),\ldots (u,v),(g(u,v),g(v,u)))\big), \\ &\big(D((u,v),(u,v),\ldots (u,v),(f(x,y),f(y,x))) + D((x,y),(x,y),\ldots (x,y),(g(u,v),g(v,u)))\big)\Bigg\}\end{aligned} \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \leqslant u$ and $y \geqslant v$,

(iii) if f or g is continuous.

Then f and g have a coupled common fixed point in X.

Proof. Let (x_0, y_0) be a given point in $X \times X$, satisfying (i). Write $x_1 = f(x_0, y_0)$, $y_1 = f(y_0, x_0)$, $x_2 = g(x_1, y_1)$, $y_2 = g(y_1, x_1)$. Define the sequences $\{x_n\}$ and $\{y_n\}$ inductively

$$\begin{aligned}x_{2n+1} = f(x_{2n}, y_{2n}),\ y_{2n+1} = f(y_{2n}, x_{2n}) \\ x_{2n+2} = g(x_{2n+1}, y_{2n+1}),\ y_{2n+2} = g(y_{2n+1}, x_{2n+1}) \\ \text{for all } n \in \mathbb{N}\end{aligned} \tag{2.2}$$

Since $x_0 \leqslant f(x_0, y_0)$ and $y_0 \geqslant f(y_0, x_0)$ and since (f, g) has mixed weakly monotone property, we have

$x_1 = f(x_0, y_0) \leqslant g(f(x_0, y_0), f(y_0, x_0)) = g(x_1, y_1) = x_2 \implies x_1 \leqslant x_2$

and $x_2 = g(x_1, y_1) \leqslant f(g(x_1, y_1), g(y_1, x_1)) = f(x_2, y_2) = x_3 \implies x_2 \leqslant x_3$

also $y_1 = f(y_0, x_0) \geqslant g(f(y_0, x_0), f(x_0, y_0)) = g(y_1, x_1) = y_2 \implies y_1 \geqslant y_2$

and $y_2 = f(y_1, x_1) \geqslant f(g(y_1, x_1), g(x_1, y_1)) = f(y_2, x_2) = y_3 \implies y_2 \geqslant y_3$.

By induction,

$$\begin{aligned}i.e,\ x_0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant x_{n+1} \leqslant \ldots \\ y_0 \geqslant y_1 \geqslant y_2 \geqslant \ldots \geqslant y_n \geqslant y_{n+1} \geqslant \ldots \\ \text{for all } n \in \mathbb{N}\end{aligned} \tag{2.3}$$

Now we show that these sequences are Cauchy.
Define $D_n : X \times X \to X$ by

$$\begin{aligned}D_n &= D((x_n, y_n), (x_n, y_n), \ldots, (x_n, y_n), (x_{n+1}, y_{n+1})) \\ &= A(x_n, x_n, \ldots, x_n, x_{n+1}) + A(y_n, y_n, \ldots, y_n, y_{n+1}) \text{ for all } x_i, y_i \in X, i, j = 1, 2, \ldots, n.\end{aligned}$$

Now

$$
\begin{aligned}
D_{2n+1} =& A(x_{2n+1}, x_{2n+1}, \dots, x_{2n+1}, x_{2n+2}) + A(y_{2n+1}, y_{2n+1}, \dots, y_{2n+1}, y_{2n+2}) = \\
=& A(f(x_{2n}, y_{2n}), f(x_{2n}, y_{2n}), \dots, f(x_{2n}, y_{2n}), g(x_{2n+1}, y_{2n+1})) + \\
&+ A(f(y_{2n}, x_{2n}), f(y_{2n}, x_{2n}), \dots, f(y_{2n}, x_{2n}), g(y_{2n+1}, x_{2n+1})) \leqslant
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \alpha \max \Big\{ \Big[\big(1 + D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \dots, (x_{2n}, y_{2n}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n})))\big) \times \\
&\times \frac{(D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), \dots, (x_{2n+1}, y_{2n+1}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1}))))}{(1 + D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \dots, (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})))} \Big], \\
&D\big((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \dots, (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})\big), \\
&\big(D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \dots, (x_{2n}, y_{2n}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n}))) + \\
&+ (D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), \dots, (x_{2n+1}, y_{2n+1}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1}))))\big), \\
&\big((D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), \dots, (x_{2n+1}, y_{2n+1}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n})))) + \\
&+ (D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \dots, (x_{2n}, y_{2n}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1}))))\big) \Big\} \leqslant \\
&\leqslant \alpha \max \{ D\big((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), \dots, (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})\big), \\
&D\big((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \dots, (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})\big), \\
&\big(D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \dots, (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) + \\
&+ (D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), \dots, (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})))\big), \\
&\big(D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), \dots, (x_{2n}, y_{2n}), (x_{2n+2}, y_{2n+2}))\big) \}.
\end{aligned}
$$

By using Lemma 1.6, we have

$$
\begin{aligned}
D_{2n+1} \leqslant \alpha \big\{ &(n-1)b[A(x_{2n}, x_{2n}, \dots, x_{2n}, x_{2n+1}) + A(y_{2n}, y_{2n}, \dots, y_{2n}, y_{2n+1})] + \\
&+ b^2[A(x_{2n+1}, x_{2n+1}, \dots, x_{2n+1}, x_{2n+2}) + A(y_{2n+1}, y_{2n+1}, \dots, y_{2n+1}, y_{2n+2})] \big\}.
\end{aligned}
\tag{2.4}
$$

Similarly, we get

$$
\begin{aligned}
&A(y_{2n+1}, y_{2n+1}, \dots, y_{2n+1}, y_{2n+2}) + A(x_{2n+1}, x_{2n+1}, \dots, x_{2n+1}, x_{2n+2}) \leqslant \\
&\leqslant \alpha\{(n-1)b[A(y_{2n}, y_{2n}, \dots, y_{2n}, y_{2n+1}) + A(x_{2n}, x_{2n}, \dots, x_{2n}, x_{2n+1})] + \\
&\quad + b^2[A(y_{2n+1}, y_{2n+1}, \dots, y_{2n+1}, y_{2n+2}) + A(x_{2n+1}, x_{2n+1}, \dots, x_{2n+1}, x_{2n+2})]\}.
\end{aligned}
\tag{2.5}
$$

From (2.4) and (2.5) we have,

$$
\begin{aligned}
2D_{2n+1} = &2[A(x_{2n+1}, x_{2n+1}, \dots, x_{2n+1}, x_{2n+2}) + A(y_{2n+1}, y_{2n+1}, \dots, y_{2n+1}, y_{2n+2})] \leqslant \\
&\leqslant 2\alpha\{(n-1)b[A(x_{2n}, x_{2n}, \dots, x_{2n}, x_{2n+1}) + A(y_{2n}, y_{2n}, \dots, y_{2n}, y_{2n+1})] + \\
&\quad + b^2[A(x_{2n+1}, x_{2n+1}, \dots, x_{2n+1}, x_{2n+2}) + A(y_{2n+1}, y_{2n+1}, \dots, y_{2n+1}, y_{2n+2})]\}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
D_{2n+1} \leqslant \alpha\{&(n-1)b[A(x_{2n}, x_{2n}, \dots, x_{2n}, x_{2n+1}) + A(y_{2n}, y_{2n}, \dots, y_{2n}, y_{2n+1})] + \\
&+ b^2[A(x_{2n+1}, x_{2n+1}, \dots, x_{2n+1}, x_{2n+2}) + A(y_{2n+1}, y_{2n+1}, \dots, y_{2n+1}, y_{2n+2})]\}.
\end{aligned}
\tag{2.6}
$$

It gives that

$$
D_{2n+1} \leqslant \frac{\alpha(n-1)b}{1 - \alpha b^2} D_{2n}. \tag{2.7}
$$

Put $\beta = \dfrac{\alpha(n-1)b}{1-\alpha b^2}$, then $0 \leqslant \beta < 1$.
From (2.7),

$$D_{2n+1} \leqslant \beta D_{2n}.$$

Similarly we can show that

$$D_{2n+2} \leqslant \beta D_{2n+1} \text{ for } \; n = 0, 1, 2, \dots .$$

Hence

$$D_{n+1} \leqslant \beta D_n.$$

Therefore

$$D_{n+1} \leqslant \beta^{n+1} D_0. \tag{2.8}$$

Define

$$\begin{aligned} D_{n,m} &= D(\underbrace{(x_n, y_n), (x_n, y_n), \dots, (x_n, y_n)}_{(n-1)-times}, (x_m, y_m)) = \\ &= A(\underbrace{x_n, x_n, \dots, x_n}_{(n-1)-times}, x_m) + A(\underbrace{y_n, y_n, \dots, y_n}_{(n-1)-times}, y_m). \end{aligned}$$

Now we have to show that $D_{n,m}$ is a Cauchy sequence.
By Lemma 1.6, for all $n, m \in \mathbb{N}, \; n \leqslant m$, we have

$$\begin{aligned} D_{n+1,m+1} = \; & A(x_{n+1}, x_{n+1}, \dots, x_{n+1}, x_{m+1}) + A(y_{n+1}, y_{n+1}, \dots, y_{n+1}, y_{m+1}) \leqslant \\ \leqslant \; & b(n-1)[A(x_{n+1}, x_{n+1}, \dots, x_{n+1}, x_{n+2}) + A(y_{n+1}, y_{n+1}, \dots, y_{n+1}, y_{n+2})] + \\ + \; & b^2[A(x_{n+2}, x_{n+2}, \dots, x_{n+2}, x_{m+1}) + A(y_{n+2}, y_{n+2}, \dots, y_{n+2}, y_{m+1})] = \\ = \; & b(n-1)D_{n+1} + b^2 b(n-1)[A(x_{n+2}, x_{n+2}, \dots, x_{n+2}, x_{n+3}) + \\ + \; & A(y_{n+2}, y_{n+2}, \dots, y_{n+2}, y_{n+3})] + \\ + \; & b^2 b^2[A(x_{n+3}, x_{n+3}, \dots, x_{n+3}, x_{m+1}) + A(y_{n+3}, y_{n+3}, \dots, y_{n+3}, y_{m+1})] = \\ = \; & b(n-1)D_{n+1} + b^3(n-1)D_{n+2} + b^5(n-1)D_{n+3} \cdots + \\ + \; & b^{2(m-n)-3}(n-1)[A(x_{m-1}, x_{m-1}, \dots, x_{m-1}, x_m) + A(y_{m-1}, y_{m-1}, \dots, y_{m-1}, y_m)] + \\ + \; & b^{2(m-n)-1}(n-1)[A(x_m, x_m, \dots, x_m, x_{m+1}) + A(y_m, y_m, \dots, y_m, y_{m+1})]. \end{aligned}$$

From (2.8), we have that

$$\begin{aligned} D_{n+1,m+1} \; & \leqslant \; b(n-1)[\beta^{n+1} + b^2\beta^{n+2} + b^4\beta^{n+3} + \dots + b^{2(m-n)-2}\beta^m]D_0 \leqslant \\ & \leqslant \; b(n-1)\beta^{n+1}[1 + b^2\beta + (b^2\beta)^2 + \dots + (b^2\beta)^{(m-n-1)}]D_0 = \\ & = \; b(n-1)\beta^{n+1}[1 + \gamma + \gamma^2 + \dots + \gamma^{(m-n-1)}]D_0 \leqslant \\ & \leqslant \; b(n-1)\beta^{n+1}\left(\frac{1}{1-\gamma}\right)D_0 \\ & \to \; 0 \text{ as } n \to \infty. \end{aligned}$$

Thus

$$\lim_{n,m\to\infty} A(x_n, x_n, \dots, x_n, x_m) = \lim_{n,m\to\infty} A(y_n, y_n, \dots, y_n, y_m) = 0.$$

Therefore $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences in X.
By the completeness of X, there exists $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.
Therefore $D_{n,m}$ is a Cauchy sequence.
Now we show that (x, y) is a coupled fixed point of f and g.
Without loss of generality, we may suppose that f is continuous, we have

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} f(x_{2n}, y_{2n}) = f\left(\lim_{n\to\infty} x_{2n}, \lim_{n\to\infty} y_{2n}\right) = f(x, y)$$

and

$$y = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} f(y_{2n}, x_{2n}) = f\left(\lim_{n\to\infty} y_{2n}, \lim_{n\to\infty} x_{2n}\right) = f(y, x).$$

Thus (x, y) is a coupled fixed point of f.
From (2.1), taking $x = u$ *and* $y = v$, we have,

$$\begin{aligned}
& A(x, x, , \dots, x, g(x, y)) + A(y, y, \dots, y, g(y, x)) = \\
= \;& A(f(x, y), f(x, y), \dots, f(x, y), g(x, y)) + A(f(y, x), f(y, x), \dots, f(y, x), g(y, x)) \leqslant \\
\leqslant \;& \alpha \max\Bigg\{\Bigg[(1 + D((x, y), (x, y) \dots (x, y), (x, y)))\frac{(D((x, y), (x, y) \dots (x, y), (g(x, y), g(y, x))))}{(1+ D((x, y), (x, y) \dots (x, y), (x, y)))}\Bigg], \\
& D\big((x, y), (x, y), \dots, (x, y), (x, y)\big), \big(D((x, y), (x, y), \dots, (x, y), (x, y)) + \\
& +D((x, y), (x, y), \dots, (x, y), (g(x, y), g(y, x)))\big), \big(D((x, y), (x, y), \dots, (x, y), (x, y)) + \\
& +D((x, y), (x, y), \dots, (x, y), (g(x, y), g(y, x)))\big)\Bigg\} \leqslant \\
\leqslant \;& \alpha b\, ((g(x, y), g(y, x)), (g(x, y), g(y, x)), \dots, (g(x, y), g(y, x)), (x, y)).
\end{aligned}$$

Since $\alpha b < 1$, we have $(g(x, y), g(y, x)) = (x, y)$.
Therefore $g(x, y) = x$ and $g(y, x) = y$.
Therefore (x, y) is a coupled fixed point of g.
Thus (x, y) is a coupled common fixed point of f and g. □

Theorem 2.2. *Let $(X, \leqslant, A)$ be a partially ordered, complete A_b-metric space and $f, g : X \times X \to X$ be the mappings such that*
(i) the pair (f, g) has mixed weakly monotone property on X and there exists $x_0, y_0 \in X$ such that $x_0 \leqslant f(x_0, y_0), f(y_0, x_0) \leqslant y_0$ or $x_0 \leqslant g(x_0, y_0), g(y_0, x_0) \leqslant y_0$,
(ii) there is an α such that $\alpha b^2((n-1)b + 1) < 1$ and

$$A(f(x, y), f(x, y), \dots, f(x, y), g(u, v)) + A(f(y, x), f(y, x), \dots, f(y, x), g(v, u)) \leqslant \alpha M$$

where

$$\begin{aligned}
M \;=\; & \max\Bigg\{\Bigg[(1 + D((x, y), (x, y), \dots, (x, y), (f(x, y), f(y, x)))) \\
& \frac{(D((u, v), (u, v), \dots, (u, v), (g(u, v), g(v, u))))}{(1 + D((x, y), (x, y), \dots, (x, y), (u, v)))}\Bigg], \\
& D\big((x, y), (x, y), \dots, (x, y), (u, v)\big), \big(D((x, y), (x, y), \dots, (x, y), (f(x, y), f(y, x))) + \\
& +D((u, v), (u, v), \dots, (u, v), (g(u, v), g(v, u)))\big), \big(D((u, v), (u, v), \dots, (u, v), (f(x, y), f(y, x))) + \\
& +D((x, y), (x, y), \dots, (x, y), (g(u, v), g(v, u)))\big)\Bigg\}
\end{aligned}$$

for all $x, y, u, v \in X$ with $x \leqslant u$ and $y \geqslant v$,
(iii) X has the following properties

(a) if $\{x_n\}$ is an increasing sequence with $x_n \to x$, then $x_n \leqslant x$ for all $n \in \mathbb{N}$,
(b) if $\{y_n\}$ is a decreasing sequence with $y_k \to y$, then $y \leqslant y_n$ for all $n \in \mathbb{N}$.
Then f and g have coupled common fixed points in X.

Proof. Suppose X satisfies (a) and (b), by (2.3) we get $x_n \leqslant x$ and $y_n \geqslant y$ for all $n \in \mathbb{N}$. Applying Lemmas 1.5 and 1.6, we have

$$\begin{aligned} & D((x,y),(x,y),\ldots,(x,y),(f(x,y),f(y,x))) \leqslant \\ & \leqslant\ b(n-1)D((x,y),(x,y),\ldots,(x,y),(x_{2n+2},y_{2n+2})+ \\ & \quad +\ b^2D((x_{2n+2},y_{2n+2}),(x_{2n+2},y_{2n+2}),\ldots,(x_{2n+2},y_{2n+2}),(f(x,y),f(y,x))) = \\ & =\ b(n-1)D((x,y),(x,y),\ldots,(x,y),(x_{2n+2},y_{2n+2}))+ \\ & \quad +\ b^2D((g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})),\ldots \\ & \quad \ldots,(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})),(f(x,y),f(y,x))). \end{aligned} \tag{2.9}$$

By (2.1), we get

$$\begin{aligned} & A\big((g(x_{2n+1},y_{2n+1})),(g(x_{2n+1},y_{2n+1})),\ldots,(g(x_{2n+1},y_{2n+1})),(f(x,y))\big) + \\ + \quad & A\big((g(y_{2n+1},x_{2n+1})),(g(y_{2n+1},x_{2n+1})),\ldots,(g(y_{2n+1},x_{2n+1})),(f(y,x))\big) \leqslant \\ \leqslant \quad & \alpha \max\Bigg\{\Bigg[\big(1+D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),\ldots,,(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))\big)\times \\ & \times\frac{(D((x,y),(x,y),\ldots,(x,y),(f(x,y),f(y,x))))}{(1+D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),\ldots,(x_{2n+1},y_{2n+1}),(x,y)))}\Bigg], \\ & D\big((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),\ldots,(x_{2n+1},y_{2n+1}),(x,y)\big), \\ & \big(D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),\ldots,(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+ \\ & +D((x,y),(x,y),\ldots,(x,y),(f(x,y),f(y,x)))\big),\big(D((x,y),(x,y),\ldots,(x,y),(x_{2n+2},y_{2n+2}))+ \\ & +D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),\ldots,(x_{2n+1},y_{2n+1}),(f(x,y),f(y,x)))\big)\Bigg\}. \end{aligned}$$

Taking the limit as $n \to \infty$ in (2.9), we obtain

$$D\big((x,y),(x,y),\ldots,(x,y),(f(x,y),f(y,x))\big) \leqslant b^2\alpha\ D\big((x,y),(x,y),\ldots,(x,y),(f(x,y),f(y,x))\big).$$

Since $b^2\alpha < 1$, we have $D((x,y),(x,y),\ldots,(x,y),(f(x,y),f(y,x))) = 0$.
That is, $f(x,y) = x$ and $f(y,x) = y$. Therefore (x,y) is a coupled fixed point of f.
Similarly we can show that $g(x,y) = x$ and $g(y,x) = y$. Hence $f(x,y) = x = g(x,y)$ and $f(y,x) = y = g(y,x)$.
Thus (x,y) is a coupled common fixed point of f and g. □

Theorem 2.3. *Suppose Theorem 2.1 or Theorem 2.2 satisfied, if further $\{x_n\}$ is an increasing sequence with $x_n \to x$ and $x_n \leqslant u$ for each n, then $x \leqslant u$. Then f and g have a unique coupled common fixed points. Further more, any fixed point of f is a fixed point of g, and conversely.*

Proof. Suppose the given condition holds. Let (x,y) and $(u,v) \in X \times X$, there exist $(x^*,y^*) \in$

$X \times X$, that is, comparable to (x, y) and (u, v).

$$\begin{aligned}
&D((x,y),(x,y),\dots,(x,y),(u,v)) = \\
&= A(x,x,\dots,x,u) + A(y,y,\dots,y,u) = \\
&= A(f(x,y),f(x,y),\dots,f(x,y),g(u,v)) + A(f(y,x),f(y,x),\dots,f(y,x),g(v,u)) \leqslant \\
&\leqslant \alpha \max\Bigg\{\Bigg[(1 + D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x))))\times
\end{aligned}$$

$$\times\frac{(D((u,v),(u,v),\dots,(u,v),(g(u,v),g(v,u))))}{(1+D((x,y),(x,y),\dots,(x,y),(u,v)))}\Bigg], D((x,y),(x,y),\dots,(x,y),(u,v)),$$

$$(D((x,y),(x,y),\dots,(x,y),(f(x,y),f(y,x))) + D((u,v),(u,v),\dots,(u,v),(g(u,v),g(v,u)))),$$

$$(D((u,v),(u,v),\dots,(u,v),(f(x,y),f(y,x))) + D((x,y),(x,y),\dots,(x,y),(g(u,v),g(v,u))))\Bigg\} \leqslant$$

$$\leqslant \alpha\ (b+1)D((x,y),(x,y),\dots,(x,y),(u,v)).$$

Since $\alpha(b+1) < 1$, so that
$D((x,y),(x,y),\dots,(x,y),(u,v)) = 0$
$\implies (x,y) = (u,v) \implies x = u \textit{ and } y = v$
Suppose (x, y) and (x^*, y^*) are Coupled common fixed points such that $x \leqslant x^*$ and $y \geqslant y^*$, then $x = x^*$ and $y = y^*$.
Now

$$\begin{aligned}
&D((x,y),(x,y),\dots,(x,y),(x^*,y^*)) = A(x,x,\dots,x,x^*) + A(y,y,\dots,y,y^*) = \\
&= A(f(x,y),f(x,y),\dots,f(x,y),g(x^*,y^*)) + A(f(y,x),f(y,x),\dots,f(y,x),g(y^*,x^*)) \leqslant \\
&\leqslant \alpha(b+1)D((x,y),(x,y),\dots,(x,y),(x^*,y^*)).
\end{aligned}$$

Since $\alpha(b+1) < 1$, so that
$D((x,y),(x,y),\dots,(x,y),(x^*,y^*)) = 0$
$\implies (x,y) = (x^*,y^*)$
$\implies x = x^* \textit{ and } y = y^*$
we show that any fixed point of f is a fixed point of g, and conversely.
That is, to show that (x, y) is a fixed point of $f \iff (x, y)$ is a fixed point of g.
Suppose that (x, y) is a coupled fixed point of f

$$\begin{aligned}
&D((x,y),(x,y),\dots,(x,y),(g(x,y),g(y,x))) = \\
= \ &A(f(x,y),f(x,y),\dots,f(x,y),g(x,y)) + A(f(y,x),f(y,x),\dots,f(y,x),g(y,x)) \leqslant \\
\leqslant \ &\alpha b\ D((g(x,y),g(y,x)),(g(x,y),g(y,x)),\dots,(g(x,y),g(y,x)),(x,y)).
\end{aligned}$$

Since $\alpha b < 1$, we have
$D((g(x,y),g(y,x)),(g(x,y),g(y,x)),\dots,(g(x,y),g(y,x)),(x,y)) = 0$
$\implies (g(x,y),g(y,x)) = (x,y)$
$\implies x = g(x,y) \textit{ and } y = g(y,x)$
Therefore (x, y) is a coupled fixed point of g, and conversely. □

Taking $M = D((x,y),(x,y),\dots,(x,y),(u,v))$ and $g = f$ in Theorem 2.1, we get the following

Corollary 2.4. *Let $(X, \leqslant, A)$ be a partially ordered, complete A_b-metric space and let $f : X \times X \to X$ be the mapping such that*
(i) f has mixed weakly monotone property on X and there exists $x_0, y_0 \in X$ such that $x_0 \leqslant$

$f(x_0, y_0), f(y_0, x_0) \leqslant y_0$,
(ii) there is an α such that $\alpha < 1$ and

$$\begin{aligned} &A(f(x,y), f(x,y), \dots, f(x,y), f(u,v)) + A(f(y,x), f(y,x), \dots, f(y,x), f(v,u)) \leqslant \\ &\leqslant \alpha\ D((x,y),(x,y),\dots,(x,y),(u,v)), \end{aligned} \tag{2.10}$$

for all $x, y, u, v \in X$ with $x \leqslant u$ and $y \geqslant v$,
(iii) if f is continuous.
Then f has a coupled fixed point in X.

We give an example to demonstrate the validity of the result 2.1.

Example 2.5. *Let $(\mathbb{R}, \leqslant, A)$ be a partially ordered complete A_b-metric space with A_b-metric defined as $X = [-\infty, +\infty]$ by $A_b : X^n \to [-\infty, +\infty]$ by $A_b(x_1, x_2, \dots, x_{n-1}, x_n) = \sum_{i=1}^{n}\sum_{i<j} |x_i - x_j|^2$, for all $x_i \in X$, $i = 1, 2, \dots, n$. Then (X, A_b) is an A_b-metric space with b=2.*
Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be two maps defined by $f(x,y) = \dfrac{4x-2y+48n-2}{48n}$ and $g(x,y) = \dfrac{6x-3y+72n-3}{72n}$. Then the pair (f, g) has mixed weakly monotone property on $\mathbb{R}$

$$\begin{aligned} &A(f(x,y), f(x,y), \dots, f(x,y), g(u,v)) + A(f(y,x), f(y,x), \dots, f(y,x), g(v,u)) = \\ &= (n-1)(|f(x,y) - g(u,v)|) + (n-1)(|f(y,x) - g(v,u)|) = \\ &= (n-1)\left(\left|\frac{4x-2y+48n-2}{48n} - \frac{6u-3v+72n-3}{72n}\right|\right) + \\ &\quad + (n-1)\left(\left|\frac{4y-2x+48n-2}{48n} - \frac{6v-3u+72n-3}{72n}\right|\right) = \\ &= \frac{(n-1)}{24n}(|2(x-u)-(y-v)| + |2(y-v)-(x-u)|) \leqslant \frac{(n-1)}{24n}(3\,|x-u| + 3\,|y-v|) \leqslant \\ &\leqslant \frac{(n-1)}{8n}(|x-u| + |y-v|) = \\ &= \frac{(n-1)}{8n}\ D((x,y),(x,y),\dots,(x,y),(u,v)). \end{aligned}$$

For $n = 2$ and b=2, since $\alpha b^2((n-1)b+1) < 1 \implies \alpha < \dfrac{1}{12}$.
Then the contractive condition (2.1) is satisfied with $\alpha = \dfrac{1}{16} < \dfrac{1}{12}$ and also $(1, 1)$ is the unique coupled common fixed point of f and g.

3. Application

The following type system of integral equations:

$$\begin{aligned} u(t) &= q(t) + \int_a^b \lambda(t,s)(f_1(s,u(s)) + f_2(s,v(s)))ds, \\ v(t) &= q(t) + \int_a^b \lambda(t,s)(f_1(s,v(s)) + f_2(s,u(s)))ds, \end{aligned} \tag{3.1}$$

where the space $X = C([a,b], \mathbb{R})$ of continuous functions defined in $[a,b]$. Obviously, the space with the metric is given by

$$A(u,v) = \max_{t\in[a,b]} |u(t) - v(t)|\,,\ u, v \in C([a,b], \mathbb{R})$$

is a complete metric space.
Let $X = C([a,b],\mathbb{R})$ the natural partial order relation, that is,
$u, v \in C([a,b],\mathbb{R}),\ u \leqslant v \iff u(t) \leqslant v(t),\ t \in [a,b]$.

Theorem 3.1. *Consider the corollary 2.4 and assume that the following conditions are hold:*

(i) $f_1, f_2 : [a,b]\times\mathbb{R} \to \mathbb{R}$ *are continuous;*

(ii) $q : [a,b] \to \mathbb{R}$ *is continuous;*

(iii) $\lambda : [a,b]\times\mathbb{R} \to [0,\infty)$ *is continuous;*

(iv) there exist $c > 0$ *and* $0 \leqslant \alpha < 1$*, such that for all* $u, v \in \mathbb{R},\ v \geqslant u$,
$0 \leqslant f_1(s,v) - f_1(s,u) \leqslant c\alpha(v-u)$
$0 \leqslant f_2(s,v) - f_2(s,u) \leqslant c\alpha(v-u)$;

(v) assume that $c \max\limits_{t\in[a,b]} \int\limits_a^b \lambda(t,s)ds \leqslant 1$;

(vi) there exist $x_0, y_0 \in X$ *such that*

$$x_0(t) \geqslant q(t) + \int_a^b \lambda(t,s)(f(s,x_0(s)) + g(s,y_0(s)))ds,$$
$$y_0(t) \leqslant q(t) + \int_a^b \lambda(t,s)(f(s,y_0(s)) + g(s,x_0(s)))ds.$$

Then the system of Volterra type integral equation (3.1) has a unique solution in $X\times X$ *with* $X = C([a,b],\mathbb{R})$.

Proof. Define the mapping $F : X\times X \to X$ by

$$F(u,v)(t) = q(t) + \int_a^b \lambda(t,s)(f_1(s,u(s)) + f_2(s,v(s)))ds \tag{3.2}$$

for all $u, v \in X$ *and* $t \in [a,b]$.

Now we have to show that all the conditions of Corollary 2.4 are satisfied.

From (iv) of the Theorem 3.1, clearly F has mixed monotone property.

For $x, y, u, v \in X$ with $x \geqslant u$ and $y \leqslant v$, we have

$$\begin{aligned}
& A(F(x,y),F(x,y),\ldots,F(x,y),F(u,v)) + A(F(y,x),F(y,x),\ldots,F(y,x),F(v,u)) = \\
= \;& (n-1)\max_{t\in[a,b]}(|F(x,y)(t)-F(u,v)(t)| + |F(y,x)(t)-F(v,u)(t)|) = \\
= \;& (n-1)\max_{t\in[a,b]}\left|\int_a^b \lambda(t,s)(f_1(s,x(s))+f_2(s,y(s)))ds - \int_a^b \lambda(t,s)(f_1(s,u(s))+f_2(s,v(s)))ds\right| + \\
& +(n-1)\max_{t\in[a,b]}\left|\int_a^b \lambda(t,s)(f_1(s,y(s))+f_2(s,x(s)))ds - \int_a^b \lambda(t,s)(f_1(s,v(s))+f_2(s,u(s)))ds\right| \leqslant \\
\leqslant \;& (n-1)\max_{t\in[a,b]}\Bigg(\int_a^b |f_1(s,x(s))-f_1(s,u(s))|\,|\lambda(t,s)|\,ds + \\
& + \int_a^b |f_2(s,y(s))-f_2(s,v(s))|\,|\lambda(t,s)|\,ds + \\
& + \int_a^b |f_1(s,y(s))-f_1(s,v(s))|\,|\lambda(t,s)|\,ds + \int_a^b |f_2(s,x(s))-f_2(s,u(s))|\,|\lambda(t,s)|\,ds\Bigg) \leqslant
\end{aligned}$$

$$\begin{aligned}
&\leqslant (n-1) \max_{t\in[a,b]} c\alpha\Big(\int_a^b |x(s)-u(s)|\,|\lambda(t,s)|\,ds + \int_a^b |y(s)-v(s)|\,|\lambda(t,s)|\,ds + \\
&\quad + \int_a^b |y(s)-v(s)|\,|\lambda(t,s)|\,ds + \int_a^b |x(s)-v(s)|\,|\lambda(t,s)|\,ds \Big) \leqslant \\
&\leqslant (n-1) \left(\max_{t\in[a,b]} |x(t)-u(t)| + \max_{t\in[a,b]} |y(t)-v(t)| + \right. \\
&\quad \left. + \max_{t\in[a,b]} |y(t)-v(t)| + \max_{t\in[a,b]} |x(t)-u(t)| \right) c\alpha \int_a^b |\lambda(t,s)|\,ds \leqslant \\
&\leqslant 2(n-1) \left(\max_{t\in[a,b]} |x(t)-u(t)| + \max_{t\in[a,b]} |y(t)-v(t)| \right) c\alpha \int_a^b |\lambda(t,s)|\,ds \leqslant \\
&\leqslant 2(n-1)\ \alpha\ (A(x,x,\ldots,x,u) + A(y,y,\ldots,y,v)) = \\
&= 2(n-1)\ \alpha\ D((x,y),(x,y),\ldots,(x,y),(u,v)).
\end{aligned}$$

Therefore

$$\begin{aligned}
&A(F(x,y),F(x,y),\ldots,F(x,y),F(u,v)) + A(F(y,x),F(y,x),\ldots,F(y,x),F(v,u)) \leqslant \\
&\leqslant 2(n-1)\ \alpha D((x,y),(x,y),\ldots,(x,y),(u,v)).
\end{aligned}$$

For n=2, $\alpha < \frac{1}{2} < 1$. Which is the contractive condition in Corollary 2.4.

Thus, F has a coupled fixed point in X.

That is, the system of integral equations has a solution. □

References

[1] M.Abbas, B.Ali, Y.I.Suleiman, Generalized coupled common fixed point results in partially ordered A-metric spaces, *Fixed point theory Appl.*, **64**(2015). DOI: 10.1186/s13663-015-0309-2

[2] I.A.Bakhtin, The contraction mapping principle in almost metric space, *Functional analysis. Ulyanovsk. Gos. Ped. Inst., Ulyanovsk*, 1989, 26-37 (in Russian).

[3] Jyzef Banas, Donal O'Regan, On existence and local attractivity of solutions of a quadratic Volterra integral equation of fractional order, *J. Math. Anal. Appl.*, **345**(2008), no. 1, 573–582. DOI: j.jmaa.2008.04.050

[4] M.Bousselsal, M.Laid Kadri, Coupled Coincidence point for generalized monotone operators in partially ordered metric spaces, *Thai Journal of Mathematics*, **15**(2017), no. 2, 367–385.

[5] Nguyen Van Dung, On coupled common fixed points for mixed weakly monotone maps in partially ordered S-metric spaces, *Fixed Point Theory and Applications*, **48**(2013), 1–17. DOI: 10.1186/1687-1812-2013-48

[6] T.Gnana Bhaskar, V.Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal.*, **65**(2006), 1379–1393. DOI: 10.1016/j.na.2005.10.017

[7] M.E.Gordji, E. Akbartabar, Y.J.Cho, M.Ramezani, Coupled common fixed point theorems for mixed weakly monotone mappings in partially ordered metric spaces, *Fixed point theory and Appl.*, **95**(2012). DOI: 10.1186/1687-1812-2012-95

[8] E.Karapinar, P.Kumam, W.Sintunavarat, Coupled fixed point theorems in cone metric spaces with a c-distance and applications, *Fixed Point Theory and Appl.*, **194**(2012). DOI: 10.1186/1687-1812-2012-194

[9] E.Karapinar, P.Kumam, I.Erhan, Coupled fixed points on partially ordered G-metric spaces, *Fixed Point Theory and Appl.*, **2012**(2012), 174. DOI: 10.1186/1687-1812-2012-174

[10] P.Kumam, V.Pragadeeswarar, M.Marudai, K.Sitthithakerngkiet, Coupled Best Proximity Points in Ordered Metric Spaces, *Fixed Point Theory and Appl.*, **107**(2014). DOI: 10.1155/2014/274062

[11] N.Mlaiki, Y.Rohen, Some coupled fixed point theorem in partially ordered A_b-metric spaces, *J. Nonlinear. Sci. Appl.*, **10**(2017), 1731–1743.

[12] K.Ravibabu, Ch.Srinivasarao, Ch.Raghavendra Naidu, Coupled Fixed point and coincidence point theorems for generalized contractions in metric spaces with a partial order, *Italian Journal of pure and applied mathematics*, **39**(2018), 434–450.

[13] K.Ravibabu, Ch.Srinivasarao and Ch.Ragavendra naidu, Applications to Integral Equations with Coupled Fixed Point Theorems in Ab-Metric Space, *Thai Journal of Mathematics*, (2018), 148–167.

[14] M.Ughade, D.Turkoglu, S.R.Singh, R.D.Daheriya, Some fixed point theorems in A_b-metric space, *British J. Math. Comput. Sci.*, **19**(2016), 1–24. DOI: 10.9734/BJMCS/2016/29828

Связанные теоремы о неподвижной точке через свойство смешанной монотонности в A_b-метрических пространствах и приложения к интегральным уравнениям

К. Равибабу
Кафедра математики, G.M.R.I.T.
Раджам, Шрикакулам, Индия

Г.Н.В. Кишор
Департамент инженерной математики и гуманитарных наук
Инженерный колледж Саги Рама Кришнам Раджу
Чинамирам, Бхимаварам, Андхра-Прадеш, Индия

Ч. Шриниваса Рао
Кафедра математики, Университет Андхра
Вишакхапатнам, Индия

Ч. Рагхавендра Найду
Кафедра математики, Государственный колледж
Палаконда, Шрикакулам, Индия

Аннотация. В этой статье мы устанавливаем некоторые результаты о существовании и единственности связанных теорем об общей неподвижной точке в частично упорядоченных A_b-метрических пространствах. Приведены примеры для обоснования актуальности результатов, полученных в результате анализа существующей теоремы. Кроме того, мы также находим приложение к интегральным уравнениям через теоремы о неподвижной точке в A_b-метрических пространствах.

Ключевые слова: связанная неподвижная точка, смешанная слабомонотонность, A_b-метрическое пространство, интегральное уравнение.

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(3), 356–365

DOI: 10.17516/1997-1397-2022-15-3-356-365
УДК 517.9

Idempotent Values of Commutators Involving Generalized Derivations

Gurninder S. Sandhu*
Department of Mathematics
Patel Memorial National College
Rajpura, Punjab, India
Shakir Ali†
Department of Mathematics
Aligarh Muslim University
Aligarh, Uttar Pradesh, India

Received 24.12.2021, received in revised form 28.01.2022, accepted 20.03.2022

Abstract. In the present article, we characterize generalized derivations and left multipliers of prime rings involving commutators with idempotent values. Precisely, we prove that if a prime ring of characteristic different from 2 admits a generalized derivation G with an associative nonzero derivation g of R such that $[G(u), u]^n = [G(u), u]$ for all $u \in \{[x, y] : x, y \in L\}$, where L a noncentral Lie ideal of R and $n > 1$ is a fixed integer, then one of the following holds:

(i) R satisfies s_4 and there exists $\lambda \in C$, the extended centroid of R such that $G(x) = ax + xa + \lambda x$ for all $x \in R$, where $a \in U$, the Utumi quotient ring of R,

(ii) there exists $\gamma \in C$ such that $G(x) = \gamma x$ for all $x \in R$.

As an application, we describe the structure of left multipliers of prime rings satisfying the condition $([T^m(u), u])^n = [T^m(u), u]$ for all $u \in \{[x, y] : x, y \in L\}$, where $m, n > 1$ are fixed integers. In the end, we give an example showing that the hypothesis of our main theorem is not redundant.

Keywords: prime ring, Lie ideal, generalized derivation, GPI.

Citation: G.S. Sandhu, Shakir Ali, Idempotent Values of Commutators Involving Generalized Derivations, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 356–365.
DOI: 10.17516/1997-1397-2022-15-3-356-365.

1. Introduction

A celebrated result of Wedderburn states that: *Every finite division ring is commutative and also any Boolean ring is a commutative ring.* In 1945, Jacobson [15] generalized this result by proving the following: *Any ring in which every element satisfies an equation of the form $x^{n(x)} = x$, is commutative, where $n(x) > 1$ is an integer related to x.* In this vein Herstein [11] proved the following theorem: *If R is a ring with center $Z(R)$, and if $x^n - x \in Z(R)$ for all $x \in R$, then R is commutative, where $n > 1$ is a fixed integer*, which is of course a generalization of the classical theorem due to Jacobson. Later, Herstein [12] established the commutativity of rings that satisfy the condition $[x, y]^n = [x, y]$, where $n(x, y) > 1$ is an integer. These results

*gurninder_rs@pbi.ac.in https://orcid.org/0000-0001-8618-6325
†shakir.ali.mm@amu.ac.in https://orcid.org/0000-0001-5162-7522

have inspired the development of several techniques to explore the conditions that force a ring to be commutative, for instance, generalizing Herstein's conditions, using certain polynomial constraints, using restrictions on automorphisms, introducing identities involving derivations and generalized derivations etc. For more details and references one can see a well organized survey paper by Pinter–Lucke [20].

An additive mapping $T : R \to R$ is said to be a left (resp. right) multiplier if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) holds for all $x, y \in R$. If T is both a left as well as a right multiplier of R, then it is said to be a multiplier of R (cf.; [23] and [25] for details). An additive mapping $g : R \to R$ is called a derivation if $g(xy) = g(x)y + xg(y)$ holds for all $x, y \in R$. An additive mapping G is called a generalized derivation if there is a derivation g of R satisfying $G(xy) = G(x)y + xg(y)$ for all $x, y \in R$. Obviously, any derivation is a generalized derivation, but the converse is not true in general. A significant example is a map of the form $F(x) = ax + xb$ for all $x \in R$, where a and b are fixed element of R. Moreover, the concept of generalized derivation includes both the concepts of derivation and left multiplier. Hence, the concept of generalized derivation is a natural generalization of the concept of derivation and left multiplier. Further, generalized derivations have been primarily studied on operator algebras. Therefore, any investigation from the algebraic point of view might be interesting (see for example Hvala [13] and Lee [18], where further references can be looked). In the present paper, we describe the structure of generalized derivations and left multipliers of prime rings under some specific situations.

The study of commutators involving derivations goes back to 1957, when Posner [21] proved that a prime ring R admits a nonzero derivation d satisfying $[d(x), x] = 0$ for all $x \in R$, is commutative. Since then, this result has been generalized in many directions. In 2000, Carini and Filippis [6] studied the nilpotent values of commutators involving derivations of prime rings. Precisely, they proved that: *Let R be a prime ring of characteristic different from 2, L a non-central Lie ideal of R, d a nonzero derivation of R and $n \geqslant 1$ is a fixed integer. If $[d(x), x]^n = 0$ for all $x \in L$, then R is commutative.* In 2006, Filippis [9] extended this result to the class of generalized derivations as follows: *Let R be a prime ring of characteristic different from 2, L a noncentral Lie ideal of R and $n \geqslant 1$ is a fixed integer. If R admits a generalized derivation F associated with a derivation d such that $[F(x), x]^n = 0$ for all $x \in L$, then either R satisfies s_4, the standard identity in four noncommuting variables or there exists $a \in U$ and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$ for all $x \in R$.* Therefore, it is natural to look at the idempotent elements of the set $E = \{[\varphi(x), x] : x \in L\}$, where φ is a mapping and L is a subset of a prime ring R. Recently, Scudo and Ansari [22] considered this problem with generalized derivations of prime rings. In fact, they proved the following theorem:

Theorem 1.1. *Let R be a noncommutative prime ring with char$(R) \neq 2$, U the Utumi qotient ring of R, C the extended centroid of R and L a noncentral Lie ideal of R. If G is a generalized derivation of R with an associated derivation d of R such that $[G(u), u]^n = [G(u), u]$ for all $u \in L$, where $n > 1$ a fixed integer, then one of the following holds:*

(i) R satisfies the s_4 identity and there exists $a \in U$ and $\lambda \in C$ such that $G(x) = ax + xa + \lambda x$ for all $x \in R$.

(ii) there exists $\gamma \in C$ such that $G(x) = \gamma x$ for all $x \in R$.

In this line of investigation, Filippis et al. [10] obtained the following result on multilinear polynomials: *Let R be a prime ring with char$(R) \neq 2$, C the extended centroid of R, d a nonzero derivation of R, $f(x_1, \cdots, x_n)$ a multilinear polynomial over C, I a nonzero right ideal of R and*

$m > 1$ a fixed integer such that

$$[d(f(x_1, \cdots, x_n)), f(x_1, \cdots, x_n)]^m = [d(f(x_1, \cdots, x_n)), f(x_1, \cdots, x_n)]$$

for all $x_1, \cdots, x_n \in I$. Then either $[f(x_1, \cdots, x_n), x_{n+1}]x_{n+2}$ is an identity for I or $d(I)I = (0)$.

Very recently, Ashraf et al. [2] studied a related problem for automorphisms of prime rings. Specifically, they proved the following theorem: *Let R be a prime ring with char$(R) \neq 2, 3$ and L a noncentral Lie ideal of R. If σ is an automorphism of R such that $[\sigma(x), x]^m = [\sigma(x), x]$ for all $x \in L$, where $m > 1$ a fixed integer, then R is commutative.*

The main objective of this paper is to study the above mentioned problem for the set $[L, L] = \{[x, y \mid x, y \in L\}$, where L is a noncentral Lie-ideal of a prime ring R. In fact, we describe the structure of generalized derivations and left multipliers of prime rings with idempotent values on commutators. Precisely, we prove the following results:

Theorem 1.2. *Let $n > 1$ be a fixed integer. Next, let R be a prime ring with char$(R) \neq 2$, U the Utumi quotient ring, C is the extended centroid and L is a noncentral Lie ideal of R. If R admits a generalized derivation G associated with a derivation g such that $[G(u), u]^n = [G(u), u]$ for all $u \in \{[x, y] : x, y \in L\}$, then one of the following holds:*

(i) R satisfies s_4 and there exists $\lambda \in C$ such that $G(x) = ax + xa + \lambda x$ for all $x \in R$, where $a \in U$.

(ii) there exists $\gamma \in C$ such that $G(x) = \gamma x$ for all $x \in R$.

Further, as an application, we describe the structure of left multipliers of prime rings. In particular, we establish the following:

Theorem 1.3. *Let $m, n > 1$ be fixed integers. Next, let R be a prime ring with char$(R) \neq 2$, U the Utumi quotient ring, C is the extended centroid of R and L is a noncentral Lie ideal of R. If T is a left multiplier of R such that $([T^m(u), u])^n = [T^m(u), u]$ for all $u \in \{[x, y] : x, y \in L\}$, then there exists $\gamma \in C$ such that $T(x) = \gamma x$ for all $x \in R$.*

2. Preliminaries

A ring R is said to be a prime if for any $a, b \in R$; $aRb = (0)$ implies $a = 0$ or $b = 0$. An additive mapping $g : R \to R$ is called a derivation if $g(xy) = g(x)y + xg(y)$ for all $x, y \in R$. For a fixed element $a \in R$, a mapping $x \mapsto [a, x]$ is a well-known example of a derivation, which is called the inner derivation induced by a. By generalized derivation, we mean an additive mapping $F : R \to R$ such that $F(xy) = F(x)y + xg(y)$, where g is a derivation of R associated with F. For any $x, y \in R$, the symbol $[x, y]$ denotes the Lie product (or commutator) $xy - yx$. An additive subgroup L of R is known as Lie ideal of R if $[x, r] \in L$ for all $x \in L$ and $r \in R$. The Utumi quotient ring of R is denoted by U and C the extended centroid of R. For more detail of these objects and generalized polynomial identities, we refer the reader to [3]. By s_4, we denote the standard identity in four noncommuting variables, which is defined as follows:

$$s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)},$$

where S_4 is the symmetric group of degree 4 and $(-1)^\sigma$ is the sign of permutation $\sigma \in S_4$. It is known that by the standard PI-theory, a prime ring R satisfying s_4 can be characterized in a number of ways, as follows: Let R be a prime ring with C its extended centroid. Then the following assertions are equivalent:

- $\dim_C(RC) \leqslant 4$.
- R satisfies s_4.
- R is commutative or R embeds into $M_2(F)$, for a field F.
- R is algebraic of bounded degree 2 over C.
- R satisfies $[[x^2, y], [x, y]]$ (see [5, Lemma 1]).

In order to prove this result, we need the following remarks:

Remark 1 ([18], Theorem 3)**.** *Every generalized derivation of R can be uniquely extended to a generalized derivation of U and assumes the form that $F(x) = ax + g(x)$ for some $a \in U$ and a derivation g of U.*

Remark 2 ([7], Theorem 2)**.** *If I is a two-sided ideal of R, then I, R and U satisfy the same generalized polynomial identities with coefficients in U.*

3. Main results

We begin our discussions with the following lemma.

Lemma 1. *Let $R = M_k(C)$ be the ring of $k \times k$ matrices over a field C with char$(R) \neq 2$ and $a, b \in R$. If $k = 2$ and $n > 1$ a fixed integer such that*

$$\big([a[u,v] + [u,v]b, [u,v]]\big)^n = [a[u,v] + [u,v]b, [u,v]]$$

for all $u, v \in [R,R]$, then $b - a$ is central.

Proof. By the given hypothesis, R satisfies the generalized polynomial identity

$$\begin{aligned}&\big([a[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]b, [[x_1,x_2],[x_3,x_4]]]\big)^n-\\ &\quad -[a[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]b, [[x_1,x_2],[x_3,x_4]]].\end{aligned} \tag{1}$$

Let us assume that $b - a = \sum_{i,j=1}^{k} \alpha_{ij} e_{ij}$, where $\alpha_{ij} \in C$ and e_{ij} denotes the standard matrix unit with (i,j)-th place 1 and 0 elsewhere. For $i \neq j$, we choose $x_1 = e_{ii}$, $x_2 = e_{ij}$, $x_3 = e_{ij}$ and $x_4 = e_{ji}$. With this, we have $[x_1, x_2] = e_{ij}$, $[x_3, x_4] = e_{ii} - e_{jj}$ and hence $[[x_1,x_2],[x_3,x_4]] = -2e_{ij}$. In this view, it follows from (1) that

$$4^n\big([ae_{ij} + e_{ij}b, e_{ij}]\big)^n - 4[ae_{ij} + e_{ij}b, e_{ij}] = 0.$$

Performing the computations and using the fact that char$(R) \neq 2$ and $n > 1$, we obtain $e_{ij}(b-a)e_{ij} = 0$, where $i \neq j$. It implies that $\alpha_{ji} = 0$ for all $i \neq j$, hence $b - a$ is a diagonal matrix. For any C-automorphism ξ of R, $\xi(b-a)$ enjoys the same property as $b - a$ does; i.e.,

$$\begin{aligned}&\big([\xi(a)[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]\xi(b), [[x_1,x_2],[x_3,x_4]]]\big)^n-\\ &\quad -[\xi(a)[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]\xi(b), [[x_1,x_2],[x_3,x_4]]].\end{aligned}$$

for all $x_1, x_2, x_3, x_4 \in R$, implies that $\xi(b-a)$ is a diagonal matrix. In particular, let $\xi(x) = (1 + e_{ij})x(1 - e_{ij})$, where $i \neq j$, then we see that the (j,i)-th entry of $\xi(b-a)$ is zero, i.e.,

$$0 = (\xi(b-a))_{ij} = \alpha_{ij} - \alpha_{ii} + \alpha_{jj} - \alpha_{ji} = -\alpha_{ii} + \alpha_{jj}.$$

It implies $\alpha_{ii} = \alpha_{jj}$ with $i \neq j$. It forces that $b - a$ central element in R. ☐

Lemma 2. *Let $R = M_k(C)$ be the ring of all $k \times k$ matrices over the field C with char$(R) \neq 2$ and $q \in R$. If $k \geqslant 2$, and $n > 1$ a fixed integer such that*

$$\big([q[[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, x_4]]]\big)^n = [q[[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, x_4]]]$$

for all $x_1, x_2, x_3, x_4 \in R$, then $q \in Z(R)$.

Proof. Let $q \in R$, i.e., $q = \sum\limits_{r,s=1}^{k} q_{rs}e_{rs}$, where $q_{rs} \in C$ and e_{rs} denotes the usual matrix unit with (r, s)-th entry 1 and 0 elsewhere. For $i \neq j$, we choose $x_1 = e_{ii}$, $x_2 = e_{ij}$, $x_3 = e_{ij}$ and $x_4 = e_{ji}$. With this, we have $[[x_1, x_2], [x_3, x_4]] = -2e_{ij}$. In this view, our situation yields

$$4^n([qe_{ij}, e_{ij}])^n - 4[qe_{ij}, e_{ij}] = 0.$$

Since $n > 1$ and char$(R) \neq 2$, we get $e_{ij}qe_{ij} = 0$, where $i \neq j$. It implies that $q_{ji} = 0$ for all $i \neq j$, hence q is a diagonal matrix. With the same reasoning of Lemma 1, we find that $q \in Z(R)$. □

Proposition 1. *Let R be a noncommutative prime ring with char$(R) \neq 2$, U the Utumi quotient ring and C the extended centroid of R. If for some $a, b \in U$ and a fixed integer $n > 1$,*

$$[a[u, v] + [u, v]b, [u, v]]^n = [a[u, v] + [u, v]b, [u, v]]$$

for all $u, v \in [R, R]$, then either R satisfies s_4 and $b - a \in C$ or $a, b \in C$.

Proof. By our assumption, R satisfies the generalized polynomial identity

$$\begin{aligned} &[a[[x_1, x_2], [x_3, x_4]] + [[x_1, x_2], [x_3, x_4]]b, [[x_1, x_2], [x_3, x_4]]]^n = \\ &= [a[[x_1, x_2], [x_3, x_4]] + [[x_1, x_2], [x_3, x_4]]b, [[x_1, x_2], [x_3, x_4]]]. \end{aligned} \tag{2}$$

Let us assume that

$$\begin{aligned} \Omega(x_1, x_2, x_3, x_4) = {}&[a[[x_1, x_2], [x_3, x_4]] + [[x_1, x_2], [x_3, x_4]]b, [[x_1, x_2], [x_3, x_4]]]^n - \\ &-[a[[x_1, x_2], [x_3, x_4]] + [[x_1, x_2], [x_3, x_4]]b, [[x_1, x_2], [x_3, x_4]]]. \end{aligned}$$

Since R and U satisfy the same generalized polynomial identities (see Remark 2), we have $\Omega(x_1, x_2, x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in U$. In case C is infinite, then $\Omega(x_1, x_2, x_3, x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in U \otimes_C \overline{C}$, where $\overline{C}$ denotes the algebraic closure of C. Since U and $U \otimes_C \overline{C}$ are centrally closed (see [8, Theorem 2.5, Theorem 3.5]), we may replace R by U or $U \otimes_C \overline{C}$ according as C is finite or infinite, respectively. Therefore, we may assume that R is centrally closed over C, which is either finite or algebraically closed. If both $a, b \in C$, then we have nothing to prove. Therefore we assume that at least one of a and b is not in C. Then by Remark 2, $\Omega(x_1, x_2, x_3, x_4)$ is a nontrivial generalized polynomial identity for R. Now, with the aid of Martindale's theorem [19], R is a primitive ring having nonzero socle $\mathcal{H}$ with C as the associated division ring. In this sequel, a result due to Jacobson [14, p. 75] yields that R is isomorphic to a dense ring of linear transformations of some vector space V over C. For some positive integer k, let $\dim_C(V) = k < \infty$, then by density of R on V, $R \cong M_k(C)$. In view of our assumption $\dim_C(V) \neq 1$. Moreover, in case $\dim_C(V) = 2$, then R satisfies s_4 and $b - a \in C$ by Lemma 1.

We now assume that $\dim_C(V) \geqslant 3$. For any $v \in V$, we first show that the vectors v and bv are linearly C-dependent. In this view, we suppose that for some $0 \neq v$, the set $\{v, bv\}$ is linearly C-independent and show that a contradiction follows. Since $\dim_C(V) \geqslant 3$, there exists some $w \in V$ such that the set $\{v, bv, w\}$ is linearly C-independent. By the density of R, there exist $x_1, x_2, x_3, x_4 \in R$ such that

$$x_1v = 0; \quad x_2v = -w; \quad x_3v = 0; \quad x_4v = w;$$

$$x_1bv = v; \quad x_2bv = 0; \quad x_3bv = 0; \quad x_4bv = w;$$
$$x_1w = v; \quad x_2w = bv; \quad x_3w = v; \quad x_4w = 0.$$

With all this, our hypothesis implies that

$$\begin{aligned} 0 &= (([a[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]b,[[x_1,x_2],[x_3,x_4]]])^n - \\ &\quad -([a[[x_1,x_2],[x_3,x_4]] + [[x_1,x_2],[x_3,x_4]]b,[[x_1,x_2],[x_3,x_4]]]))v = \\ &= (2^n-2)v. \end{aligned}$$

Since $\mathrm{char}(R) \neq 2$, it leads a contradiction. Thus for any $v \in V$, the vectors v and bv are linearly C-dependent. Therefore, there exists some $\tau_v \in C$ such that $bv = \tau_v v$ for all $v \in V$. By a standard argument, one can easily check that τ_v is not depending on the choice of v, i.e., $bv = \tau v$ for all $v \in V$. In this view, we have

$$\begin{aligned} [b,u]v &= (bu)v - u(bv) \\ &= \tau uv - u\tau v \\ &= 0 \end{aligned}$$

for all $v \in V$. This argument shows that for each $u \in V$, $[b,u]$ acts faithfully as a linear transformation on the vector space V, and hence $[b,u] = 0$, i.e., $b \in Z(R)$. Now Eq. (2) implies that

$$[(a+b)[[x_1,x_2],[x_3,x_4]],[[x_1,x_2],[x_3,x_4]]]^n - [(a+b)[[x_1,x_2],[x_3,x_4]],[[x_1,x_2],[x_3,x_4]]] = 0.$$

for all $x_1,x_2,x_3,x_4 \in R$. In this case, we get $a+b \in C$ by Lemma 2. Hence, $a \in C$.

In case $\dim_C(V) = \infty$, by Wong [24, Lemma 2], R satisfies the generalized polynomial identity

$$([a[u,v]+[u,v]b,[u,v]])^n - [a[u,v]+[u,v]b,[u,v]] = 0.$$

In this case the conclusion follows from [22, Proposition]. It completes the proof. □

3.1. Proof of Theorem 1.2

It is well known that every generalized derivation G takes the form $G(x) = ax + g(x)$ for all $x \in R$, where $a \in U$ (see Remark 1). By [4, Lemma 1], there exists a nonzero ideal I of R such that $[I,I] \subseteq L$. Therefore our hypothesis gives

$$\begin{aligned} &\big([a[[x_1,x_2],[x_3,x_4]] + g([[x_1,x_2],[x_3,x_4]]),[[x_1,x_2],[x_3,x_4]]]\big)^n = \\ &\quad = [a[[x_1,x_2],[x_3,x_4]] + g([[x_1,x_2],[x_3,x_4]]),[[x_1,x_2],[x_3,x_4]]] \end{aligned}$$

for all $x_1,x_2,x_3,x_4 \in I$. In light of Remark 2, we find that R satisfies the GPI

$$\begin{aligned} &\big([a[[x_1,x_2],[x_3,x_4]] + g([[x_1,x_2],[x_3,x_4]]),[[x_1,x_2],[x_3,x_4]]]\big)^n - \\ &\quad -[a[[x_1,x_2],[x_3,x_4]] + g([[x_1,x_2],[x_3,x_4]]),[[x_1,x_2],[x_3,x_4]]]. \end{aligned} \tag{3}$$

If g is the U-inner derivation, i.e., for some $c \in U$, $g(x) = [c,x]$ for all $x \in R$. In this view, we have $G(x) = (a+c)x - xc$ for all $x \in R$. By Proposition 1, we are done.

We now assume that g is not U-inner, in this case we call g an outer derivation. On expending (3), we get

$$\begin{aligned} &\big([a[[x_1,x_2],[x_3,x_4]] + [[g(x_1),x_2],[x_3,x_4]] + [[x_1,g(x_2)],[x_3,x_4]] + [[x_1,x_2],[g(x_3),x_4]] + \\ &\quad +[[x_1,x_2],[x_3,g(x_4)]],[[x_1,x_2],[x_3,x_4]]]\big)^n - [a[[x_1,x_2],[x_3,x_4]] + [[g(x_1),x_2],[x_3,x_4]] + \\ &+[[x_1,g(x_2)],[x_3,x_4]] + [[x_1,x_2],[g(x_3),x_4]] + [[x_1,x_2],[x_3,g(x_4)]],[[x_1,x_2],[x_3,x_4]]] = 0. \end{aligned}$$

With aid of a result due to Kharchenko [16, Theorem 2], R and hence Usatisfies the GPI

$$\big([a[[x_1,x_2],[x_3,x_4]]+[[A,x_2],[x_3,x_4]]+[[x_1,B],[x_3,x_4]]+[[x_1,x_2],[M,x_4]]+ \\ +[[x_1,x_2],[x_3,N]],[[x_1,x_2],[x_3,x_4]]]\big)^n-[a[[x_1,x_2],[x_3,x_4]]+[[A,x_2],[x_3,x_4]]+ \\ +[[x_1,B],[x_3,x_4]]+[[x_1,x_2],[M,x_4]]+[[x_1,x_2],[x_3,N]],[[x_1,x_2],[x_3,x_4]]]=0.$$

In particular, we find

$$\big([a[[x_1,x_2],[x_3,x_4]],[[x_1,x_2],[x_3,x_4]]]\big)^n-[a[[x_1,x_2],[x_3,x_4]],[[x_1,x_2],[x_3,x_4]]]=0$$

for all $x_1,x_2,x_3,x_4\in R$. In view of Lemma 2, it implies that $a\in C$. Thus the above relation reduces to

$$\big([[[A,x_2],[x_3,x_4]]+[[x_1,B],[x_3,x_4]]+[[x_1,x_2],[M,x_4]]+[[x_1,x_2],[x_3,N]],[[x_1,x_2],[x_3,x_4]]]\big)^n- \\ -[[[A,x_2],[x_3,x_4]]+[[x_1,B],[x_3,x_4]]+[[x_1,x_2],[M,x_4]]+[[x_1,x_2],[x_3,N]],[[x_1,x_2],[x_3,x_4]]]=0.$$

Take $A=B=M=0$. It implies that

$$\big([[[x_1,x_2],[x_3,N]],[[x_1,x_2],[x_3,x_4]]]\big)^n=[[[x_1,x_2],[x_3,N]],[[x_1,x_2],[x_3,x_4]]]. \tag{4}$$

Since it is a polynomial identity for R, in view of a result due to Lanski [17, Lemma 1], it follows that for a suitable field F, we have $R\cong M_k(F)$, moreover R and $M_k(F)$ satisfy same generalized polynomial identity. Since R is noncommutative, $k\geqslant 2$. Choose $x_1=e_{ij}$, $x_2=e_{ji}$, $x_3=e_{jj}$, $x_4=e_{ji}$, $N=-2e_{ij}$. With this, we have $[x_1,x_2]=e_{ii}-e_{jj}$, $[x_3,\ x_4]=e_{ji}$ and $[x_3,N]=2e_{ij}$. In this view, from (4), we have

$$(-1)^n8^n(e_{ii}-e_{jj})^n=8(e_{ii}-e_{jj}). \tag{5}$$

If $n=2$, we find $8^2(e_{ii}-e_{jj})^2=8(e_{ii}-e_{jj})$. It implies that $7e_{ii}=-9e_{jj}$ with $i\neq j$, a contradiction. Now we suppose that $n>2$. Right multiply the (5) by e_{ij}, we get

$$(-1)^n8^ne_{ij}=8e_{ij},\text{ i.e., }(-1)^n8^{n-1}e_{ij}=e_{ij}$$

with $i\neq j$, a contradiction. It completes the proof.

Following are immediate consequences of Theorem 1.2.

Corollary 1 ([22], Main Theorem)**.** *Let R be a noncommutative prime ring with $char(R)\neq 2$, U the Utumi qotient ring of R, C the extended centroid of R and L a noncentral Lie ideal of R. If G is a generalized derivation of R with an associated derivation d of R such that $[G(u),u]^n=$ $=[G(u),u]$ for all $u\in L$, where $n>1$ a fixed integer, then one of the following holds:*

- *(i) R satisfies the s_4 identity and there exists $a\in U$ and $\lambda\in C$ such that $G(x)=ax+xa+\lambda x$ for all $x\in R$.*
- *(ii) there exists $\gamma\in C$ such that $G(x)=\gamma x$ for all $x\in R$.*

Corollary 2. *Let $n>1$ are fixed integers. Next, let R be a prime ring with $char(R)\neq 2$, U the Utumi quotient ring, C the extended centroid of R and L a noncentral Lie ideal of R. If T is a left multiplier of R such that $([T(u),u])^n=[T(u),u]$ for all $u\in\{[x,y]:x,y\in L\}$, then there exists $\lambda\in C$ such that $T(x)=\lambda x$ for all $x\in R$.*

Proof. It is well know that every left multiplier is generalized derivation with $g = 0$. Hence, G takes the form $G(x) = ax$ for all $x \in R$ and some $a \in U$. The given hypothesis gives that R satisfies

$$[a[[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, x_4]]]^n = [a[[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, x_4]]].$$

Set $\Omega(x_1, x_2, x_3, x_4) = [[x_1, x_2], [x_3, x_4]], [[x_1, x_2], [x_3, x_4]]$, a multilinear polynomial in the variables x_1, x_2, x_3, x_4. Thus, R satisfies

$$[a\Omega(x_1, x_2, x_3, x_4), \Omega(x_1, x_2, x_3, x_4)]^n = [a\Omega(x_1, x_2, x_3, x_4), \Omega(x_1, x_2, x_3, x_4)].$$

In light of [1, Lemma 3.10], it follows that $a \in C$, which completes the proof. $\square$

3.2. Proof of Theorem 1.3

Let $m,\ n \geqslant 1$ be fixed integers and $T : R \to R$ be left multiplier such that $[T^m(u), u]^n = [T^m(u), u]$ for all $u \in \{[x, y] : x, y \in L\}$, where L is noncentral Lie ideal of R. Then, by using induction on m, it is straightforward to check that T is a left multiplier of a ring R if and only of T^m is a left multiplier of R. Hence, direct application of Corollary 3.5 yields the required result. This completes the proof of Theorem 1.3.

We conclude this article with the following example which demonstrates that Theorem 1.2 does not holds for arbitrary rings.

Example 1. *Let $\mathbb{H}$ denotes the ring of quaternions and*

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c, d \in \mathbb{H} \right\},$$

and $L = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, d \in \mathbb{H} \right\}$ be the noncentral Lie ideal of R. It can be seen that R is not a prime ring. Let us define a mappings $g,\ G : R \to R$ such that $G \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}$ and $g \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to check that G is a nonzero generalized derivation with an associative derivation g of R and satisfying the identity $[G(u), u]^n = [G(u), u]$ for all $u \in [L, L]$. Since $\mathbb{H}$ is a noncommutative rings, it is not difficult to accomplish that R does not satisfy the identity $[[x^2, y], [x, y]]$, which is equivalent to s_4 (see [5, Lemma 1]), consequently R does not satisfy s_4. Therefore, neither R satisfies s_4 nor G takes the form $G(x) = \lambda x$ for all $x \in R$ and some $\lambda \in C$. Hence, the assumption of primeness in Theorem 1.2 can not be omitted.

The research of the second author is supported by SERB-DST Project India under Grant no. MTR/2019/000603.

References

[1] M.Ashraf, S.A.Pary, M.A.Raza, m-potent commutators involving skew derivations and multilinear polynomials, *Georgian Math. J.* (to appear). DOI: 10.1515/gmj-2020-2066

[2] M.Ashraf, M.A.Raza, S.A.Pary, Commutators having idempotent values with automorphisms in semi-prime rings, *Math. Reports*, **20(70)**(2018), no. 1, 51–57

[3] K.I.Beidar, W.S.Martindale III, A.V.Mikhalev, Rings with Generalized Identities, Pure and Applied Math, Marcel Dekker, New York, 1996.

[4] J.Bergen, I.N.Herstein, J.W.Kerr, Lie ideals and derivations of prime rings, *J. Algebra*, **71**(1981), 259–267.

[5] M.Brešar, Commuting traces of biadditive mappings, commutativity preserving mappings and Lie mappings, *Trans. Amer. Math. Soc.*, **335**(1993), no. 2, 525–546.

[6] L.Carini, V. De Filippis, Commutators with power central values on a Lie ideal, *Pacific J. Math.*, **193**(2000), no. 2, 269–278.

[7] C.L.Chuang, GPIs having coefficients in Utumi quotient rings, *Proc. Amer. Math. Soc.*, **103**(1988), 723–728.

[8] T.S.Erickson, W.S.Martindale III, J.M.Osborn, Prime nonassociative algebras, *Pacific J. Math.*, **60**(1975), 49–63.

[9] V. De Filippis, Generalized derivations and commutators with nilpotent values on Lie ideals, *Tamsui Oxf. J. Math. Sci.*, **22**(2006), 167–175.

[10] V. De Filippis, M.A.Raza, N.Rehman, Commutators with idempotent values on multilinear polynomials in prime rings, *Proc. Math. Sci.*, **127**(2017), 91–98.
DOI: 10.1007/s12044-016-0316-1

[11] I.N.Herstein, A generalization of a theorem of Jacobson, *Amer. J. Math.*, **73**(1951), no. 4, 75–762.

[12] I.N.Herstein, A condition for the commutativity of the rings, *Canad. J. Math.*, **9**(1957), 583–586.

[13] B.Hvala, Generalized derivations in rings, *Commun. Algebra*, **26**(1998), no. 4, 1147–1166.
DOI:10.1080/00927879808826190

[14] N.Jacobson, Structure of rings, Amer. Math. Soc. Colloq. Pub. 37, Providence, 1964.

[15] N.Jacobson, Structure theory for algebraic algebras of bounded degree, *Annals of Math.*, **46**(1945), 695–707.

[16] V.K.Kharchenko, Differential identities of prime rings, *Algebra Logic*, **17**(1978), 155–168.

[17] C.Lanski, An Engel condition with derivation, *Proc. Amer. Math. Soc.*, **118**(1993), no. 3, 731–734.

[18] T.K.Lee, Generalized derivations of left faithful rings, *Commun. Algebra*, **27**(1999), 4057–4073. DOI: 10.1080/00927879908826682

[19] W.S.Martindale III, Prime rings satisfying a generalized polynomial identity, *J. Algebra*, **12**(1969), 576–584.

[20] J.Pinter-Lucke, Commutativity conditions for rings 1950–2005, *Expo. Math.*, **25**(2007), 165–174.

[21] E.C.Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.*, **8**(1957), 1093–1100.

[22] G.Scudo, A.Z.Ansari, Generalized derivations on Lie ideals and power values on prime rings, *Math. Slovaca*, **65**(2015), no. 5, 975–980.

[23] J.Vukman, Centralizer on semiprime rings, *Comment. Math. Univ. Carolinae*, **42**(2001), 245–273.

[24] T.L.Wong, Derivations with power values on multilinear polynomials, *Algebra Colloq.*, **3**(1996), 369–378.

[25] B.Zalar, On centralizer of semiprime rings, *Comment. Math. Univ. Carolinae*, **32**(1991), 609–614.

Идемпотентные значения коммутаторов с обобщенными дифференцированиями

Гурниндер С. Сандху
Кафедра математики Мемориальный национальный колледж Пателя
Раджпура, Пенджаб, Индия
Шакир Али
Кафедра математики
Алигархский мусульманский университет
Алигарх, Уттар-Прадеш, Индия

Аннотация. В настоящей статье мы характеризуем обобщенные дифференцирования и левые мультипликаторы первичных колец, включающие коммутаторы с идемпотентными значениями. А именно, мы доказываем, что если первичное кольцо характеристики, отличной от 2, допускает обобщенное дифференцирование G с ассоциативным ненулевым дифференцированием g кольца R такое, что $[G(u), u]^n = [G(u), u]$ для всех $u \in \{[x, y] : x, y \in L\}$, где L — нецентральный идеал Ли R, а $n > 1$ — фиксированное целое число, то выполняется одно из следующих утверждений:

(i) R удовлетворяет s_4 и существует $\lambda \in$C, расширенный центр тяжести R, такой, что $G(x) = ax + xa + \lambda x$ для всех $x \in R$, где $a \in U$, фактор-кольцо Утуми кольца R,

(ii) существует $\lambda \in C$, такое, что $G(x) = \gamma x$ для всех $x \in R$.

В качестве приложения опишем строение левых мультипликаторов первичных колец, удовлетворяющих условию $([T^m(u), u])^n = [T^m(u), u]$ for all $u \in \{[x, y] : x, y \in L\}$, где $m, n > 1$ — фиксированные целые числа. В заключение приведем пример, показывающий, что условие нашей основной теоремы не является избыточным.

Ключевые слова: первичное кольцо, идеал Ли, обобщенный вывод, GPI.

DOI: 10.17516/1997-1397-2022-15-3-366-377
УДК 517.9

A Fixed Point Approach to Study a Class of Probabilistic Functional Equations Arising in the Psychological Theory of Learning

Ali Turab*
Department of Mathematics and Statistics
Faculty of Science and Technology
Thammasat University Rangsit Center
Pathum Thani, Thailand

Received 25.11.2021, received in revised form 30.01.2022, accepted 18.03.2022

Abstract. Many biological and learning theory models have been investigated using probabilistic functional equations. This article focuses on a specific kind of predator–prey relation in which a predator has two prey options, each with a probability of x and $1-x$, respectively. Our aim is to investigate the animal's responses in such situations by proposing a general probabilistic functional equation. The noteworthy fixed-point results are used to investigate the existence, uniqueness, and stability of solutions to the proposed functional equation. An example is also given to illustrate the importance of our results in this area of research.

Keywords: probabilistic functional equations, stability, fixed points.

Citation: A. Turab, A Fixed Point Approach to Study a Class of Probabilistic Functional Equations Arising in the Psychological Theory of Learning, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 366–377. DOI: 10.17516/1997-1397-2022-15-3-366-377.

1. Introduction and preliminaries

Various mathematical learning experiments have recently shown that the behavior of a simple learning experiment follows a stochastic model. Thus, it is not a novel idea (for detail, see [1,2]). Following 1950, however, two critical features were apparent, most notably in the Bush, Estes, and Mosteller research. First, one of the most critical characteristics of the proposed models is the inclusive nature of the learning process. Second, such models may be evaluated in such a manner that their statistical properties are revealed.

In 1976, Istrăţescu [3] examined the participation of predatory animals that feed on two different kinds of prey using the following functional equation

$$\mathscr{R}(x) = x\mathscr{R}(r + (1-r)x) + (1-x)\mathscr{R}((1-s)x), \tag{1}$$

for all $x \in \mathscr{J} = [0,1]$ and $0 < r \leqslant s < 1$, where $\mathscr{R} : \mathscr{J} \to \mathbb{R}$ is an unknown function.

The states x and $(1-x)$ to $r+(1-r)x$ and $(1-s)x$, respectively, were converted into Markov transitions to explain such behavior by $\mathbb{P}(r+(1-r)x) = x$ and $\mathbb{P}((1-s)x) = 1-x$. Sintunavarat and Turab [4] discussed the properties of the above model (1) under the experimenter-subject controlled events.

*taurusnoor@yahoo.com https://orcid.org/0000-0002-5445-9728

In a two-choice scenario, in [2,5], the authors utilized such operators to monitor the movement of a paradise fish under the reinforcement-extinction and the habit formation circumstances (for detail, see Tab. 1).

Table 1. Operators for reinforcement-extinction and habit formation model

Operators for reinforcement-extinction model			
Fish's Responses	**Outcomes (Left side)**	**Outcomes (Right side)**	**Events**
Reinforcement	rx	$rx+1-r$	E_1^{RE}
Non-reinforcement	$sx+1-s$	sx	E_2^{RE}
Operators for habit formation model			
Fish's Responses	**Outcomes (Left side)**	**Outcomes (Right side)**	**Events**
Reinforcement	rx	$rx+1-r$	E_1^{HF}
Non-reinforcement	sx	$sx+1-s$	E_2^{HF}

Berinde and Khan [6] extended the preceding concept by introducing the subsequent functional equation

$$\mathscr{R}(x) = x\mathscr{R}(\mathscr{V}_1(x)) + (1-x)\mathscr{R}(\mathscr{V}_2(x)), \tag{2}$$

for all $x \in \mathscr{J}$, where $\mathscr{V}_1, \mathscr{V}_2 : \mathscr{J} \to \mathscr{J}$ are given mappings and satisfied the following boundary conditions

$$\begin{cases} \mathscr{V}_1(1) = 1, \quad and \\ \mathscr{V}_2(0) = 0. \end{cases} \tag{3}$$

Recently, Turab and Sintunavarat [7] utilized the above ideas and suggested the functional equation stated below

$$\mathscr{R}(x) = x\mathscr{R}(\varpi_1 x + (1-\varpi_1)\Theta_1) + (1-x)\mathscr{R}(\varpi_2 x + (1-\varpi_2)\Theta_2) \ \ \forall x \in \mathscr{J}, \tag{4}$$

where $\mathscr{R} : \mathscr{J} \to \mathbb{R}$ is an unknown, $0 < \varpi_1 \leqslant \varpi_2 < 1$ and $\Theta_1, \Theta_2 \in \mathscr{J}$. The aforementioned functional equation was used to investigate a particular kind of psychological resistance in dogs who were kept in a confined enclosure.

Several additional research on the behaviors of humans and animals in probability-learning situations have yielded a variety of diverse conclusions (see [8–13]).

As a result of the previous research, we propose the following general probabilistic functional equation

$$\begin{aligned} \mathscr{R}(x) = {} & \left(\frac{w-j}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_1(x)) + \left(\frac{w-j}{k-j}\right)\left(1-\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_2(x)) + \\ & + \left(1-\frac{w-j}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_3(x)) + \left(1-\frac{w-j}{k-j}\right)\left(1-\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_4(x)), \end{aligned} \tag{5}$$

for all $x \in [j,k]$, where $0 \leqslant w \leqslant 1$, $\mathscr{R} : [j,k] \to \mathbb{R}$ is an unknown and $\tau, \mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3, \mathscr{V}_4 : [j,k] \to [j,k]$ are given mappings.

The Banach fixed point theorem will be used to establish the existence and uniqueness results of the above equation (5). Finally, we examine the stability of the suggested stochastic equation's solution.

The following stated result will be needed in later sections.

Theorem 1.1 ([14]). *Let $(\mathscr{J}, d)$ be a complete metric space and $\mathscr{R} : \mathscr{J} \to \mathscr{J}$ be a mapping defined by*

$$d(\mathscr{R}s, \mathscr{R}t) \leqslant \Lambda d(s,t) \tag{6}$$

for some $\Lambda < 1$ and for all $s, t \in \mathscr{J}$. Then $\mathscr{R}$ has precisely one fixed point. Furthermore, the Picard iteration $\{s_n\}$ in $\mathscr{J}$ which is defined by $s_n = \mathscr{R}s_{n-1}$ for all $n \in \mathbb{N}$, where $s_0 \in \mathscr{J}$, converges to the unique fixed point of $\mathscr{R}$.

2. Main results

Let $\mathscr{J} = [j,k]$ with $j < k$, where $j, k \in \mathbb{R}$. We indicate the class $\mathscr{R} : \mathscr{J} \to \mathbb{R}$ consisting of all continuous real-valued functions by $\mathscr{T}$ such that $\mathscr{R}(j) = 0$ and

$$\sup_{s \neq t} \frac{|\mathscr{R}(s) - \mathscr{R}(t)|}{|s-t|} < \infty.$$

We can see that $(\mathscr{T}, \|\cdot\|)$ is a normed space (for the detail, see [5,15]), where $\|\cdot\|$ is given by

$$\|\mathscr{R}\| = \sup_{s \neq t} \frac{|\mathscr{R}(s) - \mathscr{R}(t)|}{|s-t|} \tag{7}$$

for all $\mathscr{R} \in \mathscr{T}$.

Next, we rewrite (5) as

$$\begin{aligned} \mathscr{R}(x) \;=\; & \left(\frac{w-j}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_1(x)) + \left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_2(x)) + \\ & + \left(\frac{k-w}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_3(x)) + \left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_4(x)), \end{aligned} \tag{8}$$

where $\mathscr{R} : \mathscr{J} \to \mathbb{R}$ is an unknown function such that $\mathscr{R}(j) = 0$. Also, $\mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3, \mathscr{V}_4 : \mathscr{J} \to \mathscr{J}$ are contraction mappings with contractive coefficients b_1, b_2, b_3 and b_4 respectively. Also, the following condition holds

$$\mathscr{R}(\mathscr{V}_2(j)) = j = \mathscr{R}(\mathscr{V}_4(j)). \tag{9}$$

Furthermore, $\tau : \mathscr{J} \to \mathscr{J}$ is a non-expansive mapping with $\tau(j) = j$ and $|\tau(x)| \leqslant b_5$, for all $x \in \mathscr{J}$ with $b_5 \geqslant 0$.

Before proving the main results, we mention the following conditions here.

$(\mathcal{A}_1)$: For the mappings $\mathscr{V}_1, \mathscr{V}_2 : \mathscr{J} \to \mathscr{J}$, we have

$$|\mathscr{V}_1(u) - \mathscr{V}_2(v)| \leqslant b_6 |u-v|, \tag{10}$$

for all $u, v \in \mathscr{J}$ with $u \neq v$, where $b_6 \in [0,1)$.

$(\mathcal{A}_2)$: For the mappings $\mathscr{V}_3, \mathscr{V}_4 : \mathscr{J} \to \mathscr{J}$, we have

$$|\mathscr{V}_3(u) - \mathscr{V}_4(v)| \leqslant b_7 |u-v|, \tag{11}$$

for all $u, v \in \mathscr{J}$ with $u \neq v$, where $b_7 \in [0,1)$.

$(\mathcal{A}_3)$: For the mappings $\mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3, \mathscr{V}_4 : \mathscr{J} \to \mathscr{J}$, there exist points $u^\star, v^\star \in [j,k]$ such that

$$\mathscr{V}_1(u^\star) = \mathscr{V}_2(u^\star) \quad \text{and} \quad \mathscr{V}_3(v^\star) = \mathscr{V}_4(v^\star). \tag{12}$$

We begin with the succeeding outcome.

Theorem 2.1. *Consider the probabilistic functional equation (8) with (9). Assume that the conditions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *hold with* $\Lambda_1 < 1$, *where*

$$\begin{aligned} \Lambda_1 \;:=\; & \Bigg|\left(\frac{w-j}{k-j}\right)\left[b_1\left(1+\frac{b_5-j}{k-j}\right)+b_2\left(\frac{k-b_5}{k-j}\right)+b_6\right]+ \\ & +\left(\frac{k-w}{k-j}\right)\left[b_3\left(1+\frac{b_5-j}{k-j}\right)+b_4\left(\frac{k-b_5}{k-j}\right)+b_7\right]\Bigg|, \end{aligned} \tag{13}$$

and there is a nonempty subset $\mathscr{E}$ *of* $\mathscr{S} := \{\mathscr{R} \in \mathscr{T} | \mathscr{R}(k) \leqslant k\}$ *such that* $(\mathscr{E}, \|\cdot\|)$ *is a Banach space, where* $\|\cdot\|$ *is given in (7), and the mapping* $\mathscr{K}$ *from* $\mathscr{E}$ *defined for each* $\mathscr{R} \in \mathscr{T}$ *by*

$$\begin{aligned} (\mathscr{K}\mathscr{R})(x) \;=\; & \left(\frac{w-j}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_1(x))+\left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_2(x))+ \\ & +\left(\frac{k-w}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_3(x))+\left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_4(x)), \end{aligned} \tag{14}$$

for all $x \in \mathscr{J}$ *is a self mapping. Then* $\mathscr{K}$ *is a Banach contraction mapping with the metric* d *induced by* $\|\cdot\|$.

Proof. Let $\mathscr{R}_1, \mathscr{R}_2 \in \mathscr{E}$. For each $u, v \in \mathscr{J}$ with $u \neq v$, we obtain

$$\begin{aligned} &|\Theta_{u\neq v}| := \frac{|\mathscr{K}(\mathscr{R}_1-\mathscr{R}_2)(u)-\mathscr{K}(\mathscr{R}_1-\mathscr{R}_2)(v)|}{|u-v|} = \\ &=\Bigg|\frac{1}{u-v}\Bigg[\left(\frac{w-j}{k-j}\right)\left(\frac{\tau(u)-j}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_1(u))+\left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(u)}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_2(u)) \\ &+\left(\frac{k-w}{k-j}\right)\left(\frac{\tau(u)-j}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_3(u))+\left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(u)}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_4(u)) \\ &-\left(\frac{w-j}{k-j}\right)\left(\frac{\tau(v)-j}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_1(v))-\left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(v)}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_2(v)) \\ &-\left(\frac{k-w}{k-j}\right)\left(\frac{\tau(v)-j}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_3(v))-\left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(v)}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_4(v))\Bigg]\Bigg| = \\ &=\Bigg|\frac{1}{u-v}\left[\left(\frac{w-j}{k-j}\right)\left(\frac{\tau(u)-j}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_1(u))-\left(\frac{w-j}{k-j}\right)\left(\frac{\tau(u)-j}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_1(v))\right] \\ &+\frac{1}{u-v}\left[\left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(u)}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_2(u))-\left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(u)}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_2(v))\right] \\ &+\frac{1}{u-v}\left[\left(\frac{k-w}{k-j}\right)\left(\frac{\tau(u)-j}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_3(u))-\left(\frac{k-w}{k-j}\right)\left(\frac{\tau(u)-j}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_3(v))\right] \\ &+\frac{1}{u-v}\left[\left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(u)}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_4(u))-\left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(u)}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_4(v))\right] \\ &+\frac{1}{u-v}\left[\left(\frac{w-j}{k-j}\right)\left(\frac{\tau(u)-j}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_1(v))-\left(\frac{w-j}{k-j}\right)\left(\frac{\tau(v)-j}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_1(v))\right] \\ &+\frac{1}{u-v}\left[\left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(u)}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_2(v))-\left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(v)}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_2(v))\right] \\ &+\frac{1}{u-v}\left[\left(\frac{k-w}{k-j}\right)\left(\frac{\tau(u)-j}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_3(v))-\left(\frac{k-w}{k-j}\right)\left(\frac{\tau(v)-j}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_3(v))\right] \\ &+\frac{1}{u-v}\left[\left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(u)}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_4(v))-\left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(v)}{k-j}\right)(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_4(v))\right]\Bigg| \end{aligned}$$

As $\mathscr{V}_1 - \mathscr{V}_4$ are contraction mappings with the contractive coefficients $b_1 - b_4$, respectively. Thus, by using the definition of the norm (7), we have

$$|\Theta_{u\neq v}| \leqslant \Lambda_1 \left\| \mathscr{R}_1 - \mathscr{R}_2 \right\|,$$

where Λ_1 is defined in (13). This gives that

$$d(\mathscr{K}\mathscr{R}_1, \mathscr{K}\mathscr{R}_2) = \|\mathscr{K}\mathscr{R}_1 - \mathscr{K}\mathscr{R}_2\| \leqslant \Lambda_1 \left\| \mathscr{R}_1 - \mathscr{R}_2 \right\| = \Lambda_1 d(\mathscr{R}_1, \mathscr{R}_2).$$

As $0 < \Lambda_1 < 1$, we conclude that $\mathscr{K}$ is a Banach contraction mapping with metric d induced by $\| \cdot \|$. □

Theorem 2.2. *Consider the probabilistic functional equation (8) associated with (9). Assume that the conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold with $\Lambda_1 < 1$, where Λ_1 is defined in (13). Also, there exist a nonempty subset $\mathscr{E}$ of $\mathscr{S} := \{\mathscr{R} \in \mathscr{T} | \mathscr{R}(k) \leqslant k\}$ such that $(\mathscr{E}, \| \cdot \|)$ is a Banach space, where $\| \cdot \|$ is given in (7), and the mapping $\mathscr{K}$ from $\mathscr{E}$ defined for each $\mathscr{R} \in \mathscr{T}$ by (10) is a self mapping. Then, the functional equation (8) with (9) has a unique solution in $\mathscr{E}$. Furthermore, the iteration $\mathscr{R}_n$ in $\mathscr{E}$ can be defined by*

$$\begin{aligned}(\mathscr{R}_n)(x) = &\left(\frac{w-j}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_1(x)) + \left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_2(x)) + \\ &+ \left(\frac{k-w}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_3(x)) + \left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_4(x)), \quad (15)\end{aligned}$$

for all $n \in \mathbb{N}$, where $\mathscr{R}_0 \in \mathscr{E}$, converges to the unique solution of (8).

Proof. We get the conclusion of this theorem by combining Theorem 2.1 with the Banach fixed point theorem. □

Remark 2.3. *Our proposed probabilistic equation (5) is a generalization of the functional equations discussed in [3, 5–7].*

Here, we shall look at different conditions. If $\mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3, \mathscr{V}_4 : \mathscr{J} \to \mathscr{J}$ are contraction mappings with contractive coefficients $b_1 \leqslant b_2 \leqslant b_3 \leqslant b_4$, respectively, then by Theorems 2.1 and 2.2, the outcomes are as follows.

Corollary 2.4. *Consider the probabilistic functional equation (8) associated with (9). Assume that the conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold with $\tilde{\Lambda}_1 < 1$, where*

$$\tilde{\Lambda}_1 := \left| 2b_4 + \frac{1}{k-j}\left[(w-j)b_6 + (k-w)b_7\right] \right|, \quad (16)$$

and there is a nonempty subset $\mathscr{E}$ of $\mathscr{S} := \{\mathscr{R} \in \mathscr{T} | \mathscr{R}(k) \leqslant k\}$ such that $(\mathscr{E}, \| \cdot \|)$ is a Banach space, where $\| \cdot \|$ is given in (7), and the mapping $\mathscr{K}$ from $\mathscr{E}$ defined for each $\mathscr{R} \in \mathscr{T}$ by

$$\begin{aligned}(\mathscr{K}\mathscr{R})(x) \;=\; &\left(\frac{w-j}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_1(x)) + \left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_2(x)) + \\ &+ \left(\frac{k-w}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_3(x)) + \left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_4(x)), \quad (17)\end{aligned}$$

for all $x \in \mathscr{J}$ is a self mapping. Then $\mathscr{K}$ is a Banach contraction mapping with the metric d induced by $\| \cdot \|$.

Corollary 2.5. *Consider the probabilistic functional equation (8) associated with (9). Assume that the conditions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$ *hold with* $\tilde{\Lambda}_1 < 1$, *where* $\tilde{\Lambda}_1$ *is given in (16), and there is a nonempty subset* $\mathscr{E}$ *of* $\mathscr{S} := \{\mathscr{R} \in \mathscr{T} | \mathscr{R}(k) \leqslant k\}$ *such that* $(\mathscr{E}, \|\cdot\|)$ *is a Banach space, where* $\|\cdot\|$ *is given in (7), and the mapping* $\mathscr{K}$ *from* $\mathscr{E}$ *defined for each* $\mathscr{R} \in \mathscr{T}$ *by (17) is a self mapping. Then, the functional equation (8) with (9) has a unique solution in* $\mathscr{E}$. *Furthermore, the iteration* $\mathscr{R}_n$ *in* $\mathscr{E}$ *is defined as*

$$\begin{aligned}(\mathscr{R}_n)(x) = &\left(\frac{w-j}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_1(x)) + \left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_2(x)) + \\ &+ \left(\frac{k-w}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_3(x)) + \left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_4(x)), \end{aligned} \tag{18}$$

for all $n \in \mathbb{N}$, *where* $\mathscr{R}_0 \in \mathscr{E}$, *converges to the unique solution of (8).*

Theorem 2.6. *Consider the probabilistic functional equation (8) with (9). Assume that the condition* $(\mathcal{A}_3)$ *holds with* $\Lambda_2 < 1$, *where*

$$\begin{aligned}\Lambda_2 \quad := \quad &\left|\left(\frac{w-j}{k-j}\right)\left(b_1\left(1+\frac{b_5-j}{k-j}\right) + b_2\left(1+\frac{k-b_5}{k-j}\right)\right) + \right. \\ &\left. + \left(\frac{k-w}{k-j}\right)\left(b_3\left(1+\frac{b_5-j}{k-j}\right) + b_4\left(1+\frac{k-b_5}{k-j}\right)\right)\right|. \end{aligned} \tag{19}$$

Suppose that there is a nonempty subset $\mathscr{E}$ *of* $\mathscr{S} := \{\mathscr{R} \in \mathscr{T} | \mathscr{R}(k) \leqslant k\}$ *such that* $(\mathscr{E}, \|\cdot\|)$ *is a Banach space, where* $\|\cdot\|$ *is given in (7), and the mapping* $\mathscr{K}$ *from* $\mathscr{E}$ *defined for each* $\mathscr{R} \in \mathscr{T}$ *by*

$$\begin{aligned}(\mathscr{K}\mathscr{R})(x) = &\left(\frac{w-j}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_1(x)) + \left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_2(x)) + \\ &+ \left(\frac{k-w}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_3(x)) + \left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_4(x)), \end{aligned} \tag{20}$$

for all $x \in \mathscr{J}$ *is a self mapping. Then* $\mathscr{K}$ *is a Banach contraction mapping with the metric* d *induced by* $\|\cdot\|$.

Proof. The line of proof of this theorem is the same as Theorem 2.1. Here, we highlight those parts which are different from the previous theorem.

Let $\mathscr{R}_1, \mathscr{R}_2 \in \mathscr{E}$. For each $u, v \in \mathscr{J}$ with $u \neq v$, we obtain

$$\begin{aligned}&|\Theta_{u\neq v}| \leqslant \\ &\leqslant \frac{1}{|u-v|}\left(\frac{w-j}{k-j}\right)\left(\frac{\tau(u)-j}{k-j}\right) \times \left[\frac{|(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_1(u)) - (\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_1(v))|}{|\mathscr{V}_1(u)-\mathscr{V}_1(v)|} \times |\mathscr{V}_1(u)-\mathscr{V}_1(v)|\right] \\ &+ \frac{1}{|u-v|}\left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(u)}{k-j}\right) \times \left[\frac{|(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_2(u)) - (\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_2(v))|}{|\mathscr{V}_2(u)-\mathscr{V}_2(v)|} \times |\mathscr{V}_2(u)-\mathscr{V}_2(v)|\right] \\ &+ \frac{1}{|u-v|}\left(\frac{k-w}{k-j}\right)\left(\frac{\tau(u)-j}{k-j}\right) \times \left[\frac{|(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_3(u)) - (\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_3(v))|}{|\mathscr{V}_3(u)-\mathscr{V}_3(v)|} \times |\mathscr{V}_3(u)-\mathscr{V}_3(v)|\right] \\ &+ \frac{1}{|u-v|}\left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(u)}{k-j}\right) \times \left[\frac{|(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_4(u)) - (\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_4(v))|}{|\mathscr{V}_4(u)-\mathscr{V}_4(v)|} \times |\mathscr{V}_4(u)-\mathscr{V}_4(v)|\right] \\ &\quad + \left(\frac{1}{k-j}\right)\left(\frac{w-j}{k-j}\right) \times \left[\frac{|(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_1(v)) - (\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_1(u^\star))|}{|\mathscr{V}_1(v)-\mathscr{V}_1(u^\star)|} \times |\mathscr{V}_1(v)-\mathscr{V}_1(u^\star)|\right]\end{aligned}$$

$$+\left(\frac{1}{k-j}\right)\left(\frac{w-j}{k-j}\right)\times\left[\frac{|(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_2(u^\star))-(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_2(v))|}{|\mathscr{V}_2(u^\star)-\mathscr{V}_2(v)|}\times|\mathscr{V}_2(u^\star)-\mathscr{V}_2(v)|\right]$$
$$+\left(\frac{1}{k-j}\right)\left(\frac{w-j}{k-j}\right)\times\left[\frac{|(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_3(v))-(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_3(v^\star))|}{|\mathscr{V}_3(v)-\mathscr{V}_3(v^\star)|}\times|\mathscr{V}_3(v)-\mathscr{V}_3(v^\star)|\right]$$
$$+\left(\frac{1}{k-j}\right)\left(\frac{w-j}{k-j}\right)\times\left[\frac{|(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_4(v^\star))-(\mathscr{R}_1-\mathscr{R}_2)(\mathscr{V}_4(v))|}{|\mathscr{V}_4(v^\star)-\mathscr{V}_4(v)|}\times|\mathscr{V}_4(v^\star)-\mathscr{V}_4(v)|\right]. \quad (21)$$

Here, we discuss the following cases.

Case 1: If $v = u^\star = v^\star$, then by (21) we have

$$|\Theta_{u\neq v}| \leqslant \Lambda_2 \|\mathscr{R}_1-\mathscr{R}_2\|.$$

Case 2: If $v \neq u^\star$, $v = v^\star$, then by (21) we have

$$|\Theta_{u\neq v}| \leqslant \Lambda_2 \|\mathscr{R}_1-\mathscr{R}_2\|.$$

Case 3: If $v = u^\star$, $v \neq v^\star$, then by (21) we have

$$|\Theta_{u\neq v}| \leqslant \Lambda_2 \|\mathscr{R}_1-\mathscr{R}_2\|.$$

Case 4: If $v \neq u^\star \neq v^\star$, then by (21) we have

$$|\Theta_{u\neq v}| \leqslant \Lambda_2 \|\mathscr{R}_1-\mathscr{R}_2\|,$$

where Λ_2 is defined in (19). This gives that

$$d(\mathscr{K}\mathscr{R}_1,\mathscr{K}\mathscr{R}_2)=\|\mathscr{K}\mathscr{R}_1-\mathscr{K}\mathscr{R}_2\|\leqslant\Lambda_2\|\mathscr{R}_1-\mathscr{R}_2\|=\Lambda_2 d(\mathscr{R}_1,\mathscr{R}_2).$$

As a result of $0<\Lambda_2<1$, we can conclude that $\mathscr{K}$ is a Banach contraction mapping with metric d induced by $\|\cdot\|$. □

Theorem 2.7. *Consider the probabilistic functional equation (8) associated with (9). Assume that the condition* $(\mathcal{A}_3)$ *holds with* $\Lambda_2<1$*, where* Λ_2 *is defined in (19). Also, there is a nonempty subset* $\mathscr{E}$ *of* $\mathscr{S}:=\{\mathscr{R}\in\mathscr{T}|\mathscr{R}(k)\leqslant k\}$ *such that* $(\mathscr{E},\|\cdot\|)$ *is a Banach space, where* $\|\cdot\|$ *is given in (7), and the mapping* $\mathscr{K}$ *from* $\mathscr{E}$ *defined for each* $\mathscr{R}\in\mathscr{T}$ *by (20) is a self mapping. Then, the functional equation (8) with (9) has a unique solution in* $\mathscr{E}$*. Furthermore, the iteration* $\mathscr{R}_n$ *in* $\mathscr{E}$ *can be defined by*

$$(\mathscr{R}_n)(x)=\left(\frac{w-j}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_1(x))+\left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_2(x))+$$
$$+\left(\frac{k-w}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_3(x))+\left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_4(x)), \quad (22)$$

for all $n\in\mathbb{N}$*, where* $\mathscr{R}_0\in\mathscr{E}$*, converges to the unique solution of (8).*

Proof. By coupling the Banach fixed point theorem with Theorem 2.6, we obtain the conclusion of this theorem. □

If $\mathscr{V}_1,\mathscr{V}_2,\mathscr{V}_3,\mathscr{V}_4:\mathscr{J}\to\mathscr{J}$ are contraction mappings with contractive coefficients $b_1\leqslant b_2\leqslant b_3\leqslant b_4$, respectively, then by Theorems 2.6 and 2.7, the outcomes are as follows.

Corollary 2.8. *Consider the probabilistic functional equation (8) associated with (9). Assume that the condition* $(\mathcal{A}_3)$ *holds with* $\tilde{\Lambda}_2 := 3b_4 < 1$. *Also, there is a nonempty subset* $\mathscr{E}$ *of* $\mathscr{S} := \{\mathscr{R} \in \mathscr{T} | \mathscr{R}(k) \leqslant k\}$ *such that* $(\mathscr{E}, \|\cdot\|)$ *is a Banach space, where* $\|\cdot\|$ *is given in (7), and the mapping* $\mathscr{K}$ *from* $\mathscr{E}$ *defined for each* $\mathscr{R} \in \mathscr{T}$ *by*

$$(\mathscr{K}\mathscr{R})(x) = \left(\frac{w-j}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_1(x)) + \left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_2(x)) + \\ + \left(\frac{k-w}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_3(x)) + \left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_4(x)), \tag{23}$$

for all $x \in \mathscr{J}$ *is a self mapping. Then* $\mathscr{K}$ *is a Banach contraction mapping with the metric* d *induced by* $\|\cdot\|$.

Corollary 2.9. *Consider the probabilistic functional equation (8) associated with (9). Assume that the condition* $(\mathcal{A}_3)$ *holds with* $\tilde{\Lambda}_2 := 3b_4 < 1$. *Also, there is a nonempty subset* $\mathscr{E}$ *of* $\mathscr{S} := \{\mathscr{R} \in \mathscr{T} | \mathscr{R}(k) \leqslant k\}$ *such that* $(\mathscr{E}, \|\cdot\|)$ *is a Banach space, where* $\|\cdot\|$ *is given in (7), and the mapping* $\mathscr{K}$ *from* $\mathscr{E}$ *defined for each* $\mathscr{R} \in \mathscr{T}$ *by (23) is a self mapping. Then, the functional equation (8) with (9) has a unique solution in* $\mathscr{E}$. *Furthermore, the iteration* $\mathscr{R}_n$ *in* $\mathscr{E}$ *is defined as*

$$(\mathscr{R}_n)(x) = \left(\frac{w-j}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_1(x)) + \left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_2(x)) + \\ + \left(\frac{k-w}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_3(x)) + \left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}_{n-1}(\mathscr{V}_4(x)), \tag{24}$$

for all $n \in \mathbb{N}$, *where* $\mathscr{R}_0 \in \mathscr{E}$, *converges to the unique solution of (7).*

Remark 2.10. *The authors of [3, 4, 6] utilized the boundary conditions to prove their major findings. However, compared to them, our results are independent of such conditions.*

We now offer the following example to enhance our findings.

Example. Consider the probabilistic functional equation given below

$$\mathscr{R}(x) = wx\mathscr{R}\left(\frac{x}{16}+\frac{1}{11}\right) + w(1-x)\mathscr{R}\left(\frac{x}{6}\right) + (1-w)x\mathscr{R}\left(\frac{x}{19}+\frac{1}{33}\right) + (1-w)(1-x)\mathscr{R}\left(\frac{x}{12}\right) \tag{25}$$

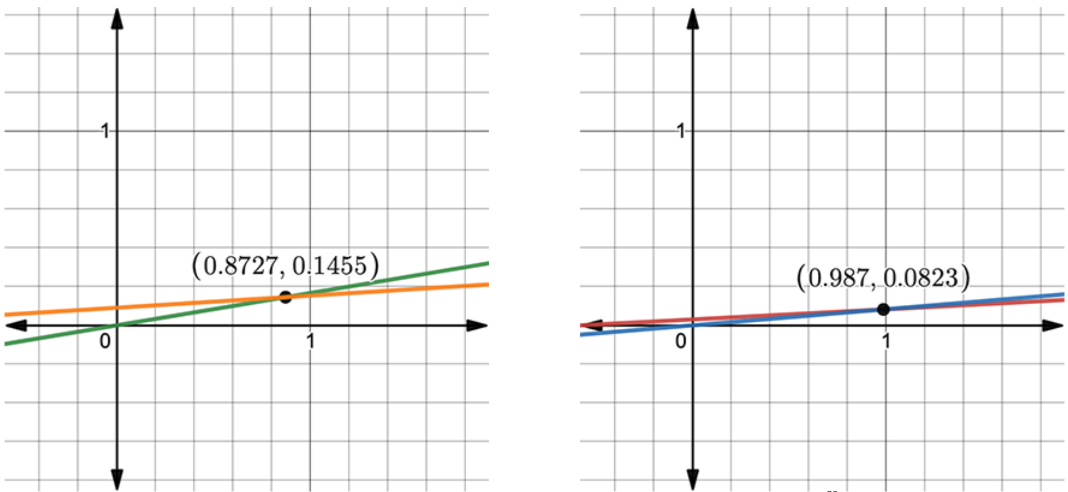
for all $x \in \mathscr{J} = [0,1]$ and $\mathscr{R} \in \mathscr{T}$. If we set the mappings $\tau, \mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3, \mathscr{V}_4 : \mathscr{J} \to \mathscr{J}$ by

$$\tau(x) = x, \quad \mathscr{V}_1(x) = \frac{x}{16}+\frac{1}{11}, \quad \mathscr{V}_2(x) = \frac{x}{6}, \quad \mathscr{V}_3(x) = \frac{x}{19}+\frac{1}{33}, \quad \mathscr{V}_4(x) = \frac{x}{12},$$

for all $x \in \mathscr{J}$. So, our equation (25) is decreased to the equation (8). It is easy to see that $\mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3, \mathscr{V}_4$ satisfy our boundary conditions (9). Also,

$$|\mathscr{V}_1 u - \mathscr{V}_1 v| \leqslant \frac{1}{16}|u-v|, \ |\mathscr{V}_2 u - \mathscr{V}_2 v| \leqslant \frac{1}{6}|u-v|, \ |\mathscr{V}_3 u - \mathscr{V}_3 v| \leqslant \frac{1}{19}|u-v|, \ |\mathscr{V}_4 u - \mathscr{V}_4 v| \leqslant \frac{1}{12}|u-v|$$

for all $u, v \in \mathscr{J}$. This implies that $\mathscr{V}_1 - \mathscr{V}_4$ are contraction mappings with coefficients $b_1 = \frac{1}{16}$, $b_2 = \frac{1}{6}$, $b_3 = \frac{1}{19}$ and $b_4 = \frac{1}{12}$ respectively. Also, there exist points $u^\star, v^\star \in [0,1]$ such that $\mathscr{V}_1(u^\star) = \mathscr{V}_2(u^\star)$ and $\mathscr{V}_3(v^\star) = \mathscr{V}_4(v^\star)$ (see Fig. 1).



(a) $\mathscr{L}_1(x) = \dfrac{x}{16} + \dfrac{1}{11}$ (orange) and $\mathscr{L}_2(x) = \dfrac{x}{6}$ (green)

(b) $\mathscr{L}_3(x) = \dfrac{x}{19} + \frac{1}{33}$ (red) and $\mathscr{L}_4(x) = \dfrac{x}{12}$ (blue)

Fig. 1. Graphs of $\mathscr{L}_1(x),\ \mathscr{L}_2(x),\ \mathscr{L}_3(x),\ and\ \mathscr{L}_4(x)$

Moreover, $\Lambda_2 = \dfrac{47w+86}{456} < 1$, for all $w \in [0,1]$, and there is a nonempty set $\mathscr{E}$ of $\mathscr{S} := \{\mathscr{R} \in \mathscr{T} | \mathscr{R}(1) \leqslant 1\}$ such that $(\mathscr{E}, \|\cdot\|)$ is a Banach space, and the mapping $\mathscr{K}$ from $\mathscr{E}$ defined in (25) for all $x \in \mathscr{J}$ is a self mapping, thus it fulfill all the requirements of Theorem 2.6, and therefore, we get the results related to the existence of the given equation (25)' solution.

If we define $\mathscr{R}_0 = x$ as a starting approximation, then by Theorem 2.7, the iteration stated below converges to a unique solution of (25):

$$
\begin{aligned}
\mathscr{R}_1(x) &= \frac{1}{10032}\left[-737wx^2 - 308x^2 + 1444wx + 1140x\right], \\
\mathscr{R}_2(x) &= \frac{wx}{28250112}\left[-8107wx^2 - 3388x^2 + 230560wx + 190784x + 352512w + 284672\right] + \\
&\quad + \frac{w(1-x)}{361152}\left[-737wx^2 - 308x^2 + 8664wx + 6840x\right] + \\
&\quad + \frac{(1-w)x}{358533648}\left[72963wx^2 - 30492x^2 + 2632146wx + 2109228x + 1539665w + 1224512\right] + \\
&\quad + \frac{(1-w)(1-x)}{1444608}\left[-737wx^2 - 308x^2 + 17328wx + 13680x,\right] \\
&\cdots \\
\mathscr{R}_n(x) &= wx\mathscr{R}_{n-1}\left(\mathscr{V}_1(x)\right) + w(1-x)\mathscr{R}_{n-1}\left(\mathscr{V}_2(x)\right) + (1-w)x\mathscr{R}_{n-1}\left(\mathscr{V}_3(x)\right) + \\
&\quad + (1-w)(1-x)\mathscr{R}_{n-1}\left(\mathscr{V}_4(x)\right)
\end{aligned}
$$

for all $n \in \mathbb{N}$.

3. Stability analysis of the proposed probabilistic functional equation

Now, we shall discuss the stability of the suggested functional equation (7) (for the details of stability, we refer [16], [17].

Theorem 3.1. *Under the hypothesis of Theorem 2.1, the equation $\mathscr{K}\mathscr{R} = \mathscr{R}$, where $\mathscr{K} : \mathscr{E} \to \mathscr{E}$ is defined as*

$$(\mathscr{K}\mathscr{R})(x) = \left(\frac{w-j}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_1(x)) + \left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_2(x)) + \\ + \left(\frac{k-w}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_3(x)) + \left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_4(x)), \quad (26)$$

for all $\mathscr{R} \in \mathscr{E}$ and $x \in \mathscr{J}$, has Hyers-Ulam-Rassias stability; that is, for a fixed function $\varphi : \mathscr{E} \to [0, \infty)$, we have that for every $\mathscr{R} \in \mathscr{E}$ with $\mathrm{d}(\mathscr{K}\mathscr{R}, \mathscr{R}) \leqslant \varphi(\mathscr{R})$, there exists a unique $\tilde{\mathscr{R}} \in \mathscr{E}$ such that $\mathscr{K}\tilde{\mathscr{R}} = \tilde{\mathscr{R}}$ and $\mathrm{d}(\mathscr{R}, \tilde{\mathscr{R}}) \leqslant \varsigma\varphi(\mathscr{R})$ for some $\varsigma > 0$.

Proof. Let $\mathscr{R} \in \mathscr{E}$ such that $d(\mathscr{K}\mathscr{R}, \mathscr{R}) \leqslant \varphi(\mathscr{R})$. From Theorem 2.1, there exists a unique $\tilde{\mathscr{R}} \in \mathscr{E}$ such that $\mathscr{K}\tilde{\mathscr{R}} = \tilde{\mathscr{R}}$. Then we have

$$d(\mathscr{R}, \tilde{\mathscr{R}}) \leqslant d(\mathscr{R}, \mathscr{K}\mathscr{R}) + d(\mathscr{K}\mathscr{R}, \tilde{\mathscr{R}}) \leqslant \varphi(\mathscr{R}) + d(\mathscr{K}\mathscr{R}, \mathscr{K}\tilde{\mathscr{R}}) \leqslant \varphi(\mathscr{R}) + \Lambda_1 d(\mathscr{R}, \tilde{\mathscr{R}}),$$

where Λ_1 is defined in (13), and so

$$d(\mathscr{R}, \tilde{\mathscr{R}}) \leqslant \varsigma\varphi(\mathscr{R}),$$

where $\varsigma := \dfrac{1}{1-\Lambda_1}$. This completes the proof. □

From the above analysis, we obtain the following result related to the Hyers-Ulam stability.

Corollary 3.2. *Under the hypothesis of Theorem 2.1, the equation $\mathscr{K}\mathscr{R} = \mathscr{R}$, where $\mathscr{K} : \mathscr{E} \to \mathscr{E}$ is defined as*

$$(\mathscr{K}\mathscr{R})(x) = \left(\frac{w-j}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_1(x)) + \left(\frac{w-j}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_2(x)) + \\ + \left(\frac{k-w}{k-j}\right)\left(\frac{\tau(x)-j}{k-j}\right)\mathscr{R}(\mathscr{V}_3(x)) + \left(\frac{k-w}{k-j}\right)\left(\frac{k-\tau(x)}{k-j}\right)\mathscr{R}(\mathscr{V}_4(x)), \quad (27)$$

for all $\mathscr{R} \in \mathscr{E}$ and $x \in \mathscr{J}$, has Hyers-Ulam stability; that is, for a fixed $\nu > 0$, we have that for every $\mathscr{R} \in \mathscr{E}$ with $\mathrm{d}(\mathscr{K}\mathscr{R}, \mathscr{R}) \leqslant \nu$, there exists a unique $\tilde{\mathscr{R}} \in \mathscr{E}$ such that $\mathscr{K}\tilde{\mathscr{R}} = \tilde{\mathscr{R}}$ and $\mathrm{d}(\mathscr{R}, \tilde{\mathscr{R}}) \leqslant \varsigma\nu$, for some $\varsigma > 0$.

Conclusion

The predator-prey paradigm, especially in a two-choice situation, is one of the most exciting frameworks in mathematical biology. A predator has two possible prey choices in these models, and the solution exists when the predator is fixed to one of them. We extended the research by introducing a generic stochastic functional equation in this paper that may cover a wide range of learning theory models in the existing literature. The existence, uniqueness, and stability of the proposed stochastic equation's solution were investigated using a fixed-point method. Our techniques do not rely on the boundary conditions discussed in [6, 9], which implies that the proposed results cover more problems than the results described in the literature. Our method is unique, and it may be used to solve a wide variety of mathematical models in the fields of mathematical psychology and learning theory.

References

[1] A.A.Bush, T.R.Wilson, Two-choice behavior of paradise fish, *J. Exp. Psych.*, **51**(1956), 315-322. DOI: 10.1037/h0044651

[2] R.Bush, F.Mosteller, Stochastic models for learning. New York: Wiley, 1955.

[3] V.I.Istrăţescu, On a functional equation, *J. Math. Anal. Appl.*, **56**(1976), no. 1, 133-136.

[4] W.Sintunavarat, A.Turab, Some particular aspects of certain type of probabilistic predator-prey model with experimenter-subject-controlled events and the fixed point method, *AIP Conference Proceedings*, **2423** (2021), 060005. DOI: 10.1063/5.0075282

[5] A.Turab, W.Sintunavarat, On analytic model for two-choice behavior of the paradise fish based on the fixed point method, *J. Fixed Point Theory Appl.*, **21**(2019), 56. DOI: 10.1007/s11784-019-0694-y

[6] V.Berinde, A.R.Khan, On a functional equation arising in mathematical biology and theory of learning, *Creat. Math. Inform.*, **24**(2015), no. 1, 9–16.

[7] A.Turab, W.Sintunavarat, On the solution of the traumatic avoidance learning model approached by the Banach fixed point theorem, *J. Fixed Point Theory Appl.*, **22**(2020), 50. DOI: 10.1007/s11784-020-00788-3

[8] W.K.Estes, J.H.Straughan, Analysis of a verbal conditioning situation in terms of statistical learning theory, *Journal of Experimental Psychology*, **47**(1954), 225-234. DOI: 10.1037/h0060989

[9] A.Turab, W. Sintunavarat, On the solutions of the two preys and one predator type model approached by the fixed point theory, Sadhana **45**(2020), 211. DOI: 10.1007/s12046-020-01468-1

[10] A.Turab, W. G.Park, W.Ali, Existence, uniqueness, and stability analysis of the probabilistic functional equation emerging in mathematical biology and the theory of learning, *Symmetry*, **13**(2021), 1313. DOI: 10.3390/sym13081313

[11] D.A.Grant, H.W.Hake, J.P.Hornseth, Acquisition and extinction of a verbal conditioned response with differing percentages of reinforcement, *Journal of Experimental Psychology*, **42**(1951), 1–5. DOI: 10.1037/h0054051

[12] L.G.Humphreys, Acquisition and extinction of verbal expectations in a situation analogous to conditioning, Journal of Experimental Psychology, **25**(1939), 294–301. DOI: 10.1037/h0053555

[13] M.E.Jarvik, Probability learning and a negative recency effect in the serial anticipation of alternative symbols, *Journal of Experimental Psychology*, **41**(1951), 291–297. DOI: 10.1037/h0056878

[14] S.Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales. Fund. Math., **3**(1922), 133–181.

[15] A.Turab, W.Sintunavarat, Corrigendum: On analytic model for two-choice behavior of the paradise fish based on the fixed point method, J. Fixed Point Theory Appl. 2019, 21:56, *J. Fixed Point Theory Appl.*, **22**(2020), 82. DOI: 10.1007/s11784-020-00818-0

[16] D.H.Hyers, G.Isac and Th.M.Rassias, Stability of functional equations in several variables, Birkhauser, Basel, 1998.

[17] J.S.Morales, E.M. Rojas, Hyers-Ulam and Hyers-Ulam-Rassias stability of nonlinear integral equations with delay, *Int. J. Nonlinear Anal. Appl.*, **2**(2011), no. 2, 1–6.
DOI: 10.22075/IJNAA.2011.47

Подход с фиксированной точкой для изучения класса вероятностных функциональных уравнений, возникающих в психологической теории обучения

Али Тураб
Департамент математики и статистики
Факультет науки и технологий
Таммасат университет Рангссит центр
Патум Тани, Таиланд

Аннотация. *Многие биологические модели и модели теории обучения были исследованы с использованием вероятностных функциональных уравнений. В этой статье основное внимание уделяется особому типу отношений хищник–жертва, в котором у хищника есть два варианта добычи, каждый с вероятностью x и $1-x$, соответственно. Наша цель состоит в том, чтобы исследовать реакцию животного в таких ситуациях, предложив общее вероятностное функциональное уравнение. Заслуживающие внимания результаты с фиксированной точкой используются для исследования существования, единственности и устойчивости решений предложенного функционального уравнения. Приведен также пример, иллюстрирующий важность наших результатов в этой области исследований.*

Ключевые слова: *вероятностные функциональные уравнения, устойчивость, неподвижные точки.*

Journal of Siberian Federal University. Mathematics & Physics 2022, 15(3), 378–384

DOI: 10.17516/1997-1397-2022-15-3-378-384
УДК 519.145

Dihedral Group of Order 8 in an Autotopism Group of a Semifield Projective Plane of Odd Order

Olga V. Kravtsova*
Siberian Federal University
Krasnoyarsk, Russian Federation

Received 10.01.2022, received in revised form 14.02.2022, accepted 01.04.2022

Abstract. We investigate the well-known hypothesis of D. R. Hughes that the full collineation group of a non-Desarguesian semifield projective plane of a finite order is solvable (the question 11.76 in Kourovka notebook was written down by N. D. Podufalov). The spread set method allows us to prove that any non-Desarguesian semifield plane of order p^N, where $p \equiv 1 \pmod 4$ is prime, does not admit an autotopism subgroup isomorphic to the dihedral group of order 8. As a corollary, we obtain the extensive list of simple non-Abelian groups which cannot be the autotopism subgroups.

Keywords: semifield plane; spread set; Baer involution; homology; autotopism.

Citation: O.V. Kravtsova, Dihedral Group of Order 8 in an Autotopism Group of a Semifield Projective Plane of Odd Order, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 378–384.
DOI: 10.17516/1997-1397-2022-15-3-378-384.

Introduction

A projective plane is called a *semifield plane* if its points and lines are coordinatized by a *semifield*, that is a non-associative ring $Q = (Q, +, \cdot)$ with identity where the equations $ax = b$ and $ya = b$ are uniquely solved for any $a, b \in Q \setminus \{0\}$. The study of finite semifields and semifield planes started more than a century ago with the first examples constructed by L. E. Dickson [1].

By the mid-1950s, some classes of finite semifield planes had been found. All of them had the common property that the collineation group (automorphism group) is solvable. So D. R. Hughes conjectured in 1959 in his report that any finite projective plane coordinatized by a non-associative semifield has the solvable collineation group. This hypothesis is presented in the monography [2, Ch. VIII, Sec. 6]; it is proved also that the hypothesis is reduced to the solvability of an autotopism group as a group fixing a triangle. In 1990 the problem was written down by N. D. Podufalov in the Kourovka notebook ([3], the question 11.76).

We represent the approach to study Hughes' problem based on the classification of finite simple groups and the theorem of J. G. Thompson on minimal simple groups. The spread set method allows us to identify the conditions when the semifield plane with certain autotopism subgroup exists. This method can be used also to construct examples, including computer calculations. The elimination of some simple groups as autotopism subgroups follows to the progress in solving the problem.

It is shown by the author in [4, 5], that an autotopism of order two has the matrix representation convenient for calculations and reasoning. These marices are used further to represent the

*ol71@bk.ru https://orcid.org/0000-0002-6005-2393

elementary abelian 2-subgroups and 2-elements in the autotopism group [6,7]. Also it was proved that any non-Desarguesian semifield plane of odd order cannot admit an autotopism subgroup isomorphic to the alternating group A_5 [8].

Here we use the spread set method to prove that any semifield plane of order p^N, p is prime and $p-1$ is divisible by 4, cannot admit an autotopism subgroup isomorphic to the dihedral group D_8 of order 8, see Theorem 2.1. The proof is based on a concretization of a geometric sense of autotopisms of order 2 and 4, it uses also the matrix representation of autotopisms of order 4. Obviously, the presence of this group in almost all simple non-Abelian groups allows us to exclude an extensive list from possible autotopism subgroups.

1. Definitions and preliminary results

We use main definitions, according [2,9], see also [6], for notifications.

Consider a linear space W, n-dimensional over the finite field $GF(p^s)$ (p be prime) and the subset of linear transformations $R \subset GL_n(p^s) \cup \{0\}$ such that:

1) R consists of p^{ns} square $(n \times n)$-matrices over $GF(p^s)$;
2) R contains the zero matrix 0 and the identity matrix E;
3) for any $A, B \in R$, $A \neq B$, the difference $A - B$ is a non-singular matrix.

The set R is called a *spread set* [2]; it is an image of an injective mapping θ from W: $R = \{\theta(y) \mid y \in W\}$. Determine the multiplication on W by the rule $x * y = x \cdot \theta(y)$ $(x, y \in W)$. Then $\langle W, +, * \rangle$ is a right quasifield of order p^{ns} [9,10]. Moreover, if R is closed under addition then $\langle W, +, * \rangle$ is a semifield. This semifield coordinatizes the projective plane π of order p^{ns} such that:

1) the affine points are the elements (x, y) of the space $W \oplus W$;
2) the affine lines are the cosets to subgroups

$$V(\infty) = \{(0, y) \mid y \in W\}, \qquad V(m) = \{(x, x\theta(m)) \mid x \in W\} \quad (m \in W);$$

3) the set of all cosets to the subgroup is the singular point;
4) the set of all singular points is the singular line;
5) the incidence is set-theoretical.

To construct and study finite semifields, we use a prime field $\mathbb{Z}_p$ as a basic field. In this case the mapping θ is presented using linear functions only; it greatly simplifies the reasoning and calculations (also computer).

The solvability of a collineation group $Aut\,\pi$ for a semifield plane is reduced [2] to the solvability of an autotopism group Λ (collineations fixing a triangle). Without loss of generality, we can assume that autotopisms are determined by linear transformations of the space $W \oplus W$:

$$\lambda : (x, y) \to (x, y) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

here the matrices A and B satisfy the condition (for instance, see [11])

$$A^{-1}\theta(m)B \in R \quad \forall \theta(m) \in R. \tag{1}$$

The collineations fixing a closed configuration have special properties. It is well-known [2], that any involutory collineation is a central collineation or a Baer collineation.

A collineation of a projective plane is called *central*, or *perspectivity*, if it fixes a line pointwise (the *axis*) and a point linewise (the *center*). If the center is incident to the axis then a collineation is called an *elation*, and a *homology* in another case. The order of any elation is a factor of the order $|\pi|$ of a projective plane, and the order of any homology is a factor of $|\pi| - 1$. All the perspectivities in an autotopism group are homologies in the case when a semifield plane is of odd order. They form the cyclic subgroups [12] which are normal in Λ, and contain three involution homologies:

$$h_1 = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}, \quad h_2 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad h_3 = \begin{pmatrix} -E & 0 \\ 0 & -E \end{pmatrix}.$$

Obviously these homologies are all in the center of Λ.

A collineation of a finite projective plane π is called a *Baer collineation* if it fixes pointwise a subplane of order $\sqrt{|\pi|}$ (*Baer subplane*). We use the following results on the matrix representation of a Baer involution $\tau \in \Lambda$ and of a spread set obtained earlier in [5].

Let π be a non-Desarguesian semifield plane of order p^N ($p > 2$ be prime). If its autotopism group Λ contains the Baer involution τ then $N = 2n$ is even and we can choose the base of $4n$-dimensional linear space over $\mathbb{Z}_p$ such that

$$\tau = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \tag{2}$$

where $L = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}$ and the Baer subplane π_τ fixed by τ is the set of points

$$\pi_\tau = \{(0, \dots, 0, x_1, \dots, x_n, 0, \dots, 0, y_1, \dots, y_n) \mid x_i, y_i \in \mathbb{Z}_p\}.$$

In this base the spread set $R \subset GL_{2n}(p) \cup \{0\}$ consists of matrices

$$\theta(V, U) = \begin{pmatrix} m(U) & f(V) \\ V & U \end{pmatrix}, \tag{3}$$

where $V \in Q$, $U \in K$; Q, K are the spread sets in $GL_n(p) \cup \{0\}$, m, f are additive injective functions from K and Q into $GL_n(p) \cup \{0\}$, $m(E) = E$. Note that throughout the article, the blocks-submatrices have the same dimension by default.

It is shown by author in [6,7], that the order of a semifield plane provides a natural restriction to the order of an elementary abelian 2-subgroup and to the order of 2-element in an autotopism group. We will use some results and so we state it here in the more convenient form.

Theorem 1.1. *Let π be a semifield plane of order p^N, p be prime, $p \equiv 1 \pmod 4$, $\tau \in \Lambda$ is a Baer involution.*

1. If α is an autotopism of order 4 and $\alpha^2 = \tau$ then the restriction of α onto the Baer subplane π_τ is a Baer involution.

2. If $\sigma \neq \tau$ is a Baer involution in $C_\Lambda(\tau)$ then the restriction of σ onto the Baer subplane π_τ is a homology if $\sigma = h_i\tau$ ($i = 1, 2, 3$) or a Baer involution.

Theorem 1.2. *Let π be a semifield plane of order p^N, p be prime, $p \equiv 1 \pmod 4$, α is an autotopism of order 4, $\tau = \alpha^2$ is a Baer involution. Then N is divisible by 4, and the base of the linear space can be chosen such that τ is (2) and*

$$\alpha = \begin{pmatrix} iL & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & iL & 0 \\ 0 & 0 & 0 & L \end{pmatrix}, \tag{4}$$

where $i \in \mathbb{Z}_p$, $i^2 = -1$. The spread set R of the plane π is formed by matrices

$$\theta(V_1, U_1, V_2, U_2) = \begin{pmatrix} m_1(U_2) & m_2(V_2) & f_1(V_1) & f_2(U_1) \\ m_3(V_2) & m_4(U_2) & f_3(U_1) & f_4(V_1) \\ \nu(U_1) & \psi(V_1) & \mu(U_2) & \varphi(V_2) \\ V_1 & U_1 & V_2 & U_2 \end{pmatrix}, \tag{5}$$

where any block-submatrix is $(N/4 \times N/4)$-dimensional, $V_1 \in Q_1$, $U_1 \in K_1$, $V_2 \in Q_2$, $U_2 \in K_2$, the matrix sets Q_1, K_1, Q_2, K_2 are the spread sets of semifield planes of order $p^{N/4}$, all the functions are additive.

Note, that α is determined up to multiplying to involution homologies h_i from the center of Λ (see the proof in [7]). If we consider certain subgroup of Λ then we can ignore these homologies.

The second statement of the theorem 1.2 is missed in [7] because obviously but here we must reconstruct it due to the importance for the main result.

Indeed, we consider the condition (1) for the autotopism α and the matrix $\theta(V, U)$ (3):

$$\begin{pmatrix} -iL & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} m(U) & f(V) \\ V & U \end{pmatrix} \begin{pmatrix} iL & 0 \\ 0 & L \end{pmatrix} = \begin{pmatrix} Lm(U)L & -iLf(V)L \\ iLVL & LUL \end{pmatrix}.$$

Then we conclude that

$$LVL \in Q, \quad LUL \in K, \quad m(LUL) = Lm(U)L, \quad f(LVL) = -Lf(V)L, \qquad \forall V \in Q, \ \forall U \in K.$$

So the semifield planes of order $p^{N/2}$ with the spreads Q and K admit the Baer involution (2) and the matrices $V \in Q$, $U \in K$ are of the same form as (3):

$$V = \begin{pmatrix} \nu(U_1) & \psi(V_1) \\ V_1 & U_1 \end{pmatrix}, \qquad U = \begin{pmatrix} \mu(U_2) & \varphi(V_2) \\ V_2 & U_2 \end{pmatrix}.$$

If we suppose that

$$m(U) = m(V_2, U_2) = \begin{pmatrix} m_1(V_2, U_2) & m_2(V_2, U_2) \\ m_3(V_2, U_2) & m_4(V_2, U_2) \end{pmatrix},$$

then from $m(-V_2, U_2) = Lm(V_2, U_2)L$ we obtain that the functions m_1, m_4 depend on the block U_2 and other functions on V_2. For the function $f(V)$ we use the condition $f(-V_1, U_1) = -Lf(V_1, U_1)L$ and complete the proof.

2. Main result

Theorem 2.1. *Any non-Desarguesian semifield plane π of order p^N, where $p > 2$ is prime and $p \equiv 1 \pmod 4$, does not admit an autotopism subgroup isomorphic to the dihedral group of order 8 without homologies.*

Proof. Let $H \simeq D_8$ be a subgroup of Λ, $H = \langle \alpha \rangle \leftthreetimes \langle \sigma \rangle$, $|\alpha| = 4$, $|\sigma| = 2$, $\sigma\alpha\sigma = \alpha^{-1}$. The autotopism $\alpha^2 = \tau$ is a Baer involution, so we can choose the base of $2N$-dimensional linear space such that τ is the matrix (2), α is the matrix (4) and the spread set consists of matrices (5).

Further, σ is a Baer involution commuting with τ, and then we have

$$\sigma = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_2 \end{pmatrix}, \quad A_1^2 = A_2^2 = B_1^2 = B_2^2 = E.$$

According the Theorem 1.1, the restriction of σ onto the Baer subplane π_τ is a Baer involution, so $A_2 \neq \pm E,\ B_2 \neq \pm E$. From the condition $\sigma\alpha\sigma = \alpha^{-1}$, we have

$$A_1 L A_1 = B_1 L B_1 = -L, \qquad A_2 L A_2 = B_2 L B_2 = L,$$

$$A_1 = \begin{pmatrix} 0 & A_{11} \\ A_{12} & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} A_{21} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & B_{11} \\ B_{12} & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} B_{21} & 0 \\ 0 & B_{22} \end{pmatrix}.$$

The restrictions of α and σ onto the Baer subplane π_τ are commuting Baer involutions and, once more from the Theorem 1.1 and [6], we can choose the base of π_τ such that $A_{21} = A_{22} = B_{21} = B_{22} = L$ and

$$\sigma = \begin{pmatrix} 0 & S & 0 & 0 & 0 & 0 & 0 & 0 \\ S^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & S & 0 & 0 \\ 0 & 0 & 0 & 0 & S^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L \end{pmatrix}.$$

Here, for compactness, $S = A_{11}$, and $A_1^2 = E$ follows $A_{12} = S^{-1}$. The equality $B_1 = A_1$ we obtain from the condition (1) for σ and $\theta(V,U) = E \in R$:

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} = \begin{pmatrix} A_1 B_1 & 0 \\ 0 & E \end{pmatrix} \in R \Rightarrow A_1 B_1 = E.$$

Now we simplify the matrix σ changing the base by the block-diagonal transition matrix

$$T = \mathrm{diag}\,(E, S, E, E, E, S, E, E).$$

This modification preserves the matrices τ and α, but allows us to write σ in the more convenient form:

$$\sigma = \begin{pmatrix} 0 & E & 0 & 0 & 0 & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L \end{pmatrix}.$$

Consider the condition (1) for the spread set (5) and the Baer involution σ. For $V_2 = U_2 = 0$ we have:

$$\begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & L & 0 \\ 0 & 0 & 0 & L \end{pmatrix} \begin{pmatrix} 0 & 0 & f_1(V_1) & f_2(U_1) \\ 0 & 0 & f_3(U_1) & f_4(V_1) \\ \nu(U_1) & \psi(V_1) & 0 & 0 \\ V_1 & U_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & L & 0 \\ 0 & 0 & 0 & L \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & f_3(U_1)L & f_4(V_1)L \\ 0 & 0 & f_1(V_1)L & f_2(U_1)L \\ L\psi(V_1) & L\nu(U_1) & 0 & 0 \\ LU_1 & LV_1 & 0 & 0 \end{pmatrix} \in R.$$

So, the matrices LU_1 and LV_1 belong to the spread sets Q_1 and K_1 for all $V_1 \in Q_1$, $U_1 \in K_1$. For instance, we have $L \in K_1$ if $V_1 = E$. The spread set K_1 of a semifield plane is closed under addition, so the non-zero degenerate matrix $L + E$ belongs to K_1, that is impossible. This contradiction proves the theorem.

Note that the absence of homologies in H is the natural condition for us because we investigate the existence problem for simple non-Abelian subgroups in the autotopism group Λ (for instance, minimal simple non-Abelian groups from the Thompson's list). Indeed, the homologies generate the normal subgroup of Λ; moreover, the involution homlogies are in the center of Λ.

Let G be a subgroup of Λ and S be the Sylow 2-subgroup of G. If two involutions in S does not commute then they generate the dihedral subgroup in S. Further, using the results of D. Goldschmidth [13] on strongly closed subgroups (see also D. Gorenstein [14, th. 4.128]), we conclude that D_8 is contained almost in all finite simple non-Abelian groups and list the exceptions.

Theorem 2.2. *Let π be a non-Desarguesian semifield plane of order p^N, where $p > 2$ is prime and $p \equiv 1 \pmod 4$. Then its autotopism group Λ does not contain a simple non-Abelian subgroup, except probably the following: $PSL(2, 2^n)$, $n \geqslant 2$, $PSU(3, 2^n)$, $n \geqslant 2$, $Sz(2^n)$, n is odd, $n > 1$, $PSL(2, q)$, $q \equiv \pm 3 \pmod 8$, J_1 or ${}^2G_2(3^n)$, n is odd, $n > 1$.*

Referring to the Thompson's list, we clarify also that the autotopism group Λ under the order condition above does not contain $PSL(2, 3^n)$, $n > 2$ is prime, $PSL(2, n)$, $n \equiv \pm 1 \pmod 8$ is prime, and $PSL(3, 3)$.

Conclusion

In order to study Hughes' problem on the solvability of the full collineation group of a finite non-Desarguesian semifield plane, the author considers it possible to use the obtained results to further investigations. The method applied will probably be useful to study other small autotopism subgroups under the conditions on the plane order.

This work is supported by the Krasnoyarsk Mathematical Center and financed by the Ministry of Science and Higher Education of the Russian Federation (Agreement no. 075-02-2022-876).

References

[1] L.E.Dickson, Linear algebras in which division is always uniquely possible, *Trans. Amer. Math. Soc.*, **7**(1906), no. 3, 370–390.

[2] D.R.Hughes, F.C.Piper, Projective planes, New-York, Springer–Verlag Publ., 1973.

[3] V.D.Mazurov, E.I.Khukhro (eds.), Unsolved Problems in Group Theory. The Kourovka Notebook. No. 19, Novosibirsk, Sobolev Inst. Math. Publ., 2018.

[4] O.V.Kravtsova, Semifield planes of even order that admit the Baer involution, *The Bulletin of Irkutsk State University. Series Mathematics*, **6**(2013), no. 2, 26–37 (in Russian).

[5] O.V.Kravtsova, Semifield planes of odd order that admit a subgroup of autotopisms isomorphic to A_4, *Russian Mathematics*, **60**(2016), no. 9, 7–22. DOI: 10.3103/S1066369X16090024

[6] O.V.Kravtsova, Elementary abelian 2-subgroups in an autotopism group of a semifield projective plane, *The Bulletin of Irkutsk State University. Series Mathematics*, **32**(2020), 49–63. DOI: 10.26516/1997-7670.2020.32.49

[7] O.V.Kravtsova, 2-elements in an autotopism group of a semifield projective plane, *The Bulletin of Irkutsk State University. Series Mathematics*, **39**(2022), 96–110. DOI: 10.26516/1997-7670.2022.39.96

[8] O.V.Kravtsova, On alternating subgroup A_5 in autotopism group of finite semifield plane, *Siberian Electronic Mathematical Reports*, **17**(2020), 47–50. DOI 10.33048/semi.2020.17.004

[9] N.D.Podufalov, On spread sets and collineations of projective planes, *Contem. Math.*, **131**(1992), part 1, 697–705.

[10] H.Luneburg, Translation planes. New-York, Springer-Verlag Publ., 1980.

[11] O.V.Kravtsova, On automorphisms of semifields and semifield planes, *Siberian Electronic Mathematical Reports*, **13**(2016), 1300–1313. DOI: 10.17377/semi.2016.13.102

[12] N.D.Podufalov, B.K.Durakov, O.V.Kravtsova, E.B.Durakov, On the semifield planes of order 16^2, *Siberian Mathematical Journal*, **37**(1996), 535–541. DOI: 10.1007/BF02104857

[13] D.M.Goldschmidth, 2-fusion in finite groups, *Ann. Math.*, **99**(1974), no 1, 70–117.

[14] D.Gorenstein, Finite simple groups. An introduction to their classification, Plenum Press, New York, 1982.

Группа диэдра порядка 8 в группе автотопизмов полуполевой проективной плоскости нечетного порядка

Ольга В. Кравцова
Сибирский федеральный университет
Красноярск, Российская Федерация

Аннотация. Изучается известная гипотеза Д. Хьюза о разрешимости полной группы автоморфизмов конечной недезарговой полуполевой проективной плоскости (также вопрос 11.76 Н. Д. Подуфалова в Коуровской тетради). Метод регулярного множества позволяет доказать, что недезаргова полуполевая плоскость порядка p^N, где p — простое, $p-1$ делится на 4, не допускает подгрупп автотопизмов, изоморфных диэдральной группе порядка 8. В качестве следствия выделяется обширный список простых неабелевых групп, не являющихся подгруппами автотопизмов.

Ключевые слова: полуполевая плоскость, регулярное множество, бэровская инволюция, гомология, автотопизм.

DOI: 10.17516/1997-1397-2022-15-3-385-396
УДК 532

Perturbation Approach for a Flow over a Trapezoidal Obstacle

May Manal Bounif*
Abdelkader Gasmi†
Laboratory of Pure and Applied Mathematics
Faculty of Mathematics and Informatics
University of Msila
Msila, Algeria

Received 04.10.2021, received in revised form 10.11.2021, accepted 04.02.2022

Abstract. In this paper, we tackle the two-dimensional and irrotational flow of inviscid and incompressible fluid over a trapezoidal obstacle. The free surface of the flow which is governed by the Bernoulli condition is determined as a part of solution of the problem. This condition renders difficult an analytical solution of the problem. Hence, our work's objective is utilize the Hilbert transformation and the perturbation technique to provide an approximate solution to this problem for large Weber numbers and various configurations of the obstacle. The obtained results demonstrate that the used method is easily applicable, and provides approximate solutions to these kinds of problems.

Keywords: free surface flow, surface tension, incompressible flow, Hilbert method, perturbation technique.

Citation: M.M. Bounif, A. Gasmi, Perturbation Approach for a Flow over a Trapezoidal Obstacle, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 385-396. DOI: 10.17516/1997-1397-2022-15-3-385-396.

Our study begins with a consideration of the steady two-dimensional and irrotational fluid flow over a trapezoidal obstacle. On the one hand, we assume the fluid is incompressible and inviscid. On the other hand, we consider the surface tension effect but neglect the effect of gravity. A major characteristic of the present problem is the nonlinear condition given through the Bernoulli equation on the free surface of an unknown shape. The latter can be identified as part of the solution to the problem. In addition, because this condition the proposed problem become difficult to solve it analytically, so it is necessary to look for an approximate solution to it.

Free-surface flow problems have been approached using different techniques and methods over the past few decades. Of these techniques and methods we can mention the series truncation technique and boundary integral method, which helps determine the free surface shape for potential flows over given obstacles. For example Forbes and Schwartz [1], determine the non-linear solutions of subcritical and supercritical flows over a semi-circular obstacle, Gasmi and Mekias [2], Gasmi and Amara [3] and Vanden-Broeck [4], studied the problems of flow over an obstruction in a channel, whilst Dias, Killer and Vanden-Broeck [5] , obtained solutions to both subcritical and supercritical free-surface flows past a triangular obstacle, Wiryanto [6] take the problem of the flow under a sluice gate, M.B. Abd-el-Malek and S. Z. Masoud [7] obtains

*maymanal.bounif@univ-msila.dz http://orcid.org/0000-0003-2530-6981
†abdelkader.gasmi@univ-msila.dz

the linear solution of the flow over a ramp, by representing the bottom in integral form using Fourier's double-integral theorem. M. B. Abd-el-Malek and S. N. Hanna [8] solved numerically the problem of the flow over a ramp with gravity effect by the Hilbert Method and the perturbation technique. M. B.Abd-el-Malek, S. N. Hanna and M. T. Kamel [9] investigated the flow over triangular bottom. Bounif and Gasmi [10] , on the other hand, examined the problem that involves a free-surface flows over a step at the bottom of a channel, they offered a solution to the problem using the perturbation method.

The method that we employ in this paper to approximate a solution of the considered problem follows three steps. Initially, we map the flow field of the physical plane onto the upper half plane using the Schwartz–Christoffel transformation. Accordingly, the Hilbert method helps us identify a system of nonlinear equation when applied to the new upper half plane's mixed-boundary value problem. Finally, the perturbation technique is utilized to provide a solution to the system for some large values of the Weber number and varied trapezoidal obstacle configurations. The employability of our method will then be clear given the acquired results, as it provides approximate solutions to the selected kind of problems.

The outline of the paper can be given in four main sections. The first of which will introduce the mathematical formulation of the present problem. Section 2 presents the approximation of equations of the problem, while Section 3 delineates the application of the perturbation technique to solve it. Finally, we show certain free streamline shapes and results in final section.

1. Formulation of problem

Let us consider the motion of a two-dimensional flow of a fluid over a trapezoidal obstacle. The fluid is assumed to be incompressible, irrotational and inviscid. The effect of gravity is neglected but we take into account the superficial tension effect. The flow we propose is uniform and has a constant discharge $U_1 h_1 = U_2 h_2$, where U_i, $i = 1, 2$ designates the velocities and h_i, $i = 1, 2$ are the depths of the flow upstream and downstream respectively. Hence, the bottom consists of the horizontal walls $A_0 A_{-1}$ and $A_1 A^{'}$ and the asymmetric polygon $A_{-1} A_{-2} \ldots A_{-N} A_N \ldots A_2 A_1$ of $2N$ angles α_i and $(2N-1)$ straight-line segments. Furthermore, we choose Cartesian coordinates with the origin in the point (see Fig. 1).

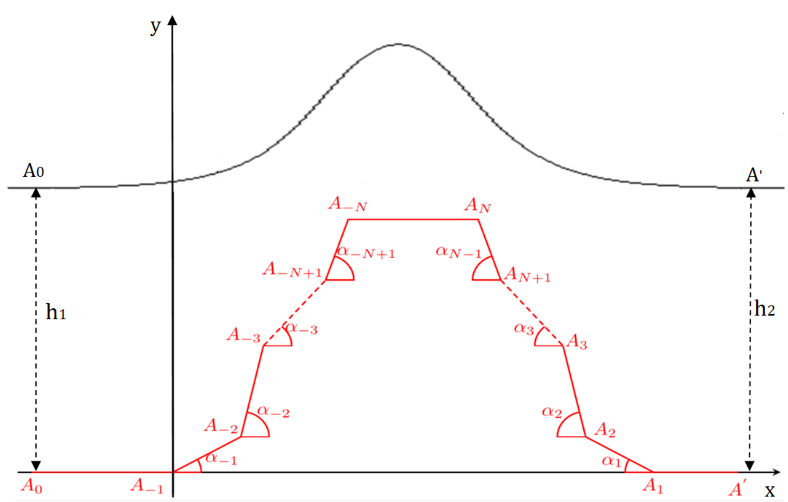


Fig. 1. Sketch of the flow and of the coordinates

The dimensionless variables are defined by choosing U_1 as the unit velocity and h_1 as the unit length. We introduce the complex potential $f(z) = \varphi(z) + i\psi(z)$, where φ is the potential function, ψ the stream function (φ and ψ are conjugate solutions of Laplace's equation) and $f(z)$ is an analytic function of z within the region of flow with complex conjugate velocity

$$\eta = \frac{df(z)}{dz} = u - iv = qe^{-i\theta}. \tag{1}$$

Let

$$\kappa = \ln \eta = \ln q - i\theta, \tag{2}$$

where κ is called the logarithmic hodograph variable. Then, from (1) and (2) we get

$$z = \int e^{-\omega} df. \tag{3}$$

Without loss of generality, we choose $\varphi = 0$ at a point A_{-1} , $\psi = 1$ on the streamline $A_0A^{'}$, and $\psi = 0$ on the streamline $A_0A_{-1}A_{-2}\dots A_{-N}A_N\dots A_1A^{'}$ (see Fig. 2). We denote the dimensionless trapezoid depth by r_i, where

$$r_i = l_i \sin(\alpha_i), \tag{4}$$

where

$$l_i = \begin{cases} |A_iA_{i-1}|, & i = -1, \dots, -N+1, \\ |A_iA_{i+1}|, & i = 1, \dots, N-1. \end{cases} \tag{5}$$

On the free-surface, where the pressure is uniform, the dimensionless form of the Bernoulli equation is given by:

$$q^2 + \frac{2}{We}\left|\frac{\partial\theta}{\partial\varphi}\right| q = 1, \tag{6}$$

where We is the adimensional parameter, known as the Weber number and defined by:

$$We = \frac{\rho U_1^2 h_1}{T}, \tag{7}$$

T is the surface tension, and ρ is the density of the fluid.

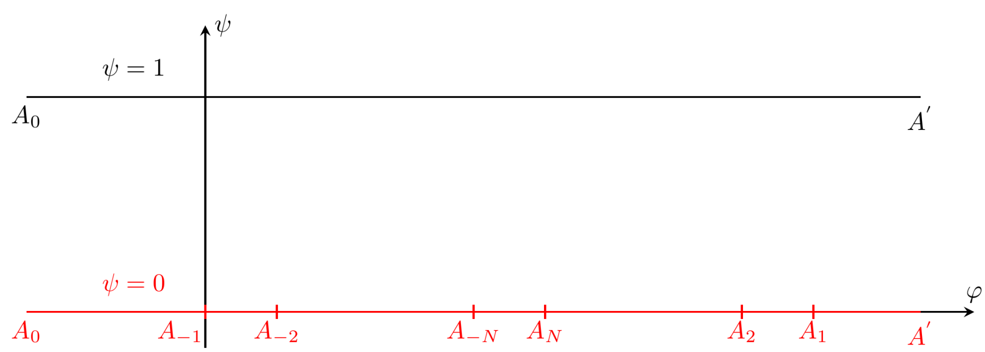


Fig. 2. The potential f plane

Using the Schwartz-Christoffel transformation, we map the potential plane f as seen in Fig. 2 onto the upper half of an auxiliary t-plane see Fig. 3.

The tranformation used is:

$$f(t) = -\frac{1}{\pi}\ln(1-t). \tag{8}$$

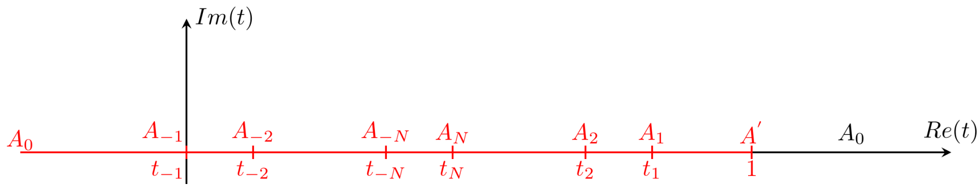


Fig. 3. The auxiliary t plane

1.1. The Hilbert method

In order to express κ as the single variable t function, we need to use the Hilbert method for the obtained mixed problem of the new plane. Hence, the solution for an analytic function $\chi(t)$ in the upper half-plane (see [11]) is given by

$$\chi(t) = \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{Im\left[\chi(s)\right]}{s-t} ds + \sum_{j=0}^{\infty} B_j t^j. \tag{9}$$

Where B_j are real constants and $p.v.$ is the principal value of the integral.
The real and imaginary parts of $\kappa(t)$ are given by

$$\begin{aligned} Im\left[\kappa(t)\right] &= -\theta(t), \\ Re\left[\kappa(t)\right] &= \ln q(t). \end{aligned} \tag{10}$$

Where

$$\theta(t) = \begin{cases} 0, & t < 0 = t_1, \\ \alpha_i, & t_i < t < t_{i-1}, \quad i = -N+1, \dots, -1, \\ -\alpha_i, & t_{i+1} < t < t_i, \quad i = 1, \dots, N-1, \\ 0, & t_N < t < 1, \\ \theta(t), & t > 1. \end{cases} \tag{11}$$

To switch the function $\kappa(t)$ to $\chi(t)$, we use an auxiliary function $H(t)$

$$H(t) = \begin{cases} \sqrt{1-t}, & t < 1, \\ -i\sqrt{t-1}, & t > 1. \end{cases} \tag{12}$$

Using (10) and (12), with $\chi(t) = \kappa(t)/H(t)$, we get

$$\chi(t) = \left\{ \begin{array}{ll} \dfrac{\ln q(t) - i\theta(t)}{\sqrt{1-t}}, & t < 1, \\ \dfrac{\ln q(t) - i\theta(t)}{-i\sqrt{t-1}}, & t > 1. \end{array} \right\} = U(t) + iV(t). \tag{13}$$

Examining the upstream condition, we have

$$B_j = 0, j = 0, 1, 2, \dots .$$

and hence

$$\chi(t) = \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{Im\left[\chi(s)\right]}{s-t} ds. \tag{14}$$

Therefore, using (13) and (14), we obtain

$$U(t) = \frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{V(s)}{s-t} ds. \tag{15}$$

$$V(t) = -\frac{1}{\pi} p.v. \int_{-\infty}^{+\infty} \frac{U(s)}{s-t} ds. \tag{16}$$

Along the real axis of the upper half-plane $Im(t) = 0$ (see Fig 3), the distribution of both real and imaginary parts of $\chi(t)$ can be recapitulated; check Tab. 1. Therefore, q_0 is defined in $t < 0$ and q_∞ is defined in $t_N < t < 1$.

Using (15), (16) and Tab. 1, we obtain the following systems of the nonlinear integral equations:

Table 1. Distribution of the flow quantities along $Im(t) = 0$

t	$U(t)$	$V(t)$
$t < 0 = t_{-1}$	$\frac{\ln q_0(t)}{\sqrt{1-t}}$	0
$t_i < t < t_{i-1}$; $i = -N+1, \ldots, -1$	$\frac{\ln q_i(t)}{\sqrt{1-t}}$	$\frac{-\alpha_i}{\sqrt{1-t}}$
$t_{i+1} < t < t_i$; $i = 1, \ldots, N-1$	$\frac{\ln q_i(t)}{\sqrt{1-t}}$	$\frac{\alpha_i}{\sqrt{1-t}}$
$t_N < t < 1$	$\frac{\ln q_\infty(t)}{\sqrt{1-t}}$	0
$t > 1$	$\frac{\theta(t)}{\sqrt{t-1}}$	$\frac{\ln q(t)}{\sqrt{t-1}}$

$$\begin{aligned}\theta(t) \quad &= \frac{\sqrt{t-1}}{\pi} p.v. \int_1^{+\infty} \frac{\ln q(s)}{(s-t)\sqrt{s-1}} ds + \sum_{i=-N+1}^{-1} \frac{2\alpha_i}{\pi} \tan^{-1}\left(\frac{(m_i - m_{i-1})\sqrt{t-1}}{t-1+m_i m_{i-1}}\right) - \\ &- \sum_{i=1}^{N-1} \frac{2\alpha_i}{\pi} \tan^{-1}\left(\frac{(m_{i+1} - m_i)\sqrt{t-1}}{t-1+m_{i+1} m_i}\right), \qquad t > 1,\end{aligned} \tag{17}$$

where

$$m_i = \sqrt{1-t_i}. \tag{18}$$

And

$$\begin{aligned}\ln(q_j(t)) \quad &= \quad \frac{\sqrt{1-t}}{\pi}\left\{p.v. \int_1^{+\infty} \frac{\ln q(s)}{(s-t)\sqrt{s-1}} ds + \sum_{i=-N+1}^{-1} \alpha_i \int_{t_i}^{t_{i-1}} \frac{ds}{(s-t)\sqrt{1-s}} - \right. \\ &- \left. \sum_{i=1}^{N-1} \alpha_i \int_{t_{i+1}}^{t_i} \frac{ds}{(s-t)\sqrt{1-s}}\right\},\end{aligned} \tag{19}$$

where $p.v.$ is the principal value of the integral and for $j = -N, \ldots, -1$, $q_j(t)$ being the flow speed in $t_j < t < t_{j-1}$, and for $j = 1, \ldots, N$, $q_j(t)$ being the flow speed in $t_{j+1} < t < t_j$.

Using (3) and (8), the coordinates of a point on the free-surface can be obtained as follows:

$$z(t) = z_\infty - \frac{1}{\pi} \int_t^{+\infty} \frac{e^{i\theta(s)}}{(1-s)q(s)} ds, \quad t > 1. \tag{20}$$

By separating the real and imaginary parts, we get:

$$x(t) = x_\infty - \frac{1}{\pi}\int_t^{+\infty} \frac{\cos\theta(s)}{(1-s)q(s)}ds, \quad t > 1, \tag{21}$$

$$y(t) = 1 - \frac{1}{\pi}\int_t^{+\infty} \frac{\sin\theta(s)}{(1-s)q(s)}ds, \quad t > 1. \tag{22}$$

2. The approximate equations

In this section, we approximate the nonlinear integral equations (6), (17), (21) and (22) , when Weber number is large.

Using the first-order Taylor development with respect to $\frac{1}{We}\left|\frac{\partial\theta}{\partial\varphi}\right|$, we can give the solution to the Bernoulli equation as follows:

$$q(t) \approx 1 - \frac{1}{We}\left|\frac{\partial\theta}{\partial\varphi}\right|. \tag{23}$$

Using the relation (8), we obtain:

$$\frac{\partial\theta}{\partial\varphi} = \frac{\partial\theta}{\partial t}\frac{\partial t}{\partial\varphi} = \pi(t-1)\frac{\partial\theta}{\partial t}, \quad t > 1. \tag{24}$$

Consequently, for $t > 1$ the flow speed is approximated by

$$q(t) \approx 1 - \frac{\pi}{We}(t-1)\frac{\partial\theta}{\partial t}(t). \tag{25}$$

which yields

$$\ln q(t) \approx -\frac{\pi}{We}(t-1)\frac{\partial\theta}{\partial t}(t), \tag{26}$$

and

$$\frac{1}{q(t)} \approx 1 + \frac{\pi}{We}(t-1)\frac{\partial\theta}{\partial t}(t). \tag{27}$$

For small angles α_i, the change in θ will be minor, thus, allowing us to approximate $\sin\theta$ by $\theta(t)$ and $\cos\theta$ by one.

Using (26), we can approximate the angle of the free surface with the horizontal (17) by

$$\theta(t) \approx -\frac{\sqrt{t-1}}{We} p.v.\int_1^{+\infty} \frac{(s-1)\frac{\partial\theta}{\partial s}(s)}{(s-t)\sqrt{s-1}}ds + \sum_{i=-N+1}^{-1} \frac{2\alpha_i}{\pi}\tan^{-1}\left(\frac{(m_i - m_{i-1})\sqrt{t-1}}{t-1+m_i m_{i-1}}\right) - \\ - \sum_{i=1}^{N-1} \frac{2\alpha_i}{\pi}\tan^{-1}\left(\frac{(m_{i+1} - m_i)\sqrt{t-1}}{t-1+m_{i+1} m_i}\right), \quad t > 1, \tag{28}$$

substituting (27) into (21) and (22), and after simplification, the free surface equations take the form:

$$\begin{aligned} x(t) &\approx x_\infty - \frac{1}{\pi}\int_t^{+\infty} \frac{1}{(1-s)}\left[1 + \frac{\pi}{We}(s-1)\frac{\partial\theta}{\partial s}(s)\right]ds \\ &\approx x_\infty - \frac{1}{\pi}\int_t^{+\infty} \frac{1}{(1-s)}ds + \frac{1}{We}\left[\lim_{s\longrightarrow\infty}\theta(s) - \theta(t)\right] \\ &\approx x_\infty - \frac{1}{\pi}\int_t^{+\infty} \frac{1}{(1-s)}ds - \frac{1}{We}\theta(t), \end{aligned} \tag{29}$$

and

$$
\begin{aligned}
y(t) &\approx 1-\frac{1}{\pi}\int_t^{+\infty}\frac{\theta(s)}{(1-s)}\left[1+\frac{\pi}{We}(s-1)\frac{\partial\theta}{\partial s}(s)\right]ds \\
&\approx 1-\frac{1}{\pi}\int_t^{+\infty}\frac{\theta(s)}{(1-s)}ds+\frac{1}{2We}\left[\lim_{s\longrightarrow\infty}\theta^2(s)-\theta(t)^2\right] \\
&\approx 1-\frac{1}{\pi}\int_t^{+\infty}\frac{\theta(s)}{(1-s)}ds-\frac{\theta^2(t)}{2We}. \qquad (30)
\end{aligned}
$$

To solve the system of the nonlinear integral equations (25), (28)–(30), we use the Perturbation technique.

3. Perturbation technique

We expand $X(t)$ in terms of the small parameters α_i

$$
X(t)=\sum_{j=-N+1}^{N-1}\sum_{k=0}^{\infty}\alpha_j^k X_{k,\alpha_j}(t). \qquad (31)
$$

Where $X(t)$ stands for $q(t)$, $\theta(t)$, $\theta^{'}(t)$, $x(t)$ and $y(t)$.

3.1. Zero-order approximation

This case corresponds to the flow far upstream, which we consider as uniform. Then, the zero-order approximation of the nonlinear integral equations (25), (28)–(30) is presented by:

- The velocity of the flow

$$
q_0(t)\approx 1-\frac{\pi}{We}(t-1)\theta_0^{'}(t)\approx 1. \qquad (32)
$$

- The velocity direction relative to the horizontal

$$
\theta_0(t)\approx -\frac{\sqrt{t-1}}{We}p.v.\int_1^{+\infty}\frac{(s-1)\theta_0^{'}(s)}{(s-t)\sqrt{s-1}}ds\approx 0. \qquad (33)
$$

- The free streamline equations:

$$
\begin{aligned}
x_0(t) &\approx x_\infty-\frac{1}{\pi}\int_t^{+\infty}\frac{1}{(1-s)}ds-\frac{1}{We}\theta_0(t) \\
&\approx x_\infty-\frac{1}{\pi}\int_t^{+\infty}\frac{1}{(1-s)}ds. \qquad (34)
\end{aligned}
$$

and

$$
y_0(t)\approx 1-\frac{1}{\pi}\int_t^{+\infty}\frac{\theta_0(s)}{(1-s)}ds-\frac{\theta_0^2(t)}{2We}\approx 1. \qquad (35)
$$

On the other hand, we have the formula:

$$
x_\infty\approx\frac{1}{\pi}p.v.\int_0^{+\infty}\frac{1}{(1-s)}ds, \qquad (36)
$$

hence

$$
x_0(t)\approx -\frac{1}{\pi}\ln(t-1). \qquad (37)
$$

3.2. First-order approximation

Now, we find the first-order approximation of the nonlinear integral equations (25), (28)–(30) by using development (31) and the zero-order approximation of the system.

Using the development (31), we can write

$$X_{1,\alpha_i}(t) \approx \frac{X(t) - X_0(t)}{\alpha_i}. \tag{38}$$

Substituting (25) and (32) into (38) yields

$$q_{1,\alpha_i}(t) \approx \frac{\pi}{We}(t-1)\theta'_{1,\alpha_i}(t). \tag{39}$$

From (28), (33) and (38) we get:

- for $i = -N+1, \ldots, -1$,

$$\theta_{1,\alpha_i}(t) \approx -\frac{\sqrt{t-1}}{We}\int_1^{+\infty} \frac{(s-1)\theta'_{1,\alpha_i}(s)}{(s-t)\sqrt{s-1}}ds + \sum_{i=-N+1}^{-1} \frac{2}{\pi}\tan^{-1}\left(\frac{(m_i - m_{i-1})\sqrt{t-1}}{t-1+m_i m_{i-1}}\right), \tag{40}$$

- for $i = 1, \ldots, N-1$,

$$\theta_{1,\alpha_i}(t) \approx -\frac{\sqrt{t-1}}{We}\int_1^{+\infty} \frac{(s-1)\theta'_{1,\alpha_i}(s)}{(s-t)\sqrt{s-1}}ds - \sum_{i=1}^{N-1} \frac{2}{\pi}\tan^{-1}\left(\frac{(m_{i+1} - m_i)\sqrt{t-1}}{t-1+m_{i+1} m_i}\right). \tag{41}$$

On the other hand, from (29), (30), (35), (37) and (38), we find:

$$x_{1,\alpha_i}(t) \approx -\frac{1}{We}\theta_{1,\alpha_i}(t), \tag{42}$$

and

$$y_{1,\alpha_i}(t) \approx -\frac{1}{\pi}\int_t^{+\infty} \frac{\theta_{1,\alpha_i}(s)}{(1-s)}ds. \tag{43}$$

From (40), (41), and for a very large value of the Weber number We, we may neglect the first term with respect to the second one. Thus, we get the first-order approximation of the velocity direction relative to the horizontal axis:

$$\theta_{1,\alpha_i}(t) \approx \frac{2}{\pi}\arctan\left(\frac{(m_i - m_{i-1})\sqrt{t-1}}{t-1+m_i m_{i-1}}\right), \quad i = -N+1, \ldots, -1, \tag{44}$$

$$\theta_{1,\alpha_i}(t) \approx -\frac{2}{\pi}\arctan\left(\frac{(m_{i+1} - m_i)\sqrt{t-1}}{t-1+m_{i+1} m_i}\right), \quad i = 1, \ldots, N-1. \tag{45}$$

Substituting (44), (45) into (42) and (43) and carrying out the integration, one finds

$$x_{1,\alpha_i}(t) \approx -\frac{2}{\pi We}\arctan\left(\frac{(m_i - m_{i-1})\sqrt{t-1}}{t-1+m_i m_{i-1}}\right), i = -N+1, \ldots, -1, \tag{46}$$

$$x_{1,\alpha_i}(t) \approx \frac{2}{\pi We}\arctan\left(\frac{(m_{i+1} - m_i)\sqrt{t-1}}{t-1+m_{i+1} m_i}\right), \quad i = 1, \ldots, N-1, \tag{47}$$

and

$$y_{1,\alpha_i}(t) \approx \frac{4(m_i - m_{i-1})}{\pi^2\sqrt{m_i m_{i-1}}}\arctan\left(\sqrt{\frac{m_i m_{i-1}}{t-1}}\right), \quad i = -N+1, \ldots, -1, \tag{48}$$

$$y_{1,\alpha_i}(t) \approx -\frac{4\,(m_{i+1}-m_i)}{\pi^2\sqrt{m_{i+1}m_i}} \arctan\left(\sqrt{\frac{m_{i+1}m_i}{t-1}}\right), \quad i=1,\dots,N-1. \tag{49}$$

Using results (35), (37), (46)–(49) and expanding (31) enables finding the approximate solutions of the free-surface flow:

$$\begin{aligned} x(t) \;&\approx\; -\frac{1}{\pi}\ln(t-1) - \sum_{i=-N+1}^{-1} \frac{2\alpha_i}{\pi We}\tan^{-1}\left(\frac{(m_i-m_{i-1})\sqrt{t-1}}{t-1+m_i m_{i-1}}\right) \\ &+ \sum_{i=1}^{N-1} \frac{2\alpha_i}{\pi We}\tan^{-1}\left(\frac{(m_{i+1}-m_i)\sqrt{t-1}}{t-1+m_{i+1}m_i}\right), \end{aligned} \tag{50}$$

and

$$\begin{aligned} y(t) \;&\approx\; 1 + \sum_{i=-N+1}^{-1} \frac{4\,(m_i-m_{i-1})\,\alpha_i}{\pi^2\sqrt{m_i m_{i-1}}}\tan^{-1}\left(\sqrt{\frac{m_i m_{i-1}}{t-1}}\right) \\ &- \sum_{i=1}^{N-1} \frac{4\,(m_{i+1}-m_i)\,\alpha_i}{\pi^2\sqrt{m_{i+1}m_i}}\tan^{-1}\left(\sqrt{\frac{m_{i+1}m_i}{t-1}}\right), \qquad t>1. \end{aligned} \tag{51}$$

4. Application example for $N = 2$ and $\alpha_{-2} = \alpha_2 = 0$

The previous approximate scheme is used to calculate the solutions and the free surface profiles for fixed values of flow with large Weber number are found throughout a range of different Weber number. The Fig. 4 presented the variation of the free surface shape with respect to the Weber number, fixed the angles values $\alpha_{-1} = \alpha_1 = \pi/6$, $l_{-2} = l_2 = 1$, and the depth of the obstacle value $r_{-1} = 0.65$.

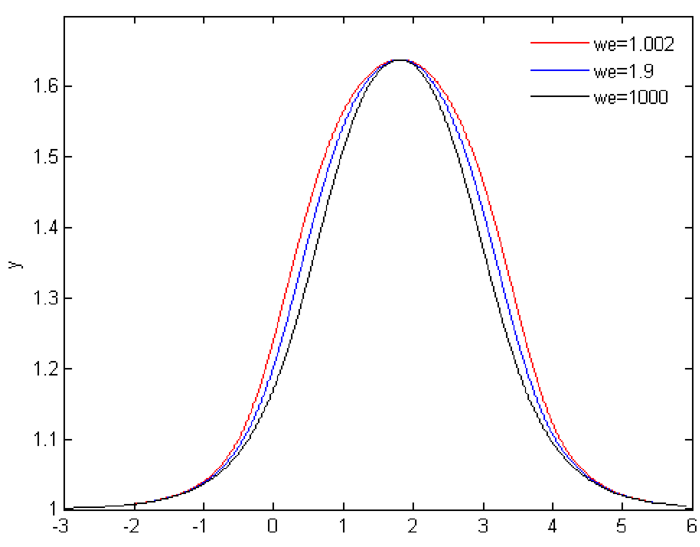


Fig. 4. Effect of Weber number on the free-surface profile at a fixed the trapezoid depth $r_{-1} = 0.65$ and the angles $\alpha_{-1} = \pi/6$, $\alpha_1 = \pi/6$

As presented in Fig. 4, the curvature of the free surface is decreased if the Weber number decreases, because this is an important characteristic property of the surface tension effects. The free-surface profiles for four different depths r_{-1} are plotted in Fig. 5 at a fixed Weber number $We = 200$, $l_{-2} = l_2 = 1$, $\alpha_{-1} = \alpha_1 = \frac{\pi}{8}$. This clarifies that increasing the depth r_{-1} results in more deviation of the free surface from the horizontal one.

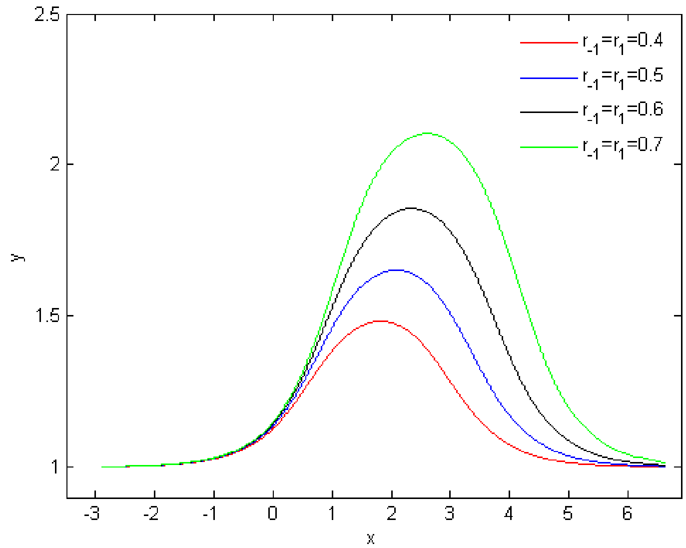


Fig. 5. Effect of the trapezoid depth r_{-1} on the free-surface profile Weber number $We = 200$ and the angles $\alpha_{-1} = \alpha_1 = \pi/8$

Fig. 6 shows the free-surface profiles for different angles α_1 at a fixed $\alpha_{-1} = \pi/8$,$r_{-1} = 0.5$, and at a fixed Weber number $We = 200$. Fig. 6 shows the free-surface profiles for four different angles α_{-1} at a fixed $\alpha_1 = \pi/6$,$r_{-1} = 0.5$ and at a fixed Weber number $We = 200$.

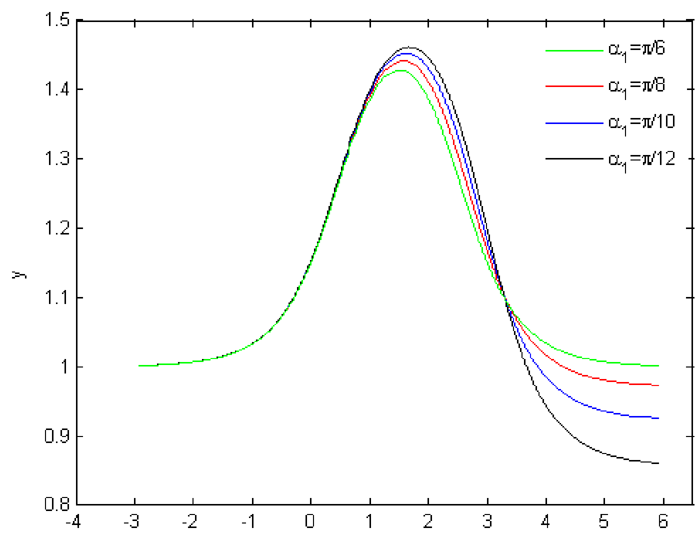


Fig. 6. Effect of the angles α_1 on the free-surface profile Weber number $We = 200$, the angle $\alpha_{-1} = \pi/6$ and the trapezoid depth $r_{-1} = 0.5$

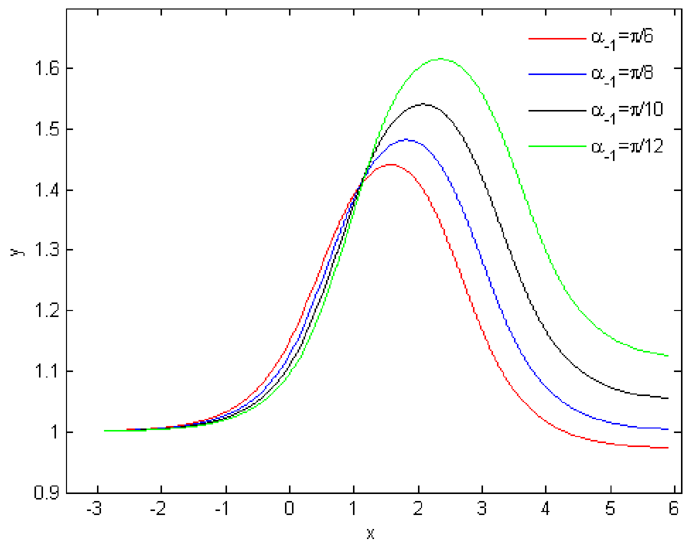


Fig. 7. Effect of the angles α_{-1} on the free-surface profile Weber number $We = 200$, the angle $\alpha_1 = \pi/8$ and the trapezoid depth $r_{-1} = 0.5$

The two Fig. 6 and 7 evidently show that the deviation of the free-surface results from the change in angles.

Conclusion

In this paper, the problem of flow over a trapezoidal obstacle is formulated as a system of nonlinear integral equations. The perturbation technique is used to give an approximate solution to this system for a large Weber number; the free surface profiles under the effect of small surface tension and bottom configurations are illustrated and plotted. The obtained results demonstrate that the used method is easily applicable, and provides approximate solutions to these kinds of problems.

References

[1] L.K.Forbes, L.W.Schwartz, Free-surface flow over a semicircular obstruction, *J. Fluid Mech*, **114**(1982), 299–314.

[2] A.Gasmi, H.Mekias, The effect of surface tension on the contraction coefficient of a jet, *J. Phys. A: Math. Gen*, **36**(2003), 851–862.

[3] A.Gasmi, A.Amara, Free-surface profile of a jet flow in U-shaped channel without gravity effects, *ASCM (Kyungshang)*, **28**(2018), no. 3, 393–400.

[4] J-M.Vanden-Broeck, J.Marc, Free-surface flow over an obstruction in a channel, *The Physics of fluids*, **30**(1987), no. 8, 2315–2317.

[5] F.Dias, J.B.Keller, J.M Vanden-Broeck, Flows over rectangular weirs, *The Physics of fluids*, **31**(1988), no. 8, 2071–2076.

[6] L.Wiryanto, Zero gravity of free-surface flow under a sluice gate, Journal of Physics: Conference Series. IOP Publishing, 2020.

[7] M.B.Abd-el Malek, S.Z.Masoud, Linearized solution of a flow over a ramp, *Appl. Math. Model*, **12**(1988), no. 4, 406–410.

[8] M.B.Abd-el Malek, S.N.Hanna, Approximate solution of a flow over a ramp for large Froude number,*J. Comput. Appl. Math*,**28**(1989), 105–117.

[9] M.B.Abd-el Malek, S.N.Hanna, M.T.Kamel, Approximate solution of gravity flow from a uniform channel over triangular bottom for large Froude number, *Appl. Math. Model*, **15**(1991), no. 1, 25–32.

[10] M.M.Bounif, A.Gasmi, First order perturbation approach for the free surface flow over a step with large Weber number, *INCAS Bulletin*, **13**(2021), no. 2, 11–19.

[11] C.S.Song, A Quasi–Linear and Linear Theory for Non-Separated and Separated Two-Dimensional, Incompressible Irrotational Flow about Lifting Bodies, St. Anthony Falls Hydraulic Laboratory, 1963.

Метод возмущений при обтекании трапециевидного препятствия

Май Манал Боуниф
Абделькадеф Газми
Лаборатория чистой и прикладной математики
Факультет математики и информатики
Университет Мсила
Мсила, Алжир

Аннотация. В этой статье мы рассматриваем двумерное и безвихревое течение невязкой и несжимаемой жидкости над трапециевидным препятствием. Свободная поверхность обтекателя регулируется условием Бернулли, которое определяется в рамках решения задачи. Это условие затрудняет аналитическое решение проблемы. Следовательно, цель нашей работы — использовать преобразование Герберта и технику возмущений, чтобы обеспечить приближенное решение этой проблемы для больших чисел Вебера и различных конфигураций препятствия. Полученные результаты показывают, что используемый метод прост в применении и дает приблизительные решения подобных задач.

Ключевые слова: свободный поверхностный поток, поверхностное натяжение, несжимаемый поток, метод Гильберта, возмущение техника.

DOI: 10.17516/1997-1397-2022-15-3-397-407
УДК 515

Coincidence Point Results and its Applications in Partially Ordered Metric Spaces

N. Seshagiri Rao*
Department of Applied Mathematics
School of Applied Natural Sciences
Adama Science and Technology University
Adama, Ethiopia

Karusala Kalyani†
Department of Mathematics
Vignan's Foundation for Science, Technology & Research
Andhra Pradesh, India

Tekle Gemechu‡
Department of Applied Mathematics
School of Applied Natural Sciences
Adama Science and Technology University
Adama, Ethiopia

Received 12.09.2021, received in revised form 29.01.2022, accepted 16.03.2022

Abstract. The purpose of this paper is to establish some common fixed point theorems for f-nondecreasing self-mapping satisfying a certain rational type contraction condition in the frame of a metric spaces endowed with partial order. Also, some consequences of the results in terms of an integral type contractions are presented in the space. Further, the monotone iterative technique has been used to find a unique solution of an integral equation.

Keywords: ordered metric space, rational contraction, compatible mappings, coincidence point, common fixed point.

Citation: N.Seshagiri Rao, K.Kalyani, Tekle Gemechu, Coincidence Point Results and its Applications in Partially Ordered Metric Spaces, J. Sib. Fed. Univ. Math. Phys., 2022, 15(3), 397–407.
DOI: 10.17516/1997-1397-2022-15-3-397-407.

Introduction

Ever since in fixed point theory and approximation theory, the classical Banach contraction principle [1] plays a vital role to acquire the unique solution of many known results. It is very important and popular tool in various disciplines of mathematics to solve the existing problems in nonlinear analysis. Later, a lot of variety of generalizations of this Banach contraction principle [1] have been taken place in a metrical fixed point theory by improving the underlying contraction condition, some of which are in [2–11]. Thereafter, vigorous research work has been noticed by weakening its hypotheses in various spaces with topological properties such as rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi-metric spaces, probabilistic metric spaces, D-metric spaces, G-metric spaces, F-metric spaces, cone metric spaces etc. Prominent works on the existence and uniqueness of a fixed point in partially ordered

*seshu.namana@gmail.com https://orcid.org/0000-0003-2409-6513
†kalyani.namana@gmail.com https://orcid.org/0000-0002-4531-5976
‡tekgem@yahoo.com https://orcid.org/0000-0003-0592-9508

metric spaces with different contractive conditions have been acquired by several researchers, the readers may refer to [12–21] and the references therein, which generate natural interest to establish usable fixed point theorems.

The concept of coupled fixed point for a certain mapping in ordered metric space was first introduced by Bhaskar and Lakshmikantham [22] and then applied their results to a periodic boundary value problem to obtain the unique solution. While, the theory of coupled coincidence point and common fixed point results was first initiated by Lakshmikantham and Ćirić [23] which generalized and extended the results of [22] by considering the monotone property of a mapping in ordered metric spaces. Some generalized results on fixed point, coupled fixed point and common fixed point under various contractive conditions in different spaces can be found from [24–37]. Recently, Seshagiri Rao et al. [38–42] and Kalyani et al. [43] have investigated some coupled fixed point theorems for the self mappings satisfying generalized rational contractions in partially ordered metric spaces.

The aim of this paper is to present some common fixed point results for a pair of self-mappings satisfying a generalized rational contraction condition in the context of complete partially ordered metric space. These results generalized and extended the results of Harjani et al. [15] and Chandok [30] in the literature. Some consequences of the main result in terms of integral contractions are presented. A numerical example has been provided to support the result obtained. Moreover, an application of the result has been given by taking the integral equation using the monotone iterative technique.

1. Mathematical preliminaries

Definition 1 ([38]). *The triple $(X, d, \preceq)$ is called a partially ordered metric space, if $(X, \preceq)$ is a partially ordered set together with (X, d) is a metric space.*

Definition 2 ([38]). *If (X, d) is a complete metric space, then the triple $(X, d, \preceq)$ is called complete partially ordered metric space.*

Definition 3 ([38]). *Let $(X, \preceq)$ be a partially ordered set. A mapping $f : X \to X$ is said to be strictly increasing (or strictly decreasing), if $f(x) \prec f(y)$ (or $f(x) \succ f(y)$), for all $x, y \in X$ with $x \prec y$.*

Definition 4 ([42]). *Let $f, T : A \to A$ be two mappings, where $A \neq \emptyset$ subset of X. Then*

(a) f and T are commutative, if $fTx = Tfx$ for all $x \in A$.

(b) f and T are compatible, if for very sequence $\{x_n\}$ with $\lim\limits_{n\to+\infty} fx_n = \lim\limits_{n\to+\infty} Tx_n = \mu$ for some $\mu \in A$, then $\lim\limits_{n\to+\infty} d(Tfx_n, fTx_n) = 0$.

(c) f and T are said to be weakly compatible if they commute only at their coincidence points (i.e., if $fx = Tx$ then $fTx = Tfx$).

(d) T is called monotone f-nondecreasing, if

$$fx \preceq fy \Rightarrow Tx \preceq Ty \ for\, all \ x, y \in X.$$

(e) A is a well ordered set, if every two elements of it are comparable.

(f) a point $x \in A$ is a common fixed (or coincidence) point of f and T, if $fx = Tx = x$ (or $fx = Tx$).

2. Main results

We begin this section with the following coincidence point theorem.

Theorem 1. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that the mappings $T, f : X \to X$ are continuous, T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfying the following contraction condition*

$$d(Tx, Ty) \leqslant \alpha \frac{d(fx, Tx)\ d(fy, Ty)}{d(fx, fy)} + \beta\,[d(fx, Ty) + d(fy, Tx)] + \gamma d(fx, fy), \tag{1}$$

for all $x, y \in X$ for which the distinct fx and fy are comparable and for some $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leqslant \alpha + 2\beta + \gamma < 1$. If there exists certain $x_0 \in X$ such that $fx_0 \preceq Tx_0$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Proof. Let $x_0 \in X$ such that $fx_0 \preceq Tx_0$. Since, $T(X) \subseteq f(X)$ then we can choose a point $x_1 \in X$ such that $fx_1 = Tx_0$. But $Tx_1 \in f(X)$, then there exists another point $x_2 \in X$ such that $fx_2 = Tx_1$. Similarly by continuing the same procedure, we construct a sequence $\{x_n\} \subseteq X$ such that $fx_{n+1} = Tx_n$ for all $n \geqslant 0$.

Again from the hypothesis, we have $fx_0 \preceq Tx_0 = fx_1$. Since T is monotone f-nondecreasing then we obtain that $Tx_0 \preceq Tx_1$. As by the similar argument, we get $Tx_1 \preceq Tx_2$, since $fx_1 \preceq fx_2$. Continuing the process, we acquire that

$$Tx_0 \preceq Tx_1 \preceq \cdots \preceq Tx_n \preceq Tx_{n+1} \preceq \ldots .$$

Case 1. Suppose that $d(Tx_n, Tx_{n+1}) = 0$ for some $n \in \mathbb{N}$, then we have $Tx_{n+1} = Tx_n$. Therefore, $Tx_{n+1} = Tx_n = fx_{n+1}$. Hence, x_{n+1} is a coincidence point of T and f in X and we have the result.

Case 2. Suppose $d(Tx_n, Tx_{n+1}) > 0$ for all $n \geqslant 0$, then from (1), we have

$$d(Tx_{n+1}, Tx_n) \leqslant \alpha \frac{d(fx_{n+1}, Tx_{n+1})\ d(fx_n, Tx_n)}{d(fx_{n+1}, fx_n)} + \beta\,[d(fx_{n+1}, Tx_n) + d(fx_n, Tx_{n+1})] + \\ + \gamma d(fx_{n+1}, fx_n),$$

which intern implies that

$$d(Tx_{n+1}, Tx_n) \leqslant \alpha d(Tx_n, Tx_{n+1}) + \beta\,[d(Tx_n, Tx_n) + d(Tx_{n-1}, Tx_{n+1})] + \gamma d(Tx_n, Tx_{n-1}).$$

Finally, we arrive at

$$d(Tx_{n+1}, Tx_n) \leqslant \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right) d(Tx_n, Tx_{n-1}).$$

Inductively, we get

$$d(Tx_{n+1}, Tx_n) \leqslant \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right)^n d(Tx_1, Tx_0).$$

Let $k = \dfrac{\beta + \gamma}{1 - \alpha - \beta} < 1$, then from the triangular inequality of a metric d for $m \geqslant n$, we have

$$d(Tx_m, Tx_n) \leqslant d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + \cdots + d(Tx_{n+1}, Tx_n) \leqslant \\ \leqslant \left(k^{m-1} + k^{m-2} + \cdots + k^n\right) d(Tx_1, Tx_0) \leqslant \frac{k^n}{1 - k} d(Tx_1, Tx_0),$$

as $m, n \to +\infty$, $d(Tx_m, Tx_n) \to 0$, which shows that the sequence $\{Tx_n\}$ is a Cauchy sequence in X. Therefore from the completeness of X, there exists a point $\mu \in X$ such that $Tx_n \to \mu$ as $n \to +\infty$. Further the continuity of T implies that

$$\lim_{n\to+\infty} T(Tx_n) = T\left(\lim_{n\to+\infty} Tx_n\right) = T\mu.$$

Since $fx_{n+1} = Tx_n$ and then $fx_{n+1} \to \mu$ as $n \to +\infty$. Furthermore, from the compatibility of the mappings T and f, we have

$$\lim_{n\to+\infty} d(Tfx_n, fTx_n) = 0.$$

By the triangular inequality, we have

$$d(T\mu, f\mu) = d(T\mu, Tfx_n) + d(Tfx_n, fTx_n) + d(fTx_n, f\mu),$$

on taking $n \to +\infty$ and from the fact that T and f are continuous, we obtain that $d(T\mu, f\mu) = 0$. Thus, $T\mu = f\mu$. Hence, μ is a coincidence point of T and f in X. □

We have the following consequences from Theorem 1.

Corollary 1. *Suppose $(X, d, \preceq)$ be a complete partially ordered metric space. Let the mappings $T, f : X \to X$ are continuous, T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfies*

$$d(Tx, Ty) \leqslant \alpha \frac{d(fx, Tx)\ d(fy, Ty)}{d(fx, fy)} + \beta \left[d(fx, Ty) + d(fy, Tx)\right],$$

for all $x, y \in X$ for which the distinct fx and fy are comparable and where $\alpha, \beta \in [0, 1)$ such that $0 \leqslant \alpha + 2\beta < 1$. If $fx_0 \preceq Tx_0$ for some $x_0 \in X$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Proof. The required proof can be obtained by setting $\gamma = 0$ in Theorem 1. □

Corollary 2. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Assume that the mappings $T, f : X \to X$ are continuous, T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfies*

$$d(Tx, Ty) \leqslant \beta \left[d(fx, Ty) + d(fy, Tx)\right] + \gamma d(fx, fy)$$

for all $x, y \in X$ for which fx, fy are comparable and $\beta, \gamma \in [0, 1)$ such that $0 \leqslant 2\beta + \gamma < 1$. If for some $x_0 \in X$ such that $fx_0 \preceq Tx_0$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

Proof. Set $\alpha = 0$ in Theorem 1. □

We extract the continuity criteria of T in Theorem 1 is still valid by assuming the following hypotheses in X:

If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Theorem 2. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose the mappings $T, f : X \to X$ are continuous, T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfies*

$$d(Tx, Ty) \leqslant \alpha \frac{d(fx, Tx)\ d(fy, Ty)}{d(fx, fy)} + \beta \left[d(fx, Ty) + d(fy, Tx)\right] + \gamma d(fx, fy) \tag{2}$$

for all $x, y \in X$ for which $fx \neq fy$ are comparable and where $\alpha, \beta, \gamma \in [0, 1)$ such that $0 \leqslant \alpha + 2\beta + \gamma < 1$. If for some $x_0 \in X$ such that $fx_0 \preceq Tx_0$ and $\{x_n\}$ is a nondecreasing *sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$. If $f(X)$ is a complete subset of X, then T and f have a coincidence point in X. Further, if T and f are weakly compatible, then T and f have a common fixed point in X.*

Moreover, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point in X.

Proof. Suppose $f(X)$ is a complete subset of X. From Theorem 1, the sequence $\{Tx_n\}$ is a Cauchy sequence and hence, $\{fx_n\}$ is also a Cauchy sequence in $(f(X), d)$ as $fx_{n+1} = Tx_n$ and $T(X) \subseteq f(X)$. Since $f(X)$ is complete, then there exists $fu \in f(X)$ such that

$$\lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} fx_n = fu.$$

Notice that the sequences $\{Tx_n\}$ and $\{fx_n\}$ are nondecreasing. Then from the hypothesis, we have $Tx_n \preceq fu$ and $fx_n \preceq fu$ for all $n \in \mathbb{N}$. But T is monotone f-nondecreasing, then we get $Tx_n \preceq T\mu$ for all n. Letting $n \to +\infty$, we obtain that $fu \preceq Tu$.

Assume that $fu \prec Tu$. Define a sequence $\{u_n\}$ by $u_0 = u$ and $fu_{n+1} = Tu_n$ for all $n \in \mathbb{N}$. An argument similar to that in the proof of Theorem 1 yields that $\{fu_n\}$ is a nondecreasing sequence and $\lim\limits_{n\to+\infty} fu_n = \lim\limits_{n\to+\infty} Tu_n = fv$ for some $v \in X$. Now, from the hypothesis, we have $\sup fu_n \preceq fv$ and $\sup Tu_n \preceq fv$, for all $n \in \mathbb{N}$. Notice that

$$fx_n \preceq fu \preceq fu_1 \preceq fu_2 \preceq \cdots \preceq fu_n \preceq \cdots \preceq fv.$$

Case 1. If $fx_{n_0} = fu_{n_0}$ for some $n_0 \geqslant 1$ then we have

$$fx_{n_0} = fu = fu_{n_0} = fu_1 = Tu.$$

Thus, u is a coincidence point of T and f in X.

Case 2. If $fx_{n_0} \neq fu_{n_0}$ for all n, then from (2), we have

$$d(fx_{n+1}, fu_{n+1}) = d(Tx_n, Tu_n) \leqslant$$
$$\leqslant \alpha \frac{d(fx_n, Tx_n)\ d(fu_n, Tu_n)}{d(fx_n, fu_n)} + \beta\left[d(fx_n, Tu_n) + d(fu_n, Tx_n)\right] + \gamma d(fx_n, fu_n).$$

On taking limit as $n \to +\infty$ in the above inequality, we obtain that

$$d(fu, fv) \leqslant (2\beta + \gamma) d(fu, fv) < d(fu, fv), \text{ since } 2\beta + \gamma < 1.$$

Therefore, we have

$$fu = fv = fu_1 = Tu.$$

Hence, we conclude that u is a coincidence point of T and f in X.

Assume that T and f are weakly compatible. Let w be the coincidence point of T and f, then we have

$$Tw = Tfz = fTz = fw, \text{ since } w = Tz = fz \text{ for some } z \in X.$$

From (2), we have

$$d(Tz, Tw) \leqslant \alpha \frac{d(fz, Tz)\ d(fw, Tw)}{d(fz, fw)} + \beta\left[d(fz, Tw) + d(fw, Tz)\right] + \gamma d(fz, fw) \leqslant$$
$$\leqslant (2\beta + \gamma) d(Tz, Tw),$$

as $2\beta + \gamma < 1$, we obtain that $d(Tz, Tw) = 0$. Thus, $Tz = Tw = fw = w$. Hence, w is a common fixed point of T and f in X.

Suppose that the set of common fixed points of T and f is well ordered. It is enough to prove that the common fixed point of T and f is unique. Assume in contrary that, $u \neq v$ be two common fixed points of T and f. Then from (2), we have

$$d(u,v) \leqslant \alpha \frac{d(fu,Tu)\ d(fv,Tv)}{d(fu,fv)} + \beta\left[d(fu,Tv)+d(fv,Tu)\right] + \gamma d(fu,fv) \leqslant$$
$$\leqslant (2\beta+\gamma)\ d(u,v) < d(u,v), \text{ since } 2\beta+\gamma<1,$$

which is a contradiction. Hence, $u = v$. Conversely, suppose T and f have only one common fixed point then the set of common fixed points of T and f being a singleton is well ordered. This completes the proof. □

We have the following results as a consequence of Theorem 2.

Corollary 3. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose the mappings $T, f : X \to X$ are continuous, T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ and satisfies the following contraction conditions for all $x, y \in X$ for which $fx \neq fy$ are comparable*

$$(i) \qquad d(Tx,Ty) \leqslant \alpha \frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)} + \beta\left[d(fx,Ty)+d(fy,Tx)\right],$$

for some $\alpha, \beta \in [0,1)$ with $0 \leqslant \alpha + 2\beta < 1$,

$$(ii) \qquad d(Tx,Ty) \leqslant \beta\left[d(fx,Ty)+d(fy,Tx)\right] + \gamma d(fx,fy),$$

where $\beta, \gamma \in [0,1)$ such that $0 \leqslant 2\beta + \gamma < 1$.

If for some $x_0 \in X$ such that $fx_0 \preceq Tx_0$ and $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$. If $f(X)$ is a complete subset of X, then T and f have a coincidence point in X. Moreover, if T and f are weakly compatible, then T and f have a common fixed point in X.

Furthermore, the set of common fixed points of T and f is well ordered if and only if T and f have one and only one common fixed point in X.

Proof. Setting $\gamma = 0$ and $\alpha = 0$ in the Theorem 2, we obtain the required proof. □

Remarks

(1) If $\beta = 0$ in Theorems 1 & 2, we obtain Theorems 2.1 & 2.3 of Chandok [30].

(2) If $f = I$ and $\beta = 0$ in Theorems 1 & 2, then we get Theorems 2.1 & 2.3 of Harjani et al. [15].

Now, we have the following consequence of Theorem 1 involving the integral type contraction.

Let Φ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(i) each φ is Lebesgue integrable function on every compact subset of $[0, +\infty)$ and

(ii) $\int_0^{\epsilon} \varphi(t)dt > 0$, for any $\epsilon > 0$.

Corollary 4. *Let $(X, d, \preceq)$ be a complete partially ordered metric space. Suppose that the mappings $T, f : X \to X$ are continuous, T is monotone f-nondecreasing, $T(X) \subseteq f(X)$ satisfies*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leqslant \alpha \int_0^{\frac{d(fx,Tx)\ d(fy,Ty)}{d(fx,fy)}} \varphi(t)dt + \beta \int_0^{d(fx,Ty)+d(fy,Tx)} \varphi(t)dt + \gamma \int_0^{d(fx,fy)} \varphi(t)dt \qquad (3)$$

for all $x, y \in X$ for which the distinct fx and fy are comparable, $\varphi(t) \in \Phi$ and there exist $\alpha, \beta, \gamma \in [0,1)$ such that $0 \leqslant \alpha + 2\beta + \gamma + < 1$. If for some $x_0 \in X$ such that $fx_0 \preceq Tx_0$ and the mappings T and f are compatible, then T and f have a coincidence point in X.

We obtain some consequences of Corollary 4 by taking $\gamma = 0$ and $\alpha = 0$.

Corollary 5. *If $\beta = 0$ in Corollary 4, we obtain the Corollary 2.5 of Chandok [30].*

Example 1. *Define a metric $d : X \times X \to [0, +\infty)$ by $d(x, y) = |x - y|$, where $X = [0, 1]$ with usual order $\leqslant$. Let us define two self mappings T and f on X by $Tx = \dfrac{x}{2}$ and $fx = \dfrac{x}{1+x}$, then T and f have a coincidence point in X.*

Proof. By definition of a metric d, it is clear that (X, d) is a complete metric space. Obviously, $(X, d, \leqslant)$ is complete partially ordered metric space with usual order. Let $x_0 = 0 \in X$, then $f(x_0) \leqslant T(x_0)$. By definitions; T, f are continuous, T is monotone f-nondecreasing and $T(X) \subseteq f(X)$.

Now for any $x, y \in X$ with $x < y$, we have

$$
\begin{aligned}
d(Tx, Ty) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x - y| <& \\
&< \frac{\alpha}{4}xy(1-y) + \frac{\beta}{2}\left[\frac{|x(2-y)-y|}{(1+x)} + \frac{|y(2-x)-x|}{(1+y)}\right] + \gamma\frac{|x-y|}{(1+x)(1+y)} < \\
&< \alpha\frac{\left|\frac{x}{1+x} - \frac{x}{2}\right| \left|\frac{y}{1+y} - \frac{y}{2}\right|}{\frac{x}{1+x} - \frac{y}{1+y}} + \beta\left[\left|\frac{x}{1+x} - \frac{y}{2}\right| + \left|\frac{y}{1+y} - \frac{x}{2}\right|\right] + \gamma\frac{|x-y|}{(1+x)(1+y)} < \\
&< \alpha\frac{d(fx, Tx)\; d(fy, Ty)}{d(fx, fy)} + \beta\left[d(fx, Ty) + d(fy, Tx)\right] + \gamma d(fx, fy),
\end{aligned}
$$

holds the contraction condition in Theorem 1 for some α, β, γ in $[0,1)$ such that $0 \leqslant \alpha+2\beta+\gamma < 1$. Therefore T and f have a coincidence point $0 \in X$. □

Similarly the following is one more example of main Theorem 1.

Example 2. *A distance function $d : X \times X \to [0, +\infty)$ by $d(x, y) = |x - y|$, where $X = [0, 1]$ with usual order $\leqslant$. Define the two self mappings T and f on X by $Tx = x^3$ and $fx = x^4$, then T and f have two coincidence points 0, 1 in X with $x_0 = \dfrac{1}{3}$.*

3. Application

In this section, we discuss a unique solution of the integral equation by the method of upper and lower solutions. The monotone iterative technique is the one to find the minimal and a maximal solution between the lower and upper solutions which validate the maximal principle.

Let $\Lambda \in \mathbb{R}^n$ be a bounded and open set and $\mathcal{H} = \mathscr{L}(\Lambda)^2$ be a Hilbert space with usual inner product and norm then a linear operation $\mathscr{L} : D(\mathscr{L}) \subset \mathcal{H} \to \mathcal{H}$ is said to be a valid maximum principle if there exists some $\lambda \in \mathbb{R}$ such that

$$
\begin{gathered}
\mathscr{L}u + \lambda u \geqslant 0 \text{ on } \Lambda \text{ implies that } u \geqslant 0 \text{ on } \Lambda, u \in D(\mathscr{L}), \\
\text{where } u \geqslant 0 \text{ on } \Lambda \text{ if } u(x) \geqslant 0 \;\text{ for a.e.} x \in \mathbb{R}.
\end{gathered} \tag{4}
$$

Now consider the first order periodic boundary value problem of an integral equation given in [18].

$$u(x) = \int_0^M f(x, u(x), [\mathscr{K}u](x)])dx \quad \text{for a.e.} x \in \Lambda = (0, M), \;\; u(0) = u(M)$$

or

$$u'(x) = f(x, u(x), [\mathscr{K}u](x)) \;\; \text{for a.e.} x \in (0, M), \;\; u(0) = u(M). \tag{5}$$

where f is Caratheodary function, $\mathscr{K}$ is an integral operator

$$[\mathscr{K}u](x) = \int_0^M k(x, y)u(y)dy \tag{6}$$

with kernal $\mathscr{K} \in \mathscr{L}^2(\Lambda \times \Lambda)$. It is clear that for any $u \in \mathcal{H} = \mathscr{L}^2(\Lambda \times \Lambda)$ then $\mathscr{K}u \in \mathcal{H}$.

For a solution of (5), first we study the linear problem for $\lambda \neq 0$

$$u' + \lambda u + \delta \mathscr{K} u = \sigma, \;\; u(0) = u(M). \tag{7}$$

Its known that u is a solution of (7) if and only if

$$u(x) = \int_0^M g(x, y)[\sigma(y) - \delta \mathscr{K} u(y)]dy = w(x) + \int_0^M \mathscr{R}(x, y)u(y)dy, \tag{8}$$

where

$$w(x) = \int_0^M g(x, y)\sigma(y)dy,$$

and

$$\mathscr{R}(x, y) = -\delta \int_0^M g(x, z)\mathscr{K}(z, y)dz.$$

Let us define the linear operator $\mathscr{L} : D(\mathscr{L}) \subset \mathcal{H} \to \mathcal{H}$, where $D(\mathscr{L}) = \{u \in \mathcal{H}^1(\Lambda) : u(0) = = u(M)\}$ as

$$[\mathscr{L}u](x) = u'(x) + \delta[\mathscr{K}u](x). \tag{9}$$

Similarly, let $\mathscr{N} : D(\mathscr{N}) \subset \mathcal{H} \to \mathcal{H}$ is a nonlinear operator, where

$$[\mathscr{N}u](x) = f(x, u(x), [\mathscr{K}u](x)) + \delta[\mathscr{K}u](x). \tag{10}$$

Hence, (5) is equivalent to $\mathscr{L}u = \mathscr{N}u$, $u \in D(\mathscr{L}) \cap D(\mathscr{N})$ and $D(\mathscr{L}) \subset \mathscr{L}^\infty(\Lambda) \subset D(\mathscr{N})$, where $\mathscr{K} \in \mathscr{L}^\infty(\Lambda \times \Lambda)$. Suppose that $\lambda \neq 0$ and follow the conditions of [44], we have

$$||\mathscr{K}||_2 < \frac{(2|\lambda|)^{\frac{1}{2}}|1 - e^{-\lambda M}|}{|\delta|(M(1 - e^{-2\lambda M}))} = d_1. \tag{11}$$

From Lemma 5.1 of [18] followed by above condition, the equation (7) has a unique solution $u \in \mathcal{H}$ for each $\sigma \in \mathcal{H}$ and $G : (\mathscr{L} + \lambda I)^{-1} : \mathcal{H} \to \mathcal{H}$ is continuous, where

$$u(x) = [G\sigma](x) = \int_0^M g(x, y)\sigma(y)dy.$$

Hence from Theorem 2.2 in [44] shows that the maximum principal (4) is valid for $\lambda > 0$ whenever $\mathscr{K} \in \mathscr{L}^\infty(\Lambda \times \Lambda)$ and

$$||\mathscr{K}||_\infty < \frac{\lambda^2}{|\delta|(e^{\lambda M} + \lambda M - 1)} = d_2. \tag{12}$$

From (11) and (12), we get $D(\mathscr{L}) \subset D(\mathscr{N})$. The functions $\alpha, \beta \in D(\mathscr{L})$ are said to be lower and upper solution of (5) if $\alpha'(x) \leqslant f(x, u(x), [\mathscr{K}u](x)) \leqslant \beta'(x)$ for a.e. $x \in \mathbb{R}$.

Now suppose there exist the constants m, δ, λ with $0 < m \leqslant \lambda$ such that (12) is satisfied and the following inequality holds.

$$\begin{aligned} \mathscr{f}(x, u(x), [\mathscr{K}u](x)) - \mathscr{f}(x, v(x), [\mathscr{K}v](x)) \geqslant \\ \geqslant -m(u(x) - v(x)) - \delta([\mathscr{K}u](x)) - [\mathscr{K}v](x)), \end{aligned} \tag{13}$$

whenever $x \in \Lambda$, $\alpha(x) \leqslant v(x) \leqslant u(x) \leqslant \beta(x)$. Then applying Theorem 3.1 of [18] it is possible to approximate the external solutions of (5) by monotone iterates between the lower solution α and the upper solution β.

The authors do thankful to the editor and anonymous referees for their valuable suggestions and comments which improved the contents of the paper.

References

[1] S.Banach, Sur les operations dans les ensembles abstraits et leur application aux equations untegrales, *Fund. Math.*, **3**(1922), 133–181.

[2] B.K.Dass, S.Gupta, An extension of Banach contraction principle through rational expression, *Indian J. Pure Appl. Math.*, **6**(1975), 1455–1458.

[3] S.K.Chetterjee, Fixed point theorems, *C.R. Acad. Bulgara Sci.*, **25**(1972), 727–730.

[4] M.Edelstein, On fixed points and periodic points under contraction mappings, *J. Lond. Math. Soc.*, **37**(1962), 74–79.

[5] G.C.Hardy, T.Rogers, A generalization of fixed point theorem of S. Reich, *Can. Math. Bull.*, **16**(1973), 201–206.

[6] D.S.Jaggi, Some unique fixed point theorems, *Indian J. Pure Appl. Math.*, **8**(1977), 223–230.

[7] R.Kannan, Some results on fixed points-II, *Am. Math. Mon.*, **76**(1969), 71–76.

[8] S.Reich, Some remarks concerning contraction mappings, *Can. Math. Bull.*, **14**(1971), 121–124.

[9] P.L.Sharma, A.K.Yuel, A unique fixed point theorem in metric space, *Bull. Cal. Math. Soc.*, **76**(1984), 153–156.

[10] D.R.Smart, Fixed Point Theorems, Cambridge University Press, Cambridge, 1974.

[11] C.S.Wong, Common fixed points of two mappings, *Pac. J. Math.*, **48**(1973), 299–312.

[12] R.P.Agarwal, M.A.El-Gebeily, D.O'Regan, Generalized contractions in partially ordered metric spaces, *Appl. Anal.*, **87**(2008), 1–8.

[13] A.Amini-Harandi, H.Emami, A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations, *Nonlinear Anal., Theory Methods Appl.*, **72**(2010), 2238–2242.

[14] C.Ankush, B.Damjanovic, D.Lakshmi Kanta, Fixed point results on metric spaces via simulation functions, *Filomat*, **31**(2017), no.11, 3365–3375. DOI: 10.2298/FIL1711365C.

[15] J.Harjani, B.Lopez, K.Sadarangani, A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space, *Abstr. Appl. Anal.*, **2010**(2010), Article ID 190701, 8 pages.

[16] S.Hong, Fixed points of multivalued operators in ordered metric spaces with applications, *Nonlinear Anal., Theory Methods Appl.*, **72**(2010), 3929–3942.

[17] J.J.Nieto, R.R.Lopez, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, *Order*, **22**(2005), 223–239.

[18] J.J.Nieto, An abstract monotone iterative technique, *Nonlinear Analysis, Theory Methods and Applications*, **28**(1997), no.12, 1923–1933.

[19] A.C.M.Ran, M.C.B.Reurings, A fixed point theorem in partially ordered sets and some application to matrix equations, *Proc. Am. Math. Soc.*, **132**(2004), 1435–1443.

[20] E.S.Wolk, Continuous convergence in partially ordered sets, *Gen. Topol. Appl.*, **5**(1975), 221–234.

[21] X.Zhang, Fixed point theorems of multivalued monotone mappings in ordered metric spaces, *Appl. Math. Lett.*, **23**(2010), 235–240.

[22] T.G.Bhaskar, V.Lakshmikantham, Fixed point theory in partially ordered metric spaces and applications, *Nonlinear Anal., Theory Methods Appl.*, **65**(2006), 1379–1393.

[23] V.Lakshmikantham, L.J.Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric space, *Nonlinear Anal.*, **70**(2009). 4341–4349.
DOI: 10.1016/j.na.2008.09.020

[24] M.Arshad, A.Azam, P.Vetro, Some common fixed results in cone metric spaces, *Fixed Point Theory Appl.*, **2009**(2009), Article ID 493965.

[25] M.Arshad, J.Ahmad, E.Karapinar, Some common fixed point results in rectangular metric spaces, *Int. J. Anal.*, **2013**(2013), Article ID 307234.

[26] I.Altun, B.Damjanovic, D.Djoric, Fixed point and common fixed point theorems on ordered cone metric spaces, *Appl. Math. Lett.*, **23**(2010), 310–316.

[27] Y.J.Cho, M.H.Shah, N.Hussain, Coupled fixed points of weakly F-contractive mappings in topological spaces, *Appl. Math. Lett.*, **24**(2011), 1185–1190. DOI: 10.1016/j.aml.2011.02.004

[28] L.J.Ćirić, N.Cakić, M Rajović, J.S.Ume, Monotone generalized nonlinear contractions in partially ordered metric spaces, *Fixed Point Theory Appl.*, (2008), no.11, Article ID 131294.

[29] S.Chandok, Some common fixed point results for generalized weak contractive mappings in partially ordered metrix spaces, *Journal of Nonlinear Anal. Opt.*, **4**(2013), 45–52.

[30] S.Chandok, Some common fixed point results for rational type contraction mappings in partially ordered metric spaces, *Mathematica Bohemica*, **138**(2013), no. 4, 407–413.

[31] E.Graily, S.M.Vaezpour, R.Saadati, Y.J.Cho, Generalization of fixed point theorems in ordered metric spaces concerning generalized distance, *Fixed Point Theory Appl.*, **2011**(2011), no. 30. DOI: 10.1186/1687-1812-2011-30

[32] E.Karapinar, Couple fixed point theorems for nonlinear contractions in cone metric spaces, *Comput. Math. Appl.*, **59**(2010), no. 12, 3656–3668. DOI: 10.1016/j.camwa.2010.03.062

[33] M.Ozturk, M.Basarir, On some common fixed point theorems with rational expressions on cone metric spaces over a Banach algebra, *Hacet. J. Math. Stat.*, **41**(2012), no. 2, 211–222.

[34] F.Sabetghadam, H.P.Masiha, A.H.Sanatpour, Some coupled fixed point theorems in cone metric spaces, *Fixed point Theory Appl.*, **2009**(2009), no. 8, Article ID 125426.

[35] B.Samet, H.Yazidi, Coupled fixed point theorems in partially ordered ϵ-chainable metric spaces, *J. Math. Comput. Sci.*, **1**(2010), 142–151.

[36] W.Shatanawi, Partially ordered cone metric spaces and coupled fixed point results, *Comput. Math. Appl.*, **60**(2010), 2508–2515. DOI: 10.1016/j.camwa.2010.08.074

[37] W.Sintunavarat, Y.J.Cho, P.Kumam, Common fixed point theorems for c-distance in ordered cone metric space, *Comput. Math. Appl.*, **62**(2011), 1969–1978. DOI: 10.1016/j.camwa.2011.06.040

[38] N.Seshagiri Rao, K.Kalyani, Generalized Contractions to Coupled Fixed Point Theorems in Partially Ordered Metric Spaces, *Journal of Siberian Federal University. Mathematics & Physics*, **13**(2020), no. 4, 492–502. DOI: 10.17516/1997-1397-2020-13-4-492-502

[39] N.Seshagiri Rao, K.Kalyani, Coupled fixed point theorems with rational expressions in partially ordered metric spaces, *The Journal of Analysis*, **28**(2020), no. 4, 1085–1095. DOI: 10.1007/s41478-020-00236-y

[40] N.Seshagiri Rao, K.Kalyani, Kejal Khatri, Contractive mapping theorems in Partially ordered metric spaces, *CUBO*, **22**(2020), no. 2, 203–214.

[41] N.Seshagiri Rao, K.Kalyani, Unique fixed point theorems in partially ordered metric spaces, *Heliyon* **6**(2020), no. 11, e05563. DOI: 10.1016/j.heliyon.2020.e05563

[42] N.Seshagiri Rao, K.Kalyani, Generalized fixed point results of rational type contractions in partially ordered metric spaces, BMC Research Notes, **14**:390 (2021) 13 pages. DOI: 10.1186/s13104-021-05801-7

[43] K.Kalyani, N.Seshagiri Rao, Belay Mituku, On fixed point theorems of monotone functions in Ordered metric spaces, *The Journal of Analysis*, 2021, 14 pages. DOI: 10.1007/s41478-021-00308-7

[44] E.Liz, J.J.Nieto, Periodic boundary value problem for integro-differentail equations with general kernel, Dynamical Systems and Applications, **3**(1994), 297–304.

Результаты точки совпадения и их приложения в частично упорядоченных метрических пространствах

Н. Сешагири Рао
Научно-технический университет Адамы
Адама, Эфиопия

Карусала Кальяни
Фонд науки, технологий и исследований Вигнан
Прадеш, Индия

Текле Гемечу
Научно-технический университет Адамы
Адама, Эфиопия

Аннотация. Целью данной работы является установление некоторых общих теорем о неподвижной точке для f-неубывающего отображения в себя, удовлетворяющего некоторому условию сжатия рационального типа в репере метрических пространств, наделенных частичным порядком. Также в пространстве представлены некоторые следствия результатов в терминах сжатий интегрального типа. Кроме того, метод монотонной итерации был использован для нахождения единственного решения интегрального уравнения.

Ключевые слова: упорядоченное метрическое пространство, рациональное сжатие, согласованные отображения, точка совпадения, общая неподвижная точка.