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Some Systems of Transcendental Equations

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Abstract. Several examples of transcendental systems of equations are considered. Since the number of roots of such systems, as a rule, is infinite, it is necessary to study power sums of the roots of negative degree. Formulas for finding residue integrals, their relation to power sums of a negative degree of roots and their relation to residue integrals (multidimensional analogs of Waring's formulas) are obtained. Calculations of multidimensional numerical series are given.

Keywords: transcendental systems of equations, power sums of roots, residue integral.

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Introduction

Based on the multidimensional logarithmic residue, for systems of non-linear algebraic equations in $\mathbb{C}^n$ formulas for finding power sums of the roots of a system without calculating the roots themselves were earlier obtained (see [1–3]). For different types of systems such formulas have different forms. Based on this, a new method for the study of systems of algebraic equations in $\mathbb{C}^n$ have been constructed. It arose in the work of L. A. Aizenberg [1], its development was continued in monographs [2–4]. The main idea is to find power sums of roots of systems (for positive powers) and then, to use one-dimensional or multidimensional recurrent Newton formulas (see [5]). Unlike the classical method of elimination, it is less labor-intensive and does not increase the multiplicity of roots. It is based on the formula (see [1]) for a sum of the values of an arbitrary polynomial in the roots of a given systems of algebraic equations without finding the roots themselves.

For systems of transcendental equations, formulas for the sum of the values of the roots of the system, as a rule, cannot be obtained, since the number of roots of a system can be infinite and a series of coordinates of such roots can be diverging. Nevertheless, such transcendental systems of equations may very well arise, for example, in the problems of chemical kinetics [6, 7]. Thus, this is an important task to consider such systems.

In the works [8–21] power sums of roots in a negative power are considered for various systems of non-algebraic (transcendental) equations. To compute these power sums, a residue integral is used, the integration is carried out over skeletons of polycircles centered at the origin. Note that this residue integral is not, generally speaking, a multidimensional logarithmic residue or a Grothendieck residue. For various types of lower homogeneous systems of functions included in the system, formulas are given for finding residue integrals, their relationship with power sums of the roots of the system to a negative degree are established.

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The paper [12] investigated more complex systems in which the lower homogeneous parts are decomposed into linear factors and integration cycles in residue integrals are constructed from these factors. In [11], a system is studied that arises in the Zel'dovich–Semenov model (see [6,7]) in chemical kinetics.

The object of this study is some systems of transcendental equations in which the lower homogeneous parts of the functions included in the system form a non-degenerate system of algebraic equations: formulas are found for calculating the residue integrals, power sums of roots for a negative power, their relationship with the residue integrals are established. See [21].

1. General systems of transcendental equations

In this section we follow the paper [22].

Let $f_1(z), \ldots, f_n(z)$ be a system of functions holomorphic in a neighborhood of the origin in the multidimensional complex space $\mathbb{C}^n$, $z = (z_1, \ldots, z_n)$.

We expand the functions $f_1(z), \ldots, f_n(z)$ in Taylor series in a neighborhood of the origin and consider a system of equations of the form

$$f_j(z) = P_j(z) + Q_j(z) = 0, \quad i = 1, \ldots, n, \tag{1}$$

where P_j is the lowest homogeneous part of the Taylor expansion of the function $f_j(z)$. The degree of all monomials (with respect to the totality of variables) included in P_j, is equal to m_j, $j = 1, \ldots, n$. In the functions $Q_j l$, the degrees of all monomials are strictly greater than m_j.

The expansion of the functions Q_j, P_j, $j = 1, \ldots, n$ in a neighborhood of zero in Taylor series that converges absolutely and uniformly in this neighborhood has the form

$$Q_j(z) = \sum_{\|\alpha\| > m_j} a^j_\alpha z^\alpha, \tag{2}$$

$$P_j(z) = \sum_{\|\beta\| = m_j} b^j_\beta z^\beta, \tag{3}$$

$$j = 1, \ldots, n,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ are multi-indexes, i.e. α_j and β_j are non-negative integers, $j = 1, \ldots, n$, $\|\alpha\| = \alpha_1 + \ldots + \alpha_n$, $\|\beta\| = \beta_1 + \ldots + \beta_n$, and monomials $z^\alpha = z_1^{\alpha_1} \cdot z_2^{\alpha_2} \cdots z_n^{\alpha_n}$, $z^\beta = z_1^{\beta_1} \cdot z_2^{\beta_2} \cdots z_n^{\beta_n}$.

In what follows, we will assume that the system of polynomials $P_1(z), \ldots, P_n(z)$ is *nondegenerate*, that is, its common zero is only the point 0, the origin.

Consider an open set (a special analytic polyhedron) of the form

$$D_P(r_1, \ldots, r_n) = \{z : \ |P_j(z)| < r_j, \ i = j, \ldots, n\},$$

where $r_1, \ldots, r_n$ are positive numbers. Its *skeleton* has the form

$$\Gamma_P(r_1, \ldots, r_n) = \Gamma_P(r) = \{z : \ |P_j(z)| = r_j, \ j = 1, \ldots, n\}.$$

Let us start with a statement.

Lemma 1. *The next equality is true*

$$J_\gamma = \frac{1}{(2\pi i)^n} \int\limits_{\Gamma_P} \frac{1}{z_1^{\gamma_1+1} \cdot z_2^{\gamma_2+1} \cdots z_n^{\gamma_n+1}} \cdot \frac{df_1}{f_1} \wedge \frac{df_2}{f_2} \wedge \ldots \wedge \frac{df_n}{f_n} =$$

$$= \frac{(-1)^n}{(2\pi i)^n} \int\limits_{\Gamma_{\tilde{P}}} w_1^{\gamma_1+1} \cdot w_2^{\gamma_2+1} \cdots w_n^{\gamma_n+1} \cdot \frac{d\tilde{f}_1}{\tilde{f}_1} \wedge \frac{d\tilde{f}_2}{\tilde{f}_2} \wedge \ldots \wedge \frac{d\tilde{f}_n}{\tilde{f}_n} = (-1)^n \tilde{J}_\gamma.$$

For what follows, we need a generalized formula for transforming the Grothendieck residue (see [23]).

Theorem 1. *Let $h(w)$ be a holomorphic function, and the polynomials $f_k(w)$ and $g_j(w)$, $j,k=1,\dots,n$, are related by*

$$g_j=\sum_{k=1}^{n}a_{jk}f_k,\quad j=1,2,\dots,n,$$

the matrix $A=\|a_{jk}\|_{j,k=1}^{n}$ consists of polynomials. Consider the cycles

$$\Gamma_f=\{w:\ |f_j(w)|=r_j,\ j=1,\dots,n\},$$

$$\Gamma_g=\{w:\ |g_j(z)|=r_j,\ j=1,\dots,n\},$$

where all $r_j>0$. Then the equality

$$\int\limits_{\Gamma_f}h(w)\frac{dw}{f^{\alpha}}=\sum_{K,\sum\limits_{s=1}^{n}k_{sj}=\beta_s}\frac{\beta!}{\prod\limits_{s,j=1}^{n}(k_{sj})!}\int\limits_{\Gamma_g}h(w)\frac{\det A\prod\limits_{s,j=1}^{n}a_{sj}^{k_{sj}}\,dw}{g^{\beta}},\tag{4}$$

holds. Here $\beta!=\beta_1!\beta_2!\dots\beta_n$, $\beta=(\beta_1,\beta_2,\dots,\beta_n)$, the summation in the formula is over all non-negative integer matrices $K=\|k_{sj}\|_{s,j=1}^{n}$ with the conditions that the sum $\sum\limits_{s=1}^{n}k_{sj}=\alpha_j$, then $\beta_j=\sum\limits_{j=1}^{n}k_{js}$. Here$f^{\alpha}=f_1^{\alpha_1}\cdots f_n^{\alpha_n}$, $g^{\beta}=g_1^{\beta_1}\cdots g_n^{\beta_n}$.

Theorem 2. *The next formulas are valid*

$$\sum_{j=1}^{p}\frac{1}{z_{j1}^{\gamma_1+1}\cdot z_{j2}^{\gamma_2+1}\cdots z_{jn}^{\gamma_n+1}}=$$

$$=(2\pi i)^n)\int\limits_{\Gamma_{\tilde{P}}}w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\cdot\frac{d\tilde{f}_1}{\tilde{f}_1}\wedge\frac{d\tilde{f}_2}{\tilde{f}_2}\wedge\ldots\wedge\frac{d\tilde{f}_n}{\tilde{f}_n}=$$

$$=\sum_{\|\alpha\|\leqslant\|\gamma\|+n}\frac{(-1)^{n+\|\alpha\|}}{(2\pi i)^n}\int\limits_{\Gamma_{\tilde{P}}}w_1^{\gamma_1+1}\cdot w_2^{\gamma_2+1}\cdots w_n^{\gamma_n+1}\times$$

$$\times\frac{\tilde{\Delta}\cdot\tilde{Q}_1^{\alpha_1}\cdot\tilde{Q}_2^{\alpha_2}\cdots\tilde{Q}_n^{\alpha_n}dw_1\wedge dw_2\wedge\ldots\wedge dw_n}{\tilde{P}_1^{\alpha_1+1}\cdot\tilde{P}_2^{\alpha_2+1}\cdots\tilde{P}_n^{\alpha_n+1}}=$$

$$=\sum_{\|K\|\leqslant\|\gamma\|+n}\frac{(-1)^{\|K\|+n}\prod\limits_{s=1}^{n}\left(\sum\limits_{j=1}^{n}k_{sj}\right)!}{\prod\limits_{s,j=1}^{n}(k_{sj})!}\mathfrak{M}\left[\frac{w^{\gamma+I}\cdot\tilde{\Delta}\cdot\det A\cdot Q^{\alpha}\prod\limits_{s,j=1}^{n}a_{sj}^{k_{sj}}}{\prod\limits_{j=1}^{n}w_j^{\beta_jN_j+\beta_j+N_j}}\right],$$

where $\|K\|=\sum\limits_{s,j=1}^{n}k_{sj}$, and the functional $\mathfrak{M}$ assigns its free term to the Laurent polynomial.

In fact, in Theorem 2, analogs of the classical Waring formulas for finding power sums of roots of a system of algebraic equations are obtained.

2. Examples

Example 1. Consider a system of equations in two complex variables

$$\begin{cases} f_1(z_1, z_2) = z_1 z_2 + b_1 z_1 + b_2 z_2 = 0, \\ f_2(z_1, z_2) = 1 + a_1 z_1 + a_2 z_2 = 0. \end{cases} \tag{5}$$

Let us replace the variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will take the form

$$\begin{cases} \widetilde{f}_1 = 1 + b_2 w_1 + b_1 w_2 = 0, \\ \widetilde{f}_2 = w_1 w_2 + a_2 w_1 + a_1 w_2 = 0. \end{cases} \tag{6}$$

The Jacobian of the system (6) $\widetilde{\Delta}$ is equal to

$$\widetilde{\Delta} = \begin{vmatrix} b_2 & b_1, \\ w_2 + a_2 & w_1 + a_1 \end{vmatrix} = b_2 w_1 - b_1 w_2 + (a_1 b_2 - a_2 b_1).$$

Note that

$$\begin{cases} \widetilde{Q}_1 = 1, \\ \widetilde{Q}_2 = a_1 w_2 + a_2 w_1. \end{cases} \tag{7}$$

$$\begin{cases} \widetilde{P}_1 = b_1 w_2 + b_2 w_1, \\ \widetilde{P}_2 = w_1 w_2. \end{cases} \tag{8}$$

Let us calculate $\det A$:
Since

$$w_1^2 = a_{11}\widetilde{P}_1 + a_{12}\widetilde{P}_2,$$
$$w_2^2 = a_{21}\widetilde{P}_1 + a_{22}\widetilde{P}_2,$$

where $\widetilde{P}_1 = b_1 w_2 + b_2 w_1$, $\widetilde{P}_2 = w_1 w_2$.

Therefore, the elements of a_{ii} are equal

$$a_{11} = \frac{w_1}{b_2}, a_{12} = -\frac{b_1}{b_2},$$

$$a_{21} = \frac{w_2}{b_1}, a_{22} = -\frac{b_2}{b_1}.$$

Therefore,

$$\det A = \frac{w_2}{b_2} - \frac{w_1}{b_1} = \frac{w_2 b_1 - w_1 b_2}{b_1 b_2}.$$

By Theorem 2

$$J_{(0,0)} = \sum_{\|K\| = k_{11}+k_{12}+k_{21}+k_{22} \leqslant 2} \frac{(-1)^{\|K\|} \cdot (k_{11}+k_{12})! \cdot (k_{21}+k_{22})!}{k_{11}! \cdot k_{12}! \cdot k_{21}! \cdot k_{22}!} \times$$
$$\times \mathfrak{M}\left[\frac{\widetilde{\Delta} \cdot \det A \cdot \widetilde{Q}_1^{k_{11}+k_{21}} \cdot \widetilde{Q}_2^{k_{12}+k_{22}} \cdot a_{11}^{k_{11}} \cdot a_{12}^{k_{12}} \cdot a_{21}^{k_{21}} \cdot a_{22}^{k_{22}}}{w_1^{2(k_{11}+k_{12})} \cdot w_2^{2(k_{21}+k_{22})}}\right],$$

$$J_{(0,0)} = \sum_{\|K\| = k_{11}+k_{12}+k_{21}+k_{22} \leqslant 2} \frac{(-1)^{\|K\|} \cdot (k_{11}+k_{12})! \cdot (k_{21}+k_{22})!}{k_{11}! \cdot k_{12}! \cdot k_{21}! \cdot k_{22}!} \times$$

$$\times\mathfrak{M}\left[\frac{(-1)^{k_{12}+k_{22}}(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1^{1+k_{21}+k_{22}-k_{12}}\cdot b_2^{1+k_{11}+k_{12}-k_{22}}}\right]\left[\frac{(a_1w_2+a_2w_1)^{k_{12}+k_{22}}}{w_1^{k_{11}+2k_{12}}\cdot w_2^{k_{21}+2k_{22}}}\right].$$

Calculate the value of the sums

$(0,0,0,0)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1b_2}\right]=0,$$

$(1,0,0,0)$:

$$\mathfrak{M}\left[\frac{-(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1b_2^2\cdot w_1}\right]=\frac{a_1b_2-a_2b_1}{b_1b_2},$$

$(0,1,0,0)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)}{b_2^2\cdot w_1^2}\right]=$$

$$=\frac{-a_2(a_1b_2-a_2b_1)}{b_2}=-a_1a_2+\frac{a_2^2b_1}{b_2},$$

$(0,0,1,0)$:

$$\mathfrak{M}\left[\frac{-(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1^2b_2\cdot w_2}\right]=-\frac{a_1b_2-a_2b_1}{b_1b_2},$$

$(0,0,0,1)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)}{b_1^2\cdot w_2^2}\right]=$$

$$=\frac{a_1(a_1b_2-a_2b_1)}{b_1}=-a_1a_2+\frac{a_1^2b_2}{b_1},$$

$(2,0,0,0)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1b_2^3\cdot w_1^2}\right]=-\frac{1}{b_1b_2},$$

$(0,2,0,0)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)^2\cdot b_1}{b_2^3\cdot w_1^4}\right]=-\frac{a_2^2b_1}{b_2},$$

$(0,0,2,0)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1^3b_2\cdot w_2^2}\right]=-\frac{1}{b_1b_2},$$

$(0,0,0,2)$:

$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)^2\cdot b_2}{b_1^3\cdot w_2^4}\right]=-\frac{a_1^2b_2}{b_1},$$

$(1,1,0,0)$:

$$\mathfrak{M}\left[\frac{-2(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)}{b_2^3\cdot w_1^3}\right]=\frac{2a_2}{b_2},$$

$(1,0,1,0):$
$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)}{b_1^2b_2^2\cdot w_1w_2}\right]=\frac{2}{b_1b_2},$$
$(1,0,0,1):$
$$\mathfrak{M}\left[\frac{-(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)}{b_1^2b_2\cdot w_1w_2^2}\right]=\frac{a_2}{b_2}-\frac{2a_1}{b_1},$$
$(0,1,1,0):$
$$\mathfrak{M}\left[\frac{-(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)}{b_1b_2^2\cdot w_1^2w_2}\right]=\frac{a_1}{b_1}-\frac{2a_2}{b_2},$$
$(0,1,0,1):$
$$\mathfrak{M}\left[\frac{(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)^2}{b_1b_2\cdot w_1^2w_2^2}\right]=-\frac{a_1^2b_2}{b_1}-\frac{a_2^2b_1}{b_2}+2a_1a_2,$$
$(0,0,1,1):$
$$\mathfrak{M}\left[\frac{-2(w_2b_1-w_1b_2)\cdot(b_2w_1-b_1w_2+a_1b_2-a_2b_1)\cdot(a_1w_2+a_2w_1)}{b_1^3\cdot w_2^3}\right]=\frac{2a_1}{b_1}.$$

Therefore
$$J_{(0,0)}=2a_1a_2+\frac{a_2}{b_2}+\frac{a_1}{b_1}-\frac{a_1^2b_2}{b_1}-\frac{a_2^2b_1}{b_2}.$$

Example 2. Consider a system of equations in two complex variables
$$\begin{cases}\widetilde{f}_1=1+b_1w_2+b_2w_1=0,\\ \widetilde{f}_2=w_1w_2+a_1w_2+a_2w_1=0.\end{cases}\tag{9}$$

Let $u=w_1w_2$, then $w_1=\dfrac{u}{w_2}$, substitute in our system
$$\begin{cases}1+b_1w_2+\dfrac{b_2u}{w_2}=0,\\ u+a_1w_2+\dfrac{a_2u}{w_2}=0.\end{cases}\tag{10}$$
multiply each equation of the system by w_2
$$\begin{cases}w_2+b_1w_2^2+b_2u=0,\\ w_2u+a_1w_2^2+a_2u=0.\end{cases}\tag{11}$$
Now we multiply the first equation of the system by a_1, and the second by b_1 and subtract one from the other
$$w_2(b_1u-a_1)-u(a_1b_2-a_2b_1)=0,$$
so
$$w_2=\frac{u(a_1b_2-a_2b_1)}{b_1u-a_1}.$$

Let us substitute this into the first equation of the system and get rid of the denominator and the second variable
$$b_2u(b_1u-a_1)^2+u(a_1b_2-a_2b_1)(b_1u-a_1)+b_1^2u^2(a_1b_2-a_2b_1)^2=0.$$

We get

$$b_1^2 b_2 u^2 + (b_1(a_1b_2 - a_2b_1)^2 + b_1(a_1b_2 - a_2b_1) - 2a_1b_1b_2)u + a_1^2 b_2 - a_1(a_1b_2 - a_2b_1) = 0.$$

By the generalized Vieta theorem

$$J_{(0,0)} = \sum_{j=1}^{p} w_{j1} \cdot w_{j2} = -\frac{(a_1b_2 - a_2b_1)^2}{b_1b_2} + \frac{a_1}{b_1} + \frac{a_2}{b_2}.$$

Example 3. Consider a system (13) of equations in two complex variables (example 1).

Recall the well-known expansions of the sine into an infinite product and a power series:

$$\frac{\sin\sqrt{z}}{\sqrt{z}} = \prod_{k=1}^{\infty}\Big(1 - \frac{z}{k^2\pi^2}\Big) = \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{(2k+1)!},$$

which uniformly and absolutely converge on the complex plane and have the order of growth 1/2.

Consider the system of equations

$$\begin{cases} f_1(z_1, z_2) = z_1z_2 + b_1z_1 + b_2z_2 = 0, \\ f_2(z_1, z_2) = \dfrac{\sin\sqrt{a_1z_1 + a_2z_2}}{\sqrt{a_1z_1 + a_2z_2}} = \prod\limits_{s=1}^{\infty}\left(1 - \dfrac{a_1z_1 + a_2z_2}{s^2\pi^2}\right) = 0. \end{cases} \tag{12}$$

Using the formula obtained above and the known expansion of the series, we obtain that the integral $J_{0,0}$ is equal to the sum of the series

$$J_{(0,0)} = -\sum_{s=1}^{\infty}\frac{a_1^2b_2}{\pi^4s^4b_1} - \sum_{s=1}^{\infty}\frac{a_2^2b_1}{\pi^4s^4b_2} + \sum_{s=1}^{\infty}\frac{a_2}{\pi^2s^2b_2} + \sum_{s=1}^{\infty}\frac{a_1}{\pi^2s^2b_1} + \sum_{s=1}^{\infty}\frac{2a_1a_2}{\pi^4s^4} =$$

$$= -\frac{(a_1b_2 - a_2b_1)^2}{90b_1b_2} + \frac{a_1b_2 + a_2b_1}{6b_1b_2}.$$

Example 4. Consider a system of equations in two complex variables

$$\begin{cases} f_1(z_1, z_2) = a_1z_1 - a_2z_2 + z_1^2z_2 = 0, \\ f_2(z_1, z_2) = b_1z_1 + b_2z_2 + z_1z_2^2 = 0. \end{cases} \tag{13}$$

Let us replace the variables $z_1 = \dfrac{1}{w_1}$, $z_2 = \dfrac{1}{w_2}$. Our system will take the form

$$\begin{cases} \widetilde{f}_1 = -a_2w_1^2 + a_1w_1w_2 + 1 = 0, \\ \widetilde{f}_2 = b_2w_1w_2 + b_1w_2^2 + 1 = 0. \end{cases} \tag{14}$$

The Jacobian of the system (6) $\widetilde{\Delta}$ is equal to

$$\widetilde{\Delta} = \begin{vmatrix} -2a_2w_1 + a_1w_2 & a_1w_1 \\ b_2w_2 & 2b_1w_2 + b_2w_1 \end{vmatrix} = -2a_2b_2w_1^2 - 4a_2b_1w_1w_2 + 2a_1b_1w_2^2.$$

Note that

$$\begin{cases} \widetilde{Q}_1 = 1, \\ \widetilde{Q}_2 = 1. \end{cases} \tag{15}$$

$$\begin{cases} \widetilde{P}_1 = -a_2 w_1^2 + a_1 w_1 w_2 = 0, \\ \widetilde{P}_2 = b_2 w_1 w_2 + b_1 w_2^2 = 0. \end{cases} \tag{16}$$

Calculate $\det A$:

$$Res = \begin{vmatrix} -a_2 & a_1 & 0 & 0 \\ 0 & -a_2 & a_1 & 0 \\ 0 & b_2 & b_1 & 0 \\ 0 & 0 & b_2 & b_1 \end{vmatrix}.$$

$\Delta = a_2 b_1 (a_2 b_1 + a_1 b_2)/.$

Let us determine the minors:

$$\widetilde{\Delta}_1 = \begin{vmatrix} -a_2 & a_1 & 0 \\ b_2 & b_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = -a_2 b_1^2 - a_1 b_1 b_2, \qquad \widetilde{\Delta}_2 = -\begin{vmatrix} a_1 & 0 & 0 \\ b_2 & b_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = -a_1 b_1^2,$$

$$\widetilde{\Delta}_3 = \begin{vmatrix} a_1 & 0 & 0 \\ -a_2 & a_1 & 0 \\ 0 & b_2 & b_1 \end{vmatrix} = a_1^2 b_1, \qquad \widetilde{\Delta}_4 = -\begin{vmatrix} a_1 & 0 & 0 \\ -a_2 & a_1 & 0 \\ b_2 & b_1 & 0 \end{vmatrix} = 0.$$

$$\Delta_1 = -\begin{vmatrix} 0 & -a_2 & a_1 \\ 0 & b_2 & b_1 \\ 0 & 0 & b_2 \end{vmatrix} = 0, \qquad \Delta_2 = \begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & b_2 & b_1 \\ 0 & 0 & b_2 \end{vmatrix} = -a_2 b_2^2,$$

$$\Delta_3 = -\begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & -a_2 & a_1 \\ 0 & 0 & b_2 \end{vmatrix} = -a_2^2 b_2, \qquad \Delta_4 = \begin{vmatrix} -a_2 & a_1 & 0 \\ 0 & -a_2 & a_1 \\ 0 & b_2 & b_1 \end{vmatrix} = a_2^2 b_1 + a_1 a_2 b_2.$$

Therefore, the elements of a_{ii} are equal

$$a_{11} = \frac{1}{\Delta}\left(\widetilde{\Delta}_1 w_1 + \widetilde{\Delta}_2 w_2\right) = \frac{1}{\Delta}\left((-a_2 b_1^2 - a_1 b_1 b_2) w_1 - a_1 b_1^2 w_2\right),$$

$$a_{12} = \frac{1}{\Delta}\left(\widetilde{\Delta}_3 w_1 + \widetilde{\Delta}_4 w_2\right) = \frac{a_1^2 b_1 w_1}{\Delta},$$

$$a_{21} = \frac{1}{\Delta}\left(\Delta_1 w_1 + \Delta_2 w_2\right) = \frac{-a_2 b_2^2 w_2}{\Delta},$$

$$a_{22} = \frac{1}{\Delta}\left(\Delta_3 w_1 + \Delta_4 w_2\right) = \frac{1}{\Delta}\left(-a_2^2 b_2 w_1 + (a_2^2 b_1 + a_1 a_2 b_2) w_2\right).$$

Then

$$w_1^3 = a_{11}\widetilde{P}_1 + a_{12}\widetilde{P}_2,$$
$$w_2^3 = a_{21}\widetilde{P}_1 + a_{22}\widetilde{P}_2,$$

were $\widetilde{P}_1 = -a_2 w_1^2 + a_1 w_1 w_2$, $\widetilde{P}_2 = b_2 w_1 w_2 + b_1 w_2^2$.

It is not difficult to make sure that

$$\Delta(1,3) = -\begin{vmatrix} -a_2 & a_1 \\ 0 & b_2 \end{vmatrix} = a_2 b_2,$$

$$\Delta(1,4) = \begin{vmatrix} -a_2 & a_1 \\ b_2 & b_1 \end{vmatrix} = -a_2 b_1 - a_1 b_2,$$

$$\Delta(2,4) = -\begin{vmatrix} a_1 & 0 \\ b_2 & b_1 \end{vmatrix} = -a_1 b_1,$$

$$\Delta(2,3) = \begin{vmatrix} a_1 & 0 \\ 0 & b_2 \end{vmatrix} = a_1 b_2.$$

Let us calculate now $\det A$:

$$\det A = \frac{1}{\Delta}\left(\Delta(1,3)w_1^2 + \Delta(2,4)w_2^2 + (\Delta(2,3)+\Delta(1,4))w_1w_2\right) = \\ = \frac{1}{\Delta}\left(a_2b_2w_1^2 - a_2b_1w_1w_2 - a_1b_1w_2^2\right).$$

By Theorem 2

$$J_{(0,0)} = \sum_{\|K\|=k_{11}+k_{12}+k_{21}+k_{22}\leqslant 2} \frac{(-1)^{\|K\|}\cdot(k_{11}+k_{12})!\cdot(k_{21}+k_{22})!}{k_{11}!\cdot k_{12}!\cdot k_{21}!\cdot k_{22}!}\times \\ \times\mathfrak{M}\left[\frac{\widetilde{\Delta}\cdot\det A\cdot a_{11}^{k_{11}}\cdot a_{12}^{k_{12}}\cdot a_{21}^{k_{21}}\cdot a_{22}^{k_{22}}}{w_1^{3(k_{11}+k_{12})+1}\cdot w_2^{3(k_{21}+k_{22})+1}}\right].$$

We calculate the value of the sums by denoting $\bar{\Delta} = a_2b_1 + a_1b_2$,
$(0,0,0,0)$:

$$\mathfrak{M}\left[\frac{\widetilde{\Delta}\cdot\det A}{w_1\cdot w_2}\right] = 0,$$

$(1,0,0,0)$:

$$-\mathfrak{M}\left[\frac{a_{11}\cdot\widetilde{\Delta}\cdot\det A}{w_1^4\cdot w_2}\right] = \frac{-2a_1a_2^2b_1^2b_2^2 - 2a_2^2b_1b_2(a_2b_1^2 + a_1b_1b_2)}{\Delta^2} = -\frac{2a_1b_2^2}{\bar{\Delta}^2} - \frac{2b_2}{\bar{\Delta}},$$

$(0,1,0,0)$:

$$-\mathfrak{M}\left[\frac{a_{12}\cdot\widetilde{\Delta}\cdot\det A}{w_1^4\cdot w_2}\right] = -\frac{a_1^2b_1(2a_2^2b_1b_2 - 4a_2^2b_1b_2)}{\Delta^2} = \frac{2a_1^2b_2}{\bar{\Delta}^2},$$

$(0,0,1,0)$:

$$-\mathfrak{M}\left[\frac{a_{21}\cdot\widetilde{\Delta}\cdot\det A}{w_1\cdot w_2^4}\right] = \frac{a_2b_2^2(4a_1a_2b_1^2 - 2a_1a_2b_1^2)}{\Delta^2} = \frac{2a_1b_2^2}{\bar{\Delta}^2},$$

$(0,0,0,1)$:

$$-\mathfrak{M}\left[\frac{a_{22}\cdot\widetilde{\Delta}\cdot\det A}{w_1\cdot w_2^4}\right] = -\frac{2a_1^2a_2^2b_1^2b_2 + 2a_1a_2b_1^2(a_2^2b_1 + a_1a_2b_2)}{\Delta^2} = -\frac{2a_1^2b_2}{\bar{\Delta}^2} - \frac{2a_1}{\bar{\Delta}},$$

$(2,0,0,0)$:

$$\mathfrak{M}\left[\frac{a_{11}^2\cdot\widetilde{\Delta}\cdot\det A}{w_1^7\cdot w_2}\right] = 0,$$

$(0,2,0,0)$:

$$\mathfrak{M}\left[\frac{a_{12}^2\cdot\widetilde{\Delta}\cdot\det A}{w_1^7\cdot w_2}\right] = 0,$$

$(0,0,2,0)$:

$$\mathfrak{M}\left[\frac{a_{21}^2\cdot\widetilde{\Delta}\cdot\det A}{w_1\cdot w_2^7}\right] = 0,$$

$(0,0,0,2):$

$$\mathfrak{M}\left[\frac{a_{22}^2\cdot\widetilde{\Delta}\cdot\det A}{w_1\cdot w_2^7}\right]=0,$$

$(1,1,0,0):$

$$2\mathfrak{M}\left[\frac{a_{11}a_{12}\cdot\widetilde{\Delta}\cdot\det A}{w_1^7\cdot w_2}\right]=0,$$

$(1,0,1,0):$

$$-\mathfrak{M}\left[\frac{a_{11}a_{12}\cdot\widetilde{\Delta}\cdot\det A}{w_1^4\cdot w_2^4}\right]=0,$$

$(0,1,1,0):$

$$-\mathfrak{M}\left[\frac{a_{12}a_{21}\cdot\widetilde{\Delta}\cdot\det A}{w_1^4\cdot w_2^4}\right]=0,$$

$(0,1,0,1):$

$$\mathfrak{M}\left[\frac{a_{12}a_{22}\cdot\widetilde{\Delta}\cdot\det A}{w_1^4\cdot w_2^4}\right]=0,$$

$(0,0,1,1):$

$$2\mathfrak{M}\left[\frac{a_{21}a_{22}\cdot\widetilde{\Delta}\cdot\det A}{w_1\cdot w_2^7}\right]=0,$$

$(1,0,0,1):$

$$\mathfrak{M}\left[\frac{a_{11}a_{22}\cdot\widetilde{\Delta}\cdot\det A}{w_1^4\cdot w_2^4}\right]=0.$$

Therefore

$$J_{(0,0)}=-\frac{2a_1b_2^2}{\bar{\Delta}^2}-\frac{2a_1^2b_2}{\bar{\Delta}^2}-\frac{2b_2}{\bar{\Delta}}-\frac{2a_1}{\bar{\Delta}}+\frac{2a_1^2b_2}{\bar{\Delta}^2}+\frac{2a_1b_2^2}{\bar{\Delta}^2}=$$
$$=-\frac{2a_1b_2(a_1+b_2)}{\bar{\Delta}^2}-\frac{2(a_1+b_2)}{\bar{\Delta}}+\frac{2a_1b_2(a_1+b_2)}{\bar{\Delta}^2},$$

$$J_{(0,0)}=-\frac{2(a_1+b_2)}{\bar{\Delta}}. \tag{17}$$

Example 5. Consider a system of equations in two complex variables

$$\begin{cases}\widetilde{f}_1=-a_2w_1^2+a_1w_1w_2+1=0,\\ \widetilde{f}_2=b_2w_1w_2+b_1w_2^2+1=0.\end{cases} \tag{18}$$

Let $u=w_1w_2$, then $w_2=\dfrac{u}{w_1}$, substitute in our system

$$\begin{cases}-a_2w_1^2+a_1u+1=0,\\ b_2u+b_1\frac{u^2}{w_1^2}+1=0.\end{cases} \tag{19}$$

We get rid of the denominators

$$\begin{cases}-a_2w_1^2+a_1u+1=0,\\ b_2uw_1^2+b_1u^2+w_1^2=0.\end{cases} \tag{20}$$

Calculate $\det \Delta$:

$$\Delta = \begin{vmatrix} -a_2 & 0 & a_1u+1 & 0 \\ 0 & -a_2 & 0 & a_1u+1 \\ b_2u+1 & 0 & b_1u^2 & 0 \\ 0 & b_2u+1 & 0 & b_1u^2 \end{vmatrix},$$

$$\Delta = u^4(a_1^2b_2^2 + 2a_1a_2b_1b_2 + a_2^2b_1^2)+$$
$$+u^3(2a_1a_2b_1 + 2a_2b_1b_2 + 2a_1^2b_2 + 2a_1b_2^2) + u^2(2a_2b_1 + a_1^2 + 4a_1b_2 + b_2^2) + u(2a_1 + 2b_2) + 1.$$

By the generalized Vieta theorem

$$J_{(0,0)} = \sum_{j=1}^{p} w_{j1}^{\gamma_1+1} \cdot w_{j2}^{\gamma_2+1} = -\frac{a_3}{a_4} = \frac{2a_1a_2b_1 + 2a_2b_1b_2 + 2a_1^2b_2 + 2a_1b_2^2}{a_1^2b_2^2 + 2a_1a_2b_1b_2 + a_2^2b_1^2} =$$
$$= -2\frac{a_2b_1(a_1+b_2) + a_1b_2(a_1+b_2)}{\bar{\Delta}^2} = -2\frac{(a_1+b_2)(a_1b_2+a_2b_1)}{\bar{\Delta}^2} = -\frac{2(a_1+b_2)}{\bar{\Delta}}. \quad (21)$$

Example 6. Recall the well-known expansions of the sine into an infinite product and a power series:

$$\frac{\sin\sqrt{z}}{\sqrt{z}} = \prod_{k=1}^{\infty}\left(1 - \frac{z}{k^2\pi^2}\right) = \sum_{k=0}^{\infty}\frac{(-1)^k z^k}{(2k+1)!},$$

which uniformly and absolutely converge on any compact from the complex plane and have the order of growth 1/2.

Consider the system of equations

$$\begin{cases} f_1(w_1,w_2) = \dfrac{\sin\sqrt{-a_1w_1w_2 + a_2w_1^2}}{\sqrt{-a_1w_1w_2 + a_2w_1^2}} = \prod\limits_{k=1}^{\infty}\left(1 - \dfrac{-a_1w_1w_2 + a_2w_1^2}{k^2\pi^2}\right) = 0, \\ f_2(w_1,w_2) = \dfrac{\sin\sqrt{b_1w_2^2 + b_2w_1w_2}}{\sqrt{b_1w_2^2 + b_2w_1w_2}} = \prod\limits_{s=1}^{\infty}\left(1 - \dfrac{b_1w_2^2 + b_2w_1w_2}{s^2\pi^2}\right) = 0. \end{cases} \quad (22)$$

Each of the functions of this system decomposes into an infinite product of functions from the system (18).

Therefore, the integral $J_{0,0}$ is equal to the sum of the series

$$J_{(0,0)} = \sum_{k,s=1}^{\infty}\frac{2\pi^2(a_1s^2 - b_2k^2)}{a_1b_2 + a_2b_1}.$$

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Некоторые системы трансцендентных уравнений

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Аннотация. Рассмотрены различные примеры систем трансцендентных уравнений. Так как число корней таких систем, как правило, бесконечно, то необходимо изучить степенные суммы корней в отрицательной степени. Получены формулы для нахождения вычетных интегралов, их связь со степенными суммами корней в отрицательной степени, многомерные аналоги формул Варинга. Вычислены суммы многомерных числовых рядов.

Ключевые слова: трансцендентные системы уравнений, степенные суммы корней, вычетные интегралы.

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Convolutional Integro-Differential Equations in Banach Spaces With a Noetherian Operator in the Main Part

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Abstract. An initial-value problem for an integro-differential equation of convolution type with a finite index operator for the higher order derivative in Banach spaces is considered. The equations under consideration model the evolution of the processes with "memory" when the current state of the system is influenced not only by the entire history of observations but also by the factors that have formed it and that remain relevant to the current moment of observation. Solutions are constructed in the class of generalized functions with a left bounded support with the use of the theory of fundamental operator functions of degenerate integro-differential operators in Banach spaces. A fundamental operator function that corresponds to the equation under consideration is constructed. Using this function the generalized solution is restored. The relationship between the generalized solution and the classical solution of the original initial-value problem is studied. Two examples of initial-boundary value problems for the integro-differential equations with partial derivatives are considered.

Keywords: Banach space, generalized function, Jordan set, Noetherian operator, fundamental operator-function.

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Introduction

Let us consider the following initial-value problem

$$Bu^{(N)}(t) = Au(t) + \int_0^t \big(\alpha(t-s)A + \beta(t-s)B\big)u(s)ds + f(t), \tag{1}$$

$$u(0) = u_0,\ u'(0) = u_1,\ \dots\ , u^{(N-1)}(0) = u_{N-1}, \tag{2}$$

where A, B are closed linear operators with compact domains of definition which act from the Banach space E_1 into the Banach space E_2, operator B is Noetherian operator [1, 2], $\alpha(t)$, $\beta(t)$ are sufficiently smooth numerical functions, $f(t)$ is a sufficiently smooth function with the values in E_2.

Some initial-boundary value problems of mathematical physics can be reduced to problem (1)–(2), for example, vibration of plates in visco-elastic media [3]. It is known that in the class of functions $C^N(t \geqslant 0; E_1)$ the initial-value problem (1)–(2) can be solved only if initial conditions (2) and function $f(t) are compatible$. Meanwhile, in the case of constructing solutions

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in $K'_+(E_1)$, in a space of distributions with a left bounded support, there are no requirements of such compatibility. Some results of corresponding studies in the case of differential equations, i.e., when $\alpha(t) = \beta(t) \equiv 0$, can be found in [4] and [5] for the space $C^N(t \geqslant 0; E_1)$ and in [6] for $K'_+(E_1)$. Below integro-differential equations of form (1) are studied.

1. Auxiliary data and designations

1.1. Jordan sets of Noetherian operators

Let us assume that the following condition is satisfied for operators A and B:
(A) $D(B) \subset D(A)$, $\overline{D(A)} = \overline{D(B)} = E_1$, $\dim N(B) = n$, $\dim N(B^*) = m$, $n \neq m$, operator B is normally solvable, i.e., $\overline{R(B)} = R(B)$.

The following designations are used: $\{\varphi_i\}_{i=1}^n \in E_1$ is the core basis $N(B)$ of operator B, $\{\phi_j\}_{j=1}^m \in E_2^*$ is the core basis $N(B^*)$ of the conjugate operator B^*, $\{z_j\}_{j=1}^m \in E_2$ and $\{\gamma\}_{i=1}^n \in E_1^*$ are the systems of elements and functionals that are bi-orthogonal to these bases, i.e., $\langle \varphi_i, \gamma_k \rangle = \delta_{ik}$, $i, k = 1, \dots, n$ and $\langle z_k, \phi_j \rangle = \delta_{kj}$, $k, j = 1, \dots, m$ (see [2], p. 168, Lemma 4). Let us now construct the projectors with the use of these systems of elements and functionals

$$P = \sum_{i=1}^n P_i = \sum_{i=1}^n \langle \cdot, \gamma_i \rangle \varphi_i : E_1 \to E_1, \quad Q = \sum_{j=1}^m Q_i = \sum_{j=1}^m \langle \cdot, \phi_j \rangle z_j : E_2 \to E_2.$$

It has been proved [7] that there is only one bounded pseudo-inverse operator $B^+ \in \mathcal{L}(E_2, E_1)$ which has the following properties

$$D(B^+) = R(B) \oplus \{z_1, \dots, z_m\} \equiv E_2, \quad R(B^+) = N(P) \cap D(B),$$

$$BB^+ = I - Q \text{ on } D(B^+), \quad B^+B = I - P \text{ on } D(B).$$

Furthermore, $N(B^+) = \{z_1, \dots, z_m\}$, the following operator equalities are valid $BB^+B = B$, $B^+BB^+ = B^+$, and operator AB^+ is bounded due to condition **(A)**.

The conjugate operator $B^{+*} \in \mathcal{L}(E_1^*, E_2^*)$ has similar set of properties, i.e.,

$$N(B^{+*}) = \{\gamma_1, \dots, \gamma_n\}, \quad B^*B^{+*}B^* = B^*, \quad B^{+*}B^*B^{+*} = B^{+*}, \quad B^{+*} = B^{*+},$$

$$D(B^{+*}) = R(B^*) \oplus \{\gamma_1, \dots, \gamma_n\} \equiv E_1^*, \quad R(B^{+*}) = N(Q^*) \cap D(B^*),$$

$$B^*B^{*+} = I - P^* \text{ on } D(B^{*+}), \quad B^{*+}B^* = I - Q^* \text{ on } D(B^*),$$

here

$$P^* = \sum_{i=1}^n P_i^* = \sum_{i=1}^n \langle \varphi_i, \cdot \rangle \gamma_i : E_1^* \to E_1^*, \quad Q^* = \sum_{j=1}^m Q_i^* = \sum_{j=1}^m \langle z_j, \cdot \rangle \phi_j : E_2^* \to E_2^*.$$

Following [5,8], let us introduce the systems of adjoined elements and functionals:

$$\varphi_i^{(j)} = (B^+A)^{j-1}\varphi_i^{(1)}, \ i = 1, \dots, n, \ j \geqslant 2, \quad \varphi_i^{(1)} = \varphi_i,$$

$$\phi_i^{(j)} = (B^{+*}A^*)^{j-1}\phi_i^{(1)}, \ i = 1, \dots, m, \ j \geqslant 2, \quad \phi_i^{(1)} = \phi_i.$$

The following inclusions $\varphi_i^{(j)} \in N(P)$ and $\phi_i^{(j)} \in N(Q^*)$ are valid for the adjoined elements and functionals. Due to their constructions and properties of operators B^+ and B^{+*} the following equalities are satisfied $\langle\varphi_i^{(j)},\gamma_k\rangle=0,\ i,k=1,\dots,n,\ j\geqslant 2$ and $\langle z_k,\phi_i^{(j)}\rangle=0, i,k=1,\dots,m,\ j\geqslant 2$.

Next, let us introduce condition [5,8]
(B) Elements $\varphi_i^{(j)}$ satisfy the system of equations and inequalities

$$B\varphi_i^{(j)}=A\varphi_i^{(j-1)},\ i=1,\dots,n,\ j=1,\dots,p_i,$$

$$B\varphi_i^{(p_i+1)}\neq A\varphi_i^{(p_i)},\ i=1,\dots,n,$$

$$rang\left\|\left\langle A\varphi_i^{(p_i)},\phi_k^{(1)}\right\rangle\right\|_{i=1,\dots,n,\,k=1,\dots,m}=\min(n,m)=l.$$

Condition **(B)** means that the system of elements $\left\{\varphi_i^{(j)},\ i=1,\dots,n,\ j=1,\dots,p_i\right\}$ forms a complete A-Jordan set of operator B [1,4]. It has been shown in [5,8] that bases $\{\varphi_i\}_{i=1}^n$, $\{\phi_j\}_{j=1}^m$ and the corresponding to them systems $\{z_j\}_{j=1}^m$, $\{\gamma_i\}_{i=1}^n$ can be chosen such that the following equalities are satisfied

$$\left\langle A\varphi_i^{(j)},\phi_k\right\rangle=\begin{cases}0, & i=1,\dots,n,\ j=1,\dots,p_i-1,\ k=1,\dots,m,\\ \delta_{ik}, & j=p_i,\ i,k=1,\dots,l,\end{cases}$$

$$\left\langle \varphi_i,A^*\phi_k^{(j)}\right\rangle=\begin{cases}0, & k=1,\dots,m,\ j=1,\dots,p_k-1,\ i=1,\dots,n,\\ \delta_{ik}, & j=p_k,\ i,k=1,\dots,l.\end{cases}$$

Therefore, if condition **(B)** is satisfied, then (without any restriction of the generality) one can state that $z_i=A\varphi_i^{(p_i)}$, $\gamma_k=A^*\phi_k^{(p_k)}$, $i,k=1,\dots,l$. In this case, matrices $\left\|\left\langle A\varphi_i^{(p_i)},\phi_k^{(1)}\right\rangle\right\|$ and $\left\|\left\langle \varphi_i^{(1)},A^*\phi_k^{(p_k)}\right\rangle\right\|$, where $i=1,\dots,n,\ k=1,\dots,m$, have the same full rank l. Both matrices are equivalent to a rectangular matrix with the unit main rank minor of order l and with zeros at the rest of the places. Next, let us suppose that all such transformations of the bases are fulfilled. Along with projector Q of space E_2, we introduce another projector

$$\tilde{Q}=\sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle\cdot,\phi_i^{(j)}\right\rangle A\varphi_i^{(p_i+1-j)}. \tag{3}$$

Furthermore, if $n>m$, then we assume that $\phi_i^{(1)}=0$ when $i=m+1,\dots,n$, and other functionals $\phi_i^{(j)}\in E_2^*$, $i=m+1,\dots,n$, $j=2,\dots,p_i$ are arbitrary ("free parameters"). Hence, for $n>m$

$$Q\tilde{Q}=Q,\quad Q(I-\tilde{Q})=0\ \text{ and }\ Q(AB^+)^k(I-\tilde{Q})=0,\ \forall k\in N. \tag{4}$$

Respectively, when $n<m$, we have the following relations $\forall k\in N$

$$\begin{cases}Q_j\tilde{Q}=Q_j, & Q_j(I-\tilde{Q})=0, & Q_j(AB^+)^k(I-\tilde{Q})=0, & j=1,\dots,n,\\ Q_j\tilde{Q}=0, & Q_j(I-\tilde{Q})=Q_j, & Q_j(AB^+)^k(I-\tilde{Q})=Q_j(AB^+)^k, & j=n+1,\dots,m.\end{cases} \tag{5}$$

Hence, the following auxiliary lemma is valid.

Lemma 1. *If $n<m$ then*

$$AB^+\left[I-\tilde{Q}\right]B-\left[I-\tilde{Q}\right]A\equiv 0, \tag{6}$$

$$B^+\left[I-\tilde{Q}\right]B+\sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle\cdot,A^*\phi_i^{(j)}\right\rangle\varphi_i^{(p_i+1-j)}=I. \tag{7}$$

Proof. Actually we have

$$AB^{+}\left[I-\tilde{Q}\right]B-\left[I-\tilde{Q}\right]A=AB^{+}B-AB^{+}\tilde{Q}B-A+\tilde{Q}A=$$
$$=A(I-P)-AB^{+}\tilde{Q}B-A+\tilde{Q}A=-AP-AB^{+}\tilde{Q}B+\tilde{Q}A\equiv 0.$$

Due to the choice of the bases we have

$$AP+AB^{+}\tilde{Q}B=\sum_{i=1}^{n}\left\langle \cdot, A^{*}\phi_i^{(p_i)}\right\rangle A\varphi_i^{(1)}+\sum_{i=1}^{n}\sum_{j=2}^{p_i}\left\langle \cdot, B^{*}\phi_i^{(j)}\right\rangle AB^{+}A\varphi_i^{(p_i+1-j)}=$$
$$=\sum_{i=1}^{n}\left\langle \cdot, A^{*}\phi_i^{(p_i)}\right\rangle A\varphi_i^{(1)}+\sum_{i=1}^{n}\sum_{j=1}^{p_i-1}\left\langle \cdot, A^{*}\phi_i^{(j)}\right\rangle AB^{+}A\varphi_i^{(p_i+1-j)}=\tilde{Q}A.$$

Then equality (6) is proved.

Another relation is proved similarly:

$$B^{+}\left[I-\tilde{Q}\right]B+\sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle \cdot, A^{*}\phi_i^{(j)}\right\rangle \varphi_i^{(p_i+1-j)}=$$
$$=I-P-\sum_{i=1}^{n}\sum_{j=2}^{p_i}\left\langle \cdot, B^{*}\phi_i^{(j)}\right\rangle B^{+}A\varphi_i^{(p_i+1-j)}+\sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle \cdot, A^{*}\phi_i^{(j)}\right\rangle \varphi_i^{(p_i+1-j)}=$$
$$=I-\sum_{i=1}^{n}\left\langle \cdot, A^{*}\phi_i^{(p_i)}\right\rangle \varphi_i^{(1)}-\sum_{i=1}^{n}\sum_{j=1}^{p_i-1}\left\langle \cdot, A^{*}\phi_i^{(j)}\right\rangle \varphi_i^{(p_i+1-j)}+\sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle \cdot, A^{*}\phi_i^{(j)}\right\rangle \varphi_i^{(p_i+1-j)}=I.$$

□

1.2. Generalized functions in Banach spaces

Let E be a Banach space and E^* be the conjugate Banach space. A set of finite infinitely differentiable functions $s(t)$ with the values in E^* will be called the main space $K(E^*)$. Set $K(E^*)$ is provided with some topology by introducing the concept of convergence.

Definition. *Sequence $s_n(t)\in K(E^*)$ converges to zero in $K(E^*)$ when*
a) $\exists R>0$ such that $\forall n\in N$ inclusion $supp s_n(t)\subset[-R;R] is valid$;
b) $\forall k\in N$ the sequence of functions $\|s_n^{(k)}(t)\|\to 0$ for $n\to\infty$ uniformly in $[-R;R]$.

Let any linear continuous functional in $K(E^*)$ be called the generalized function. The set of generalized functions $K'(E)$ is a linear space, and it is provided with some topology by introducing weak ("pointwise") convergence. The concepts of support, derivative, linear replacement of the variables in space $K'(E)$ are introduced as in the classical Sobolev-Schwarz theory [9,10]. The Sobolev-Schwarz space distributions is traditionally denoted by $\mathcal{D}'(R^1)$ [9,10]. The space of main functions is denoted by $\mathcal{D}(R^1)$. The space of generalized functions (distributions) with the left bounded support will be denoted by $K'_+(E)$, similarly to $\mathcal{D}'_+(R^1)$ (see [9,10]).

Assume that E_1, E_2 are Banach spaces, $\mathcal{K}(t)\in\mathcal{L}(E_1,E_2)$ are strongly continuous operator function of bounded operators, $g(t)\in\mathcal{D}'(R^1)$. Hence the expression of the form $\mathcal{K}(t)g(t)$ will be called the generalized operator function. The operation of convolution and the fundamental operator function are the key concepts for further consideration.

Definition. *If $v(t)\in K'_+(E_1)$ then the generalized function of class $K'_+(E_2)$ which acts according to the rule*

$$(\mathcal{K}(t)g(t)*v(t),s(t))=\big(g(t),\big(v(y),\mathcal{K}^*(t)s(t+y)\big)\big),\ \ \forall s(t)\in K(E_2),$$

under the assumption that $\mathcal{K}^*(t) \in \mathcal{L}(E_2^*, E_1^*)$ exists almost everywhere in R^1, is called the convolution of the generalized operator function $\mathcal{K}(t)g(t)$ and the distribution $v(t) \in K'_+(E_1)$.

Due to restrictions imposed on the supports, the operation of convolution introduced above always exists. Furthermore, it has the property of transitivity. In particular, the equality

$$A\delta^{(i)}(t) * \mathcal{K}(t)g(t) * v(t) = \left(A\mathcal{K}(t)g(t)\right)^{(i)} * v(t),$$

is valid, where $A \in \mathcal{L}(E_2, E_3)$, $\delta(t)$ is the Dirac delta-function, $R(\mathcal{K}(t)) \subset D(A)\, \forall t \in R^1$. This equality is satisfied by definition for the closed operator A.

For $t < 0$ we set $\tilde{u}(t) = u(t)\theta(t)$, where $\theta(t)$ is the Heaviside function. Then solution of initial-value problem (1)–(2) $u(t) \in C^N(t \geqslant 0, E_1)$ satisfies the convolution equation

$$\begin{gathered}\big(B\delta^{(N)}(t) - A\delta(t) - \big(\alpha(t)A + \beta(t)B\big)\theta(t)\big) * \tilde{u}(t) = \\ = f(t)\theta(t) + Bu_{N-1}\delta(t) + Bu_{N-2}\delta'(t) + \ldots + Bu_1\delta^{(N-2)}(t) + Bu_0\delta^{(N-1)}(t).\end{gathered} \tag{8}$$

Then relation (8) is initial-value problem (1)–(2) written in terms of generalized functions.

Definition. *The fundamental operator function for the integro-differential operator*

$$\mathcal{L}_N(\delta(t)) = B\delta^{(N)}(t) - A\delta(t) - \big(\alpha(t)A + \beta(t)B\big)\theta(t) =$$

$$= \left(\delta^{(N)}(t) - \beta(t)\theta(t)\right)B - \big(\delta(t) + \alpha(t)\theta(t)\big)A$$

defined in class $K'_+(E_2)$ is a generalized operator function $\mathcal{E}_N(t)$ which satisfies the following relations

$$\mathcal{L}_N(\delta(t)) * \mathcal{E}_N(t) * v(t) = v(t), \;\; \forall v(t) \in K'_+(E_2), \tag{9}$$

$$\mathcal{E}_N(t) * \mathcal{L}_N(\delta(t)) * u(t) = u(t), \;\; \forall u(t) \in K'_+(E_1). \tag{10}$$

Relation (9) implies that the following function of class $K'_+(E_1)$

$$\tilde{\tilde{u}}(t) = \mathcal{E}_N(t) * \left(f(t)\theta(t) + Bu_{N-1}\delta(t) + Bu_{N-2}\delta'(t) + \ldots + Bu_0\delta^{(N-1)}(t)\right) \tag{11}$$

is a solution of equation (8). Accordingly, equality (10) guarantees uniqueness of solution (11) in class $K'_+(E_1)$. Indeed, if function $h(t) \in K'_+(E_1)$ is a solution of equation (8) then due to (10) we have

$$h(t) = \mathcal{E}_N(t) * \mathcal{L}_N(\delta(t)) * h(t) = \mathcal{E}_N(t) * (\mathcal{L}_N(\delta(t)) * h(t)) =$$

$$= \mathcal{E}_N(t) * \left(f(t)\theta(t) + Bu_{N-1}\delta(t) + Bu_{N-2}\delta'(t) + \ldots + Bu_1\delta^{(N-2)}(t) + Bu_0\delta^{(N-1)}(t)\right) = \tilde{\tilde{u}}(t).$$

Therefore, fundamental operator function allows one to solve the problem of existence and uniqueness of the solution of original initial-value problem (1)-(2), firstly, in class $K'_+(E_1)$. Then using analysis of representation (11) for the generalized solution, we obtain theorems on solvability of initial-value problem (1)–(2) in the class of functions $C^N(t \geqslant 0, E_1)$, i.e., we obtain classical solutions and conditions of their existence in the form of simple corollaries.

1.3. Auxiliary convolution-operator equalities

Let us introduce the following designations:

$\Lambda(t)$ is the resolvent of core $(-\alpha(t)\theta(t))$;

$\mathcal{R}(t)$ is the resolvent of core $k(t)\theta(t) = \dfrac{t^{N-1}}{(N-1)!}\theta(t) * \beta(t)\theta(t)$;

$$\mathcal{U}_N(AB^+t) = \sum_{k=1}^{\infty} \frac{t^{kN-1}}{(kN-1)!}\theta(t) * \big(\delta(t) + \mathcal{R}(t)\theta(t)\big)^k * \big(\delta(t) + \alpha(t)\theta(t)\big)^{k-1}(AB^+)^{k-1}; \tag{12}$$

$$\mathcal{V}_N(t) =$$
$$= \sum_{i=1}^{n} \left[\sum_{k=0}^{p_i-1} \left\{ \sum_{j=1}^{p_i-k} \left\langle \cdot, \phi_i^{(j)} \right\rangle \varphi_i^{(p_i-k+1-j)} \right\} \Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)^k * \Big(\delta(t) + \Lambda(t)\theta(t)\Big)^{k+1} \right], \tag{13}$$

here degree k of the generalized function is understood as its k-tuple convolution with itself, zero degree of the generalized function is the Dirac delta-function $\delta(t)$.

The following two statements are valid.

Lemma 2.

$$\mathcal{L}_N(\delta(t)) * B^+\mathcal{U}_N(AB^+t) = (I-Q)\delta(t) - Q\mathcal{U}_N(AB^+t) * \big(\delta(t) + \alpha(t)\theta(t)\big)AB^+, \tag{14}$$

$$\mathcal{L}_N(\delta(t)) * \mathcal{V}_N(t) = -\tilde{Q}\delta(t). \tag{15}$$

Proof. Since

$$\mathcal{L}_N(\delta(t)) * B^+\mathcal{U}_N(AB^+t) = \Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)B * B^+\mathcal{U}_N(AB^+t) -$$
$$-\Big(\delta(t) + \alpha(t)\theta(t)\Big)A * B^+\mathcal{U}_N(AB^+t),$$

we can sequentially find each of the terms. We have

$$\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)B * B^+\mathcal{U}_N(AB^+t) = (I-Q)\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) * \mathcal{U}_N(AB^+t),$$

but $\forall k \in N$

$$\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) * \frac{t^{kN-1}}{(kN-1)!}\theta(t) * \big(\delta(t) + \mathcal{R}(t)\theta(t)\big)^k =$$
$$= (\delta(t) - k(t)\theta(t)) * \frac{t^{(k-1)N-1}}{((k-1)N-1)!}\theta(t) * \big(\delta(t) + \mathcal{R}(t)\theta(t)\big)^k =$$
$$= \frac{t^{(k-1)N-1}}{((k-1)N-1)!}\theta(t) * \big(\delta(t) + \mathcal{R}(t)\theta(t)\big)^{k-1}.$$

Therefore,

$$\begin{aligned}\big(\delta^{(N)}(t) - \beta(t)\theta(t)\big)B * B^+\mathcal{U}_N(AB^+t) &= \\ = (I-Q)\big(I\delta(t) + \mathcal{U}_N(AB^+t) * \big(\delta(t) + \alpha(t)\theta(t)\big)AB^+\big).\end{aligned} \tag{16}$$

One can similarly obtain

$$\Big(\delta(t) + \alpha(t)\theta(t)\Big)A * B^+\mathcal{U}_N(AB^+t) = \mathcal{U}_N(AB^+t) * \big(\delta(t) + \alpha(t)\theta(t)\big)AB^+. \tag{17}$$

Subtracting equality (17) from equation (16), we obtain desired equality (14).

Equality (15) is proved in a similar way

$$\mathcal{L}_N(\delta(t)) * \mathcal{V}_N(t) = \Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)\, B * \mathcal{V}_N(t) - \Big(\delta(t) + \alpha(t)\theta(t)\Big) A * \mathcal{V}_N(t),$$

and because $B\varphi_i^{(1)} = 0$ we have

$$\Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)\, B * \mathcal{V}_N(t) = \tag{18}$$

$$= \sum_{i=1}^{n} \left[\sum_{k=0}^{p_i-2} \left\{ \sum_{j=1}^{p_i-k-1} \left\langle \cdot, \phi_i^{(j)} \right\rangle B\varphi_i^{(p_i-k+1-j)} \right\} \Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)^{k+1} * \Big(\delta(t) + \Lambda(t)\theta(t)\Big)^{k+1} \right].$$

Since

$$\Big(\delta(t) + \alpha(t)\theta(t)\Big) * \Big(\delta(t) + \Lambda(t)\theta(t)\Big)^{k+1} = \Big(\delta(t) + \Lambda(t)\theta(t)\Big)^{k},$$

we obtain

$$\Big(\delta(t) + \alpha(t)\theta(t)\Big) A * \mathcal{V}_N(t) = \tilde{Q}\delta(t) +$$

$$+ \sum_{i=1}^{n} \left[\sum_{k=1}^{p_i-1} \left\{ \sum_{j=1}^{p_i-k} \left\langle \cdot, \phi_i^{(j)} \right\rangle A\varphi_i^{(p_i-k+1-j)} \right\} \Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)^{k} * \Big(\delta(t) + \Lambda(t)\theta(t)\Big)^{k} \right]$$

Then we obtain the following relation

$$\Big(\delta(t) + \alpha(t)\theta(t)\Big) A * \mathcal{V}_N(t) = \tilde{Q}\delta(t) + \tag{19}$$

$$+ \sum_{i=1}^{n} \left[\sum_{k=0}^{p_i-2} \left\{ \sum_{j=1}^{p_i-k-1} \left\langle \cdot, \phi_i^{(j)} \right\rangle A\varphi_i^{(p_i-k-j)} \right\} \Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big)^{k+1} * \Big(\delta(t) + \Lambda(t)\theta(t)\Big)^{k+1} \right].$$

Subtracting equation (19) from equation (18) and taking equality $B\varphi_i^{(p_i-k+1-j)} = A\varphi_i^{(p_i-k-j)}$ into account, we obtain relation (15). □

Lemma 3. *When $n < m$ the following equalities are satisfied:*

$$B^{+}\mathcal{U}_N(AB^{+}t)\left[I - \tilde{Q}\right] * \mathcal{L}_N(\delta(t)) = B^{+}\left[I - \tilde{Q}\right] B\delta(t), \tag{20}$$

$$\mathcal{V}_N(t) * \mathcal{L}_N(\delta(t)) = -\sum_{i=1}^{n}\sum_{j=1}^{p_i} \left\langle \cdot, A^{*}\phi_i^{(j)} \right\rangle \varphi_i^{(p_i+1-j)}\delta(t). \tag{21}$$

Proof. Now we can sequentially find

$$B^{+}\mathcal{U}_N(AB^{+}t)\left[I - \tilde{Q}\right] * \Big(\delta^{(N)}(t) - \beta(t)\theta(t)\Big) B = B^{+}\left[I - \tilde{Q}\right] B\delta(t) +$$

$$+ B^{+}\sum_{k=2}^{\infty} \frac{t^{(k-1)N-1}}{((k-1)N-1)!}\theta(t) * \big(\delta(t) + \mathcal{R}(t)\theta(t)\big)^{k-1} * \big(\delta(t) + \alpha(t)\theta(t)\big)^{k-1} (AB^{+})^{k-1}\left[I - \tilde{Q}\right] B =$$

$$= B^{+}\left[I - \tilde{Q}\right] B\delta(t) + B^{+}\mathcal{U}_N(AB^{+}t) * \big(\delta(t) + \alpha(t)\theta(t)\big) AB^{+}\left[I - \tilde{Q}\right] B,$$

$$B^{+}\mathcal{U}_N(AB^{+}t)\left[I - \tilde{Q}\right] * \Big(\delta(t) + \alpha(t)\theta(t)\Big) A = B^{+}\mathcal{U}_N(AB^{+}t) * \Big(\delta(t) + \alpha(t)\theta(t)\Big)\left[I - \tilde{Q}\right] A.$$

Subtracting the second equality from the first one and taking into account (6) from Lemma 1, we obtain (20).

The second equality of this Lemma is proved similarly:

$$\mathcal{V}_N(t) * \mathcal{L}_N(\delta(t)) =$$

$$= \sum_{i=1}^{n} \left[\sum_{k=0}^{p_i-2} \left\{ \sum_{j=2}^{p_i-k} \left\langle \cdot, B^*\phi_i^{(j)} \right\rangle \varphi_i^{(p_i-k+1-j)} \right\} \left(\delta^{(N)}(t) - \beta(t)\theta(t)\right)^{k+1} * \left(\delta(t) + \Lambda(t)\theta(t)\right)^{k+1} \right] -$$

$$- \sum_{i=1}^{n} \left[\sum_{k=1}^{p_i-1} \left\{ \sum_{j=1}^{p_i-k} \left\langle \cdot, A^*\phi_i^{(j)} \right\rangle \varphi_i^{(p_i-k+1-j)} \right\} \left(\delta^{(N)}(t) - \beta(t)\theta(t)\right)^{k} * \left(\delta(t) + \Lambda(t)\theta(t)\right)^{k} \right] -$$

$$- \sum_{i=1}^{n} \sum_{j=1}^{p_i} \left\langle \cdot, A^*\phi_i^{(j)} \right\rangle \varphi_i^{(p_i+1-j)} \delta(t) =$$

$$= \sum_{i=1}^{n} \left[\sum_{k=0}^{p_i-2} \left\{ \sum_{j=1}^{p_i-k-1} \left\langle \cdot, B^*\phi_i^{(j+1)} - A^*\phi_i^{(j)} \right\rangle \varphi_i^{(p_i-k-j)} \right\} \cdot \right.$$

$$\left. \left(\delta^{(N)}(t) - \beta(t)\theta(t)\right)^{k+1} * \left(\delta(t) + \Lambda(t)\theta(t)\right)^{k+1} \right] - \sum_{i=1}^{n} \sum_{j=1}^{p_i} \left\langle \cdot, A^*\phi_i^{(j)} \right\rangle \varphi_i^{(p_i+1-j)} \delta(t).$$

Because $B^*\phi_i^{(j+1)} = A^*\phi_i^{(j)}$, we obtain equality (21). □

2. Theorems on fundamental operator functions and their applications

Theorem 1. *Assume that conditions* **(A)** *and* **(B)**, $n > m$ *are satisfied. Hence the integro-differential operator* $\mathcal{L}_N(\delta(t))$ *has the fundamental operator function*

$$\mathcal{E}_N(t) = B^+\mathcal{U}_N(AB^+t) \left[I - \tilde{Q}\right] - \mathcal{V}_N(t), \tag{22}$$

in class $K'_+(E_2)$ *(components* $\tilde{Q}, \mathcal{U}_N(AB^+t), \mathcal{V}_N(t)$ *are defined in (3), (12), (13)).*

Proof. Let us demonstrate the validity of identity (9) from the definition of the fundamental operator function. Taking into account relations (14), (15) and (4), we have

$$\mathcal{L}_N(\delta(t)) * \mathcal{E}_N(t) = \left(I - Q\right) \left[I - \tilde{Q}\right] \delta(t) - Q\mathcal{U}_N(AB^+t) * \left(\delta(t) + \alpha(t)\theta(t)\right) AB^+ \left[I - \tilde{Q}\right] + \tilde{Q}\delta(t) =$$

$$= I\delta(t) - Q\left[I - \tilde{Q}\right]\delta(t) - Q\mathcal{U}_N(AB^+t) * \left(\delta(t) + \alpha(t)\theta(t)\right) AB^+ \left[I - \tilde{Q}\right] = I\delta(t),$$

because

$$Q\mathcal{U}_N(AB^+t) * \left(\delta(t) + \alpha(t)\theta(t)\right) AB^+ \left[I - \tilde{Q}\right] =$$

$$= \sum_{k=1}^{\infty} \frac{t^{kN-1}}{(kN-1)!}\theta(t) * \left(\delta(t) + \mathcal{R}(t)\theta(t)\right)^k * \left(\delta(t) + \alpha(t)\theta(t)\right)^k Q(AB^+)^k \left[I - \tilde{Q}\right] = 0.$$

Therefore, it has been proved that function (11) is a solution of convolution equation (8). The existence of free functionals in the projector $\tilde{Q}$ (see (3)) shows that solution (11) is represented by a multi-parametric function, therefore, the solution is not unique. So, in the given case, there is no sense to verify identity (10) from the definition of the fundamental operator function, the identity is not satisfied. □

Theorem 2. *Assume that conditions* **(A)** *and* **(B)**, $n < m$ *are satisfied. Hence operator function (22) is fundamental operator function for the integro-differential operator* $\mathcal{L}_N(\delta(t))$ *in the subclass of generalized functions from* $K'_+(E_2)$ *such that*

$$Q_J\mathcal{U}_N(AB^+t) * v(t) \equiv 0, \quad j = n+1, \ldots, m. \tag{23}$$

Proof. Repeating the process of reasoning in the proof of Theorem 1 and using relations (5), we obtain

$$\mathcal{L}_N(\delta(t)) * \mathcal{E}_N(t) = I\delta(t) - \sum_{j=n+1}^{m} Q_j\delta(t) * \Big(I\delta(t) + \mathcal{U}_N(AB^+t) * \Big(\delta(t) + \alpha(t)\theta(t)\Big)AB^+\Big) =$$
$$= I\delta(t) - (\delta(t) - k(t)\theta(t)) * \frac{d^N}{dt^N}\left(\sum_{j=n+1}^{m} Q_j\mathcal{U}_N(AB^+t)\right).$$

According to condition (23), we obtain that relation (9) is satisfied.

On the other hand, taking into account relations (20) and (21), we have

$$\mathcal{E}_N(t) * \mathcal{L}_N(\delta(t)) = B^+\left[I - \tilde{Q}\right]B\delta(t) + \sum_{i=1}^{n}\sum_{j=1}^{p_i}\left\langle \cdot, A^*\phi_i^{(j)}\right\rangle \varphi_i^{(p_i+1-j)}\delta(t).$$

Now, using equality (7), we obtain $\mathcal{E}_N(t) * \mathcal{L}_N(\delta(t)) = I\delta(t)$, i.e., from the definition of the fundamental operator function equality (10) is satisfied under given conditions. □

If $n = m$ in condition (**A**), i.e., operator B is a Fredholm operator [1, 2] then Theorem 1 assumes the following form.

Theorem 3. *If* $n = m$ *in conditions* **(A)** *and* **(B)** *then integro-differential operator* $\mathcal{L}_N(\delta(t))$ *has the fundamental operator function of the form*

$$\mathcal{E}_N(t) = \Gamma\mathcal{U}_N(A\Gamma t)\left[I - \tilde{Q}\right] - \mathcal{V}_N(t), \tag{24}$$

in class $K'_+(E_2)$, *where* $\Gamma = \left(B + \sum_{i=1}^{n}\langle\cdot,\gamma_i\rangle z_i\right)^{-1} \in \mathcal{L}(E_2, E_1)$ *is the Trenogin-Schmidt operator (see in [1, 2]).*

Relation (24) assumes the most compact form when $p_1 = \ldots = p_n = 1$. In this case

$$\mathcal{E}_N(t) = \Gamma\mathcal{U}_N(A\Gamma t)\left[I - \sum_{i=1}^{n}\langle\cdot,\phi_i^{(1)}\rangle A\varphi_i^{(1)}\right] - \sum_{i=1}^{n}\langle\cdot,\phi_i^{(1)}\rangle\varphi_i^{(1)}\Big(\delta(t) + \Lambda(t)\theta(t)\Big).$$

Generalized solution (11) of initial-value problem (1)–(2) turns out to be a regular generalized function which transforms equation (1) into an identity. Conditions wherein a regular function satisfies initial conditions (2) are the conditions of solvability for initial-value problem (1)–(2) in the classical sense. So, we can postulate the following statement.

Theorem 4. *If the lengths of all the A-Jordan chains (see in [1, 2]) are equal to 1 under the conditions of Theorem 3 then initial-value problem (1)–(2) has unique solution (11) in class* $C^N(t \geqslant 0, E_1)$ *if and only if the following conditions are satisfied*

$$\left\langle Au_j + f^{(j)}(0) + \Lambda(0)f^{(j-1)}(0) + \cdots + \Lambda^{(j-2)}(0)f'(0) + \Lambda^{(j-1)}(0)f(0), \psi_i^{(1)}\right\rangle = 0,$$
$$j = 0, 1, \ldots, N-1, \quad i = 1, \ldots, n.$$

Let us use Theorem 4 in order to study the following two initial-boundary value problems from the theory of vibrations in visco-elastic media [3].

Example 1. Consider the following equation

$$(\lambda-\Delta)\,u_t-(\mu-\Delta)\,u-\int_0^t g(t-\tau)\,(\gamma-\Delta)\,u(\tau,\bar{x})d\tau=f(t,\bar{x}), \tag{25}$$

where $g(t),\ f(t,\bar{x})$ are given functions, $u=u(t,\bar{x})$ is the required function, $\bar{x}\in\Omega\subset R^m$ is the bounded domain with an infinitely smooth boundary $\partial\Omega,\ \Delta$ is the Laplace operator, $u=u(t,\bar{x})$ is defined on a cylinder $R_+\times\Omega$ and satisfies the following initial and boundary conditions

$$u\Big|_{t=0}=u_0(\bar{x}),\ \ \bar{x}\in\Omega;\ \ u\Big|_{\bar{x}\in\partial\Omega}=0,\ \ t\geqslant 0. \tag{26}$$

Cauchy-Dirichlet problem (25)–(26) is reduced to initial-value problem (1)–(2) when the Banach spaces E_1 and E_2 are Sobolev spaces, i.e.,

$$E_1\equiv\left\{v(\bar{x})\in W_2^2(\Omega):\ v\Big|_{\partial\Omega}=0\right\},\ \ E_2\equiv W_2(\Omega), \tag{27}$$

and operators A and B are defined as follows

$$B=\lambda-\Delta,\quad A=\mu-\Delta,\quad \lambda\in\sigma(\Delta),\quad \mu\neq\lambda. \tag{28}$$

In this case,

$$\alpha(t)=\frac{\gamma-\mu}{\lambda-\mu}g(t),\quad \beta(t)=\frac{\lambda-\gamma}{\lambda-\mu}g(t),$$

operator B is a Fredholm operator, the dimension n of the core of operator B is equal to the multiplicity of the eigenvalue $\lambda\in\sigma(\Delta)$ of the Laplace operator; lengths of all $A-$Jordan chains of the elements of core $\varphi_i\in B,\ i=1,\ldots,n$ are equal to 1, here $\lambda\varphi_i=\Delta\varphi_i,\ \varphi_i\Big|_{\bar{x}\in\partial\Omega}=0$. All conditions of Theorem 4 are satisfied. Hence, the following statement is true.

Theorem 5. *Assume that for Cauchy-Dirichlet problem (25)–(26) spaces E_1 and E_2 are defined in (27), operators A and B are defined in (28) and $\lambda\in\sigma(\Delta)$. Cauchy-Dirichlet problem (25)–(26) is unequivocally solvable in class $C^1(t\geqslant 0,E_1)$ if and only if initial-boundary value conditions (26) and function $f(t,\bar{x})$ satisfy the following relations*

$$\Big\langle(\mu-\lambda)u_0(\bar{x})+f(0,\bar{x}),\varphi_i(\bar{x})\Big\rangle=0,\quad i=1,\ldots,n.$$

Example 2. Consider the following equation

$$(\lambda-\Delta)\,u_{tt}-(\mu-\Delta)\,u-\int_0^t g(t-\tau)\,(\gamma-\Delta)\,u(\tau,\bar{x})d\tau=f(t,\bar{x}), \tag{29}$$

with initial and boundary conditions

$$u\Big|_{t=0}=u_0(\bar{x}),\quad u_t\Big|_{t=0}=u_1(\bar{x}),\quad \bar{x}\in\Omega;\quad u\Big|_{\bar{x}\in\partial\Omega}=0,\ \ t\geqslant 0. \tag{30}$$

Spaces E_1 and E_2 are defined in (27), operators A and B are defined in (28). Then the following statement is valid.

Theorem 6. *Assume that for Cauchy-Dirichlet problem (29)–(30) spaces E_1 and E_2 are defined in (27), operators A and B are defined in (28) and $\lambda \in \sigma(\Delta)$. Hence Cauchy-Dirichlet problem (29)–(30) is unequivocally solvable in clsss $C^2(t \geqslant 0, E_1)$ if and only if initial-boundary value conditions (30) and function $f(t, \bar{x})$ satisfy the following relations*

$$\Big\langle (\mu-\lambda)^2 u_1(\bar{x}) + (\mu-\lambda) f'_t(0,\bar{x}) + (\gamma-\mu) g(0) f(0,\bar{x}), \varphi_i(\bar{x}) \Big\rangle = 0,$$

$$\Big\langle (\mu-\lambda) u_0(\bar{x}) + f(0,\bar{x}), \varphi_i(\bar{x}) \Big\rangle = 0, \quad i = 1, \ldots, n.$$

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Сверточные интегро-дифференциальные уравнения в банаховых пространствах с нетеровым оператором в главной части

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Аннотация. В работе исследуется задача Коши для интегро-дифференциального уравнения сверточного типа с оператором конечного индекса при старшей производной в банаховых пространствах. Рассматриваемые уравнения моделируют эволюцию процессов с "памятью", когда на текущее состояние системы влияет не только вся история наблюдений, но и формировавшие ее факторы, остающиеся актуальными на текущий момент наблюдений. Решения строятся в классе обобщенных функций с ограниченным слева носителем методами теории фундаментальных оператор-функций вырожденных интегро-дифференциальных операторов в банаховых пространствах. Построена фундаментальная оператор-функция, соответствующая рассматриваемому уравнению, с помощью которой восстановлено обобщенное решение, исследована связь между обобщенным и классическим решениями исходной задачи Коши. Абстрактные результаты проиллюстрированы на примерах начально-краевых задач для интегро-дифференциальных уравнений в частных производных прикладного характера.

Ключевые слова: банахово пространство, обобщенная функция, жорданов набор, нетеров оператор, фундаментальная оператор-функция.

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On Ground States for the SOS Model with Competing Interactions

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Abstract. We study periodic and weakly periodic ground states for the SOS model with competing interactions on the Cayley tree of order two and three. Further, we study non periodic ground states for the SOS model with competing interactions on the Cayley tree of order two.

Keywords: Cayley tree, SOS model, periodic and weakly periodic ground states.

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Introduction

It is known that a phase diagram of Gibbs measures for a Hamiltonian is close to the phase diagram of isolated (stable) ground states of this Hamiltonian. At low temperatures, a periodic ground state corresponds to a periodic Gibbs measure, (see [1, 2]). It leads us to investigate the problem of description of periodic and weakly periodic ground states. For the Potts model with competing interactions on the Cayley tree of order $k = 2$ periodic ground states are studied in [3] (see also [4]). The notion of a weakly periodic ground state is introduced in [5]. For the Ising model with competing interactions, weakly periodic ground states are described in [1, 5]. Such states for the Potts model for normal subgroups of index 2 are studied in [6, 7]. For the Potts model with competing interactions, such states for normal subgroups of index 4 are studied in [8] and in this work also studied periodic ground states for normal subgroups of index 4 (see also [9]). In [10] for the Potts model, with competing interactions and countable spin values, on a Cayley tree of order three periodic ground states are studied.

In [11] finite-range lattice models on Cayley trees with two basic properties: the existence of only a finite number of ground states and with a Peierls type condition are considered and the

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notion of a contour for the model on the Cayley tree is defined. Also using a contour argument the existence of different Gibbs measures is shown.

A q-component models on a Cayley tree is investigated in [12] and using a contour argument the existence of q different Gibbs measures for several q-component models is shown.

In [13] for the SOS model with $m = 2$ on the Cayley tree order of $k = a+b+2$ the existence of at least two non periodic Gibbs measures is proved. In [14] an infinite system of functional equations for the Ising model with competing interactions and countable spin values $0, 1, \ldots$ and non zero field on a Cayley tree of order two is investigated. In [15] the authors proved the existence of weakly periodic Gibbs measures for the Ising model on the Cayley tree of order $k = 2$ with respect to a normal divisor of index 4.

In this paper, we study periodic and weakly periodic ground states for the SOS model with competing interactions on a Cayley tree of order $k = 2$ and $k = 3$. Moreover, in the case $k = 2$ the existence of a countable set of non periodic ground states is proved.

1. Preliminaries

Let $\Gamma^k = (V, L)$ be the Cayley tree of order k, i.e., an infinite tree such that exactly $k+1$ edges are incident to each vertex. Here V is the set of vertices and L is the set of edges of Γ^k. Let G_k denote the free product of $k+1$ cyclic groups $\{e; a_i\}$ of order 2 with generators $a_1, a_2, a_3, \ldots a_{k+1}$, i.e., let $a_i^2 = e$ (see [4]).

The group of all left (right) shifts on G_k is isomorphic to the group G_k. Each transformation S on the group G_k induces a transformation $\tilde{S}$ on the vertex set V of the Cayley tree Γ^k. In the sequel, we identify V with G_k.

The following assertion is quite obvious (see also [4]).

Theorem 1.1. The group of left (right) shifts on the right (left) representation of the Cayley tree is the group of translations.

By the group of translations we mean the automorphism group of the Cayley tree regarded as a graph. Recall (see, for example, [4]) that a mapping ψ on the vertex set of a graph G is called an automorphism of G if ψ preserves the adjacency relation, i.e., the images $\psi(u)$ and $\psi(v)$ of vertices u and v are adjacent if and only if u and v are adjacent.

For an arbitrary vertex $x_0 \in V$, we put

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \{x \in V | d(x, x^0) \leqslant n\},$$

where $d(x, y)$ is the distance between x and y in the Cayley tree, i.e., the number of edges of the path between x and y.

For each $x \in G_k$, let $S(x)$ denote the set of immediate successors of x, i.e., if $x \in W_n$ then

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}.$$

For each $x \in G_k$, let $S_1(x)$ denote the set of all neighbors of x, i.e., $S_1(x) = \{y \in G_k : \langle x, y\rangle \in L\}$. The set $S_1(x) \setminus S(x)$ is a singleton. Let $x_\downarrow$ denote the (unique) element of this set.

Let us assume that the spin values belong to the set $\Phi = \{0, 1, 2, \ldots m\}$. A function $\sigma : x \in V \to \sigma(x) \in \Phi$ is called configuration on V. The set of all configurations coincides with the set $\Omega = \Phi^V$.

Consider the quotient group $G_k/G_k^* = \{H_1, H_2, \ldots, H_r\}$, where G_k^* is a normal subgroup of index r with $r \geqslant 1$.

Definition 1.1. *A configuration $\sigma(x)$ is called G_k^*-periodic, if $\sigma(x) = \sigma_i$ for all $x \in G_k$ with $x \in H_i$. A G_k-periodic configuration is called translation invariant.*

The period of a periodic configuration is the index of the corresponding normal subgroup.

Definition 1.2. *A configuration $\sigma(x)$ is called G_k^*-weakly periodic, if $\sigma(x) = \sigma_{ij}$ for all $x \in G_k$ with $x_\downarrow \in H_i$ and $x \in H_j$.*

The Hamiltonian of the model SOS model with competing interactions has a form:

$$H(\sigma) = -J_1 \sum_{\langle x,y\rangle \in L} |\sigma(x) - \sigma(y)| - J_2 \sum_{\substack{x,y\in V:\\ d(x,y)=2}} |\sigma(x) - \sigma(y)|, \tag{1}$$

where $(J_1, J_2) \in \mathbb{R}^2$.

2. Ground states

In this section, we study ground states for the SOS model on a Cayley tree. For a pair of configurations σ and φ which coincide almost everywhere, i.e., everywhere except finitely many points, we consider the relative Hamiltonian $H(\sigma, \varphi)$ describing the energy differences of the two configurations σ and φ :

$$\begin{aligned} H(\sigma,\varphi) = &- J_1 \sum_{\langle x,y\rangle \in L} (|\sigma(x) - \sigma(y)| - |\varphi(x) - \varphi(y)|) - \\ &- J_2 \sum_{\substack{x,y\in V:\\ d(x,y)=2}} (|\sigma(x) - \sigma(y)| - |\varphi(x) - \varphi(y)|), \end{aligned} \tag{2}$$

where $(J_1, J_2) \in \mathbb{R}^2$.

Let M be the set of all unit balls with vertices in V, i.e. $M = \{\{x\} \cup S_1(x) : \forall x \in V\}$. A restriction of a configuration σ to the ball $b \in M$ is a bounded configuration and it is denoted by σ_b.

We define the energy of the configuration σ_b on b by the following formula

$$U(\sigma_b) \equiv U(\sigma_b, J_1, J_2) = -\frac{1}{2} J_1 \sum_{\substack{\langle x,y\rangle:\\ x,y\in b}} |\sigma(x) - \sigma(y)| - J_2 \sum_{\substack{x,y\in b:\\ d(x,y)=2}} |\sigma(x) - \sigma(y)|, \tag{3}$$

where $(J_1, J_2) \in \mathbb{R}^2$.

The following assertion is known (see [4]).

Lemma 2.1. *Relative Hamiltonian (2) has the form:*

$$H(\sigma, \varphi) = \sum_{b\in M} (U(\sigma_b) - U(\varphi_b)).$$

The existence of a countable set of non periodic ground states on the Cayley tree of order two

We consider the case $k = 2$.

Let $m = 2$. It is easy to see that $U(\sigma_b) \in \{U_i : i = 1, \ldots, 10\}$ for σ_b, where

$$U_1 = 0, \quad U_2 = -\frac{1}{2}J_1 - 2J_2, \quad U_3 = -J_1 - 2J_2, \quad U_4 = -\frac{3}{2}J_1, \quad U_5 = -J_1 - 4J_2,$$

$$U_6 = -2J_1 - 4J_2, \quad U_7 = -3J_1, \quad U_8 = -\frac{3}{2}J_1 - 4J_2, \quad U_9 = -2J_1 - 2J_2, \quad U_{10} = -\frac{5}{2}J_1 - 2J_2.$$

Definition 2.1. *The configuration φ is called the ground state for the Hamiltonian (1) if $U(\varphi_b) = \min\{U_1, U_2, U_3, \dots, U_{10}\}$ for any $b \in M$.*

Let

$$A_m = \{(J_1, J_2) \in R^2 \mid U_m = \min_{1 \leqslant k \leqslant 10}\{U_k\}\}.$$

It is easy to check that

$$A_1 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; J_2 \leqslant -\frac{1}{4}J_1\},$$
$$A_2 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; J_2 = -\frac{1}{4}J_1\},$$
$$A_3 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\},$$
$$A_4 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 \leqslant 0\},$$
$$A_5 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; J_2 \geqslant -\frac{1}{4}J_1\},$$
$$A_6 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0; J_2 \geqslant \frac{1}{4}J_1\},$$
$$A_7 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0; J_2 \leqslant \frac{1}{4}J_1\},$$
$$A_8 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 \geqslant 0\},$$
$$A_9 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\}\},$$
$$A_{10} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0; J_2 = \frac{1}{4}J_1\}$$

and $\bigcup_{i=1}^{10} A_i = \mathbb{R}^2$.

In [16] periodic ground states are studied for SOS model on Cayley tree order of 2. In this subsection we shall prove the existence of a countable set of non periodic ground states on the Cayley tree of order two. The next subsection we study periodic and weakly periodic ground states for the model (1) on the Cayley tree of order three.

Let c_b denote the center of a unit ball b. We put

$$C_i = \{\sigma_b : U(\sigma_b) = U_i\}, i = \overline{1, 10},$$

$$B^{(i)} =\mid \{x \in S_1(c_b) : \varphi_b(x) = i\} \mid, \text{ for } i = 0, 1, 2$$

and $D_i = \Omega_i \cup \tilde{\Omega}_i$, where

$$\Omega_i = \{\sigma_b : \sigma_b(c_b) = 0,\ |x \in b\backslash\{c_b\} : \sigma_b(x) = 2| = i;\ \ |x \in b\backslash\{c_b\} : \sigma_b(x) = 1| = 0\},$$

$$\tilde{\Omega}_i = \{\tilde{\sigma}_b : |\tilde{\sigma}(x) - \sigma(x)| = 2, |x \in b\backslash\{c_b\} : \tilde{\sigma}_b(x) = 1| = 0, x \in b\},\ i = 0, 1, 2, 3, \text{ i.e.,}$$

$$\tilde{\sigma}(x) = \begin{cases} 2, \text{ if } \sigma(x) = 0 \\ 0, \text{ if } \sigma(x) = 2 \end{cases}.$$

For $A_i, A_j, i \neq j$ we have

$$A_i \cap A_j = \begin{cases} A_2 \text{ if } i = 1, j = 5, \\ A_4 \text{ if } i = 1, j = 7, \\ A_8 \text{ if } i = 5, j = 6, \\ A_{10} \text{ if } i = 6, j = 7. \end{cases} \tag{4}$$

Fix $J = (J_1, J_2) \in \mathbb{R}^2$ and denote

$$N_J(\sigma_b) = |\{j : \sigma_b \in C_j\}|.$$

Using (4) one can prove

Lemma 2.2. *For any $b \in M$ and σ_b we have*

$$N_J(\sigma_b) = \begin{cases} 10, & \text{if } J = (0;0) \\ 3, & \text{if } J \in A_i \setminus \{(0,0)\},\ i = 2,4,8,10 \\ 1, & \text{otherwise} \end{cases} . \tag{5}$$

Let GS(H) be the set of all ground states of the Hamiltonian (1).

Theorem 2.1. *(i) If $J = (0;0)$ then $GS(H) = \Omega$.*

(ii) If $J \in A_i \setminus \{(0,0)\}$, $i = 2,8,10$ then there exists a countable set of non periodic ground states.

Proof. The assertion (i) is trivial.

Prove (ii):

a) if $J \in A_2 \setminus \{(0,0)\}$ then the minimum points of $U(\sigma_b)$ would belong to the classes C_1, C_2 and C_5;

b) if $J \in A_8 \setminus \{(0,0)\}$ then the minimum points of $U(\sigma_b)$ would belong to the classes C_5, C_6 and C_8;

c) if $J \in A_{10} \setminus \{(0,0)\}$ then the minimum points of $U(\sigma_b)$ would belong to the classes C_6, C_7 and C_{10}.

Below we define the configurations of classes C_1, C_5, C_6 and C_7 which satisfying the condition $|x \in b\backslash\{c_b\} : \sigma_b(x) = 1| = 0$,

$$\begin{cases} \sigma_b^{(0)}(c_b) = 0, \quad |x \in b\backslash\{c_b\} : \sigma_b^{(0)}(x) = 2| = 0 \text{ and} \\ \tilde{\sigma}_b^{(0)}(c_b) = 2, \quad |x \in b\backslash\{c_b\} : \quad \tilde{\sigma}_b^{(0)}(x) = 0| = 0,\ \sigma^{(0)}, \tilde{\sigma}^{(0)} \in C_1, \\ \sigma_b^{(1)}(c_b) = 0, \quad |x \in b\backslash\{c_b\} : \sigma_b^{(1)}(x) = 2| = 1 \text{ and} \\ \tilde{\sigma}_b^{(1)}(c_b) = 2, \quad |x \in b\backslash\{c_b\} : \quad \tilde{\sigma}_b^{(1)}(x) = 0| = 1,\ \sigma^{(1)}, \tilde{\sigma}^{(1)} \in C_5, \\ \sigma_b^{(2)}(c_b) = 0, \quad |x \in b\backslash\{c_b\} : \sigma_b^{(2)}(x) = 2| = 2 \text{ and} \\ \tilde{\sigma}_b^{(2)}(c_b) = 2, \quad |x \in b\backslash\{c_b\} : \quad \tilde{\sigma}_b^{(2)}(x) = 0| = 2,\ \sigma^{(2)}, \tilde{\sigma}^{(2)} \in C_6, \\ \sigma_b^{(3)}(c_b) = 0, \quad |x \in b\backslash\{c_b\} : \sigma_b^{(3)}(x) = 2| = 3 \text{ and} \\ \tilde{\sigma}_b^{(3)}(c_b) = 2, \quad |x \in b\backslash\{c_b\} : \quad \tilde{\sigma}_b^{(2)}(x) = 0| = 3, \sigma^{(3)}, \tilde{\sigma}^{(3)} \in C_7 \end{cases} . \tag{6}$$

Thus any ground state $\varphi \in D_i$ must satisfy

$$\varphi_b \in \{\sigma_b^{(i)}, \tilde{\sigma}_b^{(i)}, \sigma_b^{(i+1)}, \tilde{\sigma}_b^{(i+1)}\}, \quad i = 0,1,2,\ b \in M. \tag{7}$$

Now we shall construct ground states $\varphi \in D_i$ which satisfying (7).

Note that the configurations σ_b and $\sigma_{b'}$ $(b, b' \in M)$ are the same up to a motion in G_k so we shall omit b. Thus configuration $\sigma^{(i)}$ is the configuration such that on any unit ball $b \in M$ the condition (6) is satisfied.

Suppose two unit balls b and b' are neighbors, i.e., they have a common edge. We shall then say that the two bounded configurations σ_b and $\sigma_{b'}$ are compatible if they coincide on the common edge of the balls b and b'. Denote by $B(b)$ the set of all neighbor balls of b.

Denote $\bar{\Omega}_i = \{\sigma^{(i)}, \tilde{\sigma}^{(i)}, \sigma^{(i+1)}, \tilde{\sigma}^{(i+1)}\}$, $i = 0, 1, 2$. For any $\omega, \nu \in \bar{\Omega}_i$ denote by $n(\omega, \nu)$ the number of possibilities to set up the configuration ν as a compatible configuration (with ω) around (i.e., on neighboring balls of the ball on which ω is given) the configuration ω. Clearly $n(\omega, \nu) \in \{0, 1, 2, 3\}$, for any $\omega, \nu \in \bar{\Omega}_i, i = 0, 1, 2$.

Denote

$$N_i = \begin{pmatrix} n(\sigma^{(i)}, \sigma^{(i)}) & n(\sigma^{(i)}, \tilde{\sigma}^{(i)}) & n(\sigma^{(i)}, \sigma^{(i+1)}) & n(\sigma^{(i)}, \tilde{\sigma}^{(i+1)}) \\ n(\tilde{\sigma}^{(i)}, \sigma^{(i)}) & n(\tilde{\sigma}^{(i)}, \tilde{\sigma}^{(i)}) & n(\tilde{\sigma}^{(i)}, \sigma^{(i+1)}) & n(\tilde{\sigma}^{(i)}, \tilde{\sigma}^{(i+1)}) \\ n(\sigma^{(i+1)}, \sigma^{(i)}) & n(\sigma^{(i+1)}, \tilde{\sigma}^{(i)}) & n(\sigma^{(i+1)}, \sigma^{(i+1)}) & n(\sigma^{(i+1)}, \tilde{\sigma}^{(i+1)}) \\ n(\tilde{\sigma}^{(i+1)}, \sigma^{(i)}) & n(\tilde{\sigma}^{(i+1)}, \tilde{\sigma}^{(i)}) & n(\tilde{\sigma}^{(i+1)}, \sigma^{(i+1)}) & n(\tilde{\sigma}^{(i+1)}, \tilde{\sigma}^{(i+1)}) \end{pmatrix}.$$

It is easy to see that

$$N_0 = \begin{pmatrix} 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 \\ 2 & 0 & 2 & 1 \\ 0 & 2 & 1 & 2 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 2 & 1 & 2 & 0 \\ 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 \end{pmatrix}.$$

Consider 3 sets $\mathbf{Q}_i = \{Q\}$, $(i = 0, 1, 2)$ of matrices $Q = \{q(u, v)\}_{u,v \in \bar{\Omega}_i}$ such that

$$q(u, v) \in \{0, 1, \dots, n(u, v)\}, \quad \sum_{v \in \bar{\Omega}_i} q(u, v) = 3, \forall u \in \bar{\Omega}_i.$$

$q(u, \sigma^{(i)}) + q(u, \sigma^{(j)}) = n(u, \sigma^{(i)})$, $q(u, \tilde{\sigma}^{(i)}) + q(u, \tilde{\sigma}^{(j)}) = n(u, \tilde{\sigma}^{(i)})$, and $q(u, v) = 0$ if and only if $q(v, u) = 0$, $u, v \in \bar{\Omega}_i$.

Using matrices N_i we have

$$\mathbf{Q}_0 = \left\{ Q = \begin{pmatrix} a & 0 & 3-a & 0 \\ 0 & b & 0 & 3-b \\ c & 0 & 2-c & 1 \\ 0 & d & 1 & 2-d \end{pmatrix} \right\},$$

here $a, b \in \{0, 1, 2, 3\}$; $c, d \in \{0, 1, 2\}$; $a = 3$ iff $c = 0$; $b = 3$ iff $d = 0$.

For $i = 1$ we get

$$\mathbf{Q}_1 = \left\{ Q = \begin{pmatrix} a_1 & b_1 & 2-a_1 & 1-b_1 \\ b_2 & a_2 & 1-b_2 & 2-a_2 \\ c_1 & d_1 & 1-c_1 & 2-d_1 \\ d_2 & c_2 & 2-d_2 & 1-c_2 \end{pmatrix} \right\},$$

here $a_1, a_2, d_1, d_2 \in \{0, 1, 2\}$; $b_1, b_2, c_1, c_2 \in \{0, 1\}$; $a_1 = 2$ iff $c_1 = 0$; $a_2 = 2$ iff $c_2 = 0$; $b_1 = 0$ iff $b_2 = 0$; $b_1 = 1$ iff $d_2 = 0$; $b_2 = 1$ iff $d_1 = 0$; $d_1 = 2$ iff $d_2 = 2$.

For $i = 2$ we obtain

$$\mathbf{Q}_2 = \left\{ Q = \begin{pmatrix} 1 & a & 0 & 2-a \\ b & 1 & 2-b & 0 \\ 0 & c & 0 & 3-c \\ d & 0 & 3-d & 0 \end{pmatrix} \right\},$$

here $a, b \in \{0,1,2\}$; $c, d \in \{0,1,2,3\}$; $a = 0$ iff $b = 0$; $a = 2$ iff $d = 0$; $b = 2$ iff $c = 0$; $c = 3$ iff $d = 3$.

For a given $\xi \in \bar{\Omega}_i$ and $Q = \{q(u,v)\}_{u,v\in\bar{\Omega}_i} \in \mathbf{Q}_i$ we recurrently construct a ground state $\varphi^{Q,\xi}$ by the following way: fix a ball $b \in M$ and put on b the configuration $\varphi^{Q,\xi} := \xi$. On balls taken from $B(b)$ we set exactly $q(\xi,\omega)$ copies of ω for any $\omega \in \bar{\Omega}_i$. Thus configurations $\varphi^{Q,\xi}_{b'}, b' \in B(b)$ are defined. Using these configurations, we define configurations on the balls $B(b') \setminus \{b\}, (b' \in B(b))$ putting $q(\varphi^{Q,\xi}_{b'}, v)$ copies of $v \in \bar{\Omega}_i \setminus \xi$ and $q(\varphi^{Q,\xi}_{b'}, \xi) - 1$ copies of ξ which are compatible with $\varphi^{Q,\xi}_{b'}$. Further, on the balls $B(b'') \setminus \{b'\}, (b'' \in B(b)), b' \in B(b)$ we set $q(\varphi^{Q,\xi}_{b''}, \tau)$ copies of $\tau \in \bar{\Omega}_i \setminus \{\varphi^{Q,\xi}_{b'}\}$ and $q(\varphi^{Q,\xi}_{b''}, \varphi^{Q,\xi}_{b'}) - 1$ copies of $\varphi^{Q,\xi}_{b'}$ which are compatible with $\varphi^{Q,\xi}_{b''}$. Repeating this construction one can obtain a ground state $\varphi^{Q,\xi}$ such that

$$\varphi^{Q,\xi}_b \in \bar{\Omega}_i, \quad |\{b' \in B(b) : \varphi^{Q,\xi}_b = \omega,\ \varphi^{Q,\xi}_{b'} = \nu\}| = q(\omega,\nu),$$

for any $b \in M$ and $\omega, \nu \in \bar{\Omega}_i$.

In general, the ground state $\varphi^{Q,\xi}$ is non periodic (see example below). It is easy to see that

$$\varphi^{Q_i,\sigma^{(j)}}_b \equiv \sigma^{(j)}, \quad \varphi^{Q_i,\tilde{\sigma}^{(j)}}_b \equiv \tilde{\sigma}^{(j)}, \quad j = i,\ i+1,\ i = 0,1,2,$$

where

$$Q_i = \begin{pmatrix} 3-i & i & 0 & 0 \\ i & 3-i & 0 & 0 \\ 0 & 0 & 2-i & i+1 \\ 0 & 0 & i+1 & 2-i \end{pmatrix}. \tag{8}$$

Now using the ground states $\varphi^{Q,\xi}$ we shall construct an infinite set of ground states by the following way: one can choose $\xi \neq \eta,\ \xi, \eta \in \bar{\Omega}_i$ and $Q_1, Q_2 \in \mathbf{Q}_1$ such that for configurations $\varphi^{Q_1,\xi}, \varphi^{Q_2,\eta}$ there are infinitely many $b \in M$ on which $\varphi^{Q_1,\xi}_b$ and $\varphi^{Q_2,\eta}_{b'}$ are compatible for some $b' \in B(b)$. Indeed it is sufficient to take $\xi \neq \eta$ such that $q_1(\xi,\eta)q_2(\xi,\eta) \neq 0$ (see example below).

Denote

$$M_1 \equiv M_1^{\xi\eta}(Q_1,Q_2) = \{b \in M :\ \varphi^{Q_1,\xi}_b$$

is compatible $\varphi^{Q_2,\eta}_{b'}$ for some $b' \in B(b)\}$;

$$\aleph_1 = \{n \in \{0,1,2,\dots\} :\ \exists b \in M_1 \text{ such that } |c_b| = n\};$$

$$V^{(y)} = \{z \in V : y < z\}.$$

Fix $m \in \aleph_1$ and denote

$$\tilde{W}_m = \{x \in W_m :\ \exists b \in M_1 \text{ such that } c_b = x\}.$$

Consider the configuration

$$\varphi^{Q_1,Q_2,\xi,\eta}_m(x) = \begin{cases} \varphi^{Q_1,\xi}(x) \text{ if } x \in V_m \cup \{V^{(y)}, y \in W_m \backslash \tilde{W}_m\} \\ \varphi^{Q_2,\eta}(x) \text{ if } x \in V^{(y)}, y \in \tilde{W}_m \end{cases}.$$

Clearly $\varphi_m^{Q_1,Q_2,\xi,\eta}, m \in \aleph_1$ is a ground state and the number of such ground states is infinite, since $|\aleph_1| = \infty$. This finishes the proof of Theorem 2.1. □

Remark 2.1 Let $J \in A_4 \backslash (0,0)$. $\bar{\Omega}_3 = \{\sigma_b^{(0)}, \tilde{\sigma}_b^{(0)}, \sigma_b^{(3)}, \tilde{\sigma}_b^{(3)}\}$ are periodic ground states such that on any $b \in M$ the bounded configurations $\sigma_b^{(0)}, \tilde{\sigma}_b^{(0)} \in C_1$ and $\sigma_b^{(3)}, \tilde{\sigma}_b^{(3)} \in C_7$, i.e., $\sigma_b^{(0)}$, $\tilde{\sigma}_b^{(0)}$ are translation-invariant and $\sigma_b^{(3)}, \tilde{\sigma}_b^{(3)}$ are periodic with period 2. $\bar{\Omega}_3 = \{\sigma_b^{(0)}, \tilde{\sigma}_b^{(0)}, \sigma_b^{(3)}, \tilde{\sigma}_b^{(3)}\}$ and $\mathbf{Q}_3$ contains the unique matrix

$$Q_3 = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

Example. Take matrices

$$Q'_2 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}, \quad Q''_2 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

and $\xi = \sigma^{(2)}$, $\eta = \sigma^{(3)}$. The configurations $\varphi^{Q'_2,\xi}$, $\varphi^{Q''_2,\eta}$ and $\varphi^{Q'_2,Q''_2\xi,\eta}$ are represented in Fig. 1 a), b) and c), respectively.

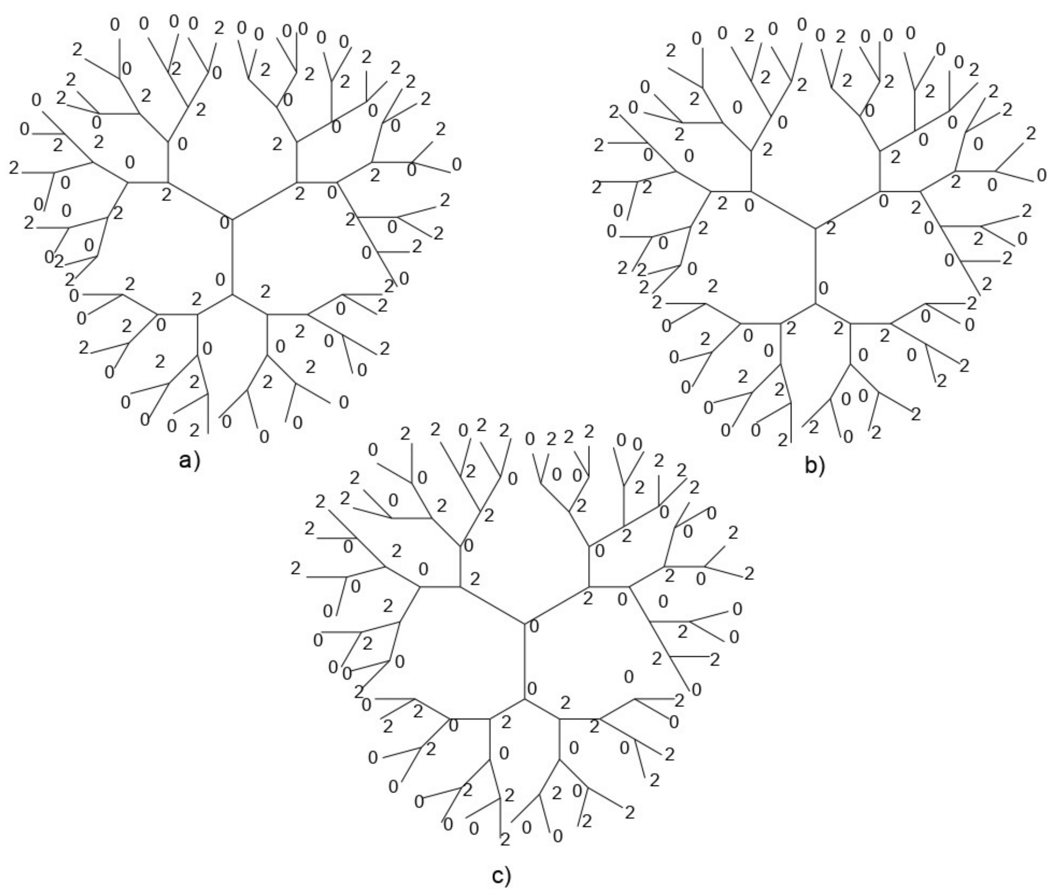

Fig. 1. Ground states

Periodic and weakly periodic ground states on the Cayley tree of order three

We consider the case $k = 3$.

Let $m = 2$. By (3) for any σ_b we have $U(\sigma_b) \in \{U_1, U_2, U_3, \dots, U_{15}\}$, where

$$U_1 = 0, \quad U_2 = -\frac{1}{2}J_1 - 3J_2, \quad U_3 = -J_1 - 4J_2, \quad U_4 = -J_1 - 6J_2,$$

$$U_5 = -\frac{3}{2}J_1 - 3J_2, \quad U_6 = -2J_1 - 8J_2, \quad U_7 = -3J_1 - 6J_2, \quad U_8 = -2J_1 - 6J_2,$$

$$U_9 = -\frac{5}{2}J_1 - 7J_2, \quad U_{10} = -\frac{3}{2}J_1 - 7J_2, \quad U_{11} = -2J_1, U_{12} = -4J_1,$$

$$U_{13} = -\frac{7}{2}J_1 - 3J_2, \quad U_{14} = -\frac{5}{2}J_1 - 3J_2, \quad U_{15} = -3J_1 - 4J_2.$$

Definition 2.2. *The configuration φ is called the ground state for the Hamiltonian (1), if $U(\varphi_b) = \min\{U_1, U_2, U_3, \dots, U_{15}\}$ for $\forall b \in M$.*

Let $A_m = \{(J_1, J_2) \in \mathbb{R}^2 \mid U_m = \min_{1 \leqslant k \leqslant 15}\{U_k\}\}$. It is easy to check that

$$A_1 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; J_2 \leqslant -\frac{1}{6}J_1\},$$
$$A_2 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; J_2 = -\frac{1}{6}J_1\},$$
$$A_3 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\},$$
$$A_4 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; -\frac{1}{6}J_1 \leqslant J_2 \leqslant -\frac{1}{2}J_1\},$$
$$A_5 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\},$$
$$A_6 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_2 \geqslant \frac{1}{2}|J_1|\},$$
$$A_7 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0; \frac{1}{6}J_1 \leqslant J_2 \leqslant \frac{1}{2}J_1\},$$
$$A_8 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\},$$
$$A_9 = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0; J_2 = \frac{1}{2}J_1\},$$
$$A_{10} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \leqslant 0; J_2 = -\frac{1}{2}J_1\},$$
$$A_{11} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 \leqslant 0\},$$
$$A_{12} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0; J_2 \leqslant \frac{1}{6}J_1\},$$
$$A_{13} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 \geqslant 0, J_2 = \frac{1}{6}J_1\},$$
$$A_{14} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\},$$
$$A_{15} = \{(J_1, J_2) \in \mathbb{R}^2 \mid J_1 = 0; J_2 = 0\}$$

and $\bigcup_{i=1}^{15} A_i = \mathbb{R}^2$.

Let c_b be the center of a unit ball b. We put

$$C_i = \{\sigma_b : U(\sigma_b) = U_i\}, i = \overline{1, 15}$$

and

$$B^{(i)} =| \{x \in S_1(c_b) : \varphi_b(x) = i\} |,$$

for $i = 0, 1, 2$.

Let $H_A = \{x \in G_k : \sum_{i \in A} \omega_x(a_i) - \text{even}\}$, where $\omega_x(a_i)$ is the number of a_i in the word x.

Note, that H_A is a normal subgroup of index two (see [4]). Let $G_k/H_A = \{H_A, G_k \backslash H_A\}$ be the quotient group. Denote $H_0 = H_A$, $H_1 = G_k \backslash H_A$.

Periodic Ground States for the case $k = 3$

In this section, we shall study H_0-periodic ground states. We note that each H_0 periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_1, & \text{if } x \in H_0 \\ \sigma_2, & \text{if } x \in H_1 \end{cases}, \tag{9}$$

where $\sigma_i \in \Phi = \{0, 1, 2\}, i = 1, 2$.

Theorem 2.2. *Let $k = 3$. The configuration (9) is H_0-periodic ground state iff one of the following conditions holds:*

a) $|A| = 1$.

i) $| \sigma_1 - \sigma_2 |= 0$, *and* $(J_1, J_2) \in A_1$.

ii) $| \sigma_1 - \sigma_2 |= 1$, *and* $(J_1, J_2) \in A_2$.

iii) $| \sigma_1 - \sigma_2 |= 2$, *and* $(J_1, J_2) \in A_4$.

b) $|A| = 2$.

i) If $| \sigma_1 - \sigma_2 |= 1$,*then there is not a* H_0*-periodic ground state;*

ii) $| \sigma_1 - \sigma_2 |= 2$, *and* $(J_1, J_2) \in A_6$.

c) $|A| = 3$.

i) If $| \sigma_1 - \sigma_2 |= 1$, *then there is not a* H_0*-periodic ground state;*

ii) $| \sigma_1 - \sigma_2 |= 2$, *and* $(J_1, J_2) \in A_7$.

d) $|A| = 4$.

i) $| \sigma_1 - \sigma_2 |= 1$, *and* $(J_1, J_2) \in A_{11}$.

ii) $| \sigma_1 - \sigma_2 |= 2$, *and* $(J_1, J_2) \in A_{12}$.

Proof: a) i) Let us consider the following configuration

$$\varphi(x) = \begin{cases} i, & \text{if } x \in H_0 \\ i, & \text{if } x \in H_1 \end{cases},$$

where $i = 0, 1, 2$. We denote the center of $b \in M$ by c_b. Let $c_b \in H_0$, then we have

$$\varphi_b(c_b) = i, \ B^{(i)} = 4.$$

Hence, $\varphi_b(x) \in C_1$, i.e. if $(J_1, J_2) \in A_1$ then the corresponding configuration is a ground state.

ii) Now we consider the following configuration

$$\varphi(x) = \begin{cases} i, & \text{if } x \in H_0 \\ j, & \text{if } x \in H_1 \end{cases},$$

where $| i - j |= 1$.

1) Assume that $c_b \in H_0$

$$\varphi_b(c_b) = i, B^{(i)} = 3, B^{(j)} = 1.$$

Hence, $\varphi_b(x) \in C_2$.

2) Let $c_b \in H_1$, then one has

$$\varphi_b(c_b) = i, B^{(i)} = 3, B^{(j)} = 1.$$

Hence, $\varphi_b(x) \in C_2$.

We conclude that, if $(J_1, J_2) \in A_2$ then the corresponding periodic configuration $\varphi(x)$ is a H_0-periodic ground state.

iii) Let us consider the following configuration

$$\varphi(x) = \begin{cases} i, & \text{if } x \in H_0 \\ j, & \text{if } x \in H_1 \end{cases},$$

where $\mid i - j \mid = 2$.

1) Assume that $c_b \in H_0$

$$\varphi_b(c_b) = i, B^{(i)} = 3, B^{(j)} = 1.$$

Hence, $\varphi_b(x) \in C_4$.

2) Let $c_b \in H_1$, then one has

$$\varphi_b(c_b) = j, B^{(j)} = 3, B^{(i)} = 1.$$

Hence, $\varphi_b(x) \in C_4$.

We conclude that if $(J_1, J_2) \in A_4$ then the corresponding periodic configuration $\varphi(x)$ is a H_0-periodic ground state.

The proofs of assertions b), c) and d) of Theorem 2.2 are similar to the proof of assertion a). This finishes the proof of Theorem 2.2. □

Remark 2.2 In the case c), the H_0 periodic ground states coincides with the $G_k^{(2)}$-periodic ground states, where $G_k^{(2)} = \{x \in G_k : |x| \text{ is even}\}$.

Weakly Periodic Ground States for the $k = 3$

In this section, we describe H_A-weakly periodic ground states, where H_A is a normal subgroup of index two. Due to the definition of weakly periodic configuration, we infer that each H_A-weakly periodic configuration has the following form:

$$\sigma(x) = \begin{cases} \sigma_{00}, & \text{if } x_\downarrow \in H_0, x \in H_0 \\ \sigma_{01}, & \text{if } x_\downarrow \in H_0, x \in H_1 \\ \sigma_{10}, & \text{if } x_\downarrow \in H_1, x \in H_0 \\ \sigma_{11}, & \text{if } x_\downarrow \in H_1, x \in H_1 \end{cases}, \tag{10}$$

where $\sigma_{ij} \in \Phi,\ i, j = 0, 1$.

In the sequel, we write $\sigma = (\sigma_{00}, \sigma_{01}, \sigma_{10}, \sigma_{11})$ for such a weakly periodic configuration $\sigma(x)$, $x \in G_k$.

Theorem 2.3. *Let $k = 3$ and $|A| = 1$. Then for the SOS model there is no H_A-weakly periodic (non periodic) ground state.*

Proof. Consider (10). If $\sigma_{00} = \sigma_{01} = \sigma_{10} = \sigma_{11}$, then corresponding configurations are translation-invariant. Translation-invariant ground states for this case are studied in Theorem 2.2. It is easy to see that in the case $\sigma_{00} = \sigma_{10}$ and $\sigma_{01} = \sigma_{11}$ the H_A-weakly periodic configurations (10) are periodic configurations which are studied in Theorem 2.2.

Now we consider the cases $\sigma_{00} \neq \sigma_{10}$ or $\sigma_{01} \neq \sigma_{11}$.

Let

$$\varphi(x) = \begin{cases} 0, \text{ if } x_\downarrow \in H_0, x \in H_0 \\ 0, \text{ if } x_\downarrow \in H_0, x \in H_1 \\ 1, \text{ if } x_\downarrow \in H_1, x \in H_0 \\ 0, \text{ if } x_\downarrow \in H_1, x \in H_1 \end{cases}.$$

Let $c_b \in H_0$, we have the following possible cases:

a) $c_{b\downarrow} \in H_0$ and $\varphi_b(c_{b\downarrow}) = 0$, then $\varphi_b(c_b) = 0$, $B^{(0)} = 4$, $\varphi_b(c_b) \in C_1$,

b) $c_{b\downarrow} \in H_0$ and $\varphi_b(c_{b\downarrow}) = 1$, then $\varphi_b(c_b) = 0$, $B^{(0)} = 3$, $B^{(1)} = 1$, $\varphi_b(c_b) \in C_2$,

c) $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = 1$, then there is not any H_A-weakly periodic ground state,

d) $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = 0$, then $\varphi_b(c_b) = 1$, $B^{(0)} = 4$, $\varphi_b(c_b) \in C_{11}$.

Let $c_b \in H_1$, we have the following possible cases:

a) $c_{b\downarrow} \in H_0$ and $\varphi_b(c_{b\downarrow}) = 0$, then $\varphi_b(c_b) = 0$, $B^{(0)} = 4$, $\varphi_b(c_b) \in C_1$,

b) $c_{b\downarrow} \in H_0$ and $\varphi_b(c_{b\downarrow}) = 1$, then there is not any H_A-weakly periodic ground state,

c) $c_{b\downarrow} \in H_1$ and $\varphi_b(c_{b\downarrow}) = 0$, then $\varphi_b(c_b) = 0$, $B^{(0)} = 3$, $B^{(1)} = 1$, $\varphi_b(c_b) \in C_2$.

We conclude that the configuration φ is a ground state on the set

$$A_1 \cap A_2 \cap A_{11} = \{(J_1, J_2) \in \mathbb{R}^2 : J_1 = J_2 = 0\}.$$

Therefore, if $J_1 \neq 0$ and $J_2 \neq 0$ then the weakly periodic configuration φ is not a weakly periodic ground state.

By similar way we can prove that all H_A-weakly periodic (non periodic) configurations are not ground states.

This finishes the proof of Theorem 2.3. □

Remark 2.3. 1) Theorem 2.3 shows that for the SOS model with competing interactions, every H_A-weakly periodic ground state is either H_A-periodic or translation-invariant.

2) The fact that for $k = 3$ there exists a set of countable non-periodic ground states can be proved in the same manner as in Theorem 2.1.

3) For the $k > 3$ by the same manner as in Theorem 2.1 periodic (and weakly periodic) ground states could be studied.

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Основные состояния для модели SOS с конкурирующими взаимодействиями

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Аннотация. В работе для нормального делителя индекса 2 изучены слабо-периодические основные состояния для модели SOS с конкурирующими взаимодействиями на дереве Кэли порядка 2 и порядка 3. Далее изучены непериодические основные состояния для модели SOS с конкурирующими взаимодействиями на дереве Кэли второго порядка.

Ключевые слова: дерево Кэли, SOS-модель, периодические и слабо-периодические основные состояния.

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Inverse Scattering and Loaded Modified Korteweg-de Vries Equation

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Abstract. The Cauchy problem for the loaded modified Korteweg-de Vries equation in the class of "rapidly decreasing" functions is considered in this paper. The main result of this work is a theorem on the evolution of the scattering data of the Dirac operator. Potential of the operator is the solution to the loaded modified Korteweg-de Vries equation. The obtained equalities allow one to apply the method of the inverse scattering transform to solve the Cauchy problem for the loaded modified Korteweg-de Vries equation.

Keywords: loaded modified KdV equation, inverse scattering method, "rapidly decreasing" functions, soliton, evolution of the scattering data.

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Introduction

The study of non-linear waves in the viscoelastic tube is of interest, since system of such tubes is a model of the vessels of the blood circulatory system. Understanding the wave processes in the blood circulatory system can help predict the development of diseases [5].

In arterial mechanics, a widely used model assumes that artery is a thin-walled pre-stressed elastic tube with a variable radius (or with a stenosis) and blood is considered as an ideal fluid [3].

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The governing equation that models weakly non-linear waves in such fluid-filled elastic tubes is the modified Korteweg–de Vries equation

$$u_t - 6u^2 u_x + u_{xxx} - h(t)u_x = 0,$$

where t is a scaled coordinate along the axis of the vessel after static deformation characterizing axisymmetric stenosis on the surface of the arterial wall; x is a variable that depends on time and coordinates along the axis of the vessel; $h(t)$ is a form of stenosis and $u(x,t)$ characterizes the average axial velocity of the fluid.

Let us assume that form of stenosis $h(t)$ is proportional to $u(0,t)$ and consider the following loaded modified Korteweg-de Vries equation

$$u_t - 6u^2 u_x + u_{xxx} - \gamma(t)u(0,t)u_x = 0, \tag{1}$$

where $u = u(x,t)$ is unknown real value function ($x \in R, \;\; t \geqslant 0$), and $\gamma(t)$ is arbitrary continuous function. Equation (1) is considered with initial condition

$$u|_{t=0} = u_0(x), \tag{2}$$

where real value function $u_0(x)$ has the following properties:

1. $\int\limits_{-\infty}^{\infty} (1 + |x|)\, |u_0(x)|\, dx \; < \infty.$
2. The equation $L\,(0)\,y \;\equiv\; \begin{pmatrix} i\dfrac{d}{dx} & u_0 \\ u_0 & -i\dfrac{d}{dx} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \;=\; \xi \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad x \;\in\; R^1$ has N simple eigenvalues and it does not have spectral singularities.

Let us assume that function $u(x,t)$ is sufficiently smooth, tends to its limit rapidly enough when $x \to \pm\,\infty$, and it satisfies the condition

$$\int_{-\infty}^{\infty} \left((1 + |x|)\, |u(x,t)| + \sum_{j=1}^{3} \left| \frac{\partial^j u(x,t)}{\partial x^j} \right| \right) dx \; < \infty, \; t \geqslant 0. \tag{3}$$

Note that completed integrability of the modified Korteweg-de Vries (mKdV) equation was established in the class of "rapidly decreasing" functions using the method of the inverse scattering problem [11]. The evolution equations for non-linear waves which differ by small terms from equations soluble by the inverse scattering method (KdV, NSE, mKdV) were considered [4]. A perturbation theory scheme was formulated. It is based on the inverse scattering method. The term "loaded equation" was introduced by A. M. Nakhushev [7]. The most general definition of a loaded equation was given and various loaded equations were classified in detail. Loaded differential equations, the loaded part of which contains only the value of the desired solution at fixed points of the domain were considered [2, 9, 12].

The goal of this paper is to study the integration of the loaded mKdV equation in the class of "rapidly decreasing" functions in terms of inverse scattering problem.

1. Uniqueness of the solution

In this part we use the method given in [6].

Theorem 1. *If problem* (1)–(2) *has solution then it is unique.*

Proof. Let $v(x,t)$ be another solution of (1)–(2). Let us introduce $w(x,t) = (u(x,t) - v(x,t))_x$. Then we obtain

$$\begin{aligned} w_t &= 6[(u-v)(u_x^2+v_x^2) + (u+v)(u_x+v_x)w]+ \\ &+ 3[(u-v)(u+v)(u+v)_{xx} + (u^2+v^2)w_x] - w_{xxx}+ \\ &+ \frac{\gamma(t)}{2}\left((u(0,t)+v(0,t)w_x + (u(0,t)-v(0,t)(u+v)_{xx}\right). \end{aligned} \tag{4}$$

Multiplying (4) by ω and integrating with respect to x over $(-\infty,\infty)$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2dx = 6\int_{-\infty}^{\infty}(u-v)(u_x^2+v_x^2)wdx + 6\int_{-\infty}^{\infty}(u+v)(u_x+v_x)w^2dx+ \\ &+3\int {}_{-\infty}^{\infty}(u-v)(u+v)(u+v)_{xx}wdx + 3\int_{-\infty}^{\infty}(u^2+v^2)w_xwdx - \int_{-\infty}^{\infty} w_{xxx}wdx+ \\ &+\frac{\gamma(t)}{2}(u(0,t)-v(0,t))\int_{-\infty}^{\infty}(u+v)_{xx}wdx + \frac{\gamma(t)}{2}(u(0,t)+v(0,t))\int_{-\infty}^{\infty} w_xwdx. \end{aligned} \tag{5}$$

Let us denote $\max(u_x^2+v_x^2)$ by m, $\max|(u+v)(u_x+v_x)|$ by n, $\max|(u+v)(u+v)_{xx}|$ by k, $\max\left|(u^2+v^2)_x\right|$ by l, $\max|u(0,t)+v(0,t)|$ by p and $\max|(u+v)_{xx}|$ by q. Using the Cauchy–Schwarz inequality we obtain from (5) the following inequality

$$\begin{aligned} \frac{d}{2dt}\int_{-\infty}^{\infty} w^2dx &\leqslant 6m\sqrt{\int_{-\infty}^{\infty}(u-v)^2dx}\sqrt{\int_{-\infty}^{\infty}w^2dx} + 6n\int_{-\infty}^{\infty}w^2dx+ \\ &+3k\sqrt{\int_{-\infty}^{\infty}(u-v)^2dx}\sqrt{\int_{-\infty}^{\infty}w^2dx} + \frac{3l}{2}\int_{-\infty}^{\infty}w^2dx + \frac{1}{2}\int_{-\infty}^{\infty}(w_x^2)_xdx+ \\ &+\frac{q\gamma(t)}{2}\max|u-v|\sqrt{\int_{-\infty}^{\infty}w^2dx} + \frac{\gamma(t)}{2}p\sqrt{\int_{-\infty}^{\infty}w_x^2dx}\sqrt{\int_{-\infty}^{\infty}w^2dx}. \end{aligned}$$

Here it was taken into account that w and its derivatives tend to zero as $x \to \pm\infty$. There are constants m_1, $k_1 > 0$ such that [8]

$$\sqrt{\int_{-\infty}^{\infty}(u-v)^2dx} \leqslant m_1\sqrt{\int_{-\infty}^{\infty}(u-v)_x^2dx}, \qquad \max|u-v| \leqslant k_1\sqrt{\int_{-\infty}^{\infty}(u-v)_x^2dx}.$$

Then we derive

$$\begin{aligned} \frac{d}{2dt}\int_{-\infty}^{\infty} w^2dx &\leqslant 6mm_1\sqrt{\int_{-\infty}^{\infty}w^2dx}\sqrt{\int_{-\infty}^{\infty}w^2dx} + 6n\int_{-\infty}^{\infty}w^2dx+ \\ &+3km_1\sqrt{\int_{-\infty}^{\infty}w^2dx}\sqrt{\int_{-\infty}^{\infty}w^2dx} + \frac{3l}{2}\int_{-\infty}^{\infty}w^2dx+ \\ &+\frac{qk_1\gamma(t)}{2}\sqrt{\int_{-\infty}^{\infty}w^2dx}\sqrt{\int_{-\infty}^{\infty}w^2dx}. \end{aligned}$$

Let us denote $\int\limits_{-\infty}^{\infty}\omega^2dx$ by $E(t)$ and $(12mm_1 + 12n + 6km_1 + 3l + qk_1\gamma(t))$ by $C(t)$,

$$\frac{dE(t)}{dt} \leqslant C(t)E(t).$$

This differential inequality yields

$$E(t) \leqslant E(0) \exp \int_0^t C(s) ds,$$

which implies that if $E(0) = 0$ then $E(t) = 0$ and thereby

$$w(x,t) = (u(x,t) - v(x,t))_x = 0$$

,

$$u(x,t) - v(x,t) = C.$$

Assuming $t = 0$, we obtain $C = 0$. Theorem 1 is proved. □

2. Scattering problem

Let us consider the following system of equation

$$Lv \equiv \begin{pmatrix} i\dfrac{d}{dx} & u \\ u & -i\dfrac{d}{dx} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \xi \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad -\infty < x < \infty, \tag{6}$$

with real value function $u(x)$ that satisfies the condition of "rapid decrease"

$$\int_{-\infty}^{\infty} (1 + |x|) \, |u(x)| \, dx < \infty. \tag{7}$$

The present section contains well known information on the direct and inverse scattering problem for problem (6)–(7) that is required for further consideration [1]. Condition (7) implies that system of equation (6) has the Jost solutions $\varphi(x,\xi)$ and $\psi(x,\xi)$ with asymptotic relations

$$\begin{matrix} \varphi \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-i\xi x), \\ \bar{\varphi} \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} \exp(i\xi x), \end{matrix} \quad as \;\; x \to -\infty, \tag{8}$$

$$\begin{matrix} \psi \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(i\xi x), \\ \bar{\psi} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-i\xi x), \end{matrix} \quad as \;\; x \to \infty. \tag{9}$$

For real ξ, pairs $\{\varphi, \bar{\varphi}\}$ and $\{\psi, \bar{\psi}\}$ are pairs of linearly independent solutions of equation (6). Therefore,

$$\begin{cases} \varphi = a(\xi)\bar{\psi} + b(\xi)\psi, \\ \bar{\varphi} = -\bar{a}(\xi)\psi + \bar{b}(\xi)\bar{\psi}. \end{cases} \tag{10}$$

The following equality holds

$$a(\xi) = W\{\varphi, \psi\} \equiv \varphi_1\psi_2 - \varphi_2\psi_1 \tag{11}$$

and for all real ξ

$$a(\xi)\bar{a}(\xi) + b(\xi)\bar{b}(\xi) = 1.$$

Function $a(k)$ admits analytic continuation into the upper half-plane $\operatorname{Im} k > 0$. In $\operatorname{Im} k \geqslant 0$ function $a(k)$ has asymptotic behavior $a(\xi) = 1 + O\left(\frac{1}{|\xi|}\right)$. Function $a(k)$ can have a finite number of zeroes ξ_k, $k = 1, 2, \dots, N$ in the upper half-plane $\operatorname{Im} k > 0$. Zeros ξ_k of function $a(k)$ correspond to the eigenvalues of operator L in the upper half-plane. Let us note that operator L can have spectral singularities which are in the continuous spectrum.

We suppose that operator L does not have spectral singularities and zeros of function $a(k)$ are simple:

$$\varphi(x, \xi_k) = C_k \psi(x, \xi_k), \ k = 1, 2, \dots, N. \tag{12}$$

The set $\left\{r^+(\xi) \equiv \frac{b(\xi)}{a(\xi)}, \ \xi_k, C_k, \ k = 1, 2, 3, \dots, N\right\}$ is called scattering data for system of equations (6).

The following representation for the solution $\psi(x, \xi)$ is valid

$$\psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} + \int_x^\infty \mathbf{K}\ (x, s)\, e^{i\xi\ s} ds, \tag{13}$$

where $\mathbf{K}\ (x, s) = \begin{pmatrix} K_1(x, s) \\ K_2(x, s) \end{pmatrix}$ does not depend on variable ξ and it is related to the potential function $u(x)$ as follows

$$u(x) = 2iK_1(x, x). \tag{14}$$

The components of kernel $K(x, y)$ for $y > x$ are solutions of the following Gelfand-Levitan-Marchenko (GLM) system of equation

$$\begin{cases} K_2(x, y) + \displaystyle\int_x^\infty K_1(x, s) F(s + y) ds = 0, \\ -K_1(x, y) + F(x + y) + \displaystyle\int_x^\infty K_2(x, s) F(s + y) ds = 0, \end{cases} \tag{15}$$

where $F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b(\xi)}{a(\xi)} e^{i\xi x} d\xi - i \sum_{j=1}^{N} C_j e^{i\xi_j x}$.

3. Evolution of scattering data

It is easy to verify that functions

$$h_n(x) = \frac{\frac{d}{d\xi}\left(g(x, \xi) - B_n f(x, \xi)\right)\Big|_{\xi = \xi_n}}{\dot{a}(\xi_n)}, \qquad n = 1, 2, 3, \dots, N \tag{16}$$

are solutions of the system of equations $Ly = \xi_n y$. Using (11) for $\operatorname{Im} \xi > 0$, we define the following asymptotic relations

$$\psi \sim a(\xi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(i\xi x) \quad as \quad x \to -\infty,$$
$$\varphi \sim a(\xi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-i\xi x) \quad as \quad x \to \infty.$$

Using these asymptotic relations, we obtain asymptotic relations for solutions $h_n(x)$

$$\begin{aligned} h_n(x) &\sim -C_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(i\xi_n x) \quad as \quad x \to -\infty, \\ h_n(x) &\sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp(-i\xi_n x) \quad as \quad x \to \infty, \end{aligned} \tag{17}$$

and

$$W\{\varphi_n, h_n\} \equiv \varphi_{n1}h_{n2} - \varphi_{n2}h_{n1} = -C_n, \quad n = 1, 2, 3, \dots, N,$$

where $\varphi_n = \varphi(x, \xi_n)$.

Let function $u(x,t)$ in (6) be a solution of the mKdV equation

$$u_t - 6u^2 u_x + u_{xxx} = G(x,t), \tag{18}$$

where function $G(x,t)$ is sufficiently smooth and $G(x,t) = o(1)$ when $x \to \pm\infty$, $t \geqslant 0$. Equation (18) is considered with initial condition (2). According to [10], the following Theorem is valid.

Theorem 2. *If function $u(x,t)$ is a solution of equation* (18) *in the class of functions* (3) *then the scattering data of system* (6) *with function $u(x,t)$ depend on t as follows*

$$\frac{dr^+}{dt} = 8i\xi^3\, r^+ - \frac{i}{a^2}\int_{-\infty}^{\infty} G(\varphi_1^2 + \varphi_2^2)dx, \quad \operatorname{Im}\xi = 0,$$

$$\frac{dC_n}{dt} = \left(8i\xi_n^3 - i\int_{-\infty}^{\infty} G\left(h_{n1}\psi_{n1} + h_{n2}\psi_{n2}\right)dx\right) C_n,$$

$$\frac{d\xi_n}{dt} = \frac{\int\limits_{-\infty}^{\infty} G(\varphi_{n1}^2 + \varphi_{n2}^2)dx}{2\int\limits_{-\infty}^{\infty} \varphi_{n1}\varphi_{n2}dx}, \quad n = 1,\ 2,\ \dots,\ N.$$

Here $\varphi_n(x,t)$ are normalized eigenfunctions which correspond to the eigenvalue ξ_n of system of equations (6).

Let us apply the result of Theorem 2 when

$$G(x,t) = \gamma(t)u(0,t)u_x.$$

According to Theorem 2, we have the following representation

$$\frac{dr^+}{dt} = 8i\xi^3\, r^+ - \frac{i\gamma(t)u(0,t)}{a^2}\int_{-\infty}^{\infty} u_x(\varphi_1^2 + \varphi_2^2)dx, \quad \operatorname{Im} k = 0.$$

By virtue of system of equations (6) and asymptotic relations (9), we have

$$\int_{-\infty}^{\infty} u_x(\varphi_1^2 + \varphi_2^2)dx = -2\int_{-\infty}^{\infty}\left[\varphi'_{1x}\left(i\varphi'_{2x} + \xi\varphi_2\right) + \varphi'_{2x}\left(-i\varphi'_{1x} + \xi\varphi_1\right)\right]dx =$$

$$= -2\int_{-\infty}^{\infty} \xi(\varphi_1 \cdot \varphi_2)'_x dx = -2\xi\, a(\xi)b(\xi).$$

Consequently, for $\operatorname{Im} k = 0$ we obtain

$$\frac{dr^+}{dt} = \left(8i\xi^3 + 2i\xi\,\gamma(t)u(0,t)\right)\, r^+. \tag{19}$$

From relations

$$\frac{d\xi_n}{dt} = \frac{\gamma(t)u(0,t) \int\limits_{-\infty}^{\infty} u_x(\varphi_{n1}^2 + \varphi_{n2}^2)dx}{2\int\limits_{-\infty}^{\infty} \varphi_{n1}\varphi_{n2}dx}, \qquad n = 1, 2, 3, \ldots, N,$$

and

$$\int_{-\infty}^{\infty} u_x(\varphi_{n1}^2 + \varphi_{n2}^2)dx == -2\int_{-\infty}^{\infty} \left[\varphi_{n1}'\left(i\varphi_{n2}' + \xi_n\varphi_{n2}\right) + \varphi_{n2}'\left(-i\varphi_{n1}' + \xi_n\varphi_{n1}\right)\right]dx =$$
$$= -2\xi_n\left(\varphi_{n1}\cdot\varphi_{n2}\right)\Big|_{-\infty}^{\infty} = 0,$$

we have

$$\frac{d\xi_n}{dt} = 0\,, \qquad n = 1, 2, 3, \ldots, N. \tag{20}$$

Using system of equations (6) and asymptotic relations (17), we have

$$\int_{-\infty}^{\infty} u_x\left(h_{n1}\psi_{n1} + h_{n2}\psi_{n2}\right)dx = -\xi_n\int_{-\infty}^{\infty}\left[\left(h_{n1}\psi_{n2}\right)_x' + \left(h_{n2}\psi_{n1}\right)_x'\right]dx = -\xi_n.$$

Taking into account the last expression, we obtain

$$\frac{dC_n}{dt} = \left(8i\xi_n^3 + i\gamma(t)u(0,t)\xi_n\right)C_n, \quad n = 1, 2, 3, \ldots, N. \tag{21}$$

Considering relations (19), (20) and (21), we arrive to the following theorem.

Theorem 3. *If function $u(x,t)$ is a solution of problem (1)–(3) then the scattering data of system of equations (6) with function $u(x,t)$ depend on t as follows*

$$\frac{dr^+}{dt} = \left(8i\xi^3 + 2i\xi\,\gamma(t)u(0,t)\right)\, r^+ \text{ for } \operatorname{Im}\xi = 0,$$
$$\frac{dC_n}{dt} = \left(8i\xi_n^3 + i\gamma(t)u(0,t)\xi_n\right)C_n,$$
$$\frac{d\xi_n}{dt} = 0\,, \quad n = 1, 2, 3, \ldots, N.$$

The obtained relations completely determine the evolution of the scattering data for system of equations (6). It allows us to find the solution of problem (1)–(3) by using the method of inverse scattering problem.

Example. Let us consider the following Cauchy problem:

$$u_t - 6u^2u_x + u_{xxx} - \gamma(t)u(0,t)u_x = 0,$$

$$u\,|_{t=0} = \frac{2}{\operatorname{sh}2x},$$

where $\gamma(t) = -\dfrac{1}{4\sqrt{1+t^2}}\operatorname{sh}(-8t + \operatorname{arcsh}t)$.

To find the general solution of this problem we use the method of inverse scattering problem. First of all, let us find a solution of the direct problem for the following system of equations

$$L(0)y \equiv \begin{pmatrix} i\dfrac{d}{dx} & u_0 \\ u_0 & -i\dfrac{d}{dx}\end{pmatrix}\begin{pmatrix} y_1 \\ y_2\end{pmatrix} = \xi\begin{pmatrix} y_1 \\ y_2\end{pmatrix}.$$

In this case, we have the following scattering data

$$r^+(0) = 0, \quad N = 1, \;\; \xi(0) = i, \;\; C_1 = 2.$$

According to Theorem 3, we find the evolution of scattering data depending on t :

$$r^+(\xi, t) = 0, \quad \xi_1(t) = i, \;\; C_1(t) = 2 \exp\, \delta(t),$$

where

$$\delta(t) = 8t - \int_0^t \gamma(\tau) u(0, \tau) dx.$$

Then using this scattering data, we find a solution of inverse scattering problem . Solving the GLM system of equations with $F(x) = -2i \exp\left(-x + \delta(t)\right)$, we obtain

$$K_1(x, y) = -\frac{2i \exp(-x - y + \delta(t))}{1 - \exp(-4x + 2\delta(t))}\,.$$

Applying equality (14), we obtain

$$u(x, t) = \frac{2}{\operatorname{sh}\bigl(2x - 8t + \int\limits_0^t \gamma(\tau) u(0, \tau) d\tau\bigr)}\,.$$

Putting $x = 0$ and introducing $f(t) = \int\limits_0^t \gamma(\tau) u(0, \tau) dx$, we obtain the following Cauchy problem

$$\begin{cases} \dfrac{f'(t)}{\gamma(t)} = \dfrac{2}{\operatorname{sh}(f(t) - 8t)}\,, \\ f(0) = 0. \end{cases}$$

Solving this problem with $\gamma(t) = -\dfrac{1}{4\sqrt{1 + t^2}} \operatorname{sh}(-8t + \operatorname{arcsh} t)$, we have

$$f(t) = \operatorname{arcsh} t.$$

As a result, the solution of problem under consideration is expressed as follows

$$u(x, t) = \frac{2}{\operatorname{sh}(2x - 8t + \operatorname{arcsh} t)}\,.$$

It is well known that solution of the modified Korteweg-de Vries equation

$$u_t - 6u^2 u_x + u_{xxx} = 0,$$

that satisfies the same initial condition has the form

$$u(x, t) = \frac{2}{\operatorname{sh}(2x - 8t)}.$$

The difference between solutions of the loaded modified Korteweg-de Vries equation and the modified Korteweg-de Vries equation is shown in Fig. 1.

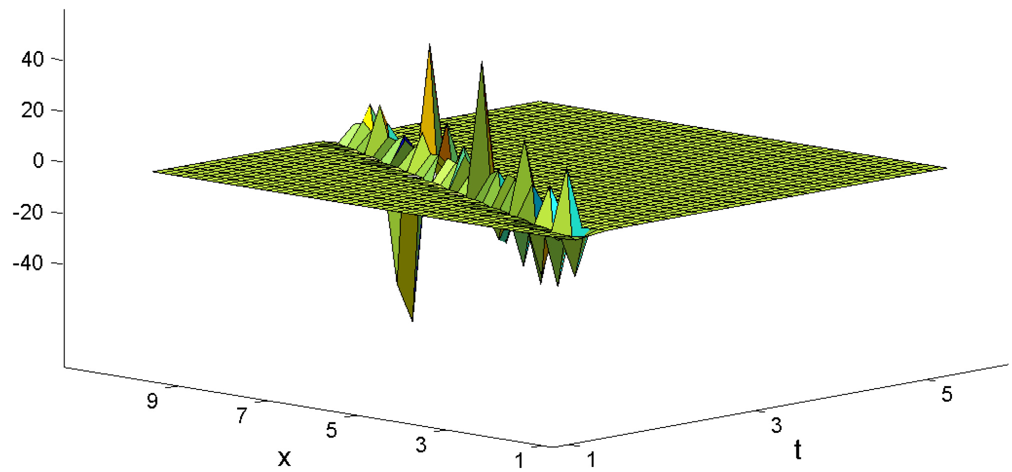

Fig. 1

Conclusion

The method of inverse scattering problem can be used to obtain solutions of the Cauchy problem for the loaded modified Korteweg-de Vries equations in the class of "rapidly decreasing" functions. Function $u(0,t)$ that appears in equations of Theorem 3 is unknown in contrast to function $u(x,0)$. If the scattering data is used to find potential $u(x,t)$ then function $u(0,t)$ is included in the solution. Therefore, we have a functional equation relating $u(x,t)$ to $u(x,0)$ which is reduced to the Cauchy problem for an ordinary differential equation of the first order. For some $\gamma(t)$ the Cauchy problem for ODE can be solved exactly and we obtain a solution of the Cauchy problem for the loaded mKdV equation.

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Метод обратной задачи рассеяния и нагруженное модифицированное уравнение Кортевега-де Фриза

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Аннотация. В данной статье мы рассматриваем задачу Коши для нагруженного модифицированного уравнения Кортевега-де Фриза в классе «быстроубывающих» функций. Основной результат настоящей работы представляет собой теорему об эволюции данных рассеяния оператора Дирака, потенциал которого является решением нагруженного модифицированного уравнения Кортевега-де Фриза. Полученные равенства позволяют применить метод обратной задачи рассеяния для решения задачи Коши для нагруженного модифицированного уравнения Кортевега-де Фриза.

Ключевые слова: нагруженное модифицированное уравнение КдФ, метод обратной задачи рассеяния, "быстроубывающие" функции, солитонное решение, эволюция данных рассеяния.

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Analytic Solvability of the Hörmander Problem and the Borel Transformation of Multiple Laurent Series

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Abstract. In this paper, the initial-boundary value problem of Hormander is formulated in the class of functions representable by Laurent series supported in rational cones. Using the Borel transformation of Laurent series we establish a connection between a differential and a difference problems and prove its global analytic solvability.

Keywords: Hörmander problem, polynomial differential operator, Borel transformation, difference operator.

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1. Introduction and preliminaries

Let $\omega = (\omega_1, \ldots, \omega_n)$ denote a multi-index, $||\omega|| = \omega_1 + \cdots + \omega_n$, $\partial = (\partial_1, \ldots, \partial_n)$, where ∂_j are derivatives with respect to the j-th variable and $c_\omega(z)$ are analytic functions of $z = (z_1, \ldots, z_n)$ in a neighborhood of zero in $\mathbb{C}^n$. Consider a polynomial differential operator of order d of the form $P(\partial, z) = \sum\limits_{||\omega|| \leqslant d} c_\omega(z)\partial^\omega$.

In the traditional formulation of the Cauchy-Kovalevskaya theorem, it is assumed that the equation is resolved with respect to the highest derivative, for example, ∂_n^d, where d is the order of the differential equation. For linear differential equations with analytic coefficients this means that $c_{(0,\ldots,0,d)}(0) \neq 0$ and the initial data are specified on the coordinate plane $z_n = 0$. In Hörmander's paper [1], a version of the Cauchy-Kovalevskaya theorem is given, where it is assumed that it is solvable with respect to an arbitrary derivative $\partial^m \mathcal{F}$, where $||m|| = d$. However, in this case, in addition to the constraint $c_m(0) \neq 0$, additional conditions must be required on the coefficients of higher-order derivatives d, and the initial data are specified on the union of the coordinate planes. Let us give an exact formulation of this result.

Let the condition

$$\sum_{||\omega||=||m||, \omega \neq m} |c_\omega(0)| < (2e)^{-||m||} |c_m(0)| \tag{1}$$

be satisfied for the equation

$$P(\partial, z)\mathcal{F} = \mathcal{G}, \tag{2}$$

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with analytic coefficients in the neighborhood of the point $z_0 = 0$.

Then equation (2) *with initial data*

$$\partial_j^k [\mathcal{F} - \Phi]|_{z_j=0} = 0, \;\; 0 \leqslant k < m_j, \;\; j = 1, \dots, n, \tag{3}$$

has a unique analytic solution in a neighborhood of zero for any given analytic functions Φ *and* $\mathcal{G}$.

In this paper, we formulate a generalization of the problem (2)–(3) for polynomial differential operators of a special form, which were considered in [2] in connection with the study of the properties of generating functions of solutions of multidimensional difference equations. The most useful classes of generating functions in enumerative combinatorial analysis (see [3]), along with rational and algebraic ones, are D-finite ones. A power series is called D-finite if it satisfies a linear differential homogeneous equation of the form (2) with polynomial coefficients. In the case of multiple power series, various approaches to the definition of D-finiteness are possible (see [4,5]), one of which is that the power series satisfies a system of linear homogeneous differential equations with polynomial coefficients. To generalize the notion of D-finiteness to Laurent series, in [2] the derivations $D = (D_{a^1}, \dots, D_{a^n})$ (see Sec. 2. below) in the ring of Laurent series supported in rational cones were defined and the corresponding definition of D-finiteness Laurent series was given.

The question naturally arises of describing the space of solutions of equations of the form (2), where the operators $P(D, z)$ are considered in a suitable way. One of the ways of such a description is to formulate an analogue of the initial-boundary value problem of Hörmander (2)–(3) instead of differential operators $P(\partial, z)$ and study its solvability. In the first section of the paper, the necessary notation and definitions are given, and sufficient conditions for global solvability of a polynomial difference operator with constant coefficients $P(D, z) = P(D)$, (i.e., the existence and uniqueness of the global solution) in the class of Laurent series supported in rational cones (Theorem 1) are proven.

The main idea of the proof of Theorem 1 is to associate the differential initial-boundary value problem of Hörmander with its difference version, and the main role in this comparison is played by the Borel transformation of Laurent series, which is defined in the second section of the paper. With its help a connection between the analytic properties of a function and its Borel transformation is established (Proposition 1), which allows to prove the existence and uniqueness of a solution to a differential initial-boundary value problem in the class of functions representable by Laurent series with supports in the rational cones in the integer lattice (in Sec. 4.).

2. Notation, definitions and formulation of the main result

Let $a^j = (a_1^j, \dots, a_n^j)$, $j = 1, \dots, n$, be *linearly independent* vectors with integer coordinates. *The rational cone* spanned by the vectors $a^1, \dots, a^n$ is the set

$$K = \left\{ x \in \mathbb{R}^n : x = \lambda_1 a^1 + \dots + \lambda_n a^n, \lambda_j \in \mathbb{R}_{\geqslant}, j = 1, \dots, n \right\},$$

where $\mathbb{R}_{\geqslant}$ is the set of non-negative real numbers.

Let A be a matrix whose determinant is not equal to zero, and the columns consist of the coordinates of the vectors a^j

$$A = \begin{pmatrix} a_1^1 & \dots & a_1^n \\ .. & .. & .. \\ a_n^1 & \dots & a_n^n \end{pmatrix}.$$

We consider only *unimodular cones*, i.e. cones for which the determinant of the matrix $\det A = 1$. Any element $x \in K \cap \mathbb{Z}^n$ can be represented as a linear combination of basis vectors $x = \lambda_1 a^1 + \cdots + \lambda_n a^n$, where $\lambda_j \in \mathbb{Z}_{\geqslant}$ is a set of non-negative integers, or in matrix form as $x = A\lambda$, where λ is a column vector.

Let us denote the rows of the inverse matrix A^{-1} by $\alpha^1, \ldots, \alpha^n$ and note that they form a *mutual basis* (see, for example, [6]) for the vectors $a^1, \ldots, a^n$, i.e. $\langle \alpha^i, a^j \rangle = \delta_{ij}$, where $\langle \alpha^i, a^j \rangle = \alpha_1^i a_1^j + \cdots + \alpha_n^i a_n^j$, and δ_{ij} is the Kronecker symbol. Also, for $x \in K$, always $\lambda_j = \langle \alpha^j, x \rangle \geqslant 0$, $j = 1, \ldots, n$.

Between the points $u, v \in \mathbb{R}^n$ we define the partial order relation $\underset{K}{\geqslant}$ as follows: $u \underset{K}{\geqslant} v \Leftrightarrow u - v \in K$, and $u \underset{K}{\ngeqslant} v$ means that $u - v \notin K$.

The cone $K^* = \{k \in \mathbb{R}^n : \langle k, x \rangle \geqslant 0, x \in K\}$ is called *dual to the cone* K, and the set of its interior points is denoted by $\overset{\circ}{K}{}^*$ and we fix the vector $\nu \in \overset{\circ}{K}{}^* \cap \mathbb{Z}^n$. For all integer points of the rational cone $x \in K \cap \mathbb{Z}^n$, the *weighted-homogeneous degree with weight* ν *(ν-degree) of the monomial* z^x is a nonnegative integer $||x||_\nu = \langle \nu, x \rangle$, and the ν-degree of the Laurent polynomial $Q(z) = \sum\limits_{x \in X} q_x z^x$ is defined by the formula $\deg_\nu Q(z) = \max\limits_{x \in X} ||x||_\nu$, where $X \subset K \cap \mathbb{Z}^n$ is a finite set of points of an n-dimensional integer lattice.

We denote the ring of formal Laurent series of the form

$$\mathcal{F}(z) = \sum_{x \in K \cap \mathbb{Z}^n} f(x) z^x \tag{4}$$

by $\mathbb{C}_K[[z]]$ and note that an operator mapping a ring into itself is called differentiation if it is linear and satisfies the usual rule for the derivative of a product (see, for example, [7]). For the Laurent series (4), taking the usual partial derivative $\partial_j = \dfrac{\partial}{\partial z_j}$ is not necessarily a derivation in the ring $\mathbb{C}_K[[z]]$, since for $x \in K \cap \mathbb{Z}^n$ the point $x - e^j$, where e^j are the unit vectors, generally speaking, may not lie in $K \cap \mathbb{Z}^n$. Derivations of the ring of Laurent series $\mathbb{C}_K[[z]]$ were defined in [2, 8], which made it possible to transfer the notion of D-finiteness of power series to Laurent series. Let us give this definition.

On the monomials z^x, $x \in K \cap \mathbb{Z}^n$ we define the operator D_{a^j} as follows

$$D_{a^j} z^x = \langle x, \alpha^j \rangle z^{x - a^j},$$

where α^j are the vectors of the mutual basis, $j = 1, \ldots, n$.

It is directly verified that in the case of a unimodular cone the operators D_{a^j}, $j = 1, \ldots, n$, are derivations of the ring $\mathbb{C}_K[[z]]$. For $\omega \in K \cap \mathbb{Z}^n$, $\omega = \lambda_1 a^1 + \cdots + \lambda_n a^n$ we define the D^ω operator as follows:

$$D^\omega = D_{a^1}^{\lambda_1} \ldots D_{a^n}^{\lambda_n},$$

where $\lambda_j = \langle \omega, \alpha^j \rangle$ and $D_{a^j}^k = \underbrace{D_{a^j} \ldots D_{a^j}}_{k \text{ times}}$. Note that for any $\omega', \omega'' \in K \cap \mathbb{Z}^n$, $D^{\omega'} D^{\omega''} = D^{\omega' + \omega''}$ is true and for $\omega = a^j$ we have $D^{a^j} z^x = \langle x, \alpha^j \rangle z^{x - a^j} = D_{a^j} z^x$, $j = 1, \ldots, n$.

Thus, the operators D^ω for $\omega \in K \cap \mathbb{Z}^n$ are derivations of the ring of series $\mathbb{C}_K[[z]]$ and their action on the monomials z^x, $x \in K \cap \mathbb{Z}^n$ is conveniently given by the following formula:

$$D^\omega z^x = \begin{cases} 0, & \text{if } x \underset{K}{\ngeqslant} \omega, x \neq \omega, \\ \dfrac{\langle x, \alpha \rangle!}{\langle x - \omega, \alpha \rangle!} z^{x - \omega}, & \text{if } x \underset{K}{\geqslant} \omega, \end{cases} \tag{5}$$

where $\langle x, \alpha \rangle ! = \langle x, \alpha^1 \rangle ! \dots \langle x, \alpha^n \rangle !$.

We consider polynomial differential operators of the form $P(D, z) = \sum\limits_{\omega \in \Omega} c_\omega(z) D^\omega$, where $\Omega \subset K \cap \mathbb{Z}^n$ is a finite set of points of the n-dimensional integer lattice and the coefficients $c_\omega(z) \in \mathbb{C}_K[[z]]$. *The characteristic polynomial* of this operator is the Laurent polynomial $P(\zeta, z) = \sum\limits_{\omega \in \Omega} c_\omega(z) \zeta^\omega$, and its support is denoted by $supp\, P = \{\omega \in \Omega : c_\omega(z) \neq 0\}$. *The order d_ν of the differential operator* $P(D, z)$ is the ν-degree $\deg_\nu P(\zeta, z)$ of the characteristic polynomial, that is $d_\nu = \max\limits_{\omega \in \Omega} ||\omega||_\nu$. In what follows, the subscript ν for d will be omitted, since $\nu \in \mathring{K}^* \cap \mathbb{Z}^n$ is fixed. Thus, the operator $P(D, z)$ of order d can be written as

$$P(D, z) = \sum_{||\omega||_\nu \leqslant d} c_\omega(z) D^\omega. \tag{6}$$

We denote by Γ_j the face of the cone K spanned by the vectors a^i, $i = 1, \dots, j-1, j+1, \dots, n$, $\Gamma_j = \{x : x = \lambda_1 a^1 + \dots + \lambda_{j-1} a^{j-1} + \lambda_{j+1} a^{j+1} + \dots + \lambda_n a^n, \lambda \in \mathbb{R}_\geqslant\}$ and by $\mathcal{F}(z)|_{z^{a^j}=0}$ the Laurent series supported by the face Γ_j of the rational cone K

$$\mathcal{F}(z)|_{z^{a^j}=0} = \sum_{x \in \Gamma_j \cap \mathbb{Z}^n} f(x) z^x. \tag{7}$$

Let the coefficients $c_\omega(z)$ of operator (6) lie in some subring $\mathcal{L}_K$ of the ring $\mathbb{C}_K[[z]]$. For m such that $||m||_\nu = d$ and $c_m(z) \neq 0$, we formulate the following analogue of the Hörmander problem (2)–(3).

For any $\Phi(z)$, $\mathcal{G}(z) \in \mathcal{L}_K$ find $\mathcal{F} \in \mathcal{L}_K$ satisfying the differential equation

$$P(D, z)\mathcal{F} = \mathcal{G}, \tag{8}$$

and the initial conditions:

$$D^{a^j k}[\mathcal{F} - \Phi]|_{z^{a^j}=0} = 0, \quad 0 \leqslant k < \langle m, \alpha^j \rangle, \quad j = 1, \dots, n. \tag{9}$$

For constant coefficients, the case $\mathcal{L}_K = \mathbb{C}_K[[z]]$ was studied in [8], and the case $K = \mathbb{R}^n_\geqslant$ was considered in [9], and the global solvability of the Cauchy–Kovalevskaya problem was proved in the class of entire functions of exponential type.

We define a subring of the ring $\mathbb{C}_K[[z]]$ of Laurent series, in which we will prove the solvability of problem (8)–(9).

Let $Exp(\mathbb{C}^n)$ be the space of entire functions $U(\xi) : \mathbb{C}^n_\xi \to \mathbb{C}$ of exponential type, that is, of entire functions satisfying the inequality $|U(\xi)| \leqslant C e^{\langle \tau, |\xi| \rangle}$, where $\tau = (\tau_1, \dots, \tau_n)$, τ_j, $C \geqslant 0$ are constants, $|\xi| = (|\xi_1|, \dots, |\xi_n|)$ (see, for example, [10]). Note that the set $\sigma_U = \left\{\tau \in \mathbb{R}^n_> : |U(\xi)| \leqslant C e^{\langle \tau, |\xi| \rangle}\right\}$ has the following property: together with each point τ_0, all points τ for which $\tau \underset{\mathbb{R}^n_\geqslant}{>} \tau_0$ also belong to it.

We denote by $\mathcal{A}$ a mapping from $\mathbb{Z}^n_\geqslant$ to $K \cap \mathbb{Z}^n$ with matrix $A = (a^j_i)_{n \times n}$. This mapping induces a mapping of rings $\mathcal{A}_* : \mathbb{C}_K[[z]] \to \mathbb{C}[[\xi]]$, which associates the Laurent series $\sum\limits_{x \in K \cap \mathbb{Z}^n} f(x) z^x$ with a power series $\sum\limits_{\lambda \in \mathbb{Z}^n_\geqslant} f(A\lambda) \xi^\lambda$, where $\xi = z^A$ and $z^A = (z^{a^1}, \dots, z^{a^n})$.

Since the mapping $\mathcal{A}_*$ is invertible in the case $\det A = 1$, we see that $\mathcal{A}_*^{-1}(Exp(\mathbb{C}^n)) = \mathcal{L}_K$ is a subring of the ring $\mathbb{C}_K[[z]]$. *The functions $\mathcal{F}$ representable by Laurent series from the subring $\mathcal{L}_K$ will be called exponential in the class of Laurent series ($\mathcal{L}_K$-exponential).*

Theorem 1. *Let the coefficients of the polynomial differential operator* (6) *be constant and* $m \in suppP \subset \{\omega : 0 \underset{K}{\leqslant} \omega \underset{K}{\leqslant} m\}$, *then for any* $\mathcal{L}_K$*-exponential functions* $\mathcal{G}$, Φ *problem* (8)–(9) *has a unique* $\mathcal{L}_K$*-exponential solution* $\mathcal{F}$.

The conditions of Theorem 1 mean, in particular, that m is the vertex of the Newton polytope of the characteristic polynomial $P(\zeta)$ and c_m is the only nonzero coefficient at the «highest derivative». In [8], problem (8)–(9) was studied under a weaker than in Theorem 1 restriction on the operator $P(D)$, namely, the condition $supp\, P \subset \{\omega : 0 \underset{K}{\leqslant} \omega \underset{K}{\leqslant} m\}$ was not required, but solutions were sought in the class of formal Laurent series.

Let us give an example of an operator satisfying the conditions of Theorem 1. Let the cone K be spanned by the vectors $a^1 = (1, -1)$, $a^2 = (-1, 2)$. Let us fix $\nu \in \mathring{K}^* \cap \mathbb{Z}^n$, for example, $\nu = (3, 2)$, $m = (0, 1)$, the set $\Omega = \{\omega : 0 \underset{K}{\leqslant} \omega \underset{K}{\leqslant} (0, 1)\} = \{(0, 0), (1, -1), (0, 1), (-1, 2)\}$. Consider the operator

$$P(D) = D^{(0,1)} + D^{(1,-1)} + D^{(-1,2)} + 1 \tag{10}$$

of the Hörmander problem (8)–(9) for operator (10) with $\mathcal{L}_K$-exponential initial data and the right-hand side will be a $\mathcal{L}_K$-exponential function, i.e. function represented by *Laurent series*.

Operator $D = (D_{a^1}, \dots, D_{a^n})$ is related to partial derivatives $\partial = (\partial_1, \dots, \partial_n)$ by formulas $z^{a^j} D_{a^j} = \langle \alpha^j, z\partial \rangle$, $j = 1, \dots, n$, where $z\partial = (z_1\partial_1, \dots, z_n\partial_n)$, therefore the polynomial differential operator (10) is expressed through $\partial = (\partial_1, \partial_2)$ as follows

$$\mathcal{P}(\partial_1, \partial_2, z) = 2z_1^2 z_2^{-1} \partial_1^2 + 3z_1 \partial_1 \partial_2 + z_2 \partial_2^2 +$$

$$+ (2z_2 + z_1^2 z_2^{-2} + 2z_1 z_2^{-1}) \partial_1 + (z_1^{-1} z_2^2 + z_1 z_2^{-1} + 1) \partial_2 + 1.$$

Note that this operator does not satisfy condition (1), which ensures the existence of an analytic solution, at any point $z = z_0$.

3. The Borel transformation of Laurent series and the connection between a differential and difference problems

In this section, we define the Borel transformation of Laurent series and prove an analogue of the Borel theorem on the connection between the analytic properties of a function and its Borel transformation (Proposition 1) in the class of $\mathcal{L}_K$-exponential functions.

For a function $f(x) : K \cap \mathbb{Z}^n \to \mathbb{C}$ of a discrete argument $x \in K \cap \mathbb{Z}^n$ we define two types of generating series (functions):

$$\mathcal{F}(z) = \sum_{x \in K \cap \mathbb{Z}^n} \frac{f(x) z^x}{\langle x, \alpha \rangle !}, \tag{11}$$

and

$$F(z) = \sum_{x \in K \cap \mathbb{Z}^n} \frac{f(x)}{z^x}. \tag{12}$$

Series (12) is called the *Borel transformation of series* (11) and in the one-dimensional case it is the classical Borel transformation of power series (see [11]). If $K = \mathbb{R}^n_{\geqslant}$, then we obtain the definition of the Borel transformation of multiple power series from [12].

Functions (11) and (12) are the *upper and lower functions of the Borel transformation*, respectively.

The definition of $\mathcal{L}_K$-exponentiality implies that $\mathcal{F}(z) \in \mathcal{L}_K$ if and only if for some $\sigma \in \mathbb{R}^n_>$ the inequality

$$|\mathcal{F}(z)| \leqslant Ce^{\langle \sigma^A, |z^A| \rangle}$$

holds, where $C \geqslant 0$ is a constant.

Let $\sigma_{\mathcal{F}}$ denote the set

$$\sigma_{\mathcal{F}} = \{\sigma \in \mathbb{R}^n_> : |\mathcal{F}(z)| \leqslant Ce^{\langle \sigma^A, |z^A| \rangle}\}$$

and call it the *type-set of the function* $\mathcal{F}$.

Note that in the case $A = E$, where E is the identity matrix, the set $\sigma_{\mathcal{F}} = \sigma_U$ and the related concept of conjugate types of entire functions were used in [12–14] to study the growth of entire functions.

The domain of convergence $\mathcal{D}_F$ of the Laurent series (12) is the open kernel of the set of those points z at which this series converges absolutely. We denote the image of the convergence domain under the projection

$$z = (z_1, \ldots, z_n) \to |z| = (|z_1|, \ldots, |z_n|) \tag{13}$$

by $|\mathcal{D}_F|$. Note that it follows from Lemma 7 in [15] that if $R = (R_1, \ldots, R_n) \in |\mathcal{D}_F|$, then series (12) also converges for all points of the set $T_K(R) = \{z \in \mathbb{C}^n : |z^{a^j}| > R^{a^j}, j = 1, \ldots, n\}$. It follows that after logarithmic projection

$$Log\ :\ z = (z_1, \ldots, z_n) \to (\log(|z_1|), \ldots, \log(|z_n|)) = Log|z|, \tag{14}$$

the set $Log|\mathcal{D}_F|$, together with each point $LogR$, also contains the affine cone $LogR + \mathring{K}^*$, where K^* is the cone dual to the cone K.

Let us give an analogue of Borel's theorem for $\mathcal{L}_K$-exponential functions.

Proposition 1. *If $\mathcal{F}(z)$ is a $\mathcal{L}_K$-exponential function with type set $\sigma_{\mathcal{F}}$ and $\mathcal{D}_F$ is the domain of convergence of its Borel transformation $F(z)$, then $\sigma_{\mathcal{F}} = |\mathcal{D}_F|$.*

Proof. When transform $\mathcal{A}_* : \mathbb{C}_K[[z]] \to \mathbb{C}[[\xi]]$ of the function $\mathcal{F}(z)$

$$\mathcal{A}_*(\mathcal{F}(z)) = \sum_{\lambda \in \mathbb{Z}^n_\geqslant} \frac{f(A\lambda)}{\lambda!}\xi^\lambda,$$

the Borel transformation is the function

$$\mathcal{A}_*(F(z)) = \sum_{\lambda \in \mathbb{Z}^n_\geqslant} \frac{f(A\lambda)}{\xi^\lambda}.$$

It follows from Borel's theorem for multiple power series (see [12], Theorem 3.3.3) that

$$\sigma_{\mathcal{A}_*(\mathcal{F})} = |\mathcal{D}_{\mathcal{A}_*(F)}|, \tag{15}$$

where $\sigma_{\mathcal{A}_*(\mathcal{F})} = \{\tau \in \mathbb{R}^n_> : |\mathcal{A}_*(\mathcal{F})| \leqslant Ce^{\langle \tau, |\xi| \rangle}\}$, and $|\mathcal{D}_{\mathcal{A}_*(F)}|$ is the image of the convergence domain under projection (13) of the function $\mathcal{A}_*(F)$. The set $|\mathcal{D}_{\mathcal{A}_*(F)}|$ possesses the property that, together with each point r, the set $\{|\xi| : |\xi_j| > r_j, j = 1, \ldots, n\}$ also belongs to it. After monomial changes $\tau = \sigma^A,\ \xi = z^A,\ r = R^A$, from equality (15) we obtain $\sigma_{\mathcal{F}} = |\mathcal{D}_F|$. □

Let us formulate a difference version of the problem (8)–(9), which we need in the proof of Theorem 1. On the complex-valued functions $f(x) = f(x_1, \dots, x_n)$ of integer variables $x_1, \dots, x_n$, we define the shift operators δ_j in the variables x_j :

$$\delta_j f(x) = f(x_1, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_n)$$

and polynomial difference operator of the form

$$P(\delta) = \sum_{||\omega||_\nu \leqslant d} c_\omega \delta^\omega,$$

where $\delta^\omega = \delta_1^{\omega_1} \dots \delta_n^{\omega_n}$ and coefficients c_ω are constant.

For m such that $||m||_\nu = d$ and $c_m \neq 0$ we formulate the following problem. For any functions g, φ of integer variables $x = (x_1, \dots, x_n)$, it is required to find a function $f(x)$ satisfying the difference equation

$$P(\delta) f(x) = g(x), \ x \in K \cap \mathbb{Z}^n \tag{16}$$

and the initial-boundary conditions

$$\delta^{a^j k}[f(x) - \varphi(x)]|_{x \in \Gamma_j \cap \mathbb{Z}^n} = 0, \quad 0 \leqslant k < \langle m, \alpha^j \rangle, \ j = 1, \dots, n, \tag{17}$$

where Γ_j is face of the cone K spanned by vectors $a^i, \ i = 1, \dots, j-1, j+1, \dots, n$.

Various versions of the statement of the problem (16)–(17) and the study of the question of its solvability were considered, for example, in [16–19].

4. Proof of the main result

In this section, we present some information from the theory of amoebas of algebraic surfaces, in order to formulate the relation between the generating function of the solution of the Cauchy problem for an inhomogeneous multidimensional difference equation and the generating function of the initial data, and also prove the main result of the work (Theorem 1).

The *Newton polytope* N_P of a polynomial $P(z) = \sum_{\omega \in \Omega} c_\omega z^\omega$ is the convex hull in $\mathbb{R}^n$ of elements of the set Ω.

The *amoeba* $\mathcal{A}_V$ *of an algebraic surface* $V = \{z \in \mathbb{C}^n : P(z) = 0\}$ is a image of the set of zeros V of the polynomial $P(z)$ under the mapping (14).

To prove Theorem 1, we need series expansions of the function $\dfrac{1}{P(z)}$ (see [20]), where $P(z)$ is the characteristic polynomial of the operator $P(D)$.

Each vertex of the Newton polytope N_P of the Laurent polynomial $P(z)$ corresponds to a non-empty connected component E_m of the complement of the amoeba $\mathbb{R}^n \backslash \mathcal{A}_V$, and in the domain $Log^{-1} E_m$ the function $\dfrac{1}{P(z)}$ expands into the Laurent series

$$\frac{1}{P(z)} = \sum_{x \in m + \Lambda_m \cap \mathbb{Z}^n} \frac{\mathcal{P}_m(x)}{z^x}, \tag{18}$$

where Λ_m is the cone constructed on the vectors $m - \omega, \ \omega \in \Omega, \ \Lambda_m \subset K$.

If the point m is a vertex of the polytope N_P, then the coefficients $\mathcal{P}_m(x)$ of the expansion (18) can be obtained as follows: at the first step, we use the expansion in a series of geometric progression

$$\frac{1}{P(z)} = \frac{1}{c_m z^m + \sum\limits_{\alpha\neq m} c_\alpha z^\alpha} = \frac{1}{c_m z^m(1 - \sum\limits_{\alpha\neq m} \tilde{c}_\alpha z^{\alpha-m})} =$$

$$= \frac{1}{c_m z^m}\sum_{k=0}^{\infty}\left(\sum_{\alpha\neq m}\tilde{c}_\alpha z^{\alpha-m}\right)^k,$$

and then, after standard transformations and reduction of similar ones, we obtain an expansion of the form $\frac{1}{P(z)} = \sum\limits_{x\in m+\Lambda_m}\frac{\mathcal{P}_m(x)}{z^x}$, where the series converges in the domain $Log^{-1}E_m$.

The *dual cone* C_m to point m of N_P is defined as follows:

$$C_m = \{s \in \mathbb{R}^n : \max_{x\in N_P}\langle s, x\rangle = \langle s, m\rangle\}.$$

Note that it is *asymptotic*, i.e. together with each point $u \in E_m$, this component also belongs to the affine cone $u + C_m \subset E_m$. If $m \underset{K}{\geqslant} \omega$ and $\Lambda_m \subset K$, then $C_m \supset K^*$; therefore, the image of the convergence domain $|\mathcal{D}_{P^{-1}}|$ in the projection (13) of the series $\frac{1}{P(z)}$, together with each point z_0, also contains points z such that $|z^{a^j}| > |z_0^{a^j}|,\ j = 1,\dots,n$.

Proof of Theorem 1. Let $\mathcal{F}(z) = \sum\limits_{x\in K\cap\mathbb{Z}^n}\frac{f(x)z^x}{\langle x,\alpha\rangle!}$ be the required solution to the problem (8)–(9) for the given initial data $\mathit{\Phi}(z) = \sum\limits_{x\in K\cap\mathbb{Z}^n, x\underset{K}{\ngeqslant} m}\frac{\varphi(x)z^x}{\langle x,\alpha\rangle!}$ and the right-hand side $\mathcal{G}(z) = \sum\limits_{x\in K\cap\mathbb{Z}^n}\frac{g(x)z^x}{\langle x,\alpha\rangle!}$. Its solvability in the class of formal Laurent series $\mathbb{C}_K[[z]]$ was proved in [8] and the proof is based on the statement that $\mathcal{F}(z)$ is a solution to the problem (8)–(9) if and only if $f(x)$ is a solution to the corresponding difference problem (16)–(17), the solvability of which was proved in [15]. To prove the solvability of the differential problem (8)–(9) in the class of L_K-exponential series, we use the fact that the generating function of the solution $f(x)$ and the data $\varphi(x)$, $g(x)$ of the difference problem (16)–(17) $F(z) = \sum\limits_{x\in K\cap\mathbb{Z}^n}\frac{f(x)}{z^x}$, $\Phi(z) = \sum\limits_{x\in K\cap\mathbb{Z}^n, x\underset{K}{\ngeqslant} m}\frac{\varphi(x)}{z^x}$ and $G(z) = \sum\limits_{x\in K\cap\mathbb{Z}^n}\frac{g(x)}{z^x}$ are the Borel transformations of the series $\mathcal{F}(z)$, $\mathit{\Phi}(z)$ and $\mathcal{G}(z)$, respectively. The formula connecting these generating functions for the homogeneous difference problem (16)–(17) is given in [2, 21], its modification for $g(x) \neq 0$ has the following form

$$F(z) = \sum_{\omega\in\Omega} c_\omega z^\omega \frac{1}{P(z)}\Phi_\omega(z) + G(z)\frac{1}{P(z)}. \tag{19}$$

By the condition of the theorem, $\mathit{\Phi}(z)$ and $\mathcal{G}(z)$ are L_K-exponential; therefore, Proposition 1 implies that $\sigma_{\mathit{\Phi}} = |\mathcal{D}_\Phi|$ and $\sigma_{\mathcal{G}} = |\mathcal{D}_G|$, where $\mathcal{D}_\Phi$, $\mathcal{D}_G$ are domains in which the series $\Phi(z)$ and $G(z)$ converge.

Let us prove that $\mathcal{F}(z)$ is an $\mathcal{L}_K$-exponential function. For the Borel transformation of the function $\mathcal{F}$, formula (19) is valid, which yields that the image under the map (13) of the convergence domain of the generating function $F(z)$ of the solution to problem (16)–(17) contains an intersection of the images of the domains of convergence of the generating functions $\Phi_\omega(z)$ of the initial data, the generating function $G(z)$ of the right-hand side and series $1/P(z)$:

$|\mathcal{D}_F| \supset \bigcap\limits_{\omega} |\mathcal{D}_{\Phi_\omega}| \cap |\mathcal{D}_G| \cap |\mathcal{D}_{P^{-1}}|$. For $\omega \in \Omega$ we have $supp\,\Phi_\omega \subset supp\,\Phi$, then the convergence domain of the series Φ_ω can only increase: $|\mathcal{D}_{\Phi_\omega}| \supseteq |\mathcal{D}_\Phi|$, therefore

$$|\mathcal{D}_F| \supset |\mathcal{D}_\Phi| \cap |\mathcal{D}_G| \cap |\mathcal{D}_{P^{-1}}|. \tag{20}$$

Since the cone K^* is asymptotic for both $Log|\mathcal{D}_\Phi|$, $Log|\mathcal{D}_G|$ and $Log|\mathcal{D}_{P^{-1}}|$, the intersection on the right-hand side of (20) is not empty.

Appling the inverse Borel transformation to $F(z)$, yields the function $\mathcal{F}(z)$, and by Proposition 1 for its image $\sigma_{\mathcal{F}}$, we have $\sigma_{\mathcal{F}} = |\mathcal{D}_F|$, that is $\sigma_{\mathcal{F}} \neq \emptyset$ and, therefore, $\mathcal{F}$ is an $\mathcal{L}_K$-exponential function. □

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Аналитическая разрешимость задачи Хермандера и преобразование Бореля кратных рядов Лорана

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Аннотация. В работе формулируется начально-краевая задача Хермандера в классе функций, представимых рядами Лорана с носителями в рациональных конусах. Преобразование Бореля рядов Лорана позволяет установить связь дифференциальной задачи с разностной и доказать теорему о ее глобальной аналитической разрешимости.

Ключевые слова: задача Хермандера, дифференциальный оператор, преобразование Бореля, разностный оператор.

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An Analogue of the Hartogs Lemma for R-Analytic Functions

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Abstract. The paper is devoted to the problem of R-analytic continuation of functions of several real variables which admit R-analytic continuation along parallel sections. We prove an analogue of the well-known Hartogs lemma for R-analytic functions.

Keywords: R-analytic functions, holomorphic functions, plurisubharmonic functions, pluripolar sets, Hartogs series.

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1. Introduction and preliminaries

In this paper we consider the R-analytic continuation of functions of several real variables that admit R-analytic continuation along parallel sections. Regarding to holomorphic functions, the first result in this direction is due to Hartogs [1]: if a holomorphic function $f('z, z_n)$ in the domain $'U \times \{|z_n| < r\} \subset \mathbb{C}^n_{'z} \times \mathbb{C}_{z_n}$, where $'z = (z_1, z_2, \ldots, z_{n-1})$, for each fixed $'z \in ('U)$ by z_n extends holomorphically to the disk $|z_n| < R$, $R > r > 0$, then it is holomorphic with respect to all variables in the domain $'U \times \{|z_n| < R\}$.

The following Forelli's theorem [2] is also directly related to Hartogs theorem: if f is infinitely smooth at a point $0 \in \mathbb{C}^n, f \in C^\infty\{0\}$, and the restrictions $f|_l$ are holomorphic in the disc $U(0,1) = l \bigcap B(0,1)$ for all complex lines $l \ni 0$, then f can be holomorphically extended to the ball $B(0,1) \subset \mathbb{C}^n$.

In a recent paper [3] A. Sadullaev proved the following analogue of Forelli's theorem for R-analytic functions.

Theorem 1. *Let a function $f(x)$, $x = (x_1, x_2, \ldots, x_n)$ be smooth in some neighborhood of the origin $0 \in \mathbb{R}^n$, $f(x) \in C^\infty\{0\}$ and let for any real line $l : x = \lambda t$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in S(0,1) \subset$*

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$\mathbb{R}^n$, $t \in \mathbb{R}$ is a parameter, the restriction $f|_l = f(\lambda t)$ is real-analytic (R-analytic) in the interval $t \in (-1,1)$. Then there is a closed pluripolar set $S \subset B(0,1)$ such that $f(x)$ is R-analytic in $B(0,1) \backslash S$, where $B(0,1) \subset \mathbb{R}^n$ is the unit ball and $S(0,1) = \partial B(0,1)$ is the unit sphere.

Note that the well-known terminology is used here, a set $S \subset \mathbb{R}^n_x$ is called pluripolar if it is pluripolar in the ambient complex space $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z$, $z = x + iy$. An example of a function $f(x_1, x_2) = \dfrac{x_1^{k+1}}{(x_2-1)^2 + x_1^2}$ shows that exact analogues of Forelli's Theorem and Hartogs' Theorem for R-analytic functions are not true. The function $f(x_1, x_2)$ is real-analytic in the domain $\mathbb{R} \times \left\{|x_2| < \frac{1}{2}\right\}$, the restriction $f\left(x_1^0, x_2\right)$ is real-analytic on the whole line $\mathbb{R}$. However, f is not real-analytic at the point $(0,1)$.

The main result of this work is

Theorem 2. *Let a function $f(x) = f('x, x_n)$ satisfy the following conditions:*

1) The function $f('x, x_n)$ is R-analtic in a polycylinder $U = ('U) \times \{|x_n| < r_n\}$, $r_n > 0$, where $'x = (x_1, x_2, \ldots, x_{n-1})$ and

$$'U = \left\{'x \in \mathbb{R}^{n-1} : |x_1| < r_1, |x_2| < r_2, \ldots, |x_{n-1}| < r_{n-1}\right\} =$$
$$= \left\{'x \in \mathbb{R}^{n-1} : -r_1 < x_1 < r_1, -r_2 < x_2 < r_2, \ldots, -r_{n-1} < x_{n-1} < r_{n-1}\right\}.$$

2) For each fixed $('x^0) \in ('U)$ the function $f\left('x^0, x_n\right)$ that is R-analytic in the interval $|x_n| < r_n$, R-analytically continues into a larger interval $|x_n| < R_n$, $R_n > r_n$.

Then there exists a closed pluripolar set $'S \subset ('U)$ such that the function $f('x, x_n)$ R-analytically with respect to all variables $('x, x_n)$ continues into the domain $('U \times \{|x_n| < R_n\}) \backslash ('S \times \{|x_n| \geqslant r_n\})$.

The proof of Theorem 2 essentially uses the method of proving Theorem 1 proposed by A. Sadullaev, namely, the embedding of a real space $\mathbb{R}^n_x \subset \mathbb{C}^n_z$, $z = x + iy$, and the natural holomorphic continuation of R-analytic functions into $\mathbb{C}^n$, the holomorphic continuation of the Hartogs series and methods of pluripotential theory (see [4–5]).

Note that using the local transformation of the pencil of lines $l \ni 0$, into parallel ones, from Theorem 2 one can obtain a proof of Theorem 1.

Real analytic functions were also studied in the work of J. Sichak [6], where he proved that if the function $f(x)$ is smooth in a domain $D \subset \mathbb{R}^n f \in C^\infty(D)$ and for each real line $l : x = x^0 + \lambda t$, $x^0 \in D$, $\lambda \in \mathbb{R}^n$, $|\lambda| = 1$, $t \in \mathbb{R}$, the restriction $f|_\ell$ is R-analytic by t in some neighborhood of zero, then $f(x)$ is R-analytic in D.

2. Domain of holomorphy of Hartogs series

Let $U = ('U) \times U_n$ be a domain in $\mathbb{C}^{n-1}_{'z} \times \mathbb{C}_{z_n}$, where U_n is a disc centered at the point $z_n = 0$ and with a radius $\delta > 0$. If the function $f('z, z_n)$ is holomorphic in U, then it can be expanded in a Hartogs series:

$$f('z, z_n) = \sum_{k=0}^{\infty} c_k('z)\, z_n^k, \qquad (1)$$

where, the coefficients $c_k('z)$ are holomorphic in $'U$ and determined by the formula

$$c_k('z) = \frac{1}{2\pi i} \int\limits_{|\xi|=\delta'} \frac{f('z, \xi)}{\xi^{k+1}} d\xi, \quad 0 < \delta' < \delta,\ k = 0, 1, 2, \ldots. \qquad (2)$$

Then, it is known that if $R('z)$ is the radius of convergence of series (1), then the function $u^*('z) = -\ln R_*('z)$ is plurisubharmonic in $'U$, and the set $\{'z \in ('U) : R_*('z) < R('z)\}$ is pluripolar. Here $R_*('z) = \varliminf_{'w \to 'z} R('w)$ is the lower regularization. Moreover, the series (1) converges uniformly on any compact subset $K \subset\subset ('U) \times \{|z_n| < R_*('z)\}$. The proof of this fact can be found, for example, in [7, 8].

The following lemma, which plays the key role in the proof of Theorem 2, is widely used in the theory of analytic continuation.

Lemma 1. *Let a function $f('z, z_n)$ be holomorphic in the domain $'U \times \{z_n \in \mathbb{C} : |z_n| < \delta\}$, $'U \subset \mathbb{C}^{n-1}$. If for each fixed $'z^0 \in ('U_0)$ from some non-pluripolar set $'U_0 \subset ('U)$ the function $f('z^0, z_n)$ of variable z_n, extends holomorphically to the larger disc $\{z_n \in \mathbb{C} : |z_n| < \Delta\}$, $\Delta \geqslant \delta > 0$, then the function $f('z, z_n)$ holomorphically extends to the domain $\left\{'z \in' U, |z_n| < \delta^{\omega^*('z,'U_0,'U)} \cdot \Delta^{1-\omega^*('z,'U_0,'U)}\right\}$, where $\omega^*('z,'U_0,'U)$ is the well-known plurisubharmonic measure of the set $'U_0$ with respect to the domain $'U$, that is defined by the following*

$$\omega^*('z,'U_0,'U) = \left(\sup\left\{u('z) \in psh('U) : u(z)|_{'U} < 1,\, u(z)|_{'U_0} \leqslant 0\right\}\right)^*.$$

Indeed, if we expand the function $f('z, z_n)$ in a Hartogs series of the form (1) in the domain $'U \times \{z_n \in \mathbb{C} : |z_n| < \delta\}$, then the function $u('z) = -\ln R_*('z)$ is plurisubharmonic in the domain $'U$ and by the conditions of the lemma $u('z)|_{'U} \leqslant -\ln \delta$, $u(z)|_{'U_0} \leqslant -\ln \Delta$. According to the theorem on two constants (see [9], p. 103), we obtain the inequality

$$u('z) \leqslant (1 - \omega^*('z,'U_0,'U)) \cdot (-\ln \Delta) + \omega^*('z,'U_0,'U) \cdot (-\ln \delta).$$

Hence it follows that

$$\ln R_*('z) \geqslant (1 - \omega^*('z,'U_0,'U)) \cdot \ln \Delta + \omega^*('z,'U_0,'U) \cdot \ln \delta,$$

or $R_*(z) \geqslant \delta^{\omega^*('z,'U_0,'U)} \cdot \Delta^{1-\omega^*('z,'U_0,'U)}$. Thus in accordance with above mentioned, the function $f('z, z_n)$ extends holomorphically to the domain

$$'U \times \{|z_n| < R_*('z)\} \supset ('U) \times \left\{|z_n| < \delta^{\omega^*('z,'U_0,'U)} \cdot \Delta^{1-\omega^*('z,'U_0,'U)}\right\}.$$

3. Proof of the main result

Without loss of generality we assume that for each fixed $'x \in ('U)$ the function $f('x, x_n)$ is R-analytic in the interval $(-R_n - \varepsilon, R_n + \varepsilon)$, $\varepsilon > 0$. The proof of the theorem will be implemented in several steps.

Step 1. We embed the real space $\mathbb{R}^n_x$ into the complex space $\mathbb{C}^n_z$, $\mathbb{R}^n_x \subset \mathbb{C}^n_z$, $z = x + iy$. Then, by definition of R-analyticity of a function $f('x, x_n)$, there exists a domain $\hat{U} \subset \mathbb{C}^n$, $\hat{U} \supset U$ and a holomorphic function $F(z) = F('z, z_n) \in O\left(\hat{U}\right)$ such that $F('z, z_n)|_U = f('x, x_n)$.

It follows that from the conditions of the theorem the function $F(z) = F('z, z_n)$ satisfies the following conditions:

1) $F(z) \in O\left(\hat{U}\right)$.

2) For each fixed $'z = ('x) \in ('U)$ the function $F('x, z_n)$ of the variable z_n, can be extended holomorphically into the ellipse of type $E_j : \dfrac{(Rez_n)^2}{R_n^2} + j^2(Imz_n)^2 < 1$, $j \in \mathbb{N}$, such that $E_j \supset\supset \{|x_n| \leqslant R_n\}$ $\forall j \in \mathbb{N}$.

We put $'\hat{U} = \hat{U} \cap \mathbb{C}^{n-1}_{'z}$ and fix a subdomain $'\hat{V} \subset\subset ('\hat{U})$ such that $'V = ('\hat{V}) \cap ('U) \neq \emptyset$. Then there is a circle $\{|z_n| < \sigma\}$, $\sigma > 0$, such that $'\hat{V} \times \{|z_n| < \sigma\} \subset \hat{U}$, i.e. the function $F(z) = F('z, z_n)$ is holomorphic with respect to the $('z, z_n)$ in $'\hat{V} \times \{|z_n| < \sigma\}$. We fix the number $j \in \mathbb{N}$ and denote by $'V_j$ the set of points $'x$ from $'V = ('\hat{V}) \cap ('U)$ for which the function $F('x, z_n)$ of variable z_n extends holomorphically into the ellipse E_j, i.e.

$$'V_j = \{'x \in ('V) : F('x, z_n) \in O(E_j)\}$$

It is obvious that

$$V_j \subset V_{j+1} \ \forall j \in \mathbb{N}$$

and

$$\bigcup_{j=1}^{\infty} ('V_j) = {}'V.$$

Step 2. Since an open non-empty subset $'V \subset \mathbb{R}^{n-1}$ is not pluripolar in $\mathbb{C}^{n-1}$, then there exists a number $j_0 \in \mathbb{N}$ such that for all $j > j_0$ the sets $'V_j \subset ('V)$ will be non-pluripolar in $\mathbb{C}^n$.

Let us fix $j \in \mathbb{N}$, $j > j_0$ and let the function $w = g_j(z_n)$ conformally maps the ellipse E_j into the unit circle $\{|w| < 1\}$, $g_j(0) = 0$. Since the function $F('z, z_n)$ is holomorphic in the neighborhood $'\hat{V} \times \{|z_n| < \sigma\}$, the function $\Phi('z, w) = F('z, g_j^{-1}(w))$ is holomorphic in the domain $'\hat{V} \times g_j^{-1}(\{|z_n| < \sigma\})$. Since $g_j(0) = 0$. there is a number $\delta_j > 0$ such that $('\hat{V}) \times \{|w| < \delta_j\} \subset ('\hat{V}) \times g_j^{-1}(\{|z_n| < \sigma\})$, i.e. the function $\Phi('z, w)$ is holomorphic in the domain $'\hat{V} \times \{|w| < \delta_j\}$. In addition, for each fixed variable $'z = ('x) \in ('V_j)$, the function $\Phi('x, w)$ of the variable w extends holomorphically to the circle $\{|w| < 1\}$.

By Lemma 1, where $\delta = \delta_j$, $\Delta = 1$, the function $\Phi('z, w)$ is holomorphic in the domain

$$\left\{'z \in{}' \hat{V}, |z_n| < \delta_j^{\omega^*('z,'V_j,'\hat{V})}\right\}.$$

Thus, if we substitute into $\Phi('z, w)$ the value $w = g_j(z_n)$, then we obtain that the function $F('z, z_n)$ extends holomorphically to the domain

$$G_j = \left\{('z, z_n) \in \mathbb{C}^n : ('z) \in ('\hat{V}), |g_j(z_n)| < \delta_j^{\omega^*('z,'V_j,'\hat{V})}\right\} \tag{3}$$

Note that if the point $'x \in ('V_j)$ is pluriregular, i.e. $\omega^*('x,' V_j,' \hat{V}) = 0$, then, according to (3), the ellipse $\{'x\} \times \{|g_j(z_n)| < 1\} \subset G_j$. Consequently, the domain G_j contains some neighborhood of the segment $\{'x\} \times [-R_n, R_n]$.

Step 3. By the construction of the domain G_j, F can be extended holomorphically to the domain $G_{'V} = \bigcup_{j=j_0}^{\infty} G_j$ as well. Let us denote by P_j the set of irregular points $'x \in ('V_j)$ and by $P_{'V} = \bigcup_{j=j_0}^{\infty} P_j$ the union of these sets $P_{'V} \subset ('V)$. It is a pluripolar set in $\mathbb{C}^{n-1}_{'z}$. For each fixed point $'z = ('x) \in ('V) \backslash P_{'V}$, the union $G_{'V} = \bigcup_{j=j_0}^{\infty} G_j$ contains a neighborhood of the segment $\{'x\} \times [-R_n, R_n]$.

Step 4. We take a sequence of domains $'\hat{V}_k \subset\subset' \hat{V}_{k+1} \subset\subset' \hat{U} : \bigcup_{k=1}^{\infty} ('\hat{V}_k) =' \hat{U}$ and put $P = \bigcup_{k=1}^{\infty} P_{'V_k}$. Then $P \subset ('U)$ is pluripolar set in $\mathbb{C}^{n-1}_{'z}$. According to Step 3, the function F extends holomorphically to the domain $G = \bigcup_{k=1}^{\infty} G_{'V_k}$, and for each fixed point $'z = ('x) \in ('U) \backslash P$ the union $G = \bigcup_{k=1}^{\infty} G_{'V_k}$ contains a neighborhood of the segment $\{'x\} \times [-R_n, R_n]$. Therefore, for such points the given function $f('x, x_n)$ is R-analytic in the set of variables in the neighborhood of the segment $\{'x\} \times [-R_n, R_n]$.

We note that the complement $S = ['U \times \{|x_n| < R_n\}] \setminus [G \bigcap \mathbb{R}^n]$ is a closed pluripolar set, $S \subset P \times \{|x_n| \geqslant r_n\}$, and the function $f('x, x_n)$ is R-analytically extended to $['U \times \{|x_n| < R_n\}] \setminus S$. The theorem is proved. □

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Об аналоге леммы Гартогса для R-аналитических функций

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Аннотация. Работа посвящена задачам R-аналитического продолжения функций многих действительных переменных, допускающих R-аналитическое продолжение на параллельные сечения. В ней доказывается аналог известной теоремы Гартогса для R-аналитических функций.

Ключевые слова: R-аналитические функции, голоморфные функции, плюрисубгармонические функции, плюриполярные множества, ряды Гартогса.

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On the Cauchy Problem for the Biharmonic Equation

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Abstract. The work is devoted to the study of continuation and stability estimation of the solution of the Cauchy problem for the biharmonic equation in the domain G from its known values on the smooth part of the boundary ∂G. The problem under consideration belongs to the problems of mathematical physics in which there is no continuous dependence of solutions on the initial data. In this work, using the Carleman function, not only the biharmonic function itself, but also its derivatives are restored from the Cauchy data on a part of the boundary of the region. The stability estimates for the solution of the Cauchy problem in the classical sense are obtained.

Keywords: biharmonic equations, Cauchy problem, ill-posed problems, Carleman function, regularized solutions, regularization, continuation formulas.

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Introducnion

Let $x = (x_1, x_2), y = (y_1, y_2) \in R^2$ and G is a bounded simply connected domain in R^2 with boundary ∂G, consisting of compact part $T = \{y_1 \in R : \ a_1 \leqslant y_1 \leqslant b_1\}$ and a smooth arc of the curve $S : y_2 = h(y_1)$ lying in the half-plane $S : y_2 = h(y_1)$. $\bar{G} = G \cup \partial G$, $\partial G = S \cup T$.

In the domain G, consider the equation

$$\Delta^2 U(y) = 0, \ \ y \in G, \tag{1}$$

where $\Delta = \dfrac{\partial^2}{\partial y_1^2} + \dfrac{\partial^2}{\partial y_2^2}$ Laplace operator.

Problem definition. It is required to find the biharmonic function $U(y) = U(y_1, y_2) \in C^4(G) \cap C^3(\bar{G})$, for which the values on the part S of the boundary ∂G are known, i.e.

$$\begin{gathered} U(y_1, y_2)|_S = f_1(y), \quad \Delta U(y_1, y_2)|_S = f_2(y), \\ \left.\frac{\partial U(y_1, y_2)}{\partial n}\right|_S = f_3(y), \quad \left.\frac{\partial(\Delta U(y_1, y_2))}{\partial n}\right|_S = f_4(y), \end{gathered} \tag{2}$$

here $f_j(y) \in C^{4-j}(S), j = 1, 2, 3, 4$ are given functions, and $\dfrac{\partial}{\partial n}$ — operator of differentiation along the outward normal to ∂G.

The considered problem (1)–(2) refers to ill-posed problems of mathematical physics. The true nature of such problems was clarified for the first time in the work of A. N. Tikhonov [4],

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and he pointed out the practical importance of unstable problems, and also showed that if the class of possible solutions is reduced to a compact set, then the stability of the solution follows from the existence and uniqueness.

Formulas that make it possible to find a solution to an elliptic equation in the case when the Cauchy data are known only on a part of the boundary of the domain are called Carleman-type formulas. In [2] Carleman established a formula giving a solution to the Cauchy–Riemann equations in a domain of a special form. Developing his idea, G. M. Goluzin and V. I. Krylov [3] derived a formula for determining the values of analytic functions from data known only on the border on the border section, already for arbitrary domains. They found a formula for restoring a solution from its values on the boundary set of positive Lebesgue measure, and also proposed a new version of the extension formula. The monograph by L. A. Aizenberg [1] is devoted to one-dimensional and multidimensional generalizations of the Carleman formula. A formula of the Carleman type, which uses the fundamental solution of a differential equation with special properties (the Carleman function), was obtained by M. M. Lavrent'ev [7, 8]. In these works, the definition of the Carleman function is given for the case when the Cauchy data are given approximately, and the a scheme of regularization of the Cauchy problem for the Laplace equation is also proposed. Using this method, Sh.Ya.Yarmukhamedov [9, 10] constructed Carleman functions for a wide class of elliptic operators defined in spatial domains of a special form, when part of the boundary of the domain is a hypersurface or a conical surface. It should de noted that the Carleman function proposed by Sh.Yarmukhamedov was also studied by M. Ikehata [11].

The Carleman matrix for the Cauchy–Riemann equation in the case when S is an arbitrary set of positive measure was constructed in [13]. In [14] in classical domains, Carleman's formulas are given that restore the values of a function inside a domain from its values given on a set of positive measure on the skeleton.

The Cauchy problem for linear elliptic differential operators has numerous applications in physics, electrodynamics, fluid mechanics (see [8, 12, 15]). It is known that if the Carleman function is constructed, then using Green's formula one can write the regularized solution explicitly. This implies that the efficiency of constructing the Carleman function is equivalent to constructing a regularized solution to the Cauchy problem.

In [16], a method is proposed for the regularization of the solution of the Cauchy problem for the Laplace equation by introducing a biharmonic operator with a small parameter, and it is shown that if a solution to the original problem exists, then the difference between the spectral expansions of the solutions of the original and regularized equations tends to zero as the parameter tends regularization to zero in the space of square-summable functions. In recent years, many numerical methods have been presented solving the Cauchy problem for elliptic equations. In the paper [18] L. Marin investigated the iterative method of fundamental solutions algorithms together with the Tikhonov regularization method.

An estimate for the conditional stability of a boundary value problem for a fourth-order elliptic type equation in rectangular domains was obtained in [19].

In [20], using the Carleman function, not only the harmonic function itself, but also its derivatives for the Laplace equation are reconstructed from the Cauchy data on a part of the boundary of the domain.

Note that when solving applied problems, one should find the approximate values of the solution $U(x)$ and its derivative $\dfrac{\partial U(x)}{\partial x_i}$, $x \in G$, $i = 1, 2$.

In this paper, we construct a family of functions $U(x,\sigma,f_{k\delta})=U_{\sigma\delta}(x)$ and $\dfrac{\partial U(x,\sigma,f_{k\delta})}{\partial x_i}=$ $=\dfrac{\partial U_{\sigma\delta}(x)}{\partial x_i}$, $k=1,2,3,4;\ i=1,2$ depending on a parameter σ and prove that with a special choice of parameter $\sigma=\sigma(\delta)$ the family $U_{\sigma\delta}(x)$ and $\dfrac{\partial U_{\sigma\delta}(x)}{\partial x_i}$ at $\delta\to 0$ converges at each point $x\in G$ to the solution $U(x)$ and its derivative $\dfrac{\partial U(x)}{\partial x_i}$, respectively. The family of functions $U(x,\sigma,f_{k\delta})$ and $\dfrac{\partial U(x,\sigma,f_{k\delta})}{\partial x_i}$, $i=1,2$ with indicated properties is said to be a regularized solution by M. M. Lavrent'ev [7]. If, under the indicated conditions, instead of the Cauchy data, their continuous approximations with a given deviation in the uniform metric are given, then an explicit regularization formula is proposed. In this case, it is assumed that the solution is bounded on the part T of the boundary.

The proof of these results is based on the construction in an explicit form of the fundamental solution of the biharmonic equation depending on a positive parameter, disappearing along with its derivatives as the parameter tends to infinity on T when the pole of the fundamental solution lies in the half-plane $y_2>0$.

1. Construction of the Carleman function

Let us define the function $\Phi_\sigma(x,y)$ (from [10]) as follows

$$-2\pi e^{\sigma x_2^2}\Phi_\sigma(x,y)=\int_0^\infty Im\left[\frac{e^{\sigma w^2}}{w-x_2}\right]\frac{udu}{\sqrt{u^2+\alpha^2}}. \tag{3}$$

Separating the imaginary part of the function $\Phi_\sigma(x,y)$, we have

$$\begin{aligned}\Phi_\sigma(x,y)=\frac{1}{2\pi}e^{-\sigma(\alpha^2+x_2^2-y_2^2)}\Bigg[\int_0^\infty \frac{e^{-\sigma u^2}\cos 2\sigma y_2\sqrt{u^2+\alpha^2}udu}{u^2+r^2}-\\ -\int_0^\infty \frac{e^{-\sigma u^2}(y_2-x_2)\sin 2\sigma y_2\sqrt{u^2+\alpha^2}}{u^2+r^2}\frac{udu}{\sqrt{u^2+\alpha^2}}\Bigg],\end{aligned} \tag{4}$$

where $y'=(y_1,0)$, $x'=(x_1,0)$, $r=|y-x|$, $\alpha=|y'-x'|$, $\alpha>0$, $\sigma>0$, $w=i\sqrt{u^2+\alpha^2}+y_2$, $u\geqslant 0$.

It the paper [10], one has proved that the function $\Phi_\sigma(x,y)$ defined by the equalities (3) with $\sigma>0$ is presentable in the from

$$\Phi_\sigma(x,y)=F(r)+G_\sigma(x,y), \tag{5}$$

where $F(r)=\dfrac{1}{2\pi}\ln\dfrac{1}{r}$, $G_\sigma(x,y)$ is harmonic function with respect to y in R^2, including $y=x$. It follows that the function $\Phi_\sigma(x,y)$ for any $\sigma>0$ in y is a fundamental solution of the Laplace equation. The fundamental solution $\Phi_\sigma(x,y)$ with the indicated property is said to be the Carleman function for the half-space [7].

Therefore, for the function $U(y)=U(y_1,y_2)\in C^4(G)\cap C^3(\bar G)$ and any $x\in G$ the following integral Green formula holds true [17]:

$$
\begin{aligned}
U(x) = &\int_{\partial G} \left[U(y) \frac{\partial \left(\Delta L(x,y) \right)}{\partial n} - \Delta L(x,y) \frac{\partial U(y)}{\partial n} \right] dS_y + \\
&+ \int_{\partial G} \left[\Delta U(y) \frac{\partial L(x,y)}{\partial n} - L(x,y) \frac{\partial \left(\Delta U(y) \right)}{\partial n} \right] dS_y, \ x \in G,
\end{aligned} \tag{6}
$$

where $L(x,y) = r^2 \ln \dfrac{1}{r}$ is the fundamental solution to the equation (1).

Since $\Phi_\sigma(x,y)$ is represented in the form (5), then in the integral representation (6) $L(x,y)$ replacing the function $L_\sigma(x,y) = r^2 \Phi_\sigma(x,y)$, we have

$$
\begin{aligned}
U(x) = &\int_{\partial G} \left[U(y) \frac{\partial \left(\Delta L_\sigma(x,y) \right)}{\partial n} - \Delta L_\sigma(x,y) \frac{\partial U(y)}{\partial n} \right] dS_y + \\
&+ \int_{\partial G} \left[\Delta U(y) \frac{\partial L_\sigma(x,y)}{\partial n} - L_\sigma(x,y) \frac{\partial \left(\Delta U(y) \right)}{\partial n} \right] dS_y, \ \ x \in G.
\end{aligned} \tag{7}
$$

2. The formula of continuation and regularization by M. M. Lavrent'ev

We denote

$$
\begin{aligned}
U_\sigma(x) = &\int_S \left[f_1(y) \frac{\partial \left(\Delta L_\sigma(x,y) \right)}{\partial n} - f_3(y) \Delta L_\sigma(x,y) \right] dS_y + \\
&+ \int_S \left[f_2(y) \frac{\partial L_\sigma(x,y)}{\partial n} - f_4(y) L_\sigma(x,y) \right] dS_y, \ \ x \in G.
\end{aligned} \tag{8}
$$

The main result of this paper is contained in the following theorem.

Theorem 1. *Let the function $U(y) = U(y_1, y_2) \in C^4(G) \cap C^3(\bar{G})$ on the part S of boundary ∂G satisfy the conditions* (2)*, and on the part T of boundary ∂G the inequality be fulfilled*

$$
|U(y)| + \left| \frac{\partial U(y)}{\partial n} \right| + |\Delta U(y)| + \left| \frac{\partial \Delta U(y)}{\partial n} \right| \leqslant M, \quad y \in T, \quad M > 0. \tag{9}
$$

Then, for any $x \in G$ and $\sigma > 0$, the estimates hold true

$$
|U(x) - U_\sigma(x)| \leqslant \varphi(\sigma, x_2) M e^{-\sigma x_2^2}, \tag{10}
$$

$$
\left| \frac{\partial U(x)}{\partial x_i} - \frac{\partial U_\sigma(x)}{\partial x_i} \right| \leqslant \varphi_i(\sigma, x_2) M e^{-\sigma x_2^2}, \ i = 1, 2, \tag{11}
$$

where

$$
\begin{aligned}
\varphi(\sigma, x_2) = &\frac{23\sqrt{\sigma\pi}}{4\sigma} + \left(\frac{3\sqrt{\sigma\pi}}{4\sigma} + 20\sqrt{\sigma\pi} + 8\sqrt{\sigma\pi}\sigma \right) x_2 + \\
&+ \left(2\sqrt{\sigma\pi} + 4\sqrt{\sigma\pi}\sigma \right) x_2^2 + \frac{9\sqrt{\sigma\pi}}{2} x_2^3 + \frac{9\sqrt{\sigma\pi}}{\sigma x_2},
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\varphi_1(\sigma, x_2) = &10 + \frac{1}{\sigma} + \frac{13\sqrt{\pi}}{\sqrt{\sigma}} + \frac{165\sqrt{\pi\sigma}}{2} + \frac{4\sqrt{\pi}\sigma^2}{\sqrt{\sigma}} + \frac{4\sqrt{\pi}\sigma^3}{\sqrt{\sigma}} + \\
&+ \left(44\sigma + \frac{\sqrt{\pi}}{4\sqrt{\sigma}} + 3 + \frac{2\sqrt{\pi}\sigma}{\sqrt{\sigma}} \right) x_2 + \left(\frac{17}{2} + \frac{9\sqrt{\pi}\sigma}{2\sqrt{\sigma}} + \frac{4\sqrt{\pi}\sigma^2}{\sqrt{\sigma}} + 4\sigma \right) x_2^2 + \\
&+ 9\sigma x_2^3 + \left(\frac{66\sqrt{\pi}}{\sqrt{\sigma}} + \frac{4\sqrt{\pi}\sigma}{\sqrt{\sigma}} + \frac{1}{2\sigma} + 8 \right) \frac{1}{x_2} + \frac{20\sqrt{\pi}}{\sqrt{\sigma} x_2^2} + 16\sigma,
\end{aligned} \tag{13}
$$

$$\varphi_2(\sigma,x_2)=\frac{21\sqrt{\pi}}{2\sqrt{\sigma}}+\frac{78\sqrt{\pi}\sigma}{\sqrt{\sigma}}+\left(\sqrt{\sigma\pi}+\frac{1}{2}+\frac{3\sqrt{\pi}}{4\sqrt{\sigma}}+\frac{4\sqrt{\pi}\sigma^2}{\sqrt{\sigma}}\right)x_2+ \\ +\left(29\sqrt{\sigma\pi}+58\sqrt{\sigma\pi}\sigma\right)x_2^2+\frac{\sqrt{\sigma\pi}}{2}x_2^3+10\sqrt{\sigma\pi}\sigma x_2^4+\frac{60\sqrt{\pi}}{\sqrt{\sigma}x_2^2}. \tag{14}$$

Proof. Let us prove the inequality (10). Denote by $I_\sigma(x)$ the difference

$$I_\sigma(x)=U(x)-U_\sigma(x).$$

Further, from (7) and (8), we have

$$I_\sigma(x)=\int_T\left[U(y)\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}-\Delta L_\sigma(x,y)\frac{\partial U(y)}{\partial n}\right]dS_y+ \\ +\int_T\left[\Delta U(y)\frac{\partial L_\sigma(x,y)}{\partial n}-L_\sigma(x,y)\frac{\partial\left(\Delta U(y)\right)}{\partial n}\right]dS_y,\quad x\in G. \tag{15}$$

From this and the inequality (9) we obtain

$$|I_\sigma(x)|=\left|\int_T\left[U(y)\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}-\Delta L_\sigma(x,y)\frac{\partial U(y)}{\partial n}\right]dS_y+\right. \\ \left.+\int_T\left[\Delta U(y)\frac{\partial L_\sigma(x,y)}{\partial n}-L_\sigma(x,y)\frac{\partial\left(\Delta U(y)\right)}{\partial n}\right]dS_y\right|\leqslant MN_\sigma(x),$$

where

$$N_\sigma(x)=\int_T\left[\left|\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}\right|+|\Delta L_\sigma(x,y)|+\left|\frac{\partial L_\sigma(x,y)}{\partial n}\right|+|L_\sigma(x,y)|\right]dS_y=J_1+J_2+J_3+J_4.$$

To show that the estimate (10) is valid, we prove the following

$$N_\sigma(x)\leqslant\varphi(\sigma,x_2)e^{-\sigma x_2^2},\ \sigma>0. \tag{16}$$

According to (4), we have

$$L_\sigma(x,y)=r^2\Phi_\sigma(x,y)=r^2\left\{\frac{1}{2\pi}e^{-\sigma(\alpha^2+x_2^2-y_2^2)}\left[\int_0^\infty\frac{e^{-\sigma u^2}\cos 2\sigma y_2\sqrt{u^2+\alpha^2}udu}{u^2+r^2}-\right.\right. \\ \left.\left.-\int_0^\infty\frac{e^{-\sigma u^2}(y_2-x_2)\sin 2\sigma y_2\sqrt{u^2+\alpha^2}}{u^2+r^2}\frac{udu}{\sqrt{u^2+\alpha^2}}\right]\right\}.$$

Hence, setting $y_2=0$ we get

$$L_\sigma(x,y)=\left((y_1-x_1)^2+x_2^2\right)\left\{\frac{1}{2\pi}e^{-\sigma x_2^2}\int_0^\infty\frac{e^{-\sigma(u^2+\alpha^2)}udu}{u^2+(y_1-x_1)^2+x_2^2}\right\}.$$

Now we estimate the following integral

$$J_1=\int_T|L_\sigma(x,y)|\,dS_y\leqslant\int_{a_1}^{b_1}\left\{\frac{(y_1-x_1)^2+x_2^2}{2\pi}e^{-\sigma x_2^2}\int_0^\infty\frac{|u|\,e^{-\sigma(u^2+(y_1-x_1)^2)}du}{u^2+(y_1-x_1)^2+x_2^2}\right\}dy_1\leqslant\frac{\sqrt{\pi}}{2\sqrt{\sigma}}e^{-\sigma x_2^2}.$$

Here, in the estimation, we used the inequality

$$\frac{(y_1-x_1)^2+x_2^2}{u^2+(y_1-x_1)^2+x_2^2}<1$$

and introduced polar coordinate systems. Considering

$$\frac{\partial L_\sigma(x,y)}{\partial n} = \frac{\partial L_\sigma(x,y)}{\partial y_1}\cos\gamma + \frac{\partial L_\sigma(x,y)}{\partial y_2}\sin\gamma,$$

get

$$\frac{\partial\left(L_\sigma(x,y)\right)}{\partial y_2} = \frac{\partial}{\partial y_2}\left[r^2\Phi_\sigma(x,y)\right] = 2(y_2-x_2)\Phi_\sigma(x,y) + r^2\frac{\partial\Phi_\sigma(x,y)}{\partial y_2},$$

here $\cos\gamma,\ \sin\gamma$ are the coordinates of the unit outward normal n at the point y of the boundary ∂G.

Further, setting $y_2=0$, we estimate the following integral

$$J_2 = \int_T\left|\frac{\partial L_\sigma(x,y)}{\partial n}\right| dS_y \leqslant \int_{a_1}^{b_1}\left\{\frac{x_2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\, e^{-\sigma(y^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du+\right.$$
$$+\frac{x_2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\left|(y_1-x_1)^2+x_2^2\right| e^{-\sigma(y^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du+$$
$$\left.+\frac{\sigma x_2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\left|(y_1-x_1)^2+x_2^2\right| e^{-\sigma(u^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du\right\} dy_1$$
$$\leqslant \left(\frac{3\sqrt{\pi}x_2}{4\sqrt{\sigma}} + \frac{\sqrt{\pi}}{4\sqrt{\sigma}} + \frac{\sqrt{\sigma\pi}}{2}x_2^3\right)e^{-\sigma x_2^2}.$$

In what follows, we need the following expressions

$$\Delta L_\sigma(x,y) = \Delta\left(r^2\Phi_\sigma(x,y)\right) = \frac{\partial}{\partial y_1^2}\left[r^2\Phi_\sigma(x,y)\right] + \frac{\partial}{\partial y_2^2}\left[r^2\Phi_\sigma(x,y)\right] =$$
$$= 4\Phi_\sigma(x,y) + 4(y_1-x_1)\frac{\partial\Phi_\sigma(x,y)}{\partial y_1} + 4(y_2-x_2)\frac{\partial\Phi_\sigma(x,y)}{\partial y_2}.$$

$$\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} = \frac{\partial}{\partial y_2}\left[\Delta\left(r^2\Phi_\sigma(x,y)\right)\right] =$$
$$= 8\frac{\partial\Phi_\sigma(x,y)}{\partial y_2} + 4(y_1-x_1)\frac{\partial^2\Phi_\sigma(x,y)}{\partial y_1\partial y_2} + 4(y_2-x_2)\frac{\partial^2\Phi_\sigma(x,y)}{\partial y_2^2},$$

where

$$\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} = \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial y_1}\cos\gamma + \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial y_2}\sin\gamma.$$

In these expressions, flat $y_2=0$, estimating, we get

$$J_3 = \int_T\left|\Delta L_\sigma(x,y)\right| dS_y \leqslant \int_{a_1}^{b_1}\left\{\frac{2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\, e^{-\sigma(u^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du+\right.$$
$$+\frac{4\sigma(y_1-x_1)^2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\, e^{-\sigma(u^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du+$$
$$+\frac{4(y_1-x_1)^2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\, e^{-\sigma(u^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du + \frac{4x_2^2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\, e^{-\sigma(u^2+(y_1-x_1)^2)}}{\left(u^2+(y_1-x_1)^2+x_2^2\right)^2}du+$$
$$\left.+\frac{4\sigma x_2}{\pi}e^{-\sigma x_2^2}\int_0^\infty \frac{|u|\, e^{-\sigma(u^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du\right\} dy_1 \leqslant \left(\frac{5\sqrt{\pi}}{\sqrt{\sigma}} + 2\sqrt{\sigma\pi}x_2^2\right)e^{-\sigma x_2^2}.$$

$$J_4 = \int_T \left| \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} \right| dS_y \leqslant \left[\left(20\sqrt{\sigma\pi} + 8\sqrt{\sigma\pi}\sigma\right) x_2 + 4\sqrt{\sigma\pi}\sigma x_2^2 + 4\sqrt{\sigma\pi} x_2^3 + \frac{9\sqrt{\sigma\pi}}{\sigma x_2} \right] e^{-\sigma x_2^2}.$$

In estimating the integrals, we used the inequality

$$\frac{|u|\,|y_1 - x_1|}{u^2 + (y_1 - x_1)^2 + x_2^2} < 1.$$

Taking into account the obtained estimates, we have

$$\int_T \left[\left| \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} \right| + |\Delta L_\sigma(x,y)| + \left| \frac{\partial L_\sigma(x,y)}{\partial n} \right| + |L_\sigma(x,y)| \right] dS_y \leqslant$$
$$\leqslant \left[\frac{23\sqrt{\sigma\pi}}{4\sigma} + \left(\frac{3\sqrt{\sigma\pi}}{4\sigma} + 20\sqrt{\sigma\pi} + 8\sqrt{\sigma\pi}\sigma \right) x_2 + \right. \tag{17}$$
$$\left. + \left(2\sqrt{\sigma\pi} + 4\sqrt{\sigma\pi}\sigma\right) x_2^2 + \frac{9\sqrt{\sigma\pi}}{2} x_2^3 + \frac{9\sqrt{\sigma\pi}}{\sigma x_2} \right] e^{-\sigma x_2^2}.$$

From (17) follows the proof of the inequality (10).

Let us prove the inequality (11). Differentiating the equalities (7) and (8) by x_i, $i = 1,2$ we get

$$\frac{\partial U(x)}{\partial x_i} = \int_{\partial G} \left[U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] - \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial x_i} \frac{\partial U(y)}{\partial n} \right] dS_y +$$
$$+ \int_{\partial G} \left[\Delta U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] - \frac{\partial L_\sigma(x,y)}{\partial x_i} \frac{\partial\left(\Delta U(y)\right)}{\partial n} \right] dS_y,$$

$$\frac{\partial U_\sigma(x)}{\partial x_i} = \int_S \left[U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] - \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial x_i} \frac{\partial\left(\Delta U(y)\right)}{\partial n} \right] dS_y +$$
$$+ \int_S \left[\Delta U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] - \frac{\partial L_\sigma(x,y)}{\partial x_i} \frac{\partial\left(\Delta U(y)\right)}{\partial n} \right] dS_y.$$

Denote by $I_{i\sigma}(x)$ the difference of the derivatives

$$I_{i\sigma}(x) = \frac{\partial U(x)}{\partial x_i} - \frac{\partial U_\sigma(x)}{\partial x_i} = \int_T \left[U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] - \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial x_i} \frac{\partial U(y)}{\partial n} \right] dS_y +$$
$$+ \int_T \left[\Delta U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] - \frac{\partial L_\sigma(x,y)}{\partial x_i} \frac{\partial\left(\Delta U(y)\right)}{\partial n} \right] dS_y.$$

From this and the inequality (9) it follows

$$|I_{i\sigma}(x)| = \left| \int_T \left[U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] - \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial x_i} \frac{\partial U(y)}{\partial n} \right] dS_y + \right.$$
$$\left. + \int_T \left[\Delta U(y) \frac{\partial}{\partial x_i} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] - \frac{\partial L_\sigma(x,y)}{\partial x_i} \frac{\partial\left(\Delta U(y)\right)}{\partial n} \right] dS_y \right| \leqslant M N_{i\sigma}(x),$$

where

$$N_{i\sigma}(x) = \int_T \left[\left| \frac{\partial}{\partial x_i} \left[\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] \right| + \left| \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial x_i} \right| + \left| \frac{\partial}{\partial x_i} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] \right| + \left| \frac{\partial L_\sigma(x,y)}{\partial x_i} \right| \right] dS_y.$$

To show that the estimate (11) is valid, we prove the following inequality

$$N_{i\sigma}(x) \leqslant \varphi_i(\sigma, x_2) e^{-\sigma x_2^2}, \;\; \sigma > 0. \tag{18}$$

For $i = 1$, we have

$$N_{1\sigma}(x) = \int_T \left[\left| \frac{\partial}{\partial x_1} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] \right| + \left| \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial x_1} \right| + \left| \frac{\partial}{\partial x_1} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] \right| + \left| \frac{\partial L_\sigma(x,y)}{\partial x_1} \right| \right] dS_y.$$

We denote

$$Z_1 = \frac{\partial L_\sigma(x,y)}{\partial x_1} = \frac{\partial}{\partial x_1} \left[r^2 \Phi_\sigma(x,y) \right] = r^2 \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - 2(y_1 - x_1)\Phi_\sigma(x,y), \tag{19}$$

$$Z_2 = \frac{\partial}{\partial x_1} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] = \frac{\partial}{\partial x_1} \left[\frac{\partial L_\sigma(x,y)}{\partial y_1} \cos\gamma + \frac{\partial L_\sigma(x,y)}{\partial y_2} \sin\gamma \right],$$

$$\begin{gathered} \frac{\partial}{\partial x_1} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] = \frac{\partial}{\partial x_1} \left[\frac{\partial}{\partial y_2} \left[r^2 \Phi_\sigma\left(x,y\right) \right] \right] = \\ = 2(y_2 - x_2) \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - 2(y_1 - x_1) \frac{\partial \Phi_\sigma(x,y)}{\partial y_2} + r^2 \frac{\partial^2 \Phi_\sigma(x,y)}{\partial x_1 \partial y_2}. \end{gathered} \tag{20}$$

$$\begin{gathered} Z_3 = \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial x_1} = \frac{\partial}{\partial x_1} \left[\Delta \left(r^2 \Phi_\sigma(x,y) \right) \right] = \\ = 4 \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - 4 \frac{\partial \Phi_\sigma(x,y)}{\partial y_1} + 4(y_1 - x_1) \frac{\partial^2 \Phi_\sigma(x,y)}{\partial x_1 \partial y_1} + 4(y_2 - x_2) \frac{\partial^2 \Phi_\sigma(x,y)}{\partial x_1 \partial y_2}. \end{gathered} \tag{21}$$

$$Z_4 = \frac{\partial}{\partial x_1} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] = \frac{\partial}{\partial x_1} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial y_1} \cos\gamma + \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial y_2} \sin\gamma \right],$$

$$\begin{gathered} \frac{\partial}{\partial x_1} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] = \frac{\partial}{\partial x_1} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial y_2} \right] = 8 \frac{\partial^2 \Phi_\sigma(x,y)}{\partial x_1 \partial y_2} - 4 \frac{\partial^2 \Phi_\sigma(x,y)}{\partial y_1 \partial y_2} + \\ + 4(y_1 - x_1) \frac{\partial^3 \Phi_\sigma(x,y)}{\partial x_1 \partial y_1 \partial y_2} + 4(y_2 - x_2) \frac{\partial^3 \Phi_\sigma(x,y)}{\partial x_1 \partial y_2^2}. \end{gathered} \tag{22}$$

When $\frac{\partial}{\partial x_1} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right]$ and $\frac{\partial}{\partial x_1} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right]$, $\cos\gamma$, $\sin\gamma$ are the coordinates of the unit outward normal n at the point y of the boundary ∂G.

In (19), (20), (21), (22), setting $y_2 = 0$ and estimating the resulting integrals, we have:

$$\begin{gathered} N_{1\sigma}(x) \leqslant \Bigg(10 + \frac{1}{\sigma} + \frac{13\sqrt{\pi}}{\sqrt{\sigma}} + \frac{165\sqrt{\sigma\pi}}{2} + 16\sigma + \frac{4\sqrt{\pi}\sigma^2}{\sqrt{\sigma}} + \frac{4\sqrt{\pi}\sigma^3}{\sqrt{\sigma}} + \\ + \left(44\sigma + \frac{\sqrt{\pi}}{4\sqrt{\sigma}} + 2\sqrt{\sigma\pi} + 3 \right) x_2 + \left(\frac{17}{2} + \frac{9\sqrt{\sigma\pi}}{2} + 4\sqrt{\sigma\pi}\sigma + 4\sigma \right) x_2^2 + \\ + 9\sigma x_2^3 + \left(\frac{66\sqrt{\pi}}{\sqrt{\sigma}} + 4\sqrt{\sigma\pi} + \frac{1}{2\sigma} + 8 \right) \frac{1}{x_2} + \frac{20\sqrt{\pi}}{\sqrt{\sigma} x_2^2} \Bigg) e^{-\sigma x_2^2}. \end{gathered} \tag{23}$$

The inequality (18) is proved for $i = 1$. Now let us prove the inequality (18) for $i = 2$.

Taking into account (15) we have

$$N_{2\sigma}(x) = \int_T \left[\left| \frac{\partial}{\partial x_2} \left[\frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right] \right| + \left| \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial x_2} \right| + \left| \frac{\partial}{\partial x_2} \left[\frac{\partial L_\sigma(x,y)}{\partial n} \right] \right| + \left| \frac{\partial L_\sigma(x,y)}{\partial x_2} \right| \right] dS_y.$$

We introduce the following designation

$$\Phi_1 = \frac{\partial L_\sigma(x,y)}{\partial x_2} = \frac{\partial}{\partial x_2} \left[r^2 \Phi_\sigma(x,y) \right] = r^2 \frac{\partial \Phi_\sigma(x,y)}{\partial x_2} - 2(y_2 - x_2)\Phi_\sigma(x,y), \tag{24}$$

$$\Phi_2 = \frac{\partial}{\partial x_2}\left[\frac{\partial L_\sigma(x,y)}{\partial n}\right] = \frac{\partial}{\partial x_2}\left[\frac{\partial L_\sigma(x,y)}{\partial y_1}\cos\gamma + \frac{\partial L_\sigma(x,y)}{\partial y_2}\sin\gamma\right],$$

$$\begin{gathered}\frac{\partial}{\partial x_2}\left[\frac{\partial L_\sigma(x,y)}{\partial n}\right] = \frac{\partial}{\partial x_2}\left[\frac{\partial}{\partial y_2}\left[r^2\Phi_\sigma(x,y)\right]\right] = \\ = -2\Phi_\sigma(x,y) + 2(y_2-x_2)\frac{\partial\Phi_\sigma(x,y)}{\partial x_2} - 2(y_2-x_2)\frac{\partial\Phi_\sigma(x,y)}{\partial y_2} + r^2\frac{\partial^2\Phi_\sigma(x,y)}{\partial x_2\partial y_2}.\end{gathered} \tag{25}$$

$$\begin{gathered}\Phi_3 = \frac{\partial(\Delta L_\sigma(x,y))}{\partial x_2} = \frac{\partial}{\partial x_2}\left[\Delta\left(r^2\Phi_\sigma(x,y)\right)\right] = 4\frac{\partial\Phi_\sigma(x,y)}{\partial x_2} + \\ 4(y_1-x_1)\frac{\partial\Phi_\sigma(x,y)}{\partial y_1\partial x_2} - 4\frac{\partial\Phi_\sigma(x,y)}{\partial y_2} + 4(y_2-x_2)\frac{\partial^2\Phi_\sigma(x,y)}{\partial x_2\partial y_2}.\end{gathered} \tag{26}$$

$$\Phi_4 = \frac{\partial}{\partial x_2}\left[\frac{\partial(\Delta L_\sigma(x,y))}{\partial n}\right] = \frac{\partial}{\partial x_2}\left[\frac{\partial(\Delta L_\sigma(x,y))}{\partial y_1}\cos\gamma + \frac{\partial(\Delta L_\sigma(x,y))}{\partial y_2}\sin\gamma\right],$$

$$\begin{gathered}\frac{\partial}{\partial x_2}\left[\frac{\partial(\Delta L_\sigma(x,y))}{\partial n}\right] = \frac{\partial}{\partial x_2}\left[\frac{\partial(\Delta L_\sigma(x,y))}{\partial y_2}\right] = 8\frac{\partial^2\Phi_\sigma(x,y)}{\partial x_2\partial y_2} - 4\frac{\partial^2\Phi_\sigma(x,y)}{\partial y_2^2} + \\ +4(y_1-x_1)\frac{\partial^3\Phi_\sigma(x,y)}{\partial x_2\partial y_1\partial y_2} + 4(y_2-x_2)\frac{\partial^3\Phi_\sigma(x,y)}{\partial x_2\partial y_2^2}.\end{gathered} \tag{27}$$

In (24), (25), (26), (27), setting $y_2 = 0$ and estimating the resulting integrals, we have

$$\begin{gathered}N_{2\sigma}(x) \leqslant \left(\frac{21\sqrt{\pi}}{2\sqrt{\sigma}} + \frac{78\sqrt{\pi}\sigma}{\sigma} + \left(\sqrt{\sigma\pi} + \frac{3\sqrt{\pi}}{4\sqrt{\sigma}} + \frac{4\sqrt{\pi}\sigma^2}{\sqrt{\sigma}} + \frac{1}{2}\right)x_2 + \right. \\ \left. + \left(29\sqrt{\sigma\pi} + \frac{58\sqrt{\pi}\sigma^2}{\sqrt{\sigma}}\right)x_2^2 + \frac{\sqrt{\sigma\pi}}{2}x_2^3 + \frac{10\sqrt{\pi}\sigma^2}{\sqrt{\sigma}}x_2^4 + \frac{60\sqrt{\pi}}{\sqrt{\sigma}x_2^2}\right)e^{-\sigma x_2^2}.\end{gathered} \tag{28}$$

The inequality (18) is proved for $i = 2$.

From (23) and (28) follows the proof of the inequality (11). Theorem 1 is proved. □

Corollary 1. *With each $x \in G$, the equality holds true*

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x), \quad \lim_{\sigma\to\infty}\frac{\partial U_\sigma(x)}{\partial x_i} = \frac{\partial U(x)}{\partial x_i}, \quad i = 1, 2.$$

Let us denote

$$\bar{G}_\varepsilon = \left\{(x_1,x_2) \in G,\ a > x_2 \geqslant \varepsilon,\ a = \max_T h(x_1),\ 0 < \varepsilon < a\right\}.$$

It is easy to see that the set $\bar{G}_\varepsilon \subset G$ is compact.

Corollary 2. *If $x \in \bar{G}_\varepsilon$, then the family of functions $\{U_\sigma(x)\}$ and $\left\{\frac{\partial U_\sigma(x)}{\partial x_i}\right\}$ converges uniformly for $\sigma \to \infty$, i.e.:*

$$U_\sigma(x) \rightrightarrows U(x), \quad \frac{\partial U_\sigma(x)}{\partial x_i} \rightrightarrows \frac{\partial U(x)}{\partial x_i}, \quad i = 1, 2.$$

It should be noted that the sets $\Pi_\varepsilon = G\backslash\bar{G}_\varepsilon$ present the boundary lever of this problem, as in the theory of singular perturbations, where there is no uniform convergence.

3. An estimate of the stability of the solution to the Cauchy problem

Consider the set

$$E=\left\{U\in C^4(G)\cap C^3(\bar G):\ |U(y)|+\left|\left\{\frac{\partial U(y)}{\partial n}\right\}\right|+|\Delta U(y)|+\left|\left\{\frac{\partial \Delta U(y)}{\partial n}\right\}\right|\leqslant M,\ M>0,\ y\in T\right\}.$$

We put

$$\max_T h(y_1)=a,\quad \max_T\sqrt{1+\left(\frac{dh}{dy_1}\right)^2}=b.$$

Theorem 2. *Let the function $U(y)\in E$, satisfy the equations* (1) *and on the part S of the boundary of the domain G the inequality*

$$|U(y)|+\left|\left\{\frac{\partial U(y)}{\partial n}\right\}\right|+|\Delta U(y)|+\left|\left\{\frac{\partial \Delta U(y)}{\partial n}\right\}\right|\leqslant\delta,\quad y\in S. \tag{29}$$

Then, for any $x\in G$ and $\sigma>0$, the following estimate holds

$$|U(x)|\leqslant\Psi(\sigma,x_2)\,M^{1-\frac{x_2^2}{a^2}}\delta^{\frac{x_2^2}{a^2}}, \tag{30}$$

where $\Psi(\sigma,x_2)=\max\left(\varphi(\sigma,x_2),\psi(\sigma,x_2)\right)$,

$$\psi(\sigma,x_2)=\frac{3b}{\sigma}+\frac{19ab\sqrt{\sigma\pi}}{\sigma}+30a^2b+\frac{97ab(a-x_2)}{2}+\frac{ab\sqrt{\sigma\pi}}{2}+\frac{3ab}{2\sigma}+4a^2b\sqrt{\sigma\pi}+$$
$$+\frac{b\sqrt{\sigma\pi}}{4\sigma}(a-x_2)+\frac{21b\sqrt{\sigma\pi}}{4\sigma}+20ab+8ab\sigma\sqrt{\sigma\pi}(a-x_2)+\frac{5ab\sqrt{\sigma\pi}(a-x_2)}{\sigma}+8a^2b\sigma(a-x_2)+$$
$$+2b(a-x_2)+16ab\sigma+\frac{4ab\sqrt{\sigma\pi}}{\sigma(a-x_2)}+\frac{2b}{\sigma(a-x_2)^2}+4a^3b\sigma\sqrt{\sigma\pi}+207ab\sqrt{\sigma\pi}+\frac{16ab\sqrt{\sigma\pi}}{(a-x_2)^2}+$$
$$+5b\sqrt{\sigma\pi}+48a^2b\sigma(a-x_2)^2+16a^3b\sigma^2+2a^3b\sigma+182a^2b\sigma+8a^2b\sigma\sqrt{\sigma\pi}+128ab\sigma(a-x_2)+$$
$$+42b+\frac{24b\sqrt{\sigma\pi}}{\sigma(a-x_2)}+4b\sigma(a-x_2)^2+16a^2b\sigma^2(a-x_2)^2+16ab\sigma(a-x_2)^3+4ab\sqrt{\sigma\pi}(a-x_2)^2+$$
$$+40ab(a-x_2)^3+\frac{8ab\sqrt{\sigma\pi}}{a-x_2}+16a^2b\sigma^2+\frac{8a^2b\sqrt{\sigma\pi}}{\sigma}(a-x_2)+4b\sqrt{\sigma\pi}(a-x_2)+$$
$$+16a^2b\sigma\sqrt{\sigma\pi}(a-x_2)+\frac{2b\sqrt{\sigma\pi}}{\sigma}(a-x_2)^2+16a^3b\sigma^2(a-x_2)+$$
$$+8ab\sigma\sqrt{\sigma\pi}(a-x_2)^2+4b(a-x_2)^2,$$

$\varphi(\sigma,x_2)$ *is determined by the formula* (12).

Proof. From Green's integral formula we have

$$\begin{aligned}U(x)=&\int_S\left[U(y)\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}-\Delta L_\sigma(x,y)\frac{\partial U(y)}{\partial n}\right]dS_y+\\&+\int_T\left[U(y)\frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n}-\Delta L_\sigma(x,y)\frac{\partial U(y)}{\partial n}\right]dS_y+\\&+\int_S\left[\Delta U(y)\frac{\partial L_\sigma(x,y)}{\partial n}-L_\sigma(x,y)\frac{\partial \Delta U(y)}{\partial n}\right]dS_y+\\&+\int_T\left[\Delta U(y)\frac{\partial L_\sigma(x,y)}{\partial n}-L_\sigma(x,y)\frac{\partial \Delta U(y)}{\partial n}\right]dS_y.\end{aligned} \tag{31}$$

From the condition (2) and the inequality (29) we obtain

$$
\begin{aligned}
|U(x)| \leqslant \Bigg| \int_S \left[f_1(y) \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} + f_3(y) \Delta L_\sigma(x,y) \right] dS_y + \\
+ \int_T \left[U(y) \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} - \Delta L_\sigma(x,y) \frac{\partial U(y)}{\partial n} \right] dS_y + \\
\int_S \left[f_2(y) \frac{\partial L_\sigma(x,y)}{\partial n} + f_4(y) L_\sigma(x,y) \right] dS_y + \\
+ \int_T \left[\Delta U(y) \frac{\partial L_\sigma(x,y)}{\partial n} - L_\sigma(x,y) \frac{\partial \left(\Delta U(y)\right)}{\partial n} \right] dS_y \Bigg| \leqslant \delta \left| U_\sigma(x) \right| + \\
+ M \left(\int_T \left| \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right| dS_y + \int_T \left| \Delta L_\sigma(x,y) \right| dS_y + \int_T \left| \frac{\partial L_\sigma(x,y)}{\partial n} \right| dS_y + \right. \\
\left. + \int_T \left| L_\sigma(x,y) \right| dS_y \right) \leqslant \delta \left| U_\sigma(x) \right| + M \varphi(\sigma, x_2) e^{-\sigma x_2^2}.
\end{aligned}
\tag{32}
$$

The estimate used here

$$
\begin{aligned}
M \left(\int_T \left| \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right| dS_y + \int_T \left| \Delta L_\sigma(x,y) \right| dS_y + \right. \\
\left. + \int_T \left| \frac{\partial L_\sigma(x,y)}{\partial n} \right| dS_y + \int_T \left| L_\sigma(x,y) \right| dS_y \right) \leqslant M \varphi(\sigma, x_2) e^{-\sigma x_2^2},
\end{aligned}
$$

proved in Theorem 1.

Next, estimate $|U_\sigma(x)|$

$$
\begin{aligned}
|U_\sigma(x)| \leqslant \int_S \left| \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right| dS_y + \int_S \left| \Delta L_\sigma(x,y) \right| dS_y + \int_S \left| \frac{\partial L_\sigma(x,y)}{\partial n} \right| dS_y + \\
+ \int_S \left| L_\sigma(x,y) \right| dS_y = A_1 + A_2 + A_3 + A_4.
\end{aligned}
$$

Estimating these integrals, we get

$$
A_1 = \int_S \left| L_\sigma(x,y) \right| dS_y \leqslant \left(\frac{b}{2\sigma} + \frac{ab\sqrt{\sigma\pi}}{\sigma} \right) e^{-\sigma\left(x_2^2 - a^2\right)},
$$

$$
\begin{aligned}
A_2 = \int_S \left| \frac{\partial L_\sigma(x,y)}{\partial n} \right| dS_y \leqslant \left[\frac{5b}{2\sigma} + \frac{2ab\sqrt{\sigma\pi}}{\sigma} + 2a^2 b + \frac{13ab}{2}(a - x_2) + \frac{2ab(a - x_2)}{\sigma} + \frac{ab\sqrt{\sigma\pi}}{2} + \right. \\
\left. + \frac{3ab}{2\sigma} + a^2 b\sqrt{\sigma\pi} + \frac{b\sqrt{\sigma\pi}}{4\sigma}(a - x_2) \right] e^{-\sigma\left(x_2^2 - a^2\right)},
\end{aligned}
$$

$$
\begin{aligned}
A_3 = \int_S \left| \Delta L_\sigma(x,y) \right| dS_y \leqslant \left[\frac{17b\sqrt{\sigma\pi}}{4\sigma} + 20ab + a^2 b\sqrt{\sigma\pi} + 8ab\sigma\sqrt{\sigma\pi}(a - x_2) + \right. \\
\left. + \frac{5ab\sqrt{\sigma\pi}}{\sigma}(a - x_2) 8a^2 b\sigma(a - x_2) + 2b(a - x_2) \right] e^{-\sigma\left(x_2^2 - a^2\right)},
\end{aligned}
$$

$$
\begin{aligned}
A_4 = \int_S \left| \frac{\partial \left(\Delta L_\sigma(x,y)\right)}{\partial n} \right| dS_y \leqslant \left[16ab\sigma + 42ab(a - x_2) + 28a^2 b + \frac{b\sqrt{\sigma\pi}}{\sigma} + \frac{4ab\sqrt{\sigma\pi}}{\sigma(a - x_2)} + \right. \\
+ \frac{2b}{\sigma(a - x_2)^2} + 4a^3 b\sigma\sqrt{\sigma\pi} + 207ab\sqrt{\sigma\pi} + \frac{16ab\sqrt{\sigma\pi}}{(a - x_2)^2} + 5b\sqrt{\sigma\pi} + 48a^2 b\sigma(a - x_2)^2 + 16a^3 b\sigma^2 +
\end{aligned}
$$

$$+2a^3b\sigma + 182a^2b\sigma + 8a^2b\sigma\sqrt{\sigma\pi} + 128ab\sigma(a-x_2) + 42b + \frac{24b\sqrt{\sigma\pi}}{\sigma(a-x_2)} + 4b\sigma(a-x_2)^2 +$$
$$+16a^2b\sigma^2(a-x_2)^2 + 16ab\sigma(a-x_2)^3 + 4ab\sqrt{\sigma\pi}(a-x_2)^2 + 40ab(a-x_2)^3 + \frac{8ab\sqrt{\sigma\pi}}{a-x_2} +$$
$$+16a^2b\sigma^2 + 2a^2b\sqrt{\sigma\pi} + \frac{8a^2b\sqrt{\sigma\pi}}{\sigma}(a-x_2) + 4b\sqrt{\sigma\pi}(a-x_2) + 16a^2b\sigma\sqrt{\sigma\pi}(a-x_2) +$$
$$+\frac{16ab\sqrt{\sigma\pi}}{\sigma} + \frac{2b\sqrt{\sigma\pi}}{\sigma}(a-x_2)^2 + 16a^3b\sigma^2(a-x_2) + 8ab\sigma\sqrt{\sigma\pi}(a-x_2)^2 + 4b(a-x_2)^2\Big] e^{-\sigma\left(x_2^2-a^2\right)}.$$

When evaluating the integrals, polar coordinates were introduced and the inequalities were used

$$\left|\sin 2\sigma y_2\sqrt{u^2+\alpha^2}\right| \leqslant \frac{4\sigma y_2\sqrt{u^2+\alpha^2}}{1+2\sigma y_2\sqrt{u^2+\alpha^2}},$$

since $|\sin x| \leqslant \dfrac{2|x|}{1+|x|}$, $x>0$.

Adding the estimates obtained, we have

$$|U_\sigma(x)| \leqslant \psi(\sigma, x_2) e^{-\sigma\left(x_2^2-a^2\right)},$$

here

$$\psi(\sigma, x_2) = \frac{3b}{\sigma} + \frac{19ab\sqrt{\sigma\pi}}{\sigma} + 30a^2b + \frac{97ab(a-x_2)}{2} + \frac{ab\sqrt{\sigma\pi}}{2} + \frac{3ab}{2\sigma} + 4a^2b\sqrt{\sigma\pi} +$$
$$+\frac{b\sqrt{\sigma\pi}}{4\sigma}(a-x_2) + \frac{21b\sqrt{\sigma\pi}}{4\sigma} + 20ab + 8ab\sigma\sqrt{\sigma\pi}(a-x_2) + \frac{5ab\sqrt{\sigma\pi}(a-x_2)}{\sigma} + 8a^2b\sigma(a-x_2) +$$
$$+2b(a-x_2) + 16ab\sigma + \frac{4ab\sqrt{\sigma\pi}}{\sigma(a-x_2)} + \frac{2b}{\sigma(a-x_2)^2} + 4a^3b\sigma\sqrt{\sigma\pi} + 207ab\sqrt{\sigma\pi} + \frac{16ab\sqrt{\sigma\pi}}{(a-x_2)^2} +$$
$$+5b\sqrt{\sigma\pi} + 48a^2b\sigma(a-x_2)^2 + 16a^3b\sigma^2 + 2a^3b\sigma + 182a^2b\sigma + 8a^2b\sigma\sqrt{\sigma\pi} + 128ab\sigma(a-x_2) +$$
$$42b + \frac{24b\sqrt{\sigma\pi}}{\sigma(a-x_2)} + 4b\sigma(a-x_2)^2 + +16a^2b\sigma^2(a-x_2)^2 + 16ab\sigma(a-x_2)^3 + 4ab\sqrt{\sigma\pi}(a-x_2)^2 +$$
$$40ab(a-x_2)^3 + \frac{8ab\sqrt{\sigma\pi}}{a-x_2} + 16a^2b\sigma^2 + +\frac{8a^2b\sqrt{\sigma\pi}}{\sigma}(a-x_2) + 4b\sqrt{\sigma\pi}(a-x_2) +$$
$$16a^2b\sigma\sqrt{\sigma\pi}(a-x_2) + \frac{2b\sqrt{\sigma\pi}}{\sigma}(a-x_2)^2 + 16a^3b\sigma^2(a-x_2) + 8ab\sigma\sqrt{\sigma\pi}(a-x_2)^2 + 4b(a-x_2)^2.$$

From the integral formula (32) and the condition (9) we obtain

$$|U(x)| \leqslant \delta e^{\sigma\left(a^2-x_2^2\right)}\psi(\sigma, x_2) + M\varphi(\sigma, x_2)e^{-\sigma x_2^2} = \Psi(\sigma, x_2)\left(Me^{-\sigma x_2^2} + \delta e^{\sigma\left(a^2-x_2^2\right)}\right). \tag{33}$$

The best estimate for the function $|U(x)|$ is obtained in the case, when

$$Me^{-\sigma x_2^2} = \delta e^{\sigma\left(a^2-x_2^2\right)}$$

or

$$\sigma = \frac{1}{a^2}\ln\frac{M}{\delta}. \tag{34}$$

Substituting the expression for σ from the equality (34) into (33) we obtain the proof of the inequality (30). Theorem 2 is proved. □

Set

$$U_{\sigma\delta}(x) = \int_S \left[f_{1\delta}(y) \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} - f_{3\delta}(y)\Delta L_\sigma(x,y) \right] dS_y + \\ + \int_S \left[f_{2\delta}(y) \frac{\partial L_\sigma(x,y)}{\partial n} - f_{4\delta}(y) L_\sigma(x,y) \right] dS_y. \qquad (35)$$

Theorem 3. *Let the function $U(y) \in E$ on S satisfy the conditions (2) and instead of the functions $f_i(y)$ their approximations $f_{i\delta}(y)$, $i = 1,2,3,4$ with a given deviation $\delta > 0$, i.e.*

$$\max_S |f_i(y) - f_{i\delta}(y)| < \delta. \qquad (36)$$

Then, for any $x \in G$ and $\sigma > 0$, the following estimate holds:

$$|U(x) - U_{\sigma\delta}(x)| \leqslant \Psi(\sigma, x_2) M^{1-\frac{x_2^2}{a^2}} \delta^{\frac{x_2^2}{a^2}}. \qquad (37)$$

Proof. From (31) and (35) we get

$$|U(x) - U_{\sigma\delta}(x)| \leqslant |I_\sigma(x)| + \delta \int_S \left\{ \left| \frac{\partial\left(\Delta L_\sigma(x,y)\right)}{\partial n} \right| + |\Delta L_\sigma(x,y)| + \left| \frac{\partial L_\sigma(x,y)}{\partial n} \right| + |L_\sigma(x,y)| \right\} dS_y.$$

From Theorems 1 and 2 we obtain

$$|U(x)| \leqslant \Psi(\sigma, x_2) \left(M e^{-\sigma x_2^2} + \delta e^{\sigma a^2 - \sigma x_2^2} \right),$$

and choosing $\sigma = \dfrac{1}{a^2} \ln \dfrac{M}{\delta}$, we obtain the proof of Theorem 3. □

Corollary 3. *For each $x \in G$ the equality*

$$\lim_{\delta \to 0} U_{\sigma\delta}(x) = U(x).$$

Corollary 4. *If $x \in \bar{G}_\varepsilon$, then the family of functions $\{U_{\sigma\delta}(x)\}$,*

$$U_{\sigma\delta}(x) \rightrightarrows U(x)$$

converges uniformly as $\delta \to 0$.

Similarly, one can obtain stability estimates for $\dfrac{\partial U(x)}{\partial x_i}$, $i = 1,2$, and the following corollaries are true:

Corollary 5. *For each $x \in G$, the equality*

$$\lim_{\delta \to 0} \frac{\partial U_{\sigma\delta}(x)}{\partial x_i} = \frac{\partial U(x)}{\partial x_i}, \quad i = 1,2.$$

Corollary 6. *If $x \in \overline{G}_\varepsilon$, then the family of functions*

$$U_{\sigma\delta}(x) \rightrightarrows U(x), \frac{\partial U_{\sigma\delta}(x)}{\partial x_i} \rightrightarrows \frac{\partial U(x)}{\partial x_i}, \quad i = 1,2$$

converge uniformly at $\delta \to 0$.

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О задаче Коши для бигармонического уравнения

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Аннотация. Работа посвящена исследованию продолжения и оценки устойчивости решения задачи Коши для бигармонического уравнения в области G по его известным значениям на гладкой части границы ∂G. Рассматриваемая задача относится к задачам математической физики, в которых отсутствует непрерывная зависимость решений от начальных данных. В данной работе с помощью функции Карлемана восстанавливается не только сама бигармоническая функция, но и ее производные по данным Коши на части границы области. Получены оценки устойчивости решения задачи Коши в классическом смысле.

Ключевые слова: бигармонические уравнения, задача Коши, некорректные задачи, функция Карлемана, регуляризованные решения, регуляризация, формулы продолжения.

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Algorithm of the Regularization Method for a Singularly Perturbed Integro-differential Equation with a Rapidly Decreasing Kernel and Rapidly Oscillating Inhomogeneity

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Abstract. In this paper, we consider a singularly perturbed integro-differential equation with a rapidly oscillating right-hand side, which includes an integral operator with a rapidly varying kernel. The main goal of this work is to generalize the Lomov's regularization method and to reveal the influence of the rapidly oscillating right-hand side and a rapidly varying kernel on the asymptotics of the solution to the original problem.

Keywords: singular perturbation, integro-differential equation, rapidly oscillating right-hand side, rapidly varying kernel, regularization, solvability of iterative problems.

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Various applied problems related to the study of the properties of media with a periodic structure can be described using differential equations with rapidly oscillating inhomogeneities. The presence of rapidly oscillating components creates serious problems for the numerical integration of such equations. Therefore, asymptotic methods are usually applied to equations of this type, the most famous of which are the Feshchenko–Shkil–Nikolenko splitting method [1–3] and the Lomov regularization method [4–17]. However, both of these methods were developed mainly for singularly perturbed differential equations that do not contain an integral operator. Note that, as far as we know, the splitting method has not been applied to integro-differential equations, and the transition from differential equations to integro-differential equations with rapidly oscillating inhomogeneities requires a significant revision of the algorithm of the regularization method itself. The integral term gives rise to new types of singularities in solutions that differ from the previously known ones, which complicates the development of the algorithm of

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the regularization method [4]. Moreover, in the problem considered below, the integral operator contains a rapidly decreasing factor. Problems of this type have been studied only in the presence of slowly varying inhomogeneities (see, for example, [12–14]. An analysis of the influence of a rapidly decreasing kernel on the asymptotic solution of problems with fast oscillations has not been performed before and is the subject of our study.

1. Problem statement

Consider the following integro-differential equation:

$$L_\varepsilon y(t,\varepsilon) \equiv \varepsilon \frac{dy}{dt} - A(t)y - \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \mu(\theta)d\theta} K(t,s)y(s,\varepsilon)ds = \\ = h_1(t) + h_2(t)\sin\frac{\beta(t)}{\varepsilon}, \quad y(0,\varepsilon) = y^0, \quad t \in [0,T], \tag{1}$$

where $A(t)$, $\mu(t)$, $h_1(t)$, $h_2(t)$, $\beta(t)$ are scalar functions, $\beta'(t) > 0$ is the frequency of a rapidly oscillating sine, y^0 is a constant number, $\varepsilon > 0$ is a small parameter. The function $\lambda_1(t) = A(t)$ is the eigenvalue of the limiting operator $A(t)$, the functions $\lambda_2(t) = -i\beta'(t)$ and $\lambda_3(t) = +i\beta'(t)$ are related to the presence of a rapidly oscillating sine in eqation (1), the function $\lambda_4(t) = \mu(t)$ characterizes the rapid change in the kernel of the integral operator.

Problem (1) will be considered under the following conditions:

1) $A(t), \mu(t)\beta(t) \in C^\infty([0,T],\mathbb{R}),\ h_1(t), h_2(t) \in ([0,T],\mathbb{C}),$
$K(t,s) \in C^\infty(\{0 \leqslant s \leqslant t \leqslant T\},\mathbb{C});$
2) $A(t) < 0\ \ \forall t \in [0,T]$.

Let us develop an algorithm for the regularization method under the specified conditions.

2. Solution space and regularization of problem (1)

We introduce the regularizing variables (см. [4])

$$\tau_j = \frac{1}{\varepsilon}\int_0^t \lambda_j(\theta)\,d\theta \equiv \frac{\psi_j(t)}{\varepsilon}, \quad j = \overline{1,4}, \tag{2}$$

and instead of the problem (1) consider the problem

$$L_\varepsilon \tilde{y}(t,\tau,\varepsilon) \equiv \varepsilon\frac{\partial\tilde{y}}{\partial t} + \sum_{j=1}^{4}\lambda_j(t)\frac{\partial\tilde{y}}{\partial\tau_j} - \lambda_1(t)\tilde{y} - \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_4(\theta)d\theta} K(t,s)\tilde{y}\Big(s,\frac{\psi(s)}{\varepsilon},\varepsilon\Big)ds = \\ = h_1(t) - \frac{1}{2i}h_2(t)\left(e^{\tau_2} - e^{\tau_3}\right),\ \ \tilde{y}(0,0,\varepsilon) = y^0, \ \ t \in [0,T] \tag{3}$$

for the function $\tilde{y} = \tilde{y}(t,\tau,\varepsilon)$, where (according to (2)) $\tau = (\tau_1,\tau_2,\tau_3,\tau_4)$, $\psi = (\psi_1,\psi_2,\psi_3,\psi_4)$. It is clear that if $\tilde{y} = \tilde{y}(t,\tau,\varepsilon)$ is the solution of the problem (3), then the vector function $\tilde{y} = \tilde{y}\left(t,\frac{\psi(t)}{\varepsilon},\varepsilon\right)$ is an exact solution of the problem (1); therefore, problem (3) is extended with respect to problem (1). However, it cannot be considered fully regularized, since the integral term

$$J\tilde{y} \equiv J\left(\tilde{y}(t,\tau,\varepsilon)\,|_{t=s,\tau=\psi(s)/\varepsilon}\right) = \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_4(\theta)d\theta} K(t,s)\,\tilde{y}\Big(s,\frac{\psi(s)}{\varepsilon},\varepsilon\Big)ds$$

has not been regularized in it.

To regularize the operator J, we introduce a class M_ε that is asymptotically invariant with respect to the operator $J\tilde{y}$ (see [4], p. 62). Let us first consider the space U of vector functions $y(t,\tau)$, representable by the sums

$$y(t,\tau) = y_0(t) + \sum_{j=1}^{4} y_j(t)\, e^{\tau_j}, \quad y_j(t) \in C^\infty\left([0,T], \mathbb{C}^1\right), \ j = \overline{0,4}. \tag{4}$$

Let us show that the class $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant under the operator Jy. The image of the operator on the element (4) of the space U has the form:

$$\begin{aligned} Jy(t,\tau) = \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_4(\theta)d\theta} K(t,s)\, y_0(s)\, ds + \sum_{j=1}^{4} \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_4(\theta)d\theta} K(t,s)\, y_j(s)\, e^{\frac{1}{\varepsilon}\int_0^s \lambda_j(\theta)d\theta} ds = \\ = \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_4(\theta)d\theta} K(t,s)\, y_0(s)\, ds + e^{\frac{1}{\varepsilon}\int_0^t \lambda_4(\theta)d\theta} \int_0^t K(t,s)\, y_4(s)\, ds + \\ + \sum_{j=1, j\neq 4}^{4} e^{\frac{1}{\varepsilon}\int_0^t \lambda_4(\theta)d\theta} \int_0^t K(t,s)\, y_j(s)\, e^{\frac{1}{\varepsilon}\int_0^s (\lambda_j(\theta)-\lambda_4(\theta))d\theta} ds. \end{aligned}$$

By applying the operation of integration by parts, we find that the image of the operator J on an element (4) of the space U can be represented as a series

$$\begin{aligned} Jy(t,\tau) = e^{\frac{1}{\varepsilon}\int_0^t \lambda_4(\theta)d\theta} \int_0^t K(t,s)\, y_4(s)\, ds + \\ + \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \left[\left(I_0^\nu (K(t,s)\, y_0(s))\right)_{s=t} e^{\frac{1}{\varepsilon}\int_0^t \lambda_4(\theta)d\theta} - \left(I_0^\nu (K(t,s)\, y_0(s))\right)_{s=0} \right] + \\ + \sum_{j=1, j\neq 4}^{4} \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \Big[\left(I_j^\nu (K(t,s)\, y_j(s))\right)_{s=t} e^{\frac{1}{\varepsilon}\int_0^t \lambda_j(\theta)d\theta} - \\ - \left(I_j^\nu (K(t,s)\, y_j(s))\right)_{s=0} e^{\frac{1}{\varepsilon}\int_0^t \lambda_4(\theta)d\theta} \Big], \quad \tau = \psi(t)/\varepsilon \end{aligned}$$

where it is indicated:

$$\begin{aligned} & I_0^0 = \frac{1}{-\lambda_4(s)}, \quad I_0^\nu = \frac{1}{-\lambda_4(s)} \frac{\partial}{\partial s} I_0^{\nu-1}, \\ & I_j^0 = \frac{1}{\lambda_j(s) - \lambda_4(s)}, \quad I_j^\nu = \frac{1}{\lambda_j(s) - \lambda_4(s)} \frac{\partial}{\partial s} I_j^{\nu-1}, \ j = \overline{1,3}, \ \nu \geqslant 1. \end{aligned}$$

It is easy to show (see, for example, [18], pp. 291—294), that this series converges asymptotically when $\varepsilon \to +0$ (uniformly over $t \in [0,T]$). This means that the class M_ε asymptotically invariant (for $\varepsilon \to +0$) with respect to the operator J.

Let us introduce operators $R_\nu : U \to U$, acting on each element $y(t,\tau) \in U$ of the form (4) according to the law:

$$R_0 y(t,\tau) = e^{\tau_4} \int_0^t K(t,s)\, y_4(s)\, ds, \tag{5_0}$$

$$\begin{aligned} R_1 y(t,\tau) = \left[\left(I_0^0 (K(t,s)\, y_0(s))\right)_{s=t} e^{\tau_4} - \left(I_0^0 (K(t,s)\, y_0(s))\right)_{s=0} \right] + \\ + \sum_{j=1}^{3} \left[\left(I_j^0 (K(t,s)\, y_j(s))\right)_{s=t} e^{\tau_j} - \left(I_j^0 (K(t,s)\, y_j(s))\right)_{s=0} e^{\tau_4} \right], \end{aligned} \tag{5_1}$$

$$R_{\nu+1}y\,(t,\tau) = (-1)^{\nu}\left[\left(I_0^{\nu}\left(K\,(t,s)\,y_0\,(s)\right)\right)_{s=t}e^{\tau_4} - \left(I_0^{\nu}\left(K\,(t,s)\,y_0\,(s)\right)\right)_{s=0}\right] + \\ + \sum_{j=1}^{3}(-1)^{\nu}\left[\left(I_j^{\nu}\left(K\,(t,s)\,y_j\,(s)\right)\right)_{s=t}e^{\tau_j} - \left(I_j^{\nu}\left(K\,(t,s)\,y_j\,(s)\right)\right)_{s=0}e^{\tau_4}\right], \quad \nu \geqslant 1. \tag{$5_{\nu+1}$}$$

Now let $\tilde{y}\,(t,\tau,\varepsilon)$ be arbitrary continuous in $(t,\tau) \in [0,T] \times \{\tau : \mathrm{Re}\tau_j \leqslant 0, \;\; j = \overline{1,4}\}$ function with asymptotic expansion

$$\tilde{y}\,(t,\tau,\varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k y_k\,(t,\tau)\,, \quad y_k\,(t,\tau) \in U \tag{6}$$

converging as $\varepsilon \to +0$ (uniformly in $(t,\tau) \in [0,T] \times \{\tau : \mathrm{Re}\tau_j \leqslant 0, j = \overline{1,4}\}$). Then image $J\tilde{y}\,(t,\tau,\varepsilon)$ of this function expands into an asymptotic series

$$J\tilde{y}\,(t,\tau,\varepsilon) = \sum_{k=0}^{\infty}\varepsilon^k J y_k\,(t,\tau) = \sum_{r=0}^{\infty}\varepsilon^r\sum_{s=0}^{r}R_{r-s}y_s\,(t,\tau)\,|_{\tau=\psi(t)/\varepsilon}.$$

This equality is the basis for introducing an extension of the operator J on series of the form (6):

$$\tilde{J}\tilde{y}\,(t,\tau,\varepsilon) \equiv \tilde{J}\left(\sum_{k=0}^{\infty}\varepsilon^k y_k\,(t,\tau)\right) \overset{def}{=} \sum_{r=0}^{\infty}\varepsilon^r\sum_{s=0}^{r}R_{r-s}y_s\,(t,\tau)\,. \tag{7}$$

Although the operator $\tilde{J}$ is defined formally, its usefulness is obvious, since in practice one usually constructs an N-th approximation of the asymptotic solution of the problem (1), in which only N-th partial sums of the series (7) that have not formal, but true meaning. Now we can write the problem completely regularized with respect to the original problem (2):

$$L_\varepsilon\tilde{y}(t,\tau,\varepsilon) \equiv \varepsilon\frac{\partial\tilde{y}}{\partial t} + \sum_{j=1}^{4}\lambda_j\,(t)\,\frac{\partial\tilde{y}}{\partial\tau_j} - \lambda_1(t)\tilde{y} - \tilde{J}\tilde{y} = \\ = h_1(t) - \frac{1}{2i}h_2(t)\left(e^{\tau_2} - e^{\tau_3}\right), \quad \tilde{y}(t,\tau,\varepsilon)|_{t=0,\tau=0} = y^0, \quad t \in [0,T], \tag{8}$$

where the operator $\tilde{J}$ has the form (7).

3. Iterative problems and their solvability in the space U

Substituting series (6) into (8) and equating the coefficients at the same powers of ε, we obtain the following iterative problems:

$$Ly_0\,(t,\tau) \equiv \sum_{j=1}^{4}\lambda_j(t)\frac{\partial y_0}{\partial\tau_j} - \lambda_1(t)y_0 - R_0y_0 = h_1(t) - \frac{1}{2i}h_2(t)\left(e^{\tau_2} - e^{\tau_3}\right),\; y_0\,(0,0) = y^0; \tag{9_0}$$

$$Ly_1\,(t,\tau) = -\frac{\partial y_0}{\partial t} + R_1y_0,\; y_1\,(0,0) = 0; \tag{9_1}$$

$$Ly_2\,(t,\tau) = -\frac{\partial y_1}{\partial t} + R_1y_1 + R_2y_0, \;\; y_0\,(0,0) = 0; \tag{9_2}$$

$$\cdots$$

$$Ly_k\,(t,\tau) = -\frac{\partial y_{k-1}}{\partial t} + R_ky_0 + \cdots + R_1y_{k-1}, \;\; y_k\,(0,0) = 0, \;\; k \geqslant 1. \tag{9_k}$$

Each of the iterative problems (9_k) can be written as

$$Ly\left(t,\tau\right)\equiv\sum_{j=1}^{3}\lambda_j\left(t\right)\frac{\partial y}{\partial\tau_j}-\lambda_1(t)y-R_0y=H\left(t,\tau\right),\quad y\left(0,0\right)=y_*\,, \tag{10}$$

where $H\left(t,\tau\right)=H_0\left(t\right)+\sum\limits_{j=1}^{3}H_j\left(t\right)e^{\tau_j}$ is a well-known function of the space U, $y_*\in\mathbb{C}$ is constant, and the operator R_0 has the form (see (6_0))

$$R_0y\equiv R_0\left(y_0\left(t\right)+\sum_{j=1}^{4}y_j\left(t\right)e^{\tau_j}\right)=e^{\tau_4}\int_0^t K\left(t,s\right)y_4\left(s\right)ds.$$

We introduce a scalar (for each $t\in[0,T]$) product in the space U :

$$< z,w >\equiv< z_0\left(t\right)+\sum_{j=1}^{4}z_j\left(t\right)e^{\tau_j},\,w_0\left(t\right)+\sum_{j=1}^{4}w_j\left(t\right)e^{\tau_j}>\overset{def}{=}$$

$$\overset{def}{=}\left(z_0\left(t\right),w_0\left(t\right)\right)+\sum_{j=1}^{4}\left(z_j\left(t\right),w_j\left(t\right)\right)$$

where $(*,*)$ denotes the usual scalar product in the complex space $\mathbb{C}$. Let us prove the following statement.

Theorem 1. *Let the conditions 1) and 2) hold and right-hand part* $H\left(t,\tau\right)=H_0\left(t\right)+$ $+\sum\limits_{j=1}^{4}H_j\left(t\right)e^{\tau_j}$ *of the equation (10) belongs to the space U. Then for the solvability of equation (10) in U it is necessary and sufficient that the identity*

$$< H_1\left(t,\tau\right),e^{\tau_1}>\equiv 0\Leftrightarrow H_1\left(t\right)\equiv 0\ \forall t\in[0,T] \tag{11}$$

is fulfilled.

Proof. We will define the solution of the equation (10) as an element (4) of the space U:

$$y\left(t,\tau\right)=y_0\left(t\right)+\sum_{j=1}^{3}y_j\left(t\right)e^{\tau_j}. \tag{12}$$

Substituting (12) into the equation (10), we will have

$$\sum_{j=1}^{4}\left[\lambda_j\left(t\right)-\lambda_1(t)\right]y_j\left(t\right)e^{\tau_j}-\lambda_1(t)y_0\left(t\right)-e^{\tau_4}\int_0^t K\left(t,s\right)y_4\left(s\right)ds=H_0\left(t\right)+\sum_{j=1}^{4}H_j\left(t\right)e^{\tau_j}.$$

Equating here separately the free terms and coefficients at the same exponents, we obtain the following equations:

$$-\lambda_1\left(t\right)y_0\left(t\right)=H_0\left(t\right), \tag{13_0}$$

$$\left[\lambda_j\left(t\right)-\lambda_1\left(t\right)\right]y_j\left(t\right)=H_j\left(t\right),j=\overline{1,3}, \tag{13_j}$$

$$\left[\lambda_4\left(t\right)-\lambda_1\left(t\right)\right]y_4\left(t\right)-\int_0^t K\left(t,s\right)y_4\left(s\right)ds=H_4(t). \tag{13_4}$$

Due to the fact that the function $\lambda_1(t) \neq 0 \; \forall t \in [0,T]$, the equation ($13_0$) has a unique solution $y_0(t) = -\lambda_1^{-1}(t)\, H_0(t)$. Since the function $[\lambda_4(t) - \lambda_1(t)] \neq 0 \; \forall t \in [0,T]$, then the equation ($13_4$) can be written as

$$y_4(t) = \int_0^t \Big([\lambda_4(t) - \lambda_1(t)]^{-1} K(t,s) \Big) y_4(s)\, ds - [\lambda_4(t) - \lambda_1(t)]^{-1} H_4(t). \tag{14}$$

Due to the smoothness of the kernel $\Big([\lambda_4(t) - \lambda_1(t)]^{-1} K(t,s) \Big)$ and heterogeneity $[\lambda_4(t) - \lambda_1(t)]^{-1} H_4(t)$ this Volterra integral equation has a unique solution $y_4(t) \in C^\infty([0,T], \mathbb{C})$.

Since $\lambda_{2,3}(t) = \pm i\beta'(t)$ are purely imaginary functions, and the function $\lambda_1(t)$ is real, then the equation (13_j) при $j = 2,3$ solvable in the space $C^\infty([0,T], \mathbb{C})$. The equation ($13_1$) is solvable in the space $C^\infty([0,T], \mathbb{C})$ if and only if the identity $H_1(t) \equiv 0 \;\; \forall t \in [0,T]$ holds. It is easy to see that this identity coincides with the identity (11). Thus, condition (11) is necessary and sufficient for the solvability of equation (10) in the space U. The theorem is proved. □

Remark 1. If identity (11) holds, then under conditions 1) and 2) the equation (10) has the following solution in the space U :

$$y(t,\tau) = y_0(t) + \alpha_1(t)\, e^{\tau_1} + H_{21}(t) e^{\tau_2} + H_{31}(t) e^{\tau_3} + y_4(t)\, e^{\tau_4}, \tag{15}$$

where $\alpha_1(t) \in C^\infty([0,T], \mathbb{C})$ is an arbitrary function, $y_0(t) = -\lambda_1^{-1}(t)\, H_0(t)$, $y_4(t)$ is the solution of the integral equation (14) and introduced the notation:

$$H_{21}(t) \equiv \frac{H_2(t)}{\lambda_2(t) - \lambda_1(t)}, \;\; H_{31}(t) \equiv \frac{H_3(t)}{\lambda_3(t) - \lambda_1(t)}.$$

4. Unique solvability of a general iterative problem in the space U. Remainder term theorem

As can be seen from (15), the solution to the equation (10) is determined ambiguously. However, if it is subject to additional conditions:

$$\begin{gathered} y(0,0) = y_*, \\ < -\frac{\partial y}{\partial t} + R_1 y + Q(t,\tau),\, e^{\tau_1} > \equiv 0 \;\forall t \in [0,T] \end{gathered} \tag{16}$$

where $Q(t,\tau) = Q_0(t) + \sum\limits_{j=1}^{4} Q_i(t)\, e^{\tau_j}$ is a known function of the space U, y_* is a constant number of the complex space $\mathbb{C}$, then problem (10) will be uniquely solvable in the space U. More precisely, the following result takes place.

Theorem 2. *Let conditions 1) and 2) hold, the right-hand side $H(t,\tau)$ of the equation (10) belongs to the space U and satisfies the orthogonality condition (11). Then equation (10) under additional conditions (16) is uniquely solvable in U.*

Proof. Under condition (11), the equation (10) has the solution (15) in the space U, where the function $\alpha_1(t) \in C^\infty([0,T], \mathbb{C})$ so far arbitrary. Subordinate (15) to the initial condition $y(0,0) = y_*$. Will have

$$
\begin{gathered}
y_* = y_0\left(0\right) + \alpha_1\left(0\right) + H_{21}(0) + \,H_{31}(0) - \frac{H_4(0)}{\lambda_4(0) - \lambda_1(0)} \quad \Leftrightarrow \\
\Leftrightarrow \;\alpha_1\left(0\right) = y_* + \lambda_1^{-1}\left(0\right) H_0\left(0\right) - H_{21}(0) - \,H_{31}(0) + \frac{H_4(0)}{\lambda_4(0) - \lambda_1(0)}.
\end{gathered} \tag{17)}
$$

Let us now subordinate the solution (15) to the second condition (16). The right-hand side of this equation is

$$
\begin{gathered}
-\frac{\partial y_0}{\partial t} + R_1 y_0 + Q\left(t,\tau\right) = -\dot{y}_0\left(t\right) - \dot{\alpha}_1(t)e^{\tau_1} - \\
-\sum_{j=2}^{3}\left(\frac{H_{j1}(t)}{\lambda_j(t) - \lambda_1(t)}\right)^{\bullet} e^{\tau_j} + \dot{y}_4(t)e^{\tau_4} + \frac{K(t,t)\alpha_1(t)}{\lambda_1\left(t\right) - \lambda_4\left(t\right)}e^{\tau_1} - \frac{K(t,0)\alpha_1(0)}{\lambda_j(0) - \lambda_4\left(0\right)} + \\
+\sum_{j=2}^{4}\left[\frac{K(t,t)y_j(t)}{\lambda_j\left(t\right) - \lambda_4\left(t\right)}e^{\tau_j} - \frac{K(t,0)y_j(0)}{\lambda_j(0) - \lambda_4\left(0\right)}\right] + Q(t,\tau).
\end{gathered} \tag{18}
$$

Now multiplying (18) scalarly by e^{τ_1}, we obtain the differential equation

$$
-\dot{\alpha}_1(t) + \frac{K(t,t)\alpha_1(t)}{[\lambda_1(t) - \lambda_4(t)]} + Q_1(t) = 0.
$$

Adding the initial condition (17) to it, we uniquely find the function $\alpha_1(t)$, and, therefore, construct the solution (15) of the problem (10) in the space U uniquely. The theorem is proved.

Applying Theorems 1 and 2 to iterative problems (9_k), we find uniquely their solutions in the space U and construct series (6). Just as in [4], we prove the following statement.

Theorem 3. *Let conditions 1)–2) be satisfied for the equation (1). Then at $\varepsilon \in (0,\varepsilon_0](\varepsilon_0 > 0$ is small enough) equation (1) has a unique solution $y(t,\varepsilon) \in C^1([0,T],\mathbb{C})$; in this case, the estimate*

$$
||y(t,\varepsilon) - y_{\varepsilon N}(t)||_{C[0,T]} \leqslant c_N \varepsilon^{N+1}, \quad N = 0,1,2,\ldots,
$$

holds true; here $y_{\varepsilon N}(t)$ is narrowing $\left(\text{at } \tau = \dfrac{\psi(t)}{\varepsilon}\right)$ of the N-th partial sum of the series (6) (with coefficients $y_k\left(t,\tau\right) \in U$ satisfying iterative problems (9_k), and the constant $c_N > 0$ does not depend on ε at $\varepsilon \in (0,\varepsilon_0]$.

5. Construction of a solution of the first iterative problem

Using Theorem 1, we will try to find a solution for the first iterative problem (9_k). Since the right-hand side $h_1(t)$ to the equation (9_0), satisfies condition (11), this equation has (according to (15)) the solution in the space U in the form

$$
y_0\left(t,\tau\right) = y_0^{(0)}\left(t\right) + \alpha_1^{(0)}\left(t\right)e^{\tau_1} + h_{21}\left(t\right)\sigma_1 e^{\tau_2} + h_{31}\left(t\right)\sigma_2 e^{\tau_3}, \tag{19}
$$

where $\alpha_1^{(0)}\left(t\right) \in C^\infty\left([0,T],\mathbb{C}\right)$ is an arbitrary function,$y_0^{(0)}\left(t\right) = -\lambda_1^{-1}\left(t\right)h_1(t)$ and introduced the notations:

$$
h_{21}\left(t\right) = -\frac{1}{2i}\frac{h_2\left(t\right)}{\lambda_2\left(t\right) - \lambda_1\left(t\right)}, \quad h_{31}\left(t\right) = \frac{1}{2i}\frac{h_2\left(t\right)}{\lambda_3\left(t\right) - \lambda_1\left(t\right)}.
$$

Submitting (19) to the initial condition $y_0\left(0,0\right) = y^0$, will have

$$
\begin{gathered}
y_0^{(0)}\left(0\right) + \alpha_1^{(0)}\left(0\right) + h_{21}\left(0\right) + h_{31}\left(0\right) = y^0 \;\; \Leftrightarrow \\
\Leftrightarrow \;\alpha_1^{(0)}\left(0\right) = y^0 + \lambda_1^{-1}\left(0\right)h_1\left(0\right) - h_{21}\left(0\right) - h_{31}\left(0\right).
\end{gathered} \tag{20}
$$

For the complete computation of the function $\alpha_1^{(0)}(t)$, we proceed to the next iterative problem (9_1). Substituting solution (19) of the equation (9_0) into it, we arrive at the following equation:

$$Ly_1(t,\tau) = -\frac{d}{dt}y_0^{(0)}(t) - \frac{d}{dt}\left(\alpha_1^{(0)}(t)\right)e^{\tau_1} + \left[\frac{K(t,t)\,\alpha_1^{(0)}(t)}{\lambda_1(t)-\lambda_4(t)}e^{\tau_1} - \frac{K(t,0)\,\alpha_1^{(0)}(0)}{\lambda_1(0)-\lambda_4(0)}\right].$$

Performing scalar multiplication here, we obtain the ordinary differential equation

$$-\frac{d\alpha_1^{(0)}(t)}{dt} + \frac{K(t,t)}{\lambda_1(t)-\lambda_4(t)}\alpha_1^{(0)}(t) = 0.$$

Adding the initial condition (20) to this equation, we find $\alpha_1^{(0)}(t)$:

$$\alpha_1^{(0)}(t) = \left[y^0 + \lambda_1^{-1}(0)\,h_1(0) - h_{21}(0) - h_{31}(0)\right]\exp\left\{\int_0^t \frac{K(\theta,\theta)}{\lambda_1(\theta)-\lambda_4(\theta)}d\theta\right\}$$

and hence the solution (19) to problem (9_0) will be found uniquely in the space U. In this case, the leading term of the asymptotics is as follows:

$$\begin{aligned} y_{\varepsilon 0}(t) &= y_0^{(0)}(t) + h_{21}(t)\,e^{-\frac{i}{\varepsilon}\int_0^t \beta'(\theta)d\theta} + h_{31}(t)\,e^{+\frac{i}{\varepsilon}\int_0^t \beta'(\theta)d\theta} + \\ &+ \left[y^0 + \lambda_1^{-1}(0)\,h_1(0) - h_{21}(0) - h_{31}(0)\right]\mathrm{e}^{\int_0^t \frac{K(\theta,\theta)}{\lambda_1(\theta)-\lambda_4(\theta)}d\theta + \frac{1}{\varepsilon}\int_0^t \lambda_1(\theta)d\theta}. \end{aligned} \tag{21}$$

From the expression (21) for $y_{\varepsilon 0}(t)$ it is seen that the construction of the leading term of the asymptotics of the solution to problem (1) is significantly influenced by both the rapidly oscillating inhomogeneity and the kernel of the integral operator.

Example. Consider the integro-differential problem

$$\varepsilon\frac{dy}{dt} = -2y + \int_0^t e^{-\frac{t-s}{\varepsilon}}\left(t - t^2 + s^2\right)y(s,\varepsilon)\,ds + t^2 + t\sin\frac{t}{\varepsilon}, \quad y(0,\varepsilon) = y^0, \quad t\in[0,T]. \tag{22}$$

Here:

$$K(t,s) = t - t^2 + s^2, \quad h_1(t) = t^2, \quad h_2(t) = t, \quad \beta(t) = t,$$

$$\lambda_1 = -2, \quad \lambda_2 = -i, \quad \lambda_3 = +i, \quad \lambda_4 = -1, \quad K(t,t) = t.$$

Using formula (21), we calculate the leading term of the asymptotic solution to problem (22):

$$\begin{aligned} y_{\varepsilon 0}(t) &= \frac{t^2}{2} - \frac{t}{2i(-i+2)}e^{-\frac{it}{\varepsilon}} + \frac{t}{2i(i+2)}e^{\frac{it}{\varepsilon}} + y^0 e^{-\frac{t^2}{2} - \frac{2t}{\varepsilon}} = \\ &= \frac{t^2}{2} + \frac{1}{5}t\left(2\,\sin\left(\frac{t}{\varepsilon}\right) - \cos\left(\frac{t}{\varepsilon}\right)\right) + y^0 e^{-\frac{t^2}{2} - \frac{2t}{\varepsilon}}. \end{aligned} \tag{23}$$

If there were no rapidly oscillating term $t\sin\dfrac{t}{\varepsilon}$ in the right-hand side of equation (22), then the leading term of the asymptotics would have the form

$$\hat{y}_{\varepsilon 0}(t) = \frac{t^2}{2} + y^0 e^{-\frac{t^2}{2} - \frac{2t}{\varepsilon}}$$

and the limiting solution of problem (22) would be the function $\bar{\bar{y}}(t) = \dfrac{t^2}{2}$. In the presence of a rapidly oscillating inhomogeneity, as can be seen from formulas (23), the exact solution $y(t,\varepsilon)$ of the problem (22), leaving the value y^0 at $t = 0$, performs (when $\varepsilon \to +0$) fast oscillations around the function $\bar{\bar{y}}(t) = \dfrac{t^2}{2}$. The formation of a term $y^0 e^{-\frac{t^2}{2} - \frac{2t}{\varepsilon}}$ is influenced by the kernel $K(t,s)$ of the integral operator (more precisely: $K(t,t)$). In its absence, the specified term would have the form $y^0 e^{-\frac{2t}{\varepsilon}}$. Influence of a rapidly decreasing factor $e^{-\frac{t-s}{\varepsilon}}$ on the leading term of the asymptotics does not affect. It will be found when constructing the following asymptotic solution $y_{\varepsilon 1}(t)$.

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Алгоритм метода регуляризации для сингулярно возмущенного интегро-дифференциального уравнения с быстро убывающим ядром и с быстро осциллирующей неоднородностью

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Аннотация. В настоящей работе рассматривается сингулярно возмущенное интегро-дифференциальное уравнение с быстро осциллирующей правой частью, которое включает интегральный оператор с быстро меняющимся ядром. Основная цель данной работы — обобщить метод регуляризации Ломова и выявить влияние быстро осциллирующей правой части и быстро меняющегося ядра на асимптотику решения исходной задачи.

Ключевые слова: сингулярное возмущение, интегро-дифференциальное уравнение, быстро осциллирующая неоднородность, быстро меняющееся ядро, регуляризация, разрешимость итерационных задач.

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A Note on Explicit Formulas for Bernoulli Polynomials

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Abstract. For $r \in \left\{1, -1, \frac{1}{2}\right\}$, we prove several explicit formulas for the n-th Bernoulli polynomial $B_n(x)$, in which $B_n(x)$ is equal to a linear combination of the polynomials x^n, $(x+r)^n, \ldots, (x+rm)^n$, where m is any fixed positive integer greater than or equal to n.

Keywords: Appell polynomial, Bernoulli polynomial, binomial coefficients, combinatorial identities.

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1. Introduction and preliminaries

Let B_n and $B_n(x)$, $n = 0, 1, 2, \ldots$, be the Bernoulli numbers and polynomials defined by the generating functions

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$
$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!},$$

respectively. These numbers and polynomials play important roles in many different branches of mathematics [14]. Since their appearance in [4], many different explicit formulas for Bernoulli numbers and polynomials have been discovered throughout the years, see for example [5,8,11,12, 15,17]. Gould's article [8] is a remarkable retrospective concerning some old explicit formulas of Bernoulli numbers and contains a rich and very interesting bibliography. It emerges from Gould's study, that over time, it often happened that the same explicit formula concerning the n-th

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Bernoulli number B_n was rediscovered several times by different authors, each of these authors thinking to have discovered a new explicit formula. In [8], Gould reported that Munch [17] found an old result and published the formula in the form

$$B_n = \frac{1}{n+1} \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^j j^n \frac{\binom{n+1}{k-j}}{\binom{n}{k}}. \tag{1}$$

In 2016, Komatsu and Pita Ruiz [12] generalized Munch's formula (1) to Bernoulli polynomials by proving that

$$B_n(x) = \frac{1}{m+1} \sum_{k=0}^{m} \sum_{j=0}^{k} (-1)^{k+j} \frac{\binom{m+1}{j}}{\binom{m}{k}} (x+k-j)^n,$$

where $0 \leqslant n \leqslant m$. This formula can be written as

$$B_n(x) = \sum_{j=0}^{m} \lambda_j (x+j)^n, \tag{2}$$

where $\lambda_j = \lambda_j(m)$ depending only on m and j is given by

$$\lambda_j(m) = \frac{1}{m+1} \sum_{k=j}^{m} (-1)^j \frac{\binom{m+1}{k-j}}{\binom{m}{k}}.$$

More generally, it is easy to see that for any triplet (A, r, m) where $A = (A_n(x))$ is a sequence of Appell polynomials, $r \neq 0$ is a complex number and $m \geqslant 0$ is an any integer, there exists an unique sequence of complex numbers $\mu_j = \mu_j(A, r, m)$ such that for $0 \leqslant n \leqslant m$, we have

$$A_n(x) = \sum_{j=0}^{m} \mu_j (x+rj)^n.$$

Thus $\mu_j(B, 1, m) = \lambda_j(m)$ with $B = (B_n(x))$.

The paper is organized as follows: In the second section we give a generalization of Theorem 2 given by Adell and Lekuona [1]. In the third section, we prove some lemmas. In the fourth and the last section, we determine different expressions of $\mu_j(B, r, m)$ for $r \in \left\{1, -1, \frac{1}{2}\right\}$ and we derive explicit formulas for Bernoulli polynomials among them the identity (2).

2. Generalization of a theorem of Adell and Lekuona

Let $A = (A_n(x))_{n \geqslant 0}$ be a sequence of polynomials and $a_n = A_n(0)$. A is called an Appell sequence [2] if $a_0 \neq 0$ and if in addition, one of the following equivalent conditions is satisfied

$$A'_n(x) = nA_{n-1}(x), \quad n \geqslant 1,$$

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^{n-k},$$

$$\sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \right) e^{xt},$$

$$A_n(x) = \Omega_A(x^n) \text{ where } \Omega_A = \sum_{k=0}^{\infty} a_k \frac{D^k}{k!}, \tag{3}$$

D being the usual differential operator. $B = (B_n(x))_{n\geqslant 0}$ is obviously an Appell sequence for which we have

$$\Omega_B = \frac{D}{e^D - 1} = \sum_{k=0}^{\infty} B_k \frac{D^k}{k!}. \tag{4}$$

We also have

$$\Omega_B = \frac{\log(1+\Delta)}{\Delta} = \sum_{k=0}^{\infty} (-1)^k \frac{\Delta^k}{k+1}, \tag{5}$$

where Δ is the difference operator. More generally, if Δ_r is the difference operator defined by

$$\Delta_r(x^n) = (x+r)^n - x^n,$$

where $r \neq 0$ is a complex number, and Ω_B can be written as a series in Δ_r. To prove this, we need the use of Stirling numbers of the first and second kind, respectively denoted $s(n,k)$ and $S(n,k)$, $k = 0, 1, \dots, n$. These numbers are defined by [6]

$$(x)_n = \sum_{k=0}^{\infty} s(n,k)\, x^k, \quad x^n = \sum_{k=0}^{\infty} S(n,k)\, (x)_k\,,$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, or equivalently by their exponential generating functions

$$\frac{\log^k(t+1)}{k!} = \sum_{n=0}^{\infty} s(n,k)\, \frac{t^n}{n!}, \quad \frac{(e^t-1)^k}{k!} = \sum_{n=0}^{\infty} S(n,k)\, \frac{t^n}{n!}. \tag{6}$$

The following lemma proves that every operator written as a serie in D can be written as a serie in Δ_r and reciprocally.

Lemma 1.2. *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ two sequences of elements of $\mathbb{C}$. Then, the following equivalences hold*

$$\sum_{k=0}^{\infty} a_k \frac{D^k}{k!} = \sum_{k=0}^{\infty} b_k \frac{\Delta_r^k}{k!} \Leftrightarrow b_n = \sum_{k=0}^{n} s(n,k)\, \frac{a_k}{r^k}, \quad n \geqslant 0$$
$$\Leftrightarrow a_n = \sum_{k=0}^{n} r^n S(n,k)\, b_k, \quad n \geqslant 0.$$

Proof. We have

$$\Delta_r = e^{rD} - 1 \quad \text{and} \quad D = \frac{1}{r} \log(1+\Delta_r)\,.$$

The lemma results from the following two relations deduced from (6):

$$\frac{D^k}{k!} = \frac{1}{r^k} \frac{\log^k(1+\Delta_r)}{k!} = \frac{1}{r^k} \sum_{n=0}^{\infty} s(n,k)\, \frac{\Delta_r^n}{n!}, \tag{7}$$

$$\frac{\Delta_r^k}{k!} = \frac{\left(e^{rD}-1\right)^k}{k!} = \sum_{n=0}^{\infty} r^n S(n,k)\, \frac{D^n}{n!}. \tag{8}$$

□

Before stating our theorem, first we prove Lemma 1.2 which is an important consequence of the relation (7). We consider the translations τ_r of $\mathbb{C}[x]$ which are the operators defined by [18, p. 195]

$$\tau_r(x^n) = (x+r)^n, n \geqslant 0.$$

Then, we have

$$\tau_r = e^{rD}$$

and

$$\Delta_r^k = (\tau_r - 1)^k = \left(e^{rD} - 1\right)^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \tau_{rj}.$$

We deduce the useful following relation

$$\Delta_r^k(x^n) = (\tau_r - 1)^k (x^n) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x+rj)^n. \tag{9}$$

Lemma 2.2. *For $r \in \mathbb{C}^*$ and $m \in \mathbb{N}$, the family of polynomials*

$$F_m = \{x^m, (x+r)^m, (x+2r)^m, \ldots, (x+mr)^m\}$$

forms a base of the $\mathbb{C}$-vectorial space

$$\mathbb{C}_m[x] := \{P(x) \in \mathbb{C}[x] : \deg P(x) \leqslant m\}.$$

Proof. Indeed, the family F_m consists of $m+1$ vectors of vector space $\mathbb{C}_m[x]$ which is of dimension $m+1$. It is therefore sufficient to prove that x^k belongs to the subspace generated by F_m for $0 \leqslant k \leqslant m$. Using the relation (7), for $0 \leqslant k \leqslant m$, we have

$$\frac{D^{m-k}}{(m-k)!}(x^m) = \frac{1}{r^{m-k}} \sum_{\ell=0}^{m} s(\ell, m-k) \frac{\Delta_r^\ell}{\ell!}(x^m).$$

Using the relation (9), we deduce the identity verified for $0 \leqslant k \leqslant m$

$$\binom{m}{k} x^k = \sum_{j=0}^{m} \left(\sum_{\ell=j}^{m} s(\ell, m-k) \frac{(-1)^{\ell-j}}{r^{m-k}\ell!} \binom{\ell}{j} \right) (x+rj)^m,$$

through this demonstration we prove that x^k belongs to the vectorial subspace generated by F_m for $0 \leqslant k \leqslant m$ and this completes the proof. □

From Lemma 2.2, we can deduce that if $A = (A_n(x))$ is a sequence of Appell polynomials, for any complex $r \neq 0$ and for any integer $m \geqslant 0$, there exists unique sequence of complex numbers $\mu_j = \mu_j(A, r, m)$ such that for $0 \leqslant n \leqslant m$

$$A_n(x) = \sum_{j=0}^{m} \mu_j (x+rj)^n. \tag{10}$$

Indeed, from the fact that $\deg A_m(x) = m$, there exists unique sequence of complex numbers $\mu_j = \mu_j(A, r, m)$ such that

$$A_m(x) = \sum_{j=0}^{m} \mu_j (x+rj)^m. \tag{11}$$

Deriving $m-n$ times each of the two members of (11), we deduce that we have also (10).
The following theorem is a generalization of Theorem 2 given in [1]. The desired theorem follows from Lemma 1.2 and the characterization (3) of Appell sequence.

Theorem 1.2. *Let $r \in \mathbb{C}^*$. $A = (A_n(x))_{n\in\mathbb{N}}$ is an Appell sequence if, and only if, there exists a sequence $(b_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{C}$ such that $b_0 \neq 0$ and*

$$A_n(x) = \Omega(x^n),$$

where

$$\Omega = \sum_{k=0}^{\infty} \frac{b_k}{k!} \Delta_r^k. \tag{12}$$

Such sequence is defined by one of the following equivalence relations

$$b_n = \sum_{k=0}^{n} s(n,k) \frac{a_k}{r^k}, \tag{13}$$

$$\frac{a_n}{r^n} = \sum_{k=0}^{n} S(n,k) b_k, \tag{14}$$

where $a_k = A_k(0)$.

Corollary 1.2. *Let $r \in \mathbb{C}^*$ et $m \in \mathbb{N}$ and let $(A_n(x))_{n\in\mathbb{N}}$ be a sequence of Appell polynomials of $\mathbb{C}[x]$. Then, there exists a unique sequence $(\mu_j)_{0\leqslant j\leqslant m}$ of $\mathbb{C}$ such that*

$$\forall n \in \{0,1,\ldots,m\}, \quad A_n(x) = \sum_{j=0}^{m} \mu_j (x+jr)^n. \tag{15}$$

Furthermore, we have

$$\mu_j = \sum_{k=j}^{m} \frac{(-1)^{k-j}}{k!} \binom{k}{j} b_k, \ 0 \leqslant j \leqslant m \quad \text{with} \quad b_k = \sum_{\ell=0}^{k} s(k,\ell) \frac{A_\ell(0)}{r^\ell}. \tag{16}$$

Proof. The existence and uniqueness of the sequence $(\mu_j)_{0\leqslant j\leqslant m}$ come from Lemma 2.2, (15) is therefore proven. We deduce (16) from Theorem 1.2 and (9) noticing that

$$A_n(x) = \sum_{k=0}^{\infty} \frac{b_k}{k!} \Delta_r^k(x^n) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k!} \binom{k}{j} b_k (x+rj)^n.$$

□

To obtain the desired expressions and explicit formulas, we give some lemmas which will be used later.

3. Lemmas

In the following lemma, we express the operator Ω_B defined by (4) in function of the operators Δ, Δ_{-1} and $\Delta_{\frac{1}{2}}$.

Lemma 1.3. *We have*

$$\Omega_B = \frac{\log(1+\Delta)}{\Delta} = \sum_{k=0}^{\infty} (-1)^k \frac{\Delta^k}{k+1}, \tag{17}$$

$$\Omega_B = \frac{\log\left(1+\Delta_{-1}\right)}{\Delta_{-1}}\left(1+\Delta_{-1}\right) = 1 - \sum_{k=1}^{\infty}(-1)^k \frac{\Delta_{-1}^k}{k\,(k+1)}, \tag{18}$$

$$\Omega_B = \sum_{k=0}^{\infty}\left(\sum_{s=0}^{k}\frac{(-1)^k}{(s+1)\,2^{k-s}}\right)\Delta_{\frac{1}{2}}^k, \tag{19}$$

$$\Omega_B = \sum_{k=0}^{\infty}\left(\sum_{s=0}^{k}\frac{(-1)^k}{k+1}\binom{k}{s}^{-1}\right)\Delta_{\frac{1}{2}}^k. \tag{20}$$

Proof. (17) is the well know relations (5) and (17) is obtained by substituting Δ by $-\dfrac{\Delta_{-1}}{1+\Delta_{-1}}$ in relation (17). We obtain (19) noticing that $D = 2\log\left(1+\Delta_{\frac{1}{2}}\right)$ and thus

$$\Omega_B = \frac{D}{e^D-1} = \frac{1}{1+\frac{1}{2}\Delta_{\frac{1}{2}}}\frac{\log\left(1+\Delta_{\frac{1}{2}}\right)}{\Delta_{\frac{1}{2}}} = \sum_{k=0}^{\infty}\left(\sum_{s=0}^{k}\frac{(-1)^k}{(s+1)\,2^{k-s}}\right)\Delta_{\frac{1}{2}}^k.$$

The relation (20) follows from (19) using the following identity [9, p. 21] or [16, Theorem 1]

$$\sum_{s=0}^{k}\frac{1}{(s+1)\,2^{-s}} = \frac{2^k}{k+1}\sum_{s=0}^{k}\binom{k}{s}^{-1}.$$

□

The following lemmas will be useful to give explicit expressions for the coefficients $\mu_j\left(B,r,m\right)$ for $r \in \left\{1,-1,\frac{1}{2}\right\}$.

Lemma 2.3. *For all integers j and m such that $0 \leqslant j \leqslant m$, we have*

$$\sum_{k=j}^{m}\binom{k}{j}x^k = \sum_{k=j}^{m}\binom{m+1}{k-j}x^k\left(1-x\right)^{m-k}. \tag{21}$$

Proof. We have

$$\begin{aligned}\sum_{k=j}^{\infty}\binom{k}{j}x^k &= \frac{x^j}{(1-x)^{j+1}} = x^j\left(1-x\right)^{-j-1}\left(x+(1-x)\right)^{m+1} = \\ &= \sum_{k=0}^{m-j}\binom{m+1}{k}x^{k+j}\left(1-x\right)^{m-k-j} + \sum_{k=m+1-j}^{m+1}\binom{m+1}{k}x^{k+j}\left(1-x\right)^{m-k-j} = \\ &= \sum_{k=j}^{m}\binom{m+1}{k-j}x^k\left(1-x\right)^{m-k} + x^{m+1}\sum_{s=0}^{j}\binom{m+1}{j-s}\frac{x^s}{(1-x)^{s+1}}. \end{aligned}\tag{22}$$

We obtain (21) by writing the equality of the polynomial parts constituted by the terms of degree less than or equal to m in each of the two members of (22). □

Lemma 3.3. *For all integers j and m such that $0 \leqslant j \leqslant m$, we have*

$$\sum_{k=j}^{m}\frac{(-1)^j}{k+1}\binom{k}{j} = \sum_{k=j}^{m}\frac{(-1)^j}{m+1}\frac{\binom{m+1}{k-j}}{\binom{m}{k}}. \tag{23}$$

Proof. By Lemma 2.3 and the identity $\int\limits_0^1 t^k (1-t)^{m-k}\, dt = \frac{1}{m+1}\binom{m}{k}^{-1}$ it follows

$$\begin{aligned}\sum_{k=j}^{m}\frac{(-1)^j}{k+1}\binom{k}{j} &= \sum_{k=j}^{m}(-1)^j\binom{k}{j}\int_0^1 t^k dt = \\ &= \sum_{k=j}^{m}(-1)^j\binom{m+1}{k-j}\int_0^1 t^k (1-t)^{m-k}\, dt = \\ &= \sum_{k=j}^{m}\frac{(-1)^j}{m+1}\frac{\binom{m+1}{k-j}}{\binom{m}{k}}.\end{aligned}$$

□

Lemma 4.3. *For all integers j and m such that $0 \leqslant j \leqslant m$, we have*

$$\sum_{k=j}^{m}\frac{(-1)^j}{k+1}\binom{k}{j} = \sum_{k=j}^{m}\frac{(-1)^k}{k+1}\binom{m+1}{k+1}. \tag{24}$$

Proof. Let $[x^j](P(x))$ be the coefficient of x^j in the polynomial $P(x) = \sum\limits_{k=0}^{m}\frac{(1-x)^k}{k+1}$. On the one hand, one have

$$[x^j](P(x)) = \sum_{k=j}^{m}\frac{(-1)^j}{k+1}\binom{k}{j},\ 0 \leqslant j \leqslant m. \tag{25}$$

On the other hand, we have

$$\begin{aligned}[x^j]\, P(x) &= [x^j]\,\frac{1}{x-1}\sum_{k=0}^{m}\int_1^x (1-t)^k\, dt = \\ &= [x^j]\,\frac{-1}{x-1}\int_1^x \frac{(1-t)^{m+1}-1}{t} dt = \\ &= [x^j]\,\frac{1}{x-1}\sum_{k=0}^{m}(-1)^k\binom{m+1}{k+1}\int_1^x t^k dt = \\ &= [x^j]\sum_{k=0}^{m}\frac{(-1)^k}{k+1}\binom{m+1}{k+1}\frac{x^{k+1}-1}{x-1} = \\ &= [x^j]\sum_{k=0}^{m}\frac{(-1)^k}{k+1}\binom{m+1}{k+1}\sum_{\ell=0}^{k}x^\ell = \\ &= \sum_{k=j}^{m}\frac{(-1)^k}{k+1}\binom{m+1}{k+1}.\end{aligned} \tag{26}$$

By writing the equality of the two expressions (25) and (26) of $[x^j](P(x))$, we obtain (24). □

4. Explicit expressions for Bernoulli polynomials

In this section, we determine different expressions of $\mu_j(B, r, m)$ for $r \in \{1, -1, \frac{1}{2}\}$.

Theorem 1.4. (Case $r = \pm 1$**).** *For all integers m, n such that $0 \leqslant n \leqslant m$, we have*

$$B_n(x) = \sum_{j=0}^{m} \left(\sum_{k=j}^{m} \frac{(-1)^j}{k+1} \binom{k}{j} \right) (x+j)^n , \tag{27}$$

$$B_n(x) = \sum_{j=0}^{m} \left(\frac{1}{m+1} \sum_{k=j}^{m} (-1)^j \frac{\binom{m+1}{k-j}}{\binom{m}{k}} \right) (x+j)^n , \tag{28}$$

$$B_n(x) = \sum_{j=0}^{m} \left(\sum_{k=j}^{m} \frac{(-1)^k}{k+1} \binom{m+1}{k+1} \right) (x+j)^n , \tag{29}$$

$$B_n(x) = \frac{x^n}{m+1} + \sum_{j=1}^{m} \left(\sum_{k=j}^{m} \frac{(-1)^{j+1}}{k(k+1)} \binom{k}{j} \right) (x-j)^n . \tag{30}$$

Proof. From the relations (17) and (9) and noticing that $\Delta = \Delta_1$, we deduce that

$$B_n(x) = \Omega_B(x^n) = \sum_{k=0}^{m} \frac{(-1)^k}{k+1} \Delta^k (x^n) = \sum_{j=0}^{m} \left(\sum_{k=j}^{m} \frac{(-1)^j}{k+1} \binom{k}{j} \right) (x+j)^n .$$

This proves (27) and also gives

$$\mu_j(B,1,m) = \sum_{k=j}^{m} \frac{(-1)^j}{k+1} \binom{k}{j}. \tag{31}$$

From Lemma 3.3 and (31), we have

$$\mu_j(B,1,m) = \sum_{k=j}^{m} \frac{(-1)^j}{m+1} \frac{\binom{m+1}{k-j}}{\binom{m}{k}},$$

and we obtain (28). From Lemma 4.3 and (31), we have

$$\mu_j(B,1,m) = \sum_{k=j}^{m} \frac{(-1)^k}{k+1} \binom{m+1}{k+1},$$

and (29) follows. By the relations (18) and (9) we have

$$B_n(x) = \Omega_B(x^n) = \left(1 - \sum_{k=1}^{m} \frac{(-1)^k \Delta_{-1}^k}{k(k+1)} \right) (x^n) = x^n - \sum_{k=1}^{m} \sum_{j=0}^{k} \frac{(-1)^j}{k(k+1)} \binom{k}{j} (x-j)^n =$$

$$= \left(1 - \sum_{k=1}^{m} \frac{1}{k(k+1)} \right) x^n - \sum_{j=1}^{m} \left(\sum_{k=j}^{m} \frac{(-1)^j}{k(k+1)} \binom{k}{j} \right) (x-j)^n ,$$

from which the identity (30) follows. □

Theorem 2.4. *(Case $r = \frac{1}{2}$). For all integers m, n such that $0 \leqslant n \leqslant m$, we have*

$$B_n(x) = \sum_{j=0}^{m} \left(\sum_{k=j}^{m} \sum_{s=0}^{k} \frac{(-1)^j}{(s+1)\, 2^{k-s}} \binom{k}{j} \right) \left(x + \frac{j}{2} \right)^n , \tag{32}$$

$$B_n(x) = \sum_{j=0}^{m} \left(\sum_{k=j}^{m} \sum_{s=0}^{k} \frac{(-1)^j \binom{k}{j}}{(k+1)\binom{k}{s}} \right) \left(x + \frac{j}{2} \right)^n . \tag{33}$$

Proof. From the relations (19) and (9), we have

$$B_n(x) = \Omega_B(x^n) = \sum_{k=0}^{m} \sum_{s=0}^{k} \frac{(-1)^k}{(s+1)2^{k-s}} \Delta_{\frac{1}{2}}^{k}(x^n) = \sum_{k=0}^{m} \sum_{s=0}^{k} \frac{(-1)^k}{(s+1)2^{k-s}} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \tau_{\frac{j}{2}}(x^n) =$$
$$= \sum_{k=0}^{m} \sum_{s=0}^{k} \sum_{j=0}^{k} \frac{(-1)^j}{(s+1)2^{k-s}} \binom{k}{j} \left(x + \frac{j}{2} \right)^n .$$

We deduce (32). By the relations (20) and (9), we have

$$B_n(x) = \Omega_B(x^n) = \sum_{k=0}^{\infty} \left(\sum_{s=0}^{k} \frac{(-1)^k}{k+1} \binom{k}{s}^{-1} \right) \Delta_{\frac{1}{2}}^{k}(x^n) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} \sum_{j=0}^{k} \frac{(-1)^j \binom{k}{j}}{(k+1)\binom{k}{s}} \left(x + \frac{j}{2} \right)^n .$$

We deduce (33). □

The identity (28) was discovered in 2016 by Komatsu and Pita Ruiz and represents a generalization to Bernoulli polynomials of identity (1) that Munch [17] proved in 1959. Their proof is based on many combinatorial identities extracted from Gould tables [9, 10]. The identity (29) represents a generalization to Bernoulli polynomials of the following identity

$$B_n = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \sum_{j=0}^{k-1} j^n ,$$

which is an identity proved by Kronecker [13] in 1883, rediscovered by Bergmann and Gould [3,8] and generalized later by Funkuhara et al. [7] in 2018.

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Заметка о явных формулах для многочленов Бернулли

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Аннотация. При $r \in \left\{1, -1, \frac{1}{2}\right\}$ доказаны несколько явных формул для n-го многочлена Бернулли $B_n(x)$, в котором $B_n(x)$ равно линейной комбинации многочленов x^n, $(x+r)^n, \ldots, (x+rm)^n$, где m — любое фиксированное натуральное число, большее или равное n.

Ключевые слова: многочлен Аппеля, многочлен Бернулли, биномиальные коэффициенты, комбинаторные тождества.

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Some Observations on Koide Formula

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Abstract. The Koide parameter for leptons and quarks is discussed. A probabilistic approach is used to verify if the results obtained in the various cases are purely coincidental.

Keywords: Koide parameter, Lepton masses, probability theory.

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In 1982 Koide discovered the following relation among lepton masses of the Standard Model [1]

$$\frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} \equiv K \approx \frac{2}{3}. \tag{1}$$

The value of K lies in the range $1/3 \leqslant K < 1$ for arbitrary masess. In fact, consider the two vectors $u = (\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau})$ and $v = (1,1,1)$, then from Cauchy–Schwarz inequality one has $\left(\sum\limits_{i=1}^{3} u_i v_i\right)^2 \leqslant \left(\sum\limits_{i=1}^{3} u_i^2\right)\left(\sum\limits_{i=1}^{3} v_i^2\right)$, being equal to $\left(\sum\limits_{i=1}^{3} u_i^2\right) \times 3$, thus obtaining the lower bound of K. The upper bound 1 for K is found when considering the limit of one mass being much larger than the other two.

According to the PDG data [2] when using the measured values for lepton masses one has the following value of K

$$K = \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3} \times (0.999991 \pm 0.000010) \tag{2}$$

which is remarkably close to $2/3$, in the center of the allowed values for K.

The same relation has been observed for the quark masses [3–5], and from [2] one obtains, respectively for the light and heavy quarks, (m_u, m_d, m_s) and (m_c, m_b, m_t), the Koide parameters

$$K^q_{light} = 0.5622 \pm 0.0010, \tag{3}$$

$$K^q_{heavy} = \frac{2}{3} \times (1.0042 \pm 0.0020). \tag{4}$$

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As pointed out by Koide himself [6, 7], there are actually two values for the parameter K, depending on the chosen mass used for the calculation of (1). The renormalization group gives the evolution of the observed mass at a scale μ from the pole mass. For the leptons this function is given by [8]

$$m(\mu) = m_{pole}\left[1 - \frac{\alpha(\mu)}{\pi}\left(1 + \frac{3}{4}\ln\frac{\mu^2}{m_{pole}^2}\right)\right] \tag{5}$$

and at the Z scale $\mu = m_Z$ one obtains the Koide parameter for pole masses

$$K_{pole} = \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3} \times (0.998103 \pm 0.000010), \tag{6}$$

which is different from (2), not even being compatible with the value of 2/3. From (5) one could observe that in absence of the logarithmic term the two Koide parameters obtained from the observed and pole masses could still be compatible. The Sumino model [9] contains a mechanism that cancels out the logarithmic term, but at the expense of having anomalies. In another paper [10] Koide and Yamashita propose a modification of Sumino model in a SUSY scenario in order to avoid some shortcomings of the original model.

We will not purse further the solution of the Koide problem by proposing another model for the electroweak sector. Our approach is to proceed the other way round to verify whether the value of K in (1) is a coincidence. Namely, given the function of (1), find out what is the probability to obtain a value at least close to 2/3 for a generic choice of masses.

Dividing numerator and denominator of (1) by the largest mass, we end up with a function of only two variables, x and y, representing the fraction of two masses with respect to the largest one, $0 \leqslant x, y \leqslant 1$. After this operation we obtain the function f

$$f(x, y) = \frac{x + y + 1}{(\sqrt{x} + \sqrt{y} + 1)^2} \tag{7}$$

which average value is given by the expression

$$\langle f\rangle(x, y) = \frac{1}{xy}\int_0^x \int_0^y f(x', y')dx'dy' \tag{8}$$

and its variance

$$\sigma_f^2(x, y) = \frac{1}{xy}\int_0^x \int_0^y [f(x', y') - \langle f\rangle(x, y)]^2 dx'dy'. \tag{9}$$

It is possible to express (8) in a closed form

$$\begin{aligned}\langle f\rangle(x, y) = \frac{1}{xy}(&-\sqrt{x}\sqrt{y}\left(\left(4 - 3\sqrt{x}\right)\sqrt{y} + 4\left(x + \sqrt{x} + 1\right) + 4y\right) + \\ &+ 4\left(x\left(x + 2\sqrt{x} + 2\right) + y\left(y + 2\sqrt{y} + 2\right) - 1\right)\ln\left(\sqrt{x} + \sqrt{y} + 1\right) - \\ &- 4\left(x\left(x + 2\sqrt{x} + 2\right) - 1\right)\ln\left(\sqrt{x} + 1\right) - 4\left(y\left(y + 2\sqrt{y} + 2\right) - 1\right)\ln\left(\sqrt{y} + 1\right))\end{aligned} \tag{10}$$

as well as (9), however the latter is a very long and not very enlightning expression.

Calculating the average value of f and its standard deviation in the full mass space for which $0 \leqslant x, y \leqslant 1$ we obtain

$$\langle f\rangle(1, 1) = 36\ln 3 - 32\ln 2 - 17,$$

$$\langle f\rangle(1, 1) \pm \sigma_f(1, 1) = 0.369 \pm 0.041. \tag{11}$$

The value of $\langle f\rangle(1,1)$ obtained is very far from $2/3$ by more than $7\sigma_f$. A random choice for the values of three masses should give therefore a result for the Koide parameter very different from (1).

This result comes out when neglecting the hierarchy of lepton masses and the hierarchies for quark masses. Measured values give the following proportions of masses

$$\begin{aligned}(m_e : m_\mu : m_\tau) &\approx (\varepsilon^2 : \varepsilon : 1),\\ (m_c : m_b : m_t) &\approx (\varepsilon^2 : \varepsilon : 1),\\ (m_u : m_d : m_s) &\approx (\varepsilon : \varepsilon : 1),\end{aligned} \tag{12}$$

showing that the hierarchies among leptons and heavy quarks are similar, and as a consequence so are their Koide parameters. Using the hierarchies found in (12) for evaluating $\langle f\rangle$ and σ_f of leptons, heavy quarks and light quarks one has the following results:

$$\langle f\rangle\left(\frac{m_e}{m_\tau},\frac{m_\mu}{m_\tau}\right)\pm\sigma_f\left(\frac{m_e}{m_\tau},\frac{m_\mu}{m_\tau}\right)=0.752\pm0.065, \tag{13}$$

$$\langle f\rangle\left(\frac{m_c}{m_t},\frac{m_b}{m_t}\right)\pm\sigma_f\left(\frac{m_c}{m_t},\frac{m_b}{m_t}\right)=0.756\pm0.051, \tag{14}$$

$$\langle f\rangle\left(\frac{m_u}{m_s},\frac{m_d}{m_s}\right)\pm\sigma_f\left(\frac{m_u}{m_s},\frac{m_d}{m_s}\right)=0.663\pm0.061. \tag{15}$$

The first two averages, for leptons and heavy quarks, have the approximate value of $3/4$, and are distant from $2/3$ by less than $2\sigma_f$. The average for light quarks is also closer than $2\sigma_f$ to the value obtained in (3).

When the hierarchy of masses like the one illustrated in (12) is taken into account the average value $\langle f\rangle$ is already much closer to the expected result of (1) for the K parameter.

Supposing that the true value of leptonic Koide parameter is in the proximity of (1) we have determined the allowed range for the fraction of the two lighter masses x, y for which (7) assumes this value.

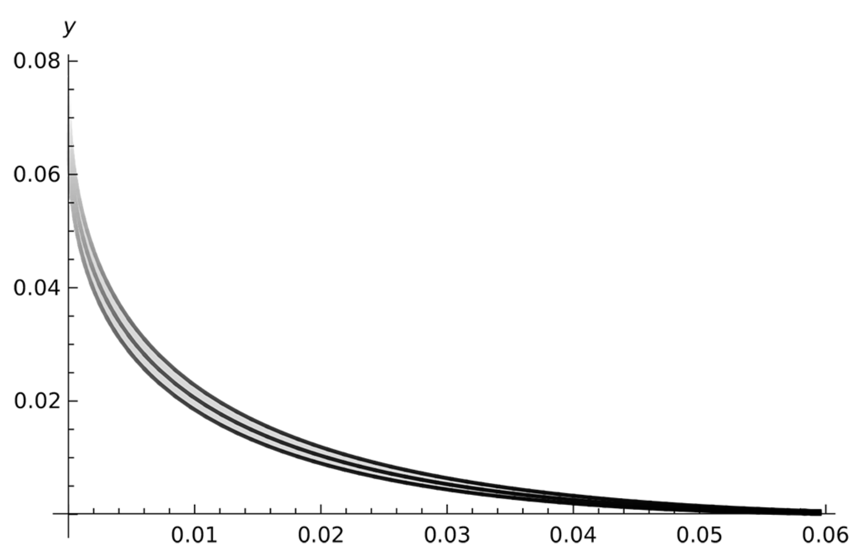

Fig. 1. Fraction of masses range of two lighter leptons for Koide parameter in the $2/3$ region. The x value of the second heaviest particle has the same upper limit found for leptons m_μ/m_τ

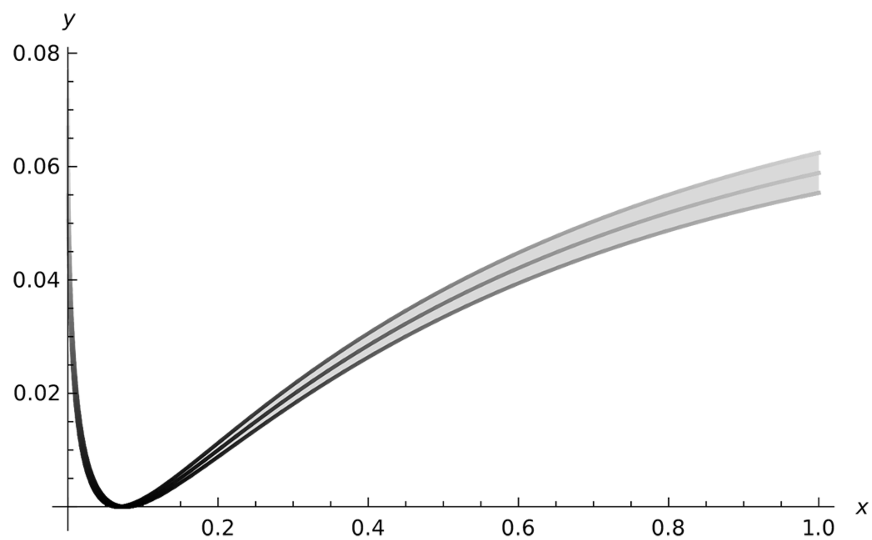

Fig. 2. Fraction of masses range of two lighter leptons for Koide parameter in the 2/3 region

In Figs. 1, 2 we show this x, y range for $K = 2/3 \pm 1\%$. The central curve represents the mean value, the shaded area the allowed fluctuation. In Fig. (1) the upper bound of x is given by the ratio m_μ/m_τ, and is clearly seen how in the allowed gray area the ratio y/x is approximately the same as found in (12). This situation is even more evident in Fig. (2) where the x axis has the $[0, 1]$ range while the y axis range is smaller by an order of magnitude.

We therefore conclude that it is possible to obtain a value close to (1) only if the random choice of masses is also constrained by the hierarchy found in (12). If the hierarchy is neglected and all mass terms are chosen to be completely independent from each other the result of 2/3 is quite different from the average value given in (11). Although there is a real discrepancy between the values found in (2) and (6) for pole and running masses respectively, it cannot be ruled out that those differences are due to other effects like missing perturbative terms of higher orders as well as model choice dependencies.

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Некоторые наблюдения над формулой Койде

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Аннотация. Обсуждается параметр Койде для лептонов и кварков. Вероятностный подход используется для проверки того, являются ли результаты, полученные в различных случаях, чисто случайными.

Ключевые слова: параметр Койде, лептонные массы, теория вероятностей.

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A Problem with Wear Involving Thermo-electro-viscoelastic Materials

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Abstract. In this paper, we consider a mathematical model of a contact problem in thermo-electro-viscoelasticity. The body is in contact with an obstacle. The contact is frictional and bilateral with a moving rigid foundation which results in the wear of the contacting surface. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed point arguments. We present a variational formulation of the problem, and we prove the existence and uniqueness of the weak solution.

Keywords: piezoelectric, temperature, thermo-electro-viscoelastic, variational inequality, wear.

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1. Introduction

In the recent years, piezoelectric contact problems have been of great interest to modern engineering. General models of electroelastic characteristics of piezoelectric materials can be found in [2, 7]. The problems of piezo- viscoelastic materials have been studied with different contact conditions within linearized elasticity in [1, 4] and with in nonlinear viscoelasticity in [9]. The modeling of these problems does not take into account the thermic effect. Mindlin [8] was the first to propose the thermo- piezoelectric model. The mathematical model which describes the frictional contact between a thermo-piezo- electric body and a conductive foundation is already addressed in the static case in [3]. Sofonea et al. considered in [6] the modeling of quasistatic viscoelastic problem with normal compliance friction and damage, they proved the existence and uniqueness of the weak solution,and they derived error estimates on the approximate solutions. In this paper,we consider a dynamic contact problem between a thermo-electro viscoelastic body and an electrically and thermally conductive rigid foundation which results in the wear of the contacting surface.

2. Problem statement

Problem P: Find a displacement field $u : \Omega \times [0.T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0.T] \to \mathbb{S}^d$, the an electric potentiel field $\varphi : \Omega \times [0.T] \to \mathbb{R}$, the an electric displacement field $D : \Omega \times [0.T] \to \mathbb{R}^d$,

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a temperature field $\theta : \Omega \times [0.T] \to \mathbb{R}$, and the wear $\omega : \Gamma_3 \times [0.T] \to \mathbb{R}_+$ such that

$$\sigma = \mathscr{A}(\varepsilon(u(t))) + \mathscr{G}(\varepsilon(\dot{u}(t))) - \xi^* E(\varphi) - \theta \mathscr{M} \quad \text{in } \Omega \times [0.T], \tag{2.1}$$

$$D = \beta E(\varphi) + \xi \varepsilon(u) - (\theta - \theta^*)\mathfrak{p} \quad \text{in } \Omega \times [0.T], \tag{2.2}$$

$$\rho \ddot{u} = Div\ \sigma + f_0 \quad \text{in } \Omega \times [0.T], \tag{2.3}$$

$$\text{div } D = q_0 \quad \text{in } \Omega \times [0.T], \tag{2.4}$$

$$\dot{\theta} - div(K \nabla \theta) = -\mathscr{M}.\nabla \dot{u} + q_1 \quad \text{in } \Omega \times [0.T], \tag{2.5}$$

$$u = 0 \text{ on } \Gamma_1 \times [0.T], \tag{2.6}$$

$$\sigma \nu = h \text{ on } \Gamma_2 \times [0.T], \tag{2.7}$$

$$\begin{cases} \sigma_\nu = -\alpha \left| \dot{u}_\nu \right|, \ \ |\sigma_\tau| = -\mu \sigma_\nu, \\ \sigma_\tau = -\lambda \left(\dot{u}_\tau - v^* \right), \ \lambda \geqslant 0, \ \dot{\omega} = -k v^* \sigma_\nu, \ k > 0 \ \text{ on } \Gamma_3 \times [0.T], \end{cases} \tag{2.8}$$

$$\varphi = 0 \text{ on } \Gamma_a \times [0.T], \tag{2.9}$$

$$D\nu = q_2 \text{ on } \Gamma_b \times [0.T], \tag{2.10}$$

$$-k_{ij} \frac{\partial \theta}{\partial \nu} \nu_j = k_e(\theta - \theta_R) - h_\tau(|\dot{u}_\tau|) \text{ on } \Gamma_3 \times [0.T], \tag{2.11}$$

$$\theta = 0 \ \text{ in } \Gamma_1 \cup \Gamma_2 \times [0.T], \tag{2.12}$$

$$u(0) = u_0, \ v(0) = v_0, \ \theta(0) = \theta_0, \ \omega(0) = \omega_0 \ \text{ in } \Omega. \tag{2.13}$$

Where (2.1), (2.2) are represent the thermo- electro-viscoelastic constitutive law of the material in which $\sigma = (\sigma_{ij})$ denotes the stress tensor, we denote $\varepsilon(u)$ (respectively; $E(\varphi) = -\nabla\varphi$, $\mathscr{A}$, $\mathscr{G}$, ξ, ξ^*, β, $\mathscr{M} = (m_{ij})$, $\mathfrak{p} = (p_i)$) the linearized strain tensor (respectively; electric field, the elasticity tensor, the viscosity nonlinear tensor, the third order piezoelectric tensor and its transpose, the electric permittivity tensor thermal expansion, pyroelectric tensor),the constant θ^*represents the reference temperature; (2.3) is represents the equation of motion where ρ represents the mass density; (2.4) is represents the equilibrium equation, we mention that $Div\sigma$, $divD$ are the divergence operators; (2.5) is represents the evolution equation of the heat field; (2.6) and (2.7) are are the displacement and traction boundary conditions; (2.8) is describes the frictional bilateral contact with wear described above on the potential contact surface Γ_3; (2.9), (2.10) are represent the electric boundary conditions; (2.11) is pointwise heat exchange condition on the contact surface, where k_{ij} are the components of the thermal conductivity tensor, v_j are the normal components of the outward unit normal v; k_e is the heat exchange coefficient, θ_R is the known temperature of the foundation; (2.12) represents the temperature boundary conditions. Finally, (2.13) is the initial data.

3. Variational formulation and preliminaries

For a weak formulation of the problem, first we itroduce some notation. The indices i, j, k, l range from 1 to d and summation over repeated indices is implied. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g: $u_{i.j} = \dfrac{\partial u_i}{\partial x_j}$. We also use the following notations

$$\begin{aligned} &H = \mathbb{L}^2(\Omega)^d = \{u = (u_i)/u_i \in \mathbb{L}^2(\Omega)\}, \\ &\mathscr{H} = \left\{\sigma = (\sigma_{ij})/\sigma_{ij} = \sigma_{ji} \in \mathbb{L}^2(\Omega)\right\}, \\ &H_1 = \{u = (u_i)/\varepsilon(u) \in \mathscr{H}\} = H^1(\Omega)^d, \\ &\mathscr{H}_1 = \{\sigma \in \mathscr{H}/Div\sigma \in H\}. \end{aligned}$$

The operators of deformation ε and divergence Div are defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad Div\sigma = (\sigma_{ij,j}).$$

The spaces $H, \mathscr{H}, H_1$ and $\mathscr{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(u,v)_H = \int_\Omega u_i v_i dx \ \forall\, u, v \in H,$$
$$(\sigma,\tau)_{\mathscr{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx \ \forall\, \sigma,\tau \in \mathscr{H},$$
$$(u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathscr{H}} \ \forall\, u,v \in H_1,$$
$$(\sigma,\tau)_{\mathscr{H}_1} = (\sigma,\tau)_{\mathscr{H}} + (Div\sigma, Div\tau)_H, \ \sigma,\tau \in \mathscr{H}_1.$$

We denote by $|\cdot|_H$ (respectively;$|\cdot|_{\mathscr{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathscr{H}1}$) the associated norm on the space H (respectively; $\mathscr{H}$, H_1 and $\mathscr{H}_1$).

Let $H_\Gamma = (H^{1/2}(\Gamma))^d$ and $\gamma : H^1(\Gamma)^d \to H_\Gamma$ be the trace map. For every element $v \in (H^1(\Gamma))^d$, we also use the notation v to denote the trace map γv of v on Γ, and we denote by v_ν and v_τ the normal and tangential components of v on Γ given by

$$v_\nu = v.\nu, \quad v_\tau = v - v_\nu \nu.$$

Similarly, for a regular (say $\mathscr{C}^1$) tensor field $\sigma : \Omega \to \mathbb{S}^d$ we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

We use standard notation for the $\mathbb{L}^p$ and the Sobolev spaces associated with Ω and Γ and, for a function $\psi \in H^1(\Omega)$ we still write ψ to denote it trace on Γ. We recall that the summation convention applies to a repeated index.

For the electric displacement field we use two Hilbert spaces

$$\mathscr{W} = \mathbb{L}^2(\Omega)^d, \quad \mathscr{W}_1 = \left\{D \in \mathscr{W}, \operatorname{div} D \in \mathbb{L}^2(\Omega)\right\}.$$

Endowed with the inner products

$$(D,E)_{\mathscr{W}} = \int_\Omega D_i E_i dx, \quad (D,E)_{\mathscr{W}_1} = (D,E)_{\mathscr{W}} + (\operatorname{div} D, \operatorname{div} E)_{\mathbb{L}^2(\Omega)}.$$

And the associated norm $|.|_{\mathscr{W}}$(respectively; $|.|_{\mathscr{W}_1}$). The electric potential field is to be found in

$$W = \left\{\psi \in H^1(\Omega), \psi = 0 \text{ on } \Gamma_a\right\}.$$

Since $meas\ (\Gamma_a) > 0$, the following Friedrichs–Poincaré's inequality holds, thus

$$|\nabla\psi|_{\mathscr{W}} \geqslant c_F\, |\psi|_{H^1(\Omega)}\, \forall\, \psi \in W, \tag{3.1}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. On W, we use the inner product given by

$$(\varphi,\psi)_W = (\nabla\varphi, \nabla\psi)_{\mathscr{W}},$$

and let $|.|_W$ be the associated norm. It follows from (3.1) that $|.|_{H^1(\Omega)}$ and $|.|_W$ are equivalent norms on W and therefore $(W, |.|_W)$ is a real Hilbert space.

Moreover, by the Sobolev trace Theorem, there exists a constant $\widetilde{c}_0$, depending only on Ω, Γ_a and Γ_3 such that

$$|\psi|_{\mathbb{L}^2(\Gamma_3)} \leqslant \widetilde{c}_0\, |\psi|_W \ \ \forall\psi \in W. \tag{3.2}$$

We define the space

$$E = \{\gamma \in H^1(\Omega)/\gamma = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}. \tag{3.3}$$

We recall that when $D \in \mathscr{W}_1$ is a sufficiently regular function,the Green's type formula holds

$$(D, \nabla\psi)_{\mathscr{W}} + (\operatorname{div} D, \psi)_{\mathbb{L}^2(\Omega)} = \int_\Gamma D\nu.\psi da\,. \tag{3.4}$$

When σ is a regular function, the following Green 's type formula holds

$$(\sigma, \varepsilon(v))_{\mathscr{H}} + (Div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \quad \forall v \in H_1. \tag{3.5}$$

Next, we define the space

$$V = \{u \in H_1 / \ u = 0 \ \text{ on } \Gamma_1\}.$$

Since $meas\ (\Gamma_1) > 0$, the following Korn's inequality holds

$$|\varepsilon(u)|_{\mathscr{H}} \geqslant c_K\, |v|_{H_1}\, \forall v \in V, \tag{3.6}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. On the space V we use the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathscr{H}}, \tag{3.7}$$

let $|.|_V$ be the associated norm. It follows by (3.6) that the norms $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V and therefore, $(V, |.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace Theorem, there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$|v|_{\mathbb{L}^2(\Gamma_3)^d} \leqslant c_0\, |v|_V\, \forall v \in V. \tag{3.8}$$

Finally, for a real Banach space $(X, |.|_X)$ we use the usual notation for the space $\mathbb{L}^p\,(0.T, X)$ and $W^{k.p}\,(0.T, X)$, where $1 \leqslant p \leqslant \infty$, $k = 1, 2, \ldots$; we also denote by $C\,(0.T, X)$ and $C^1\,(0.T, X)$ the spaces of continuous and continuously differentiable function on $[0.T]$ with values in X, with the respective norms:

$$|x|_{C(0.T,X)} = \max_{t\in[0.T]} |x\,(t)|_X\,,$$

$$|x|_{C^1(0.T,X)} = \max_{t\in[0.T]} |x\,(t)|_X + \max_{t\in[0.T]} \left|\dot{x}\,(t)\right|_X.$$

In what follows,we assume the following assumptions on the problem P.

The elasticity operator $\mathscr{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$

$$\begin{cases} (a)\ \ \exists\ L_{\mathscr{A}} > 0 \text{ such that } : |\mathscr{A}\,(x, \varepsilon_1) - \mathscr{A}\,(x, \varepsilon_2)| \leqslant L_{\mathscr{A}}\, |\varepsilon_1 - \varepsilon_2| \ \ \forall\ \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \\ \qquad a.\ e.\ x \in \Omega, \\ (c)\ \ \text{the mapping } x \to \mathscr{A}\,(x, \varepsilon) \ \text{ is lebesgue measurable in } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\ (d)\ \ \text{the mapping } x \to \mathscr{A}\,(x, 0) \in \mathscr{H}. \end{cases} \tag{3.9}$$

The viscosity operator $\mathscr{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a)\ \exists\ L_{\mathscr{G}} > 0 : |\mathscr{G}(x, \varepsilon_1) - \mathscr{G}(x, \varepsilon_2)| \leqslant L_{\mathscr{G}} |\varepsilon_1 - \varepsilon_1|, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d,\ \ p.p.\ x \in \Omega, \\ (b)\ \exists m_{\mathscr{G}} > 0 : (\mathscr{G}\,(x, \varepsilon_1) - \mathscr{G}\,(x, \varepsilon_2)\,, \varepsilon_1 - \varepsilon_2) \geqslant m_{\mathscr{G}}\, |\varepsilon_1 - \varepsilon_2|^2\,, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \\ (c)\ \text{the mapping } x \to \mathscr{G}\,(x, \varepsilon) \ \text{ is lebesgue measurable in } \Omega \text{ fo rall } \varepsilon \in \mathbb{S}^d, \\ (d)\ \text{the mapping } x \mapsto \mathscr{G}(x, 0) \in \ \mathscr{H}. \end{cases} \tag{3.10}$$

The thermal expansion tensor $\mathscr{M} = (m_{ij}) : \Omega \times \mathbb{R} \to \mathbb{R}$, and the pyroelectric tensor $\mathfrak{p} = (p_i) : \Omega \to \mathbb{R}^d$ satisfy,

$$\begin{cases} (a) \quad m_{ij} = m_{ji} \in \mathbb{L}^\infty(\Omega), \\ (b) \quad p_i \in \mathbb{L}^\infty(\Omega). \end{cases} \tag{3.11}$$

The piezoelectric tensor $\xi = (e_{ijk}) : \Omega \times S^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a)\ \xi = (e_{ijk}) : \Omega \times S^d \to \mathbb{R}^d, \\ (b)\ \xi\,(x, \tau) = (e_{ijk}\,(x)\,\tau_{jk}) \ \ \forall \tau = (\tau_{ij}) \in S^d,\ a.e.\, x \in \Omega, \\ (c)\ e_{ijk} = e_{ikj} \in \mathbb{L}^\infty\,(\Omega)\,. \end{cases} \tag{3.12}$$

The electric permittivity tensor $\beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$

$$\begin{cases} (a)\ \beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \\ (b)\ \beta\,(x, E) = (b_{ij}\,(x)\,E_j)\ \forall E = (E_i) \in \mathbb{R}^d,\ a.e.\, x \in \Omega, \\ (c)\ b_{ij} = b_{ji} \in \mathbb{L}^\infty\,(\Omega)\,, \\ (d)\ \exists\ m_\beta > 0 \text{ such that : } b_{ij}\,(x)\,E_i E_j \geqslant m_\beta\,|E|^2\ \ \forall E = (E_i) \in \mathbb{R}^d,\ x \in \Omega. \end{cases} \tag{3.13}$$

The function $h_\tau : \Gamma_3 \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies

$$\begin{cases} (a)\ \ \exists\ L_\tau > 0\ :\ |h_\tau(x, r_1) -\ h_\tau(x, r_2)| \leqslant L_h |r_1 - r_2|\ \forall\, r_1, r_2 \in \mathbb{R}_+,\ \ p.p.\, x \in \Gamma_3, \\ (b)\ \ x \longrightarrow h_\tau\,(x, r) \in \mathbb{L}^2(\Gamma_3) \text{ is lebesgue measurable in}\, \Gamma_3\ \forall r \in \mathbb{R}_+\,. \end{cases} \tag{3.14}$$

The mass density ρ satisfie

$$\rho \in \mathbb{L}^\infty(\Omega) \text{ there exists } \rho^* > 0 \text{ such that } \rho(x) \geqslant \rho^*,\ a.e.\, x \in \Omega. \tag{3.15}$$

The body forces, surface tractions,the densities of electric charges, and the functions α and μ, satisfy

$$\begin{cases} f_0 \in \mathbb{L}^2(0.T, H),\ h \in \mathbb{L}^2(0.T, \mathbb{L}^2(\Gamma_2)^d), \\ q_0 \in L^2(0.T, \mathbb{L}^2(\Omega)),\ q_2 \in L^2(0.T, \mathbb{L}^2(\Gamma_b)),\ q_1 \in L^2(0.T, \mathbb{L}^2(\Omega)),\ k_e \in \mathbb{L}^\infty(\Omega, \mathbb{R}_+). \\ \begin{cases} K = (k_{i,j});\ k_{ij} = k_{ji} \in \mathbb{L}^\infty(\Omega), \\ \forall c_k > 0, \forall (\xi_i) \in \mathbb{R}^d,\ k_{ij}\xi_i\xi_j \geqslant c_k \xi_i \xi_i. \end{cases} \\ \alpha \in \mathbb{L}^\infty(\Gamma_3),\ \alpha(x) \geqslant \alpha^* > 0,\ a.e. \text{ on } \Gamma_3, \\ \mu \in \mathbb{L}^\infty(\Gamma_3),\ \mu(x) > 0,\ a.e. \text{ on } \Gamma_3. \end{cases} \tag{3.16}$$

The initial data satisfy

$$u_0 \in V, \theta_0 \in \mathbb{L}^2(\Omega), \ \ \omega_0 \in \mathbb{L}^\infty(\Gamma_3). \tag{3.17}$$

We use a modified inner product on $H = \mathbb{L}^2(\Omega)^d$ given by

$$((u, v)) = (\rho u, v)_{\mathbb{L}^2(\Omega)^d}\, \forall\, u, v \in H.$$

That is, it is weighted with ρ. We let H be the associated norm

$$\|v\|_H = (\rho v, v)^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)^d}\ \forall v \in H.$$

We use the notation $(.,.)_{V' \times V}$ to represent the duality pairing between V' and V. Then, we have

$$(u, v)_{V' \times V} = ((u, v))\, \forall u \in H\, \forall v \in V.$$

It follows from assumption (3.15) that$\|.\|_H$ and $|.|_H$ are equivalent norms on H, and also the inclusion mapping of $(V, |.|_V)$ into $(H, \|.\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple $V \subset H = H' \subset V'$.

We define the function $f(t) \in V$ and $q : [0.T] \to W$ by

$$(f(t), v)_V = \int_\Omega f_0(t) v dx + \int_{\Gamma_2} h(t) v da \, \forall v \in V, \; t \in [0.T].$$

$$(q(t), \psi)_W = -\int_\Omega q_0(t) \psi dx + \int_{\Gamma_b} q_2(t) \psi da \, \forall \psi \in W, \; t \in [0.T].$$

for all $u, v \in V, \; \psi \in W$ and $t \in [0.T]$, and note that condition (3.14) imply that

$$f \in \mathbb{L}^2(0.T, V'), \quad q \in \mathbb{L}^2(0.T, W), \tag{3.18}$$

We consider the wear functional $j : V \times V \to \mathbb{R}$,

$$j(u, v) = \int_{\Gamma_3} \alpha \, |u_\nu| \, (\mu \, |v_\tau - v^*|) da \tag{3.19}$$

Finally, We consider $\phi : V \times V \to \mathbb{R}$,

$$\phi(u, v) = \int_{\Gamma_3} \alpha \, |u_\nu| \, v_\nu da \, \forall v \in V. \tag{3.20}$$

We define for all $\varepsilon > 0$

$$j_\varepsilon(g, v) = \int_{\Gamma_3} \alpha \, |g_\nu| \, (\mu \sqrt{|v_\tau - v^*|^2 + \varepsilon^2}) da \, \forall v \in V.$$

We define $Q : [0, T] \to E'$; $K : E \to E'$ and $R : V \to E'$ by

$$(Q(t), \mu)_{E' \times E} = \int_{\Gamma_3} k_e \theta_R(t) \mu ds + \int_\Omega q \mu dx \; \forall \mu \in E, \tag{3.21}$$

$$(K\tau, \mu)_{E' \times E} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \mu}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau \mu ds \; \forall \mu \in E, \tag{3.22}$$

$$(Rv, \mu)_{E' \times E} = \int_{\Gamma_3} h_\tau(|v_\tau|) \mu dx - \int_\Omega (\mathscr{M}.\nabla v) \mu dx \; \forall v \in V, \tau, \mu \in E. \tag{3.23}$$

Using the above notation and Green's formula, we derive the following variational formulation of mechanical problem P.

Problem PV : Find a displacement field $u : \Omega \times [0.T] \to V$, a stress field $\sigma : \Omega \times [0.T] \to \mathbb{S}^d$, the an electric potentiel field $\varphi : \Omega \times [0.T] \to \mathbb{R}$, the an electric displacement field $D : \Omega \times [0.T] \mathbb{R}^d$, a temperature field θ: $\Omega \times [0.T] \to \mathbb{R}$, and the wear $\omega : \Gamma_3 \times [0.T] \to \mathbb{R}_+$ such that

$$\begin{aligned} &(\ddot{u}(t), w - \dot{u}(t))_{V' \times V} + (\sigma(t), \varepsilon(w - \dot{u}(t)))_{\mathscr{H}} + j(\dot{u}, w) - j(\dot{u}, \dot{u}(t)) + \phi(\dot{u}, w) - \phi(\dot{u}, \dot{u}(t)) \geqslant \\ &\qquad \geqslant \big(f(t), w - \dot{u}(t)\big) \; \forall \, u, w \in V, \end{aligned} \tag{3.24}$$

$$(D(t), \nabla \psi)_{\mathbb{L}^2(\Omega)^d} + (q(t), \psi)_W = 0 \, \forall \, \psi \in W, \tag{3.25}$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{u}(t) + Q(t) \text{ on } E', \tag{3.26}$$

$$\dot{\omega} = -k v^* \sigma_\nu. \tag{3.27}$$

4. Existence and uniqueness result

Our main result which states the unique solvability of Problem are the following.

Theorem 4.1. *Let the assumptions* (3.9) − (3.17) *hold. Then, Problem PV has a unique solution* $(u, \sigma, \varphi, D, \omega)$ *which satisfies*

$$u \in C^1(0.T, H) \cap W^{1.2}(0.T, V) \cap W^{2.2}(0.T, V'), \tag{4.1}$$

$$\sigma \in \mathbb{L}^2(0.T, \mathscr{H}_1), \ Div\sigma \in \mathbb{L}^2(0.T, V'), \tag{4.2}$$

$$\varphi \in W^{1.2}(0.T, W), \tag{4.3}$$

$$D \in W^{1.2}(0.T, \mathscr{W}_1), \tag{4.4}$$

$$\theta \in W^{1.2}(0.T, E') \cap \mathbb{L}^2(0.T, E) \cap C(0.T, \mathbb{L}^2(\Omega)), \tag{4.5}$$

$$\omega \in C^1(0.T, \mathbb{L}^2(\Gamma_3)). \tag{4.6}$$

We conclude that under the assumptions (3.9)–(3.17), the mechanical problem (2.1)–(2.13) has a unique weak solution with the regularity (4.1)–(4.6).

The proof of this theorem will be carried out in several steps. It is based on arguments of first order evolution nonlinear inequalities, evolution equations, a parabolic variational inequality, and fixed point arguments.

First step: Let $g \in \mathbb{L}^2(0.T; V)$ and $\eta \in \mathbb{L}^2(0.T; V')$ are given, we deduce a variational formulation of Problem PV.

Problem $PV_{g\eta}$: Find a displacement field $u_{g\eta} : [0.T] \to V$ such that

$$\begin{cases} \big(\ddot{u}_{g\eta}(t), w - \dot{u}_{g\eta}(t)\big)_{V' \times V} + \big(\mathscr{G}\varepsilon(\dot{u}_{g\eta}(t)), w - \dot{u}_{g\eta}(t)\big)_{V' \times V} + (\eta(t), w - \dot{u}_{g\eta})_{V' \times V} j(g, w) - \\ \quad - j(g, \dot{u}_{g\eta}(t)) \geqslant (f(t), w - \dot{u}_{g\eta}(t))_{V' \times V} \ \forall\, w \in V, \ t \in [0.T], \\ u_{g\eta}(0) = u_0, \ \dot{u}_{g\eta}(0) = u_1. \end{cases} \tag{4.7}$$

We define $f_\eta(t) \in V$ for $a.e.\, t \in [0.T]$ by

$$(f_\eta(t), w)_{V' \times V} = (f(t) - \eta(t), w)_{V' \times V} \, \forall w \in V. \tag{4.8}$$

From (3.18), we deduce that

$$f_\eta \in \mathbb{L}^2(0.T, V'). \tag{4.9}$$

Let now $u_{g\eta} : [0.T] \to V$ be the function defined by

$$u_{g\eta}(t) = \int_0^t v_{g\eta}(s) ds + u_0 \, \forall t \in [0.T]. \tag{4.10}$$

We define the operator $G : V' \to V$ by

$$(Gv, w)_{V' \times V} = (\mathscr{G}\varepsilon(v(t)), \varepsilon(w))_{\mathscr{H}} \, \forall\, v, w \in V. \tag{4.11}$$

Lemma 4.2. *For all* $g \in \mathbb{L}^2(0.T, V)$ *and* $\eta \in \mathbb{L}^2(0.T, V')$, $PV_{g\eta}$ *has a unique solution with the regularity*

$$v_{g\eta} \in C(0.T, H) \cap \mathbb{L}^2(0.T, V) \ \text{and} \ \dot{v}_{g\eta} \in \mathbb{L}^2(0.T, V'). \tag{4.12}$$

Proof. The proof from nonlinear first order evolution inequalities, given in Refs (see [5,10]). □

In the second step, we use the displacement field $u_{g\eta}$ to consider the following variational problem.

Second step: We use the displacement field $u_{g\eta}$ to consider the following variational problem.

Problem $P_{\theta g\eta}$: Find $\theta_{g\eta} \in E$ such that

$$\dot{\theta}_{g\eta}(t) + K\theta_{g\eta}(t) = R\dot{u}_{g\eta}(t) + Q(t) \text{ on } E^{'}. \tag{4.13}$$

Lemma 4.3. *Under the assumptions* (3.9)–(3.17)*, the problem* $P_{\theta g\eta}$ *has a unique solution*

$$\theta_{g\eta} \in W^{1.2}(0.T, E^{'}) \cap \mathbb{L}^2(0.T, E) \cap C(0.T, \mathbb{L}^2(\Omega)).$$

Proof. Since we have the Gelfand triple $E \subset \mathbb{L}^2(\Omega) \subset E^{'}$. We use a classical result on first order evolution equations given in [11] to prove the unique solvability of (4.13). Now, we have $\theta_0 \in \mathbb{L}^2(\Omega)$. The operator K is a linear and continuous, so $a(\tau,\mu) = (K\tau,\mu)_{E'\times E}$ is bilinear, continuous and coercive, we use the continuity of $a(.,.)$ and from (3.16), we deduce that

$$a(\tau,\mu) = (K\tau,\mu)_{E'\times E} \leqslant |k|_{\mathbb{L}^\infty(\Omega)^{d\times d}} |\nabla\tau|_E |\nabla\mu|_E + |k_e|_{\mathbb{L}^\infty(\Gamma_3)} |\tau|_{\mathbb{L}^2(\Gamma_3)} |\mu|_{\mathbb{L}^2(\Gamma_3)} \leqslant C|\tau|_E |\mu|_E .$$

We have

$$a(\tau,\tau) = (K\tau,\tau)_{E'\times E} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial\tau}{\partial x_j}\frac{\partial\tau}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau^2 ds.$$

By (3.16) there exists a constants $C > 0$ such that

$$(K\tau,\tau)_{E'\times E} \geqslant C|\tau|_E^2 .$$

We have $\theta_0 \in \mathbb{L}^2(\Omega)$. Let

$$F(t) \in E' : \ (F(t),\tau)_{E'\times E} = (R\dot{u}_{g\eta}(t) + Q(t),\tau) \ \ \forall\tau \in E.$$

Under the assumptions(3.14), (3.16) we have

$$\int_0^T |R\dot{u}|_{E'}^2 \, dt < \infty, \quad \int_0^T |Q(t)|_{\mathbb{E}'}^2 \, dt < \infty, \quad \int_0^T |F|_{E'}^2 \, dt < \infty.$$

We find

$$F \in \mathbb{L}^2(0.T, E^{'}).$$

By a classical result on first order evolution equations

$$\exists\,!\theta_{g\eta} \in W^{1.2}(0.T, E^{'}) \cap \mathbb{L}^2(0.T, E) \cap C(0.T, \mathbb{L}^2(\Omega)).$$

□

Third step: We use the displacement field $u_{g\eta}$ and the temperature field $\theta_{g\eta}$ to consider the following variational problem.

Problem $PV_{g\eta}^{\varphi}$: Find an electric potential field $\varphi_{g\eta} : \Omega \times [0.T] \to W$ such that

$$\begin{aligned}(\beta\nabla\varphi_{g\eta}(t),\nabla\psi)_{\mathbb{L}^2(\Omega)^d} - (\xi\varepsilon(u_{g\eta}(t)),\nabla\psi)_{\mathbb{L}^2(\Omega)^d} - \big((\theta_{g\eta}(t) - \theta^*_{g\eta}(t))\mathfrak{p}_{\mathrm{i}},\nabla\psi\big)_{\mathbb{L}^2(\Omega)^d} = \\ = (q(t),\psi)_W \ \forall\psi \in W, \ t \in [0.T].\end{aligned} \tag{4.14}$$

We have the following result for $PV_{g\eta}^{\varphi}$

Lemma 4.4. *There exists a unique solution $\varphi_{g\eta} \in W^{1.2}(0.T, W)$ satisfies (4.14), moreover if φ_1 and φ_2 are two solutions to (4.14). Then, there exists a constants $c > 0$ sach that*

$$|\varphi_1(t) - \varphi_2(t)|_W \leqslant c\left(|u_1(t) - u_2(t)|_V + |\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}\right) \ \forall t \in [0.T]. \tag{4.15}$$

Proof. Let $t \in [0.T]$, we use the Riesz-fréchet representation theorem to define the operator $A_{g\eta} : W \to W$ by

$$\begin{aligned}(A_{g\eta}(t)\varphi, \psi)_W = (\beta\nabla\varphi_{g\eta}(t), \nabla\psi)_{\mathbb{L}^2(\Omega)^d} - (\xi\varepsilon(u_{g\eta}(t)), \nabla\psi)_{\mathbb{L}^2(\Omega)^d} - \\ - \left((\theta_{g\eta}(t) - \theta^*_{g\eta}(t))\mathfrak{p}_{\mathfrak{i}}, \nabla\psi\right)_{\mathbb{L}^2(\Omega)^d} \ \forall\, t \in [0.T]\,.\end{aligned} \tag{4.16}$$

For all φ, $\psi \in W$. Let φ_1, $\varphi_2 \in W$, then assumptions (3.11)–(3.13) imply

$$(A_{g\eta}(t)\varphi_1 - A_{g\eta}(t)\varphi_1, \varphi_1 - \varphi_2)_W \geqslant m_\beta\, |\varphi_1 - \varphi_2|^2_W\,. \tag{4.17}$$

In other hand, from (3.11)–(3.13), it results

$$(A_{g\eta}(t)\varphi_1 - A_{g\eta}(t)\varphi_1, \psi)_W \leqslant c_\beta\, |\varphi_1 - \varphi_2|_W\, |\psi|\, t_W,$$

where c_β is a positive constant which depends on β.

Thus

$$|A_{g\eta}(t)\varphi_1 - A_{g\eta}(t)\varphi_2|_W \leqslant c_\beta\, |\varphi_1 - \varphi_2|_W\,. \tag{4.18}$$

Inequalities (4.17) and (4.18) show that the operator $A_{g\eta}(t)$ is a strongly monotone, Lipschits continuous operator on W and, therefore, there exists a unique element $\varphi_{g\eta}\,(t) \in W$ such that

$$A_{g\eta}\varphi_{g\eta}(t) \ = q\,(t)\,. \tag{4.19}$$

We combine (4.16) and (4.17) and find that $\varphi_{g\eta}(t) \in W$ is the unique solution of the nonlinear variational equation (4.14).

We show that $\varphi_{g\eta} \in W^{1.2}(0.T, W)$. To this end, let $t_1, t_2 \in [0.T]$ and, for the sake of simplicity, we write $\varphi_{g\eta}\,(t_i) = \varphi_i,\ u_{g\eta}\,(t_i) = u_i,\ \theta_{g\eta}(t_i) = \theta_i,\ q\,(t_i) = q_i,$ for $i = 1.2$.

From (4.14), (3.11)–(3.13) it results

$$\begin{aligned}m_\beta\, |\varphi_1 - \varphi_2|^2_W \leqslant \\ \leqslant c(|u_1 - u_2|_V\, |\varphi_1 - \varphi_2|_W + |\theta_1 - \theta_2|_{\mathbb{L}^2(\Omega)}\, |\varphi_1 - \varphi_2|_W + |q_1 - q_2|_W\, |\varphi_1 - \varphi_2|_W).\end{aligned} \tag{4.20}$$

We find

$$\begin{aligned}|\varphi_1\,(t) - \varphi_2\,(t)|_W \leqslant \\ \leqslant c\left(|u_1\,(t) - u_2\,(t)|_V + |\theta_1\,(t) - \theta_2\,(t)|_{\mathbb{L}^2(\Omega)} + |q_1\,(t) - q_2\,(t)|_W\right) \ \forall t \in [0.T]\,.\end{aligned} \tag{4.21}$$

We also note that assumption (3.16), combined with definition imply that $q \in W^{1.2}\,(0.T, W)$. Since $u_{g\eta} \in C^1\,(0.T, V)$, $\theta_{g\eta} \in C^1\,(0.T, E)$, inequality (4.21) implies that $\varphi_{g\eta} \in W^{1.2}\,(0.T, W)$.

Let: $\eta_1, \eta_2 \in C\,(0.T, V')$, $g_1, g_2 \in C\,(0.T, V)$ and let $\varphi_{g\eta}\,(t_i) = \varphi_i,\ u_{g\eta}\,(t_i) = u_i$, we use (4.20) and arguments similar to those used in the proof of (4.15) to obtain

$$m_\beta\, |\varphi_1 - \varphi_2|_W \leqslant c(|u_1 - u_2|_V + |\theta_1 - \theta_2|_{\mathbb{L}^2(\Omega)}).$$

For all $t \in [0.T]$. This inequality leads to (4.15) which concludes the proof. □

Consider the operator

$$\begin{aligned}&\Lambda : \mathbb{L}^2(0.T, V \times V') \to \mathbb{L}^2(0.T, V \times V')\\&\Lambda(g,\eta) = (\Lambda_1(g), \Lambda_2(\eta)), \forall g \in \mathbb{L}^2(0.T, V), \forall \eta \in \mathbb{L}^2(0.T, V'),\\&\Lambda_1(g) = v_{g\eta},\\&(\Lambda_2(\eta), w)_{V'\times V} = (\mathscr{A}(\varepsilon(u(t))) - \theta\mathscr{M} - \xi^* E(\varphi), \varepsilon(w))_{\mathscr{H}} + \phi(g,w),\\&|\Lambda(g_2,\eta_2) - \Lambda(g_1,\eta_2)|^2_{\mathbb{L}^2(0.T;V\times V')} = |(\Lambda_1(g_2), \Lambda_2(\eta_2)) - (\Lambda_1(g_1), \Lambda_2(\eta_1))|^2_{\mathbb{L}^2(0.T;V\times V')} =\\&= |\Lambda_1(g_2) - \Lambda_1(g_1)|^2_{\mathbb{L}^2(0.T;V\times V')} + |\Lambda_2(\eta_2) - \Lambda_2(\eta_1)|^2_{\mathbb{L}^2(0.T;V\times V')}.\end{aligned} \tag{4.22}$$

We have the following result.

Lemma 4.5. *The mapping $\Lambda : \mathbb{L}^2(0.T, V\times V') \to \mathbb{L}^2(0.T, V\times V')$ has a unique element $(g^*, \eta^*) \in \mathbb{L}^2(0.T, V \times V')$, such that*

$$\Lambda(g^*, \eta^*) = (g^*, \eta^*). \tag{4.23}$$

Proof. Let $(g_i, \eta_i) \in \mathbb{L}^2(0.T, V \times V')$. We use the notation (u_i, φ_i). For $(g, \eta) = (g_i, \eta_i)$, $i = 1.2$. Let $t \in [0.T]$.

We have

$$\Lambda_1(g) = v_{g\eta}. \tag{4.24}$$

So

$$|g_1(t) - g_2(t)|^2_V \leqslant |v_1(t) - v_2(t)|^2_V. \tag{4.25}$$

It follows that

$$\begin{aligned}&(\dot v_1(t) - \dot v_2(t), v_1(t) - v_2(t)) + (\mathscr{G}\varepsilon(v_1(t)) - \mathscr{G}\varepsilon(v_2(t)), \varepsilon(v_1(t)) - \varepsilon(v_2(t)))+\\&+ (\eta_1(t) - \eta_2(t), v_1(t) - v_2(t)) + j(g_1, v_1(t)) - j(g_1, v_2(t)) - j(g_2, v_1(t)) + j(g_2, v_2(t)) \leqslant 0\end{aligned} \tag{4.26}$$

From the definition of the functional j given by (3.19), and using (3.8), (3.16) we have

$$j(g_2, v_2(t)) - j(g_2, v_1(t)) - j(g_1, v_2(t)) + j(g_1, v_1(t)) \leqslant C\,|g_1 - g_2|_V\,|v_1 - v_2|_V \tag{4.27}$$

Integrating the (4.26) inequality with respect to time, using the initial conditions $v_2(0) = = v_1(0) = v_0$, using (3.8), (3.10), (4.26) using Cauchy–Schwartz's inequality and the inequality $2ab \leqslant \dfrac{C}{m_{\mathscr{G}}}a^2 + \dfrac{m_{\mathscr{G}}}{C}b^2$ et $2ab \leqslant \dfrac{1}{m_{\mathscr{G}}}a^2 + m_{\mathscr{G}}b^2$, by Gronwall's inequality we find

$$|v_1(t) - v_2(t)|^2_V \leqslant C\left(\int_0^t |g_1(s) - g_2(s)|^2_V\, ds + \int_0^t |\eta_1(s) - \eta_2(s)|^2_{V'}\, ds\right). \tag{4.28}$$

So

$$|g_1 - g_2|^2_V \leqslant C\left(\int_0^t |g_1(s) - g_2(s)|^2_V\, ds + \int_0^t |\eta_1(s) - \eta_2(s)|^2_{V'}\, ds\right). \tag{4.29}$$

And, we have

$$(\Lambda_2(\eta), w)_{V'\times V} = \Big(\mathscr{A}(\varepsilon(u(t))) - \theta\mathscr{M} - \xi^* E(\varphi), \varepsilon(w)\Big)_{\mathscr{H}} + \phi(g,w). \tag{4.30}$$

From the definition of the functional ϕ given by (3.20), and using (3.8), (3.16) we have

$$\phi(g_1, v_2(t)) - \phi(g_1, v_1(t)) - \phi(g_2, v_2(t)) + \phi(g_2, v_1(t)) \leqslant C\,|g_1 - g_2|_V\,|v_1 - v_2|_V. \tag{4.31}$$

So

$$\begin{aligned}&|\eta_1(t) - \eta_2(t)|^2_{V'} \leqslant C(|u_1(t) - u_2(t)|^2_V + \textstyle\int_0^t |u_1(s) - u_2(s)|^2_V\, ds+\\&+ |\varphi_1(t) - \varphi_2(t)|^2_W + |\theta_1(t) - \theta_2(t)|^2_{\mathbb{L}^2(\Omega)} + |g_1(t) - g_2(t)|^2_V).\end{aligned} \tag{4.32}$$

By (4.26), using the inequality $2ab \leqslant \dfrac{2C}{m_{\mathscr{G}}}a^2 + \dfrac{m_{\mathscr{G}}}{2C}b^2$ and $2ab \leqslant \dfrac{2}{m_{\mathscr{G}}}a^2 + \dfrac{m_{\mathscr{G}}}{2}b^2$, we find

$$\begin{aligned} &\frac{1}{2}|v_1(t)-v_2(t)|_V^2 + m_{\mathscr{G}}\int_0^t |v_1(s)-v_2(s)|_V^2\,ds \leqslant \frac{1}{m_{\mathscr{G}}}\int_0^t |\eta_1(s)-\eta_2(s)|_{V'}^2\,ds + \\ &+\frac{m_{\mathscr{G}}}{4}\int_0^t |v_1(s)-v_2(s)|_V^2\,ds + C\times\frac{C}{m_{\mathscr{G}}}\int_0^t |g_1(s)-g_2(s)|_V^2\,ds + \\ &+ C\times\frac{m_{\mathscr{G}}}{4C}\textstyle\int_0^t |v_1(s)-v_2(s)|_V^2\,ds. \end{aligned} \tag{4.33}$$

So

$$\int_0^t |v_1(s)-v_2(s)|_V^2\,ds \leqslant C\bigg(\int_0^t |\eta_1(s)-\eta_2(s)|_{V'}^2\,ds + \int_0^t |g_1(s)-g_2(s)|_V^2\,ds\bigg). \tag{4.34}$$

By (3.26), we find

$$\begin{aligned} &\Big(\dot\theta_1(t)-\dot\theta_2(t),\theta_1(t)-\theta_2(t)\Big)_{E'\times E} + (K(\theta_1)-K(\theta_2),\theta_1(t)-\theta_2(t))_{E'\times E} = \\ &\qquad = (R(v_1)-R(v_2),\theta_1(t)-\theta_2(t))_{E'\times E}\,. \end{aligned} \tag{4.35}$$

We integrate (4.35) over [0.T] we use the initial conditions $\theta_1(0)=\theta_2(0)=\theta_0$, and we use the coercivety of K and the Lipschitz continuity of R to deduce that

$$\begin{aligned} &\frac{1}{2}|\theta_1(t)-\theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 + C\int_0^t |\theta_1(s)-\theta_2(s)|_{\mathbb{L}^2(\Omega)}^2\,ds \leqslant \\ &\leqslant C\left(\int_0^t |v_1(s)-v_2(s)|_V\,|\theta_1(s)-\theta_2(s)|_{\mathbb{L}^2(\Omega)}\,ds\right). \end{aligned}$$

using the inequality $2ab \leqslant \dfrac{1}{2}a^2 + 2b^2$, we find

$$\begin{aligned} &\frac{1}{2}|\theta_1(t)-\theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 + C\int_0^t |\theta_1(s)-\theta_2(s)|_{\mathbb{L}^2(\Omega)}^2\,ds \leqslant \\ &\leqslant \frac{C}{4}\int_0^t |v_1(s)-v_2(s)|_V\,ds + C\,|\theta_1(s)-\theta_2(s)|_{\mathbb{L}^2(\Omega)}\,ds. \end{aligned}$$

Also

$$|\theta_1(t)-\theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 \leqslant C\int_0^t |v_1(s)-v_2(s)|_V^2\,ds. \tag{4.36}$$

By (4.34), we find

$$|\theta_1(t)-\theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 \leqslant C\left(\int_0^t |\eta_1(s)-\eta_2(s)|_{V'}^2\,ds + \int_0^t |g_1(s)-g_2(s)|_V^2\,ds\right). \tag{4.37}$$

Also

$$\begin{aligned} &|u_1(t)-u_2(t)|_V^2 + \int_0^t |u_1(s)-u_2(s)|_V^2\,ds \leqslant \\ &\leqslant C\bigg(\int_0^t |v_1(s)-v_2(s)|_V^2 + \int_0^t |u_1(s)-u_2(s)|_V^2\bigg)ds. \end{aligned} \tag{4.38}$$

And

$$|u_1(t)-u_2(t)|_V^2 \geqslant 0.$$

$$\int_0^t\int_0^s |u_1(r)-u_2(r)|_V^2\,drds \geqslant 0.$$

So

$$
\begin{aligned}
&|u_1-u_2|_V^2+\int_0^t |u_1-u_2|_V^2\,ds \leqslant\\
&\qquad \leqslant C\int_0^t (|v_1(s)-v_2(s)|_V^2+|u_1(s)-u_2(s)|_V^2)ds+\int_0^t\int_0^s |u_1-u_2|_V^2\,drds,\\
&|u_1-u_2|_V^2+\int_0^t |u_1-u_2|_V^2\,ds \leqslant\\
&\qquad \leqslant C\int_0^t (|v_1(s)-v_2(s)|_V^2+|u_1(s)-u_2(s)|_V^2+\int_0^s |u_1(r)-u_2(r)|_V^2\,dr)ds
\end{aligned}
$$

by Gronwall's inequality, and using (4.34) we have

$$
|u_1-u_2|_V^2+\int_0^t |u_1-u_2|_V^2\,ds \leqslant C\bigg(\int_0^t |\eta_1(s)-\eta_2(s)|_{V'}^2\,ds+\int_0^t |g_1(s)-g_2(s)|_V^2\,ds\bigg) \tag{4.39}
$$

and using (4.29) and (4.32) we find

$$
|\Lambda(g_1,\eta_1)-\Lambda(g_2,\eta_2)|^2_{\mathbb{L}^2(0.T;V\times V')} \leqslant C\int_0^t |(g_1,\eta_1)-(g_2,\eta_2)|^2_{V\times V'}\,ds. \tag{4.40}
$$

Thus, for m sufficiently large, Λ^m is a contraction on $\mathbb{L}^2(0.T, V\times V')$ and so Λ has a unique fixed point in this Banach space. □

We consider the operator $\mathscr{L}: C(0.T,\mathbb{L}^2(\Gamma_3))\to C(0.T,\mathbb{L}^2(\Gamma_3))$,

$$
\mathscr{L}\omega(t) = -k\upsilon^*\int_0^t \sigma_\nu(s)ds\,\forall t\in[0.T]. \tag{4.41}
$$

Lemma 4.6. *The operator $\mathscr{L}: C(0.T,\mathbb{L}^2(\Gamma_3))\to C(0.T,\mathbb{L}^2(\Gamma_3))$ has a unique element $\omega^*\in C(0.T,\mathbb{L}^2(\Gamma_3))$, such that*

$$
\mathscr{L}\omega^*=\omega^*.
$$

Proof. Using (4.41), we have

$$
|\mathscr{L}\omega_1(t)-\mathscr{L}\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)} \leqslant k\upsilon^*\int_0^t |\sigma_1(s)-\sigma_2(s)|^2_{\mathscr{H}}\,ds. \tag{4.42}
$$

From (2.1),we have

$$
\begin{aligned}
|\mathscr{L}\omega_1(t)-\mathscr{L}\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)} \leqslant C\int_0^t (|u_1(t)-u_2(t)|_V^2+\int_0^t |u_1(s)-u_2(s)|_V^2\,ds+\\
+|\varphi_1(s)-\varphi_2(s)|_W^2+|\theta_1(s)-\theta_2(s)|^2_{\mathbb{L}^2(\Omega)})dt.
\end{aligned} \tag{4.43}
$$

By (4.15) and (4.36), we find

$$
\begin{aligned}
|u_1-u_2|_V^2+\int_0^t |u_1-u_2|_V^2\,ds+|\varphi_1(t)-\varphi_2(t)|^2_{\mathbb{L}^2(\Omega)}+|\theta_1(t)-\theta_2(t)|^2_{\mathbb{L}^2(\Omega)} \leqslant\\
\leqslant \int_0^t |v_1(s)-v_2(s)|_V^2\,ds.
\end{aligned} \tag{4.44}
$$

So

$$|u_1 - u_2|_V^2 + \int_0^t |u_1 - u_2|_V^2\, ds + |\varphi_1(t) - \varphi_2(t)|_W^2 + |\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 \leqslant \\ \leqslant C\left(\int_0^t |v_1(s) - v_2(s)|_V^2\, ds + |\omega_1(t) - \omega_2(t)|_{\mathbb{L}^2(\Gamma_3)}^2 \right).$$

So, we have

$$|u_1 - u_2|_V^2 + \int_0^t |u_1 - u_2|_V^2\, ds + |\varphi_1(t) - \varphi_2(t)|_W^2 + |\theta_1(t) - \theta_2(t)|_{\mathbb{L}^2(\Omega)}^2 \leqslant \\ \leqslant C\, |\omega_1(t) - \omega_2(t)|_{\mathbb{L}^2(\Gamma_3)}^2 . \tag{4.45}$$

By (4.43), we find

$$|\mathscr{L}\omega_1(t) - \mathscr{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)} \leqslant C \int_0^t |\omega_1(s) - \omega_2(s)|_{\mathbb{L}^2(\Gamma_3)}\, ds.$$

Thus, for m sufficiently large, $\mathscr{L}^m$ is a contraction on $C(0.T, \mathbb{L}^2(\Gamma_3))$ and so $\mathscr{L}$ has a unique fixed point in this Banach space. □

Now, we have all the ingredients to prove Theorem 4.1.

Existence

Let $(g^*, \eta^*) \in \mathbb{L}^2(0.T, V \times V')$ be the fixed point of Λ defined by (4.22), let $\omega^* \in C(0.T, \mathbb{L}^2(\Gamma_3))$ be the fixed point of $\mathscr{L}\omega^*$ defined by (4.41), and let $(u, \theta, \varphi) = (u_{g^*\eta^*}, \theta_{g^*\eta^*}, \varphi_{g^*\eta^*})$ be the solutions of Problems $PV_{g^*\eta^*}$, $P_{\theta g\eta}$ and $PV^{\varphi}_{g^*\eta^*}$. It results from (4.7), (4.13) and (4.16) that $(u_{g^*\eta^*}, \theta_{g^*\eta^*}, \varphi_{g^*\eta^*})$ is the solutions of Problems *PV*. Properties (4.1)–(4.6) follow from Lemmas 4.2, 4.3 and 4.4 .

Uniqueness

The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operators $\Lambda, \mathscr{L}$ defined by (4.22), (4.41), and the unique solvability of the Problem $PV_{g\eta}$, $P_{\theta g\eta}$ and $PV^{\varphi}_{g\eta}$ which completes the proof.

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Проблема износа термоэлектровязкоупругих материалов

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Аннотация. В данной работе рассматривается математическая модель контактной задачи термоэлектровязкоупругости. Тело соприкасается с препятствием. Контакт фрикционный и двусторонний с подвижным жестким основанием, что приводит к износу контактирующей поверхности. Устанавливается вариационная формулировка модели и доказывается существование единственного слабого решения задачи. Доказательство основано на классическом факте существования и единственности параболических неравенств, дифференциальных уравнений и аргументов с фиксированной точкой. Приводится вариационная постановка задачи, доказывается существование и единственность слабого решения.

Ключевые слова: пьезоэлектрик, температура, термоэлектровязкоупругость, вариационное неравенство, износ.

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Morera's Boundary Theorem in Siegel Domain of the First Kind

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Abstract. The paper considers the realization of the matrix unit polydisk in the form of a Siegel domain of the first kind and proves the boundary analogue of Morera's theorem.

Keywords: matrix unit polydisk, automorphism, Poisson kernel, holomorphic function, holomorphic extension.

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Introduction

In multidimensional complex analysis, the theory of integral representations has numerous applications in the study of functions with the one-dimensional holomorphic continuation property and in the proof of boundary analogues of Morera's theorem. Here it is worth noting the works of A. M. Kytmanov, S. G. Myslivets, S. Kosbergenov ([1–3]).The monograph [4] provides a detailed overview of the results in this direction obtained by various authors in recent years. In this article, using the properties of the Poisson integral, we prove a boundary version of Morer's theorem for one Siegel domain of the 1st kind ([5]). It asserts the possibility of a holomorphic continuation of the functions f from the boundary ∂D of the domain $D \subset \mathbb{C}^n$, provided that the integrals of f are equal to zero along the boundaries of analytical disks lying on ∂D.

The unit disc and its various multidimensional generalizations (unit n-dimensional ball, polydisk, matrix unit disc, classical domains of types according to Cartan's classification, matrix ball) are well studied: by now, many important questions of multidimensional complex analysis have been solved, such as description of automorphism groups, obtaining integral formulas of Cauchy–Szego, Bergman, Poisson type, proving necessary and sufficient conditions for holomorphic extendability of functions from the boundary, etc. Extensive results obtained in these areas are presented in the monographs [4] and [6].

Quite often, problems posed for a unit disc on a plane are transferred to the upper half-plane using the Cayley transform

$$w = \frac{i(1+z)}{1-z}.$$

In this regard, it is urgent to find multidimensional analogs of the formula for the realization of the "unit disc – upper half-plane" type ([7]). We consider the realization of the matrix unit polydisk in the form of the Siegel domain of the first kind and the transformation of the invariant Poisson kernel for such a realization.

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1. Realization of a matrix unit polydisk in the form of a Siegel domain of the 1st kind

Let be

$$D = \{U = (U_1, \dots, U_n) \in \mathbb{C}^n[m \times m] : \operatorname{Im} U_j > 0,\ j = \overline{1, n}\,\},$$

U_j is a matrix of order $[m \times m]$ with elements from $\mathbb{C}$. The skeleton of this domain is denoted

$$\Gamma = \{\operatorname{Im} Uj = 0,\ j = \overline{1, n}\,\} = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$$

Domain D is a Siegel domain of the 1st kind ([5]). In particular, for $m = n = 1$ the domain D is reduced to the upper half-plane, and the skeleton Γ coincides with the real axis.

Domain T of space $\mathbb{C}^n[m \times m]$:

$$T = \{Z = (Z_1, \dots, Z_n) :\ Z_j Z_j^* < E,\ j = \overline{1, n}\,\},$$

where Z_j is a matrix of order $[m \times m]$ with elements from $\mathbb{C}$, and E is the identity matrix of order m, is called *matrix unit polydisk*.

The boundary ∂T is a union of surfaces

$$\gamma^\nu = \{\, Z = (Z_1, \dots, Z_n) : Z_\nu Z_\nu^* = E,\ Z_\mu Z_\mu^* \leqslant E,\ \nu \neq \mu\,\},$$

each of which is a $(2nm^2 - 1)$-dimensional surface (since the $2nm^2$ coordinates of the point Z are related by one real relation $\det(E - Z_\nu Z_\nu^*) = 0$). Therefore, the entire boundary

$$\partial T = \bigcup_{\nu=1}^{n} \gamma^\nu$$

is $(2nm^2 - 1)$-dimensional. For the disc $\tau_j = Z_j Z_j^* < E$, the skeleton is the set of all unitary matrices $S_j = Z_j Z_j^* = E$ ([8, page 10]). Then the skeleton $S(T)$ of the domain T, defined as the Cartesian product of the discs τ_j, is the Cartesian product of all S_j that is,

$$S(T) = \{Z_j Z_j^* = E,\ j = \overline{1, n}\} = S_1 \times S_2 \times \cdots \times S_n \subset \partial T.$$

The dimension of the skeleton is nm^2.

For $m = n = 1$, the domain of T is reduced to a unit disc, and the skeleton of $S(T)$ is a unit circle from $\mathbb{C}$.

By $\Phi = (\Phi^1, \dots, \Phi^n)$ we denote the transformation, where

$$U_j = \Phi^j(Z) = i(E + Z_j)(E - Z_j)^{-1}, \quad j = \overline{1, n}, \tag{1}$$

which is a biholomorphic map of T onto D, with $S(T)$ being mapped to Γ.

It is known that

$$\Phi_{A_j}(Z_j) = R_j^{-1}(E - Z_j A_j^*)^{-1}(Z_j - A_j)Q_j$$

is an automorphism of the generalized circle $Z_j Z_j^* < E$ taking the point A_j to 0 ([8]). Then the mapping

$$\Phi_A(Z) = (\Phi_A^1(Z), \dots, \Phi_A^n(Z)), \quad \Phi_A^j(Z) = \Phi_{A_j}(Z_j)$$

is an automorphism of the domain T that maps the point $A = (A_1, \dots, A_n)$ to $0 = (0, \dots, 0)$.

Using the transformations Φ and Φ_A, we define the following transformation

$$\Psi_B = \Phi \circ \Phi_A \circ \Phi^{-1}, \quad B = \Phi(A)$$

which is an automorphism of the domain D taking the point B to the point $(iE, \dots, iE)$.

2. On the Poisson kernel in the Siegel domain of the 1st kind

Let $\dot{Z}$ be the volume element in $S(T)$, and $\dot{U}$ the volume element in Γ.

We have

$$\dot{Z} = \prod_{j=1}^{n} \dot{Z}_j, \quad \dot{U} = \prod_{j=1}^{n} \dot{U}_j,$$

where $\dot{Z}_j$ is the volume element in $Z_j Z_j^* = E$, and $\dot{U}_j$ is volume element in $\mathrm{Im} U_j = 0$.

Because

$$\dot{Z}_j = 2^{m^2} |\det(U_j + iE)|^{-2m} \dot{U}_j, ([8])$$

then $\dot{Z}$ can be represented as

$$\dot{Z} = \dot{Z}_1 \wedge \cdots \wedge \dot{Z}_n = 2^{nm^2} \prod_{j=1}^{n} |\det(U_j + iE)|^{-2m} \dot{U}.$$

So the following is true

Lemma 1. *The following relation holds*

$$\dot{Z} = 2^{nm^2} \prod_{j=1}^{n} |\det(U_j + iE)|^{-2m} \dot{U}.$$

The Poisson kernel for the generalized upper half-plane $\mathrm{Im} U_j > 0$ has the form (see [8])

$$P(U_j, B_j) = c \cdot \frac{\det^m(B_j - B_j^*)}{|\det(U_j - B_j^*)|^{2m}}, \quad \mathrm{Im} B_j > 0, \quad \mathrm{Im} U_j = 0,$$

here c is some constant.

Since the domain D is defined as the Cartesian product of the generalized upper half-planes, then using the formula of repeated integration (Fubini's theorem) we obtain that the Poisson kernel for the domain D will have the form:

$$P_D(U, B) = \prod_{j=1}^{n} P(U_j, B_j) = c^n \prod_{j=1}^{n} \frac{\det^m(B_j - B_j^*)}{|\det(U_j - B_j^*)|^{2m}},$$

where $B \in D$, $U \in \Gamma$.

The invariant Poisson kernel for the domain T is defined in the same way:

$$P(W, A) = \prod_{j=1}^{n} P(W_j, A_j) = \prod_{j=1}^{n} \frac{\det^m(E - A_j A_j^*)}{|\det(E - A_j W_j^*)|^{2m}},$$

where $A \in T,\ W \in S(T)$.

Пусть $B = \Phi(A)$, $U = \Phi(W)$.

Lemma 2. *Mapping Φ transforms the Poisson kernel as follows:*

$$P(\Phi^{-1}(U), \Phi^{-1}(B)) = \frac{(-2i)^{nm}}{c^n} \prod_{j=1}^{n} |\det(U_j + iE)|^{2m} P_D(U, B).$$

Proof. For each component of the mapping Φ with fixed j, we prove

$$P(\Phi_j^{-1}(U_j), \Phi_j^{-1}(B_j)) = \frac{(-2i)^m}{c} P(U_j, B_j)|\det(U_j + iE)|^{2m},$$

whence passing to the product over $j = \overline{1, n}$ we obtain the required equality. Finding the converse from (1), we have

$$W_j = (U_j + iE)^{-1}(U_j - iE), \;\; A_j = (B_j + iE)^{-1}(B_j - iE).$$

Then

$$E - A_j W_j^* = E - (B_j + iE)^{-1}(B_j - iE)(U_j + iE)(U_j - iE)^{-1} =$$
$$= (B_j + iE)^{-1}\big((B_j + iE)(U_j - iE) - (B_j - iE)(U_j + iE)\big)(U_j - iE)^{-1} =$$
$$= 2i(B_j + iE)^{-1}(U - B)(U_j - iE)^{-1}.$$

Similar

$$E - W_j A_j^* = -2i(U_j + iE)^{-1}(U - B^*)(B_j^* - iE)^{-1},$$
$$E - A_j A_j^* = -2i(B_j + iE)^{-1}(B - B^*)(B_j^* - iE)^{-1}.$$

Substituting these found into the expression for $P(W, A)$ and using the properties of the determinants of the matrices, we obtain the required equality. Lemma 2 is proved. □

3. Morera's boundary theorem

Consider the following embedding of the circle $\triangle = \{t \in \mathbb{C} : \; |t| < 1\}$ into D:

$$\{\zeta \in \mathbb{C}^n[m \times m] : \zeta_j = t\Lambda_j, \, j = 1, \ldots, n, \, t \in \triangle\}, \tag{2}$$

where $\Lambda = (\Lambda_1, \ldots, \Lambda_n) \in \Gamma$. If Ψ is an arbitrary automorphism of the domain D, then set (2) under the action of this automorphism goes to some analytic disc with boundary on Γ.

Theorem 1. *Let f be a continuous bounded function on Γ. If the function f satisfies the condition*

$$\int_{\partial\triangle} f(\Psi(\Lambda^0 t))\, dt = 0 \tag{3}$$

for all automorphisms Ψ of the domain $\mathcal{D}$ and fixed $\Lambda^0 \in \Gamma$, then the function f extends holomorphically in D to a function of class $\mathcal{H}^\infty(D)$ continuous up to Γ.

Proof. Since Γ is invariant under unitary transformations, condition (3) will hold for arbitrary $\Lambda \in \Gamma$.

Consider an automorphism Ψ_B that takes the point B from D to the point $(iE, \ldots, iE)$:

$$\Psi_B = \Phi \circ \Phi_A^{-1} \circ \Phi^{-1}.$$

Then substituting the automorphism Ψ_B into condition (3) instead of Ψ, we obtain

$$\int_{\partial\triangle} f(\Phi \circ \Phi_A^{-1} \circ \Phi^{-1}(\Lambda t))\, dt = 0. \tag{4}$$

We denote $\nu = \Phi^{-1}(\Lambda)$. Then (4) can be written as

$$\int_{\partial\triangle} f(\Phi \circ \Phi_A^{-1}(\nu t))\, dt = 0. \tag{5}$$

We parametrize the manifold S_j as follows:

$$\zeta_j = t\nu_j, \quad t = e^{i\phi}, \quad 0 \leqslant \phi \leqslant 2\pi, \ \nu_j \in S'_j$$

if $\zeta_j \in S_j$. Here S'_j is the group of special unitary matrices, that is, $\det \nu_j = 1$. Volume element $d\mu_j$ on the manifold S_j can be written in the form

$$d\mu_j = \frac{d\phi}{2\pi} \wedge d\mu_0(\nu_j) = \frac{1}{2\pi i}\frac{dt}{t} \wedge d\mu_0(\nu_j),$$

where $d\mu_0(\nu_j)$ is a differential form that defines a positive measure on S'_j.

Multiplying (5) by $d\mu_0$ and integrating over S'_j, from (5) we obtain

$$\int_{S_j} f(\Phi \circ \Phi_A^{-1}(\zeta_j))\zeta_j^k \, d\mu_j(\zeta_j) = 0. \tag{6}$$

where ζ_j^k is a component of the vector ζ_j $(k = 1, \ldots, m^2)$.

By Fubini's theorem, we obtain from this that

$$\int\limits_{S=S_1\times\cdots\times S_n} f(\Phi \circ \Phi_A^{-1}(\zeta))\zeta^k \, d\mu(\zeta) = 0, \tag{7}$$

where ζ^k is the k-th component of the vector ζ.

Let's make the change of variables $W = \Phi_A^{-1}(\zeta)$. Then (7) becomes the condition

$$\int_S f(\Phi(W))\Phi_{A,l}^k(W) \, d\mu(\Phi_A(W)) = 0. \tag{8}$$

Since ([9, Lemma 3.4])

$$d\mu_j(\Phi_A^j(W)) = P(W_j, A_j) \, d\mu_j(W_j),$$

then

$$d\mu(\Phi_A(W)) = P(W, A) \, d\mu(W).$$

Then from (8) we obtain

$$\int_S f(\Phi(W))\Phi_{A,l}^k(W) \, P(W, A) \, d\mu(W) = 0 \quad (k = 1, \ldots, m^2). \tag{9}$$

As you know ([8])

$$\Phi_{A,l}(W) = \Phi_A^l(W) = R_l^{-1}(E - W_l A_l^*)^{-1}(W_l - A_l)Q_l,$$

where R_l, Q_l are nonsingular and depend only on A_l. Therefore, if condition (9) is satisfied for the components of the mapping $\Phi_{A,l}$, then the same condition will be satisfied for the components of the mapping

$$\varphi_{A,l}(W) = (E - W_l A_l^*)^{-1}(W_l - A_l) \quad (l = \overline{1, n}).$$

Denoting the components of the mapping $\varphi_{A,l}(W)$ by $\varphi_{A,l}^{s,k}$ $(s, k = \overline{1, m})$, from (9) we get

$$\int_S f(\Phi(W))\varphi_{A,l}^{sk}(W) \, P(W, A) \, d\mu(W) = 0. \tag{10}$$

Now let's make the change of variables $U = \Phi(W)$. Then, taking into account Lemma 1 and Lemma 2, we obtain

$$\int_\Gamma f(U)\psi_{0,l}^{sk}(U)\, P_D(U,B)\, d\mu_\Gamma(U) = 0, \tag{11}$$

here $d\mu_\Gamma$ is the volume element on Γ, and

$$\psi_0 = \Phi_A \circ \Phi,$$

$$\psi_{0,l} = (iE - B_l^*)(U_l - B_l^*)^{-1}(U_l - B_l)(B_l + iE)^{-1}.$$

Since $B_l - B_l^*$ is non-singular, then condition (11) will also hold for the components of the mapping

$$\Phi_{B,l} = (U_l - B_l^*)^{-1}(U_l - B_l)(B_l - B_l^*)^{-1} \quad (l = \overline{1,n}).$$

So

$$\int_\Gamma f(U)\Phi_{B,l}^{sk}(U)\, P_D(U,B)\, d\mu_\Gamma(U) = 0 \quad (s,k = \overline{1,m},\ l = \overline{1,n}). \tag{12}$$

In [10] (see also [6, p.141]) it was proved that

$$\sum_{s,k=1}^m \bar{b}_{sk}^j \frac{\partial P(U_j,B_j)}{\partial \bar{b}_{sk}^j} + i\sum_{s=1}^m \frac{\partial P(U_j,B_j)}{\partial \bar{b}_{ss}^j} = mP(U_j,B_j)\left(\sum_{s,k=1}^m \bar{b}_{sk}^j \Phi_{B,j}^{sk} + i\sum_{s=1}^m \Phi_{B,j}^{ss}\right),$$

where

$$\prod_{j=1}^n P(U_j,B_j) = P_D(U,B),$$

and $\Phi_{B,j}^{sk}$ is the sk-component of the mapping $\Phi_{B,j}$, where $\Phi_B = (\Phi_{B,1},\ldots,\Phi_{B,n})$. From here

$$\sum_{s,k=1}^m \bar{b}_{sk}^j \frac{\partial P_D(U,B)}{\partial \bar{b}_{sk}^j} + i\sum_{s=1}^m \frac{\partial P_D(U,B)}{\partial \bar{b}_{ss}^j} =$$

$$= \sum_{s,k=1}^m \bar{b}_{sk}^j \frac{\partial \prod_{l=1}^n P(U_l,B_l)}{\partial \bar{b}_{sk}^j} + i\sum_{s=1}^m \frac{\partial \prod_{l=1}^n P(U_l,B_l)}{\partial \bar{b}_{ss}^j} =$$

$$= \prod_{l\neq j} P(U_l,B_l)\left[\sum_{s,k=1}^m \bar{b}_{sk}^j \frac{\partial P(U_j,B_j)}{\partial \bar{b}_{sk}^j} + i\sum_{s=1}^m \frac{\partial P(U_j,B_j)}{\partial \bar{b}_{ss}^j}\right] =$$

$$= \prod_{l\neq j} P(U_l,B_l)\cdot m \cdot P(U_j,B_j)\left(\sum_{s,k=1}^m \bar{b}_{sk}^j \Phi_{B,j}^{sk} + i\sum_{s=1}^m \Phi_{B,j}^{ss}\right) =$$

$$= m\, P_D(U,B)\left(\sum_{s,k=1}^m \bar{b}_{sk}^j \Phi_{B,j}^{sk} + i\sum_{s=1}^m \Phi_{B,j}^{ss}\right).$$

Hence,

$$\sum_{j=1}^n \sum_{s,k=1}^m \bar{b}_{sk}^j \frac{\partial P_D(U,B)}{\partial \bar{b}_{sk}^j} + i\sum_{j=1}^n\sum_{s=1}^m \frac{\partial P_D(U,B)}{\partial \bar{b}_{ss}^j} = m\, P_D(U,B)\left(\sum_{j=1}^n\sum_{s,k=1}^m \bar{b}_{sk}^j \Phi_{B,j}^{sk} + i\sum_{j=1}^n\sum_{s=1}^m \Phi_{B,j}^{ss}\right).$$

Taking this into account, we obtain from (12)

$$\mathfrak{d}F(B) = 0, \tag{13}$$

where

$$\mathfrak{d} = \sum_{j=1}^{n}\sum_{s,k=1}^{m} \overline{b}_{sk}^{j} \frac{\partial}{\partial \overline{b}_{sk}^{j}} + i\sum_{j=1}^{n}\sum_{s=1}^{m} \frac{\partial}{\partial \overline{b}_{ss}^{j}},$$

and

$$F(B) = \int_{\Gamma} f(U)\, P_D(U,B)\, d\mu_{\Gamma}(U)$$

is the Poisson integral of the function f. The function $F(B)$ is real-analytic in the domain D. We expand it into a Taylor series in a neighborhood of the point $I = (iE, \ldots, iE)$:

$$F(B) = \sum_{|\alpha|,|\beta|} c_{\alpha,\beta}\, (B-I)^{\alpha}\, \overline{(B-I)}^{\beta},$$

where $|\alpha| = \|\alpha_{sk}^{l}\|$, $|\beta| = \|\beta_{sk}^{l}\|$ are matrices with integer elements,

$$|\alpha| = \sum_{l=1}^{n}\sum_{s,k=1}^{m} \alpha_{sk}^{l},\ \ B^{\alpha} = \prod_{l=1}^{n}\prod_{s,k=1}^{m} b_{sk}^{l,\alpha_{sk}^{l}}.$$

Then condition (13) implies

$$\mathfrak{d}F(B) = \sum_{|\alpha|,|\beta|} |\beta| \cdot c_{\alpha,\beta}\, (B-I)^{\alpha}\, \overline{(B-I)}^{\beta},$$

hence all the coefficients $c_{\alpha,\beta}$. Hence $F(B)$ is holomorphic in D and belongs to $\mathcal{H}^{\infty}(D)$. □

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Граничная теорема Морера в области Зигеля первого рода

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Аннотация. В работе рассмотрена реализация матричного единичного поликруга в виде области Зигеля первого рода и доказывается граничный вариант теоремы Морера.

Ключевые слова: матричный единичный поликруг, автоморфизм, ядро Пуассона, голоморфная функция, голоморфное продолжение.