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Magnetic Phase Transitions in Disordered Systems: A Review of the Method of Random Exchange Fields

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Abstract. Magnetic phase transitions in disordered systems are governed by the interplay of exchange interactions, fluctuations, and structural randomness, which cannot be adequately described within classical mean-field approaches. In this work, we review the method of random exchange fields as an analytical framework for studying magnetic phase transitions in such systems. The method is based on a statistical description of local exchange fields and allows one to incorporate the effects of disorder and fluctuations while preserving analytical tractability. Its applicability is demonstrated for a wide class of magnetic materials, including spin glasses and spinel compound. The method provides a unified and physically transparent description of magnetic ordering and critical behavior in complex magnetic systems.

Keywords: magnetic phase transitions; random exchange fields; disordered magnetic systems; spin glasses; magnetic susceptibility.

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Introduction

The study of magnetic phase transitions remains one of the central problems of condensed matter physics due to both its fundamental importance and numerous applications in modern technologies, including spintronics and magnetic data storage [1, 2]. Particular interest is associated with systems characterized by structural disorder, competing exchange interactions, and reduced dimensionality, where conventional approaches often fail to provide an adequate description.

This paper is published in a special issue dedicated to Valter A. Ignatchenko, an outstanding Russian theoretical physicist whose work made a substantial contribution to the theory of condensed matter, magnetic phenomena, and randomly inhomogeneous media. Over the course of

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his scientific career, he authored more than 200 publications and developed a number of important ideas related to amorphous magnets, stochastic magnetic structures, and spin excitations in disordered magnetic systems. In this context, the present study continues the line of analytical approaches aimed at understanding the role of disorder and fluctuations in magnetic ordering.

In classical theories of magnetism, such as the Heisenberg model [3] and its mean-field approximations [4], the magnetic state is determined by the average exchange interaction. However, in real systems, especially in disordered materials, fluctuations of local exchange fields play a crucial role. This is particularly evident in spin-glass systems, where the presence of competing interactions leads to frustration and the absence of long-range magnetic order [5–8].

A number of theoretical approaches have been developed to describe disordered magnetic systems, including renormalization group methods [9], numerical simulations [10], and first-principles calculations [11]. Despite their success, these methods are often computationally demanding or do not provide simple analytical relations between microscopic parameters and macroscopic observables.

In this context, the method of random exchange fields offers an alternative analytical approach for describing magnetic phase transitions in disordered systems. The method is based on introducing a distribution of local exchange fields and allows one to take into account both the average interaction and its fluctuations within a unified framework. This approach has been successfully applied to a wide range of systems, including alloys, multilayers, nanoparticle ensembles, and amorphous materials.

In a series of works [12–18, 20], the method of random exchange fields was further developed and applied to the study of magnetic phase transitions in systems with competing interactions, structural disorder, and complex lattice structures. In particular, analytical expressions for magnetization, susceptibility, and critical temperatures were obtained, and the influence of fluctuations of exchange fields on magnetic ordering was investigated.

Recent studies have demonstrated that the method is capable of describing not only conventional magnetic transitions but also more complex phenomena, such as the transition from long-range magnetic order to spin-glass states, concentration-induced phase transitions, and the effects of interaction in ensembles of superparamagnetic particles [19, 21]. These results highlight the importance of taking into account the statistical distribution of exchange interactions when analyzing real magnetic systems.

The aim of the present work is to present and further develop the method of random exchange fields as an analytical approach to the study of magnetic phase transitions in disordered systems with competing interactions. The method is applied to a range of real magnetic systems, including spin-glass alloys and spinel compounds, with an emphasis on the role of fluctuations of local exchange fields, relaxation processes, and concentration effects. Particular attention is paid to the comparison between theoretical results and experimental data, as well as to the identification of the key parameters governing the stability of magnetic ordering.

The method has also been successfully applied to other types of disordered magnetic systems, such as amorphous alloys and ensembles of interacting nanoparticles, where fluctuations of local exchange fields play a crucial role in determining magnetic behavior.

1. Method of random exchange fields

The method of random exchange fields provides an analytical framework for describing magnetic systems with structural disorder and fluctuations of exchange interactions. In contrast to classical mean-field approaches [4], where the effective field is assumed to be uniform, this method treats the local exchange field acting on each spin as a random variable determined by the microscopic configuration of the system [12–15]. The formalism presented below follows the general ideas summarized in the dissertation abstract — including the definition of the local

exchange field distribution and the corresponding self-consistent equation for magnetization — but is rewritten here in the form appropriate for a review article.

We start from the Heisenberg Hamiltonian [3]:

$$\mathcal{H} = -\sum_{i>j} J_{ij}\mathbf{s}_i\mathbf{s}_j - g\mu_B H_{\text{ext}} \sum_i \mathbf{s}_i. \tag{1}$$

Here, $\mathcal{H}$ is the Hamiltonian of the spin system, J_{ij} is the exchange integral between spins at sites i and j, $\mathbf{s}_i$ is the spin operator at site i, g is the Land? factor, μ_B is the Bohr magneton, and H_{ext} is the external magnetic field.

Introducing magnetic moments $m_i = g\mu_B s_i$, Eq. (1) can be rewritten in the classical form:

$$\mathcal{H} = -\sum_{i>j} J_{ij} m_i m_j - H_{\text{ext}} \sum_i m_i. \tag{2}$$

In this expression, m_i is the magnetic moment associated with the spin at site i, while the other quantities have the same meaning as in Eq. (1).

In the absence of an external field, $H_{\text{ext}} = 0$, the Hamiltonian can be represented through local effective exchange fields:

$$\mathcal{H} = -\sum_i m_i H_i, \qquad H_i = \sum_j J_{ij} m_j. \tag{3}$$

Here, H_i is the local exchange field acting on the magnetic moment m_i, and the summation over j is performed over the neighboring magnetic moments interacting with site i.

In the method of random exchange fields, the local fields H_i are regarded as random variables. Their statistical properties are described by the probability density function $W(H)$, which determines the probability of finding a local exchange field equal to H [12, 15]:

$$W(H) = \int \delta\left(H - \sum_k \varphi_k(m_k, r_k)\right) \prod_k \Phi_k(m_k, r_k)\, d^3 m_k\, d^3 r_k. \tag{4}$$

Here, $W(H)$ is the distribution function of local exchange fields, $\delta(\ldots)$ is the Dirac delta function, $\varphi_k(m_k, r_k)$ is the field created at the origin by the k-th magnetic moment, m_k is the magnetic moment of the k-th ion, r_k is its position vector, and $\Phi_k(m_k, r_k)$ is the joint distribution function over magnetic moments and spatial coordinates.

For many disordered systems, the distribution $W(H)$ can be approximated by a Gaussian function:

$$W(H) = \frac{1}{\sqrt{\pi} B} \exp\left[-\frac{(H - H_0 M)^2}{B^2}\right], \tag{5}$$

with the moments of the distribution given by

$$H_0 = p \sum_k \varphi_k, \qquad B^2 = 2p\left[1 - (\langle\alpha\rangle - \langle\beta\rangle)^2\right] \sum_k \varphi_k^2. \tag{6}$$

Here, $M = \langle\alpha\rangle - \langle\beta\rangle$ is the reduced magnetic moment, averaged both thermodynamically and over configurations, p is the concentration of magnetic ions, $\langle\alpha\rangle$ is the average probability of finding a spin oriented along the chosen quantization axis, and $\langle\beta\rangle$ is the average probability of the opposite orientation.

The reduced magnetization is obtained by averaging the local response over the distribution of exchange fields:

$$M = \int \tanh\left(\frac{m_0 H}{k_B T}\right) W(H)\, dH. \tag{7}$$

Here, M is the reduced magnetization of the system, m_0 is the magnetic moment of a single magnetic ion, k_B is the Boltzmann constant, T is the absolute temperature, H is the local exchange field, and $W(H)$ is the corresponding probability density function.

Equation (7) generalizes the standard mean-field equation by explicitly taking into account the distribution of local exchange fields. In the limit $B \to 0$, the function $W(H)$ tends to a delta function, and the conventional mean-field result is recovered [4].

Near the Curie temperature, the self-consistency condition yields the equation for the critical temperature:

$$\frac{H_0}{B} \tanh\left(\frac{m_0 B}{k_B T_c}\right) = 1. \tag{8}$$

Here, T_c is the Curie temperature, H_0 is the mean exchange field, B is the fluctuation parameter, m_0 is the magnetic moment of an individual ion, and k_B is the Boltzmann constant.

Equation (8) shows that an increase in the fluctuation amplitude B leads to a decrease in T_c, which is one of the central physical consequences of the method for disordered magnetic systems.

The main advantage of the method of random exchange fields is that it introduces fluctuations directly at the level of the effective field instead of replacing the spin environment by a single averaged quantity. As a result, the method makes it possible to describe systems with strong disorder, frustration, and nonuniform local environments while preserving analytical transparency.

2. Systems with competing exchange interactions

For systems with competing exchange interactions (amorphous and disordered materials), it is convenient to introduce the parameter

$$x = \frac{K_2}{K_1}, \tag{9}$$

which characterizes the ratio of antiferromagnetic and ferromagnetic exchange constants, as well as the fraction ν of antiferromagnetic bonds.

In this case, the mean exchange field H_0 and the fluctuation parameter B are given by

$$H_0 = K_1(1-\nu)z - K_2\nu z, \qquad B = \sqrt{2\left[K_1^2(1-\nu)z + K_2^2\nu z\right]}, \tag{10}$$

where K_1 and K_2 are the ferromagnetic and antiferromagnetic exchange integrals, respectively, ν is the fraction of antiferromagnetic interactions, and z is the coordination number.

From Eq. (10), one obtains the ratio

$$\frac{H_0}{B} = \frac{(1-\nu)z - x\nu z}{\sqrt{2\left[(1-\nu)z + x^2\nu z\right]}}, \tag{11}$$

which plays a key role in determining the magnetic state of the system.

Depending on the value of this ratio, the system can be classified into different magnetic phases:

$$\begin{cases} \dfrac{H_0}{B} > 1, & \text{ferromagnetic phase,} \\ \dfrac{H_0}{B} < 1,\ H_0 > 0, & \text{ferromagnetic spin glass,} \\ \dfrac{H_0}{B} > -1,\ H_0 < 0, & \text{antiferromagnetic spin glass,} \\ \dfrac{H_0}{B} < -1, & \text{antiferromagnetic phase.} \end{cases} \tag{12}$$

Here, the sign of H_0 determines whether ferromagnetic or antiferromagnetic interactions dominate, while the magnitude of the ratio H_0/B determines whether long-range magnetic order is stable or destroyed by fluctuations.

These relations make it possible to construct the phase diagram $\nu(x)$ for systems with competing exchange interactions. The diagram includes four characteristic regions:

- A — antiferromagnetic ordered phase,
- B — antiferromagnetic spin-glass state,
- C — ferromagnetic spin-glass state,
- D — ferromagnetic phase.

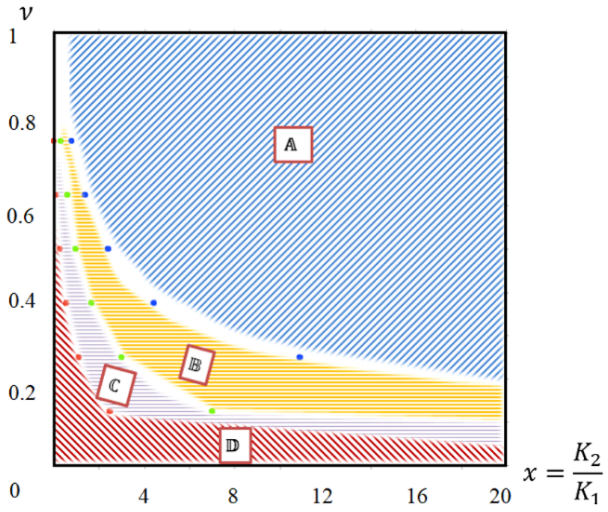


Fig. 1. Phase diagram $\nu(x)$ for systems with competing exchange interactions: A — antiferromagnetic phase, B — antiferromagnetic spin glass, C — ferromagnetic spin glass, D — ferromagnetic phase

The obtained phase diagram is in qualitative agreement with numerical results based on the model [22], taking into account the sign of the exchange interaction. This demonstrates that the method of random exchange fields provides an adequate analytical description of magnetic phase transitions in systems with competing interactions.

In particular, the method makes it possible to describe the transition from ordered magnetic phases to spin-glass states as a function of both the relative strength of competing interactions and their concentration. This is one of the key advantages of the approach when applied to disordered magnetic systems.

3. Applications to real magnetic systems

In this section, we consider several representative magnetic systems to which the method of random exchange fields has been applied. The general formalism introduced above is used here without repeating the basic equations for the magnetization, critical temperature, and susceptibility. Instead, the emphasis is placed on the specific form of the field moments, on the physical interpretation of the parameters entering the model, and on the comparison with experimental trends.

4.1. AuFe alloy and spin-glass behavior

In spin-glass systems, such as dilute AuFe alloys with concentration $p < p_c$, long-range magnetic order is absent. However, exchange-coupled groups of magnetic ions (finite clusters) are formed, which retain a nonzero magnetic moment and determine the low-temperature magnetic properties of the system.

Experimentally, it is observed that at sufficiently low temperatures the magnetization in a constant magnetic field H exhibits a logarithmic time dependence. This behavior is associated with a broad distribution of relaxation times of magnetic clusters, arising from the dispersion in their sizes and switching fields.

The characteristic relaxation time is governed by thermally activated processes and can be written as

$$\frac{1}{\tau} = f_0 \exp\left(-\frac{mH_c}{k_B T}\right), \tag{13}$$

where τ is the relaxation time, f_0 is the attempt frequency, m is the magnetic moment of the cluster, H_c is the switching (coercive) field, k_B is the Boltzmann constant, and T is the temperature.

Assuming temperature dependences typical for cluster-like magnetic objects,

$$m(T) = m_0 \left(1 - \frac{T}{\theta}\right)^{1/3}, \qquad H_c(T) = H_{c0} \left(1 - \frac{T}{\theta}\right)^{1/3}, \tag{14}$$

one obtains the relation

$$\lg(f_0 \tau) = \alpha m_0 \frac{\left(1 - \frac{T}{\theta}\right)^{2/3}}{T}, \tag{15}$$

where θ is a characteristic temperature and α is a proportionality factor determined by microscopic parameters.

A uniform distribution of clusters over the parameter αm_0 leads to a nearly uniform distribution of $\lg(\tau)$, which explains the experimentally observed logarithmic relaxation of magnetization.

For rapidly relaxing clusters ($\tau < 10^{-2}$ s, corresponding to typical measurement times), the magnetic susceptibility of a noninteracting ensemble can be estimated as

$$\chi \sim \frac{N\langle m^2 \rangle}{T}, \tag{16}$$

where N is the number of clusters and $\langle m^2 \rangle$ is the mean-square magnetic moment.

Taking into account the temperature dependence of the cluster parameters, the susceptibility can be written in the form

$$\chi = \frac{A}{T} \left(1 - \frac{T}{\theta}\right)^{1/3}, \tag{17}$$

where A is a constant determined by the concentration and magnetic moment of clusters.

A more detailed analysis of the relaxation-time distribution shows that, with decreasing temperature, a fraction of clusters becomes dynamically frozen and does not contribute to the magnetic response. As a result, the effective number of active clusters decreases, leading to a reduction of susceptibility at low temperatures.

Quantitative estimates for AuFe alloys demonstrate that a significant fraction of clusters (up to $\sim 40\%$) may become inactive within the experimental time window. This effect leads to a nonmonotonic temperature dependence of susceptibility, including the appearance of a maximum.

The obtained results can be directly compared with experimental data for AuFe alloys. In Fig. 2, the solid line corresponds to experimental measurements reported by Cannella and Mydosh [23], while the dashed curve represents the theoretical calculation within the present model.

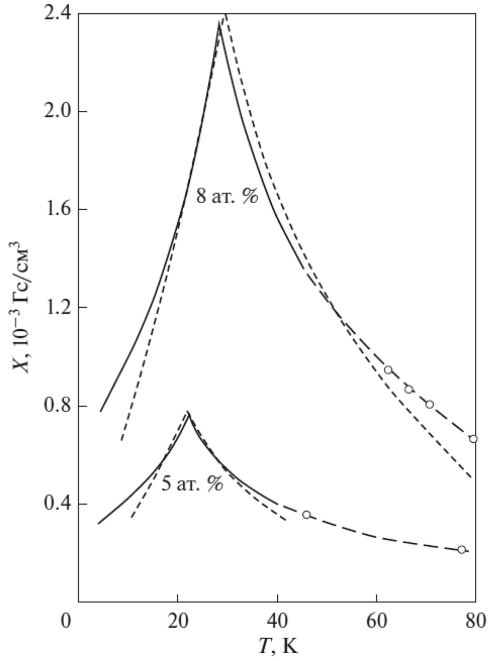


Fig. 2. Temperature dependence of the magnetic susceptibility for AuFe alloy. The position and shape of the cusp are reproduced within the method of random exchange fields

A qualitative agreement between theory and experiment is observed. In particular, the model reproduces the characteristic nonmonotonic temperature dependence of the magnetic susceptibility and the presence of a pronounced maximum. The position of the maximum is determined by the competition between thermal activation and the progressive freezing of magnetic clusters, while its amplitude is controlled by the fraction of dynamically active clusters.

The decrease of susceptibility at low temperatures is explained by the fact that a significant part of the clusters becomes frozen on the experimental time scale and therefore does not contribute to the magnetic response. This effect is naturally described within the framework of the distribution of relaxation times derived above.

Thus, the comparison with experimental data [23] confirms that the method of random exchange fields, supplemented by a description of cluster dynamics, provides an adequate description of relaxation processes and magnetic response in spin-glass systems.

4.2. Spinel system $Fe_{1-x}Zn_xCr_2S_4$

An important class of disordered magnetic systems is represented by spinel compounds of the type $Fe_{1-x}Zn_xCr_2S_4$, where substitution of magnetic Fe ions by nonmagnetic Zn ions leads to a substantial modification of magnetic properties. These systems exhibit a transition from ferrimagnetic ordering to a spin-glass state and further to antiferromagnetic behavior with increasing concentration x [24].

The magnetic structure of this compound can be described within a three-sublattice model, where Fe ions form one sublattice and Cr ions occupy two nonequivalent sublattices. The dominant interaction is the antiferromagnetic exchange between Fe^{2+} and Cr^{3+} ions, which leads to ferrimagnetic ordering at low Zn concentration. The system can be described within a three-sublattice model, where index 1 corresponds to Fe^{2+} ions, while indices 2 and 3 correspond to two nonequivalent Cr^{3+} sublattices.

Within the method of random exchange fields, the mean exchange fields acting on different sublattices can be written as

$$H_{21} = z_1 m_1 J_{21}(1-x), \qquad H_{13} = z_2 m_3 J_{13}, \tag{18}$$

where J_{ij} are the exchange integrals between sublattices, m_i are the magnetic moments, z_i are the coordination numbers, and x is the degree of substitution.

The corresponding fluctuation parameters are determined by the dispersion of exchange fields:

$$B_1 = \sqrt{2z_2\left[(m_2 J_{12})^2 + (m_3 J_{13})^2\right]}, \tag{19}$$

where B_1 characterizes the fluctuations of the local exchange field acting on the Fe sublattice.

The magnetization of each sublattice is obtained by averaging over the distribution of local fields:

$$M_1 = \frac{1}{2B_1}\int_{-B_1}^{B_1} \tanh\left[\frac{m_1}{k_B T}\left(H_1 + M_2 H_{12} + M_3 H_{13}\right)\right] dH_1, \tag{20}$$

where M_i are the reduced magnetizations of the corresponding sublattices, and H_i are the effective exchange fields.

Linearization of these equations near the ordering temperature leads to a condition that determines the N?el temperature T_N as a function of concentration x.

The calculated phase diagram $T_N(x)$ is shown in Fig. 3. The solid curve corresponds to the theoretical results obtained within the method of random exchange fields, while the experimental data are taken from Ref. [24].

A qualitative agreement between theory and experiment is observed. The model correctly reproduces the decrease of the ordering temperature with increasing Zn concentration and the existence of a concentration region where long-range magnetic order is destroyed. The transition from ferrimagnetic ordering to a spin-glass state is associated with the increasing role of fluctuations of exchange fields, which destabilize the ordered phase.

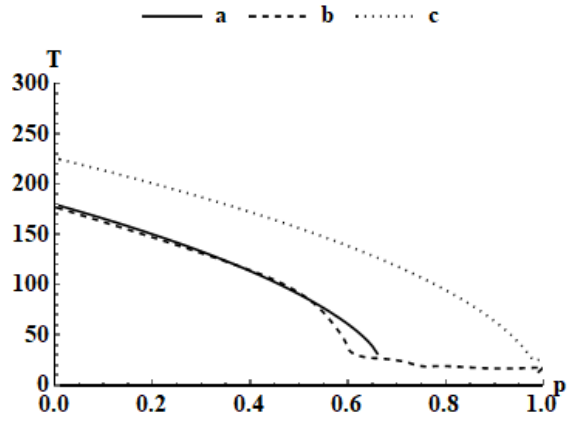


Fig. 3. Magnetic phase diagram $T_N(x)$ of the system $Fe_{1-x}Zn_xCr_2S_4$. Solid line — theoretical calculation; symbols — experimental data [24]

Thus, the method of random exchange fields provides a consistent description of magnetic phase transitions in spinel systems and allows one to relate the evolution of magnetic properties to microscopic interaction parameters and the degree of structural disorder.

4. Results and Discussion

The results obtained within the method of random exchange fields show that the magnetic behavior of disordered systems is governed by the competition between two characteristic quantities, namely the mean exchange field H_0 and the fluctuation parameter B. In the ordered regime, the average interaction dominates, whereas in disordered and frustrated systems the role of fluctuations becomes decisive. In this sense, the ratio H_0/B serves as a natural parameter controlling the stability of long-range magnetic order.

For systems with competing exchange interactions, the phase classification derived above demonstrates that the method is capable of distinguishing between ferromagnetic, antiferromagnetic, and spin-glass states within a single analytical scheme. The calculated phase diagram in the $\nu(x)$ plane shows that the transition from ordered phases to glassy states is determined not only by the sign of the dominant interaction, but also by the relative magnitude of fluctuations. This is an important result, since classical mean-field approaches usually account only for the average interaction and therefore cannot adequately describe the appearance of spin-glass regions. The qualitative agreement between the calculated diagram and earlier results for amorphous magnets [22] confirms that the method of random exchange fields captures the main physical mechanisms responsible for disorder-induced magnetic phase transitions.

A particularly illustrative example is provided by the AuFe alloy. In this system, the concentration of magnetic ions is below the percolation threshold, so that long-range magnetic order is absent, but exchange-coupled finite clusters still exist and determine the low-temperature magnetic response. The analysis of thermally activated relaxation of these clusters shows that the logarithmic time dependence of magnetization observed experimentally can be naturally explained by a broad distribution of relaxation times. The derived relations for the temperature dependence of the magnetic moment and switching field lead to a susceptibility curve with a pronounced maximum. This maximum is not associated with the onset of conventional long-range order; rather, it reflects the competition between thermal activation and the progressive freezing of magnetic clusters.

The comparison with experiment is especially important in this case. As shown in Fig. 2, the theoretical curve reproduces the main features of the experimental susceptibility reported by Cannella and Mydosh [23]. In particular, the model correctly describes the nonmonotonic temperature dependence of $\chi(T)$, the position of the maximum, and the decrease of susceptibility at lower temperatures. The latter effect is explained by the fact that a considerable fraction of clusters becomes dynamically blocked on the experimental time scale and therefore does not contribute to the measured response. Thus, the method of random exchange fields, supplemented by a description of cluster relaxation, gives a physically transparent interpretation of the experimental behavior of spin-glass alloys.

The application of the method to the spinel system $Fe_{1-x}Zn_xCr_2S_4$ demonstrates that the same formalism is also suitable for structurally ordered but compositionally disordered magnetic semiconductors. In this case, the three-sublattice description makes it possible to take into account the specific role of Fe and Cr ions and the dominant antiferromagnetic Fe–Cr exchange interaction. The introduction of nonmagnetic Zn ions reduces the effective exchange fields associated with the Fe sublattice and simultaneously enhances the relative importance of fluctuations. As a result, the ordering temperature decreases with increasing x, and the ferrimagnetic phase becomes unstable.

The calculated concentration dependence of the Néel temperature is in qualitative agreement with the experimental phase diagram reported in Ref. [24]. The theory reproduces the monotonic suppression of magnetic ordering with increasing Zn concentration and correctly describes the concentration interval in which long-range order is destroyed. This agreement is significant because it shows that the method is able to connect microscopic exchange parameters with experimentally observed phase boundaries. At the same time, small deviations between theory

and experiment at higher substitution levels may reflect additional effects not explicitly included in the present model, such as local lattice distortions, modification of superexchange pathways, and local compositional inhomogeneity.

Taken together, these results demonstrate that the method of random exchange fields provides a unified analytical description of magnetic phase transitions in systems of very different nature. In amorphous and frustrated systems it explains the emergence of spin-glass behavior through the growth of exchange-field fluctuations. In dilute cluster systems it naturally incorporates relaxation processes and the resulting time-dependent magnetic response. In multicomponent crystalline systems it describes the suppression of ordered phases under compositional substitution. An important advantage of the method is that all these effects are treated within the same framework based on the statistical distribution of local exchange fields.

Another important outcome is that the method preserves analytical transparency. Unlike purely numerical approaches, it makes it possible to identify directly how concentration, exchange parameters, and local-field fluctuations affect the phase state of the system. This is particularly useful for interpreting experimental data and for constructing phase diagrams in systems where both disorder and competing interactions are essential. Therefore, the method of random exchange fields may be regarded not only as a theoretical tool for model calculations, but also as a practical approach for analyzing real magnetic materials.

At the same time, the results indicate the natural limits of the present approximation. The use of simplified distribution functions and the neglect of some local structural effects may lead to quantitative deviations in strongly inhomogeneous systems. Nevertheless, the qualitative agreement with experimental data for AuFe alloys and $Fe_{1-x}Zn_xCr_2S_4$ shows that the method correctly describes the principal mechanisms governing magnetic phase transitions in disordered systems.

Conclusion

In this work, the method of random exchange fields has been presented as a unified analytical framework for describing magnetic phase transitions in disordered systems with competing interactions. It has been shown that the key role in determining the magnetic state is played by the interplay between the mean exchange field and its fluctuations, which are naturally incorporated within the proposed approach.

The method allows one to consistently describe different types of magnetic behavior, including long-range ordered phases and spin-glass states, within a single formalism. In particular, it provides a clear physical interpretation of the transition from ordered to disordered magnetic states as a result of increasing fluctuations of local exchange fields.

Applications of the method to real systems demonstrate its effectiveness. For AuFe alloys, the approach explains the experimentally observed nonmonotonic temperature dependence of magnetic susceptibility and the role of cluster relaxation processes. For spinel compounds of the type $Fe_{1-x}Zn_xCr_2S_4$, the method correctly reproduces the suppression of magnetic ordering with increasing substitution and shows good agreement with experimental phase diagrams.

An important advantage of the method is its analytical transparency, which makes it possible to establish direct relationships between microscopic interaction parameters and macroscopic magnetic properties. At the same time, the approach retains sufficient flexibility to describe systems with strong disorder, competing interactions, and complex magnetic structures.

Thus, the method of random exchange fields provides a physically transparent and practically useful tool for the analysis of magnetic phase transitions in a wide class of disordered materials. The results obtained indicate that further development of the method, including the refinement of distribution functions and the inclusion of additional interaction mechanisms, may extend its applicability to even more complex magnetic systems.

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Магнитные фазовые переходы в неупорядоченных системах: обзор метода случайных полей обменного взаимодействия

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Аннотация. Магнитные фазовые переходы в неупорядоченных системах зависят от обменных взаимодействий, флуктуаций и структурного беспорядка, которые не могут быть адекватно описаны в рамках классических приближений среднего поля. В настоящей работе представлен обзор метода случайных полей обменного взаимодействия как аналитического подхода к исследованию магнитных фазовых переходов в таких системах. Метод основан на статистическом описании локальных обменных полей и позволяет учитывать эффекты беспорядка и флуктуаций при сохранении аналитической разрешимости задачи. Возможности метода продемонстрированы на широком классе магнитных материалов, включая спиновые стекла и соединения со структурой шпинели. Метод случайных полей обменного взаимодействия обеспечивает единое и физически наглядное описание процессов магнитного упорядочения и критического поведения в сложных магнитных системах.

Ключевые слова: магнитные фазовые переходы, случайные поля обменного взаимодействия, неупорядоченные магнитные системы, спиновые стекла, магнитная восприимчивость.

EDN: DKVWBW
УДК 537.6

Self-consistent Approximation for the Vertex Function and its Application to Waves in Inhomogeneous Media

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Abstract. This paper presents an overview of a unified theoretical approach for analyzing resonance phenomena in stochastically inhomogeneous media based on the self-consistent approximation for the vertex function (SCA-V). It is shown that this approach removes the limitations of the traditional self-consistent approximation for the Green's function (SCA-G), also known as the Migdal approximation, the Kraichnan approximation, or the self-consistent Born approximation. The method is generalized to systems of coupled wave fields, including magnetoelastic waves, and is applied to ferromagnetic resonance in inhomogeneous thin films. The approach provides a consistent description of resonance peak broadening, the formation of fine structure in the crossing-resonance region, and asymmetry of FMR lines caused by magnetostatic waves. In general, the SCA-V approximation is a universal analytical tool for studying wave processes in stochastically inhomogeneous media.

Keywords: ferromagnetic resonance, magnetic films, magnetic anisotropy inhomogeneities, Green's functions, spin waves, magnetic dipole field, self-consistent approximation.

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Introduction

Spatial inhomogeneities of the medium parameters play a key role in the formation of wave spectra and resonance phenomena in electrical, magnetic, optical, elastic, and other physical systems. The presence of random correlated inhomogeneities leads to multiple wave scattering, modification of dispersion relations, and substantial changes in the shape, width, and structure of resonance lines. A correct theoretical description of these effects is a complex problem that goes beyond standard methods of perturbation theory.

The simplest and most widely used approach to wave analysis in stochastically inhomogeneous media is the first-order perturbation theory, i.e., the Born approximation in condensed matter physics [1, 2] or the Bourret approximation [3] in the theory of random functions [4, 5]. However, for large correlation radii of inhomogeneities, this approximation becomes unsuitable for a correct description of the shape and width of resonance lines, since multiple-scattering processes are not taken into account within its framework. To include these processes, the standard self-consistent approximation (SCA) was independently proposed in various branches of physics, also known as the Migdal approximation [6], the Kraichnan approximation [7], or the self-consistent Born approximation [1, 2]. Within this approach, all diagrams with nonintersecting correlation lines

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are summed, which makes it possible to go beyond single scattering and obtain self-consistent expressions for averaged Green's functions.

Despite the wide use of the standard self-consistent approximation in different areas of physics [1, 2, 6–16], numerous studies have shown that it has fundamental limitations. The best-known defect of the standard SCA is the distortion of resonance line shape, namely the appearance of dome-shaped peaks [2,17–19], bends on their slopes, and false central maxima [20,21]. These shortcomings are especially pronounced in media with long-wavelength inhomogeneities. In a number of problems, such distortion of resonance line shape may hinder the interpretation of experimental data or even make it impossible.

To overcome these limitations, a higher-level self-consistent approximation was developed, based on self-consistency with respect to the vertex function rather than the Green's function [22]. Within this approach, a system of two coupled nonlinear integral equations is derived for the Green's function (or self-energy) and the vertex function, which makes it possible to include a significant part of diagrams with intersecting correlation lines. It is shown that this approximation includes lower-level approximations: the standard SCA and the Bourret (Born) approximation.

The development of the self-consistent approximation for the vertex function has created a basis for the consistent analysis of more complex physical systems. Typical examples of such systems are magnetoelastic waves and ferromagnetic resonance in inhomogeneous continuous media and thin films, respectively.

Within this approach, a theory of magnetoelastic resonance in an inhomogeneous medium was developed, based on matrix Green's functions for coupled spin and elastic waves. Taking the matrix vertex function into account made it possible to correctly describe resonance peak shape, remove nonphysical artifacts of the standard self-consistent approximation, and reproduce the fine structure of the spectrum at the crossing point of the unperturbed dispersion laws of spin and elastic waves: a narrow resonance in the Green's function of spin waves and a narrow antiresonance in the Green's function of elastic waves, caused by the stochastic distribution of coupling parameters between these fields [23, 24].

Further development of the theory is associated with applying the self-consistent approximation for the vertex function to the description of ferromagnetic resonance in thin films with inhomogeneous magnetic anisotropy. It is shown that the amplitude of anisotropy fluctuations, the magnitude of their correlation radius, the presence of magnetodipole effects, and the tilt angle of the external magnetic field fundamentally affect the width, asymmetry, and position of resonance peaks [25–28]. The new self-consistent approach provides a correct description of these effects over a wide parameter range and demonstrates qualitatively better agreement with numerical modeling than the standard SCA.

Along with the terms "standard SCA" and "new SCA" that we used earlier, the papers [22–25, 29] proposed the following terms, which more accurately reflect both the mathematical content of each approximation and their key differences. We will refer to the standard SCA as the "SCA for the Green's function (SCA-G)", and to the new one as the "SCA for the vertex function (SCA-V)."

The aim of this review article is to provide a systematic exposition of a unified theoretical framework that integrates the above cycle of papers, carried out mainly in collaboration with V. A. Ignatchenko [22–29], within the sequential development of the self-consistent approximation for analyzing resonance phenomena in stochastically inhomogeneous media. The methodological foundations of this approach are presented, as well as its generalization to systems of coupled wave fields in continuous media and to problems of describing ferromagnetic resonance in thin films. Special attention is paid to the internal logic of idea development, the interrelation of results, and the universal physical mechanisms underlying the effects obtained.

1. Self-consistent approximation for the vertex function

1.1. Derivation of SCA-V

The consideration of wave processes in inhomogeneous media is conveniently formulated in terms of the averaged Green's function G. A homogeneous medium is described by the initial Green's function g, while the influence of random inhomogeneities is taken into account through the self-energy Σ, connected with the averaged Green's function G by the exact Dyson equation.

To obtain the averaged Green's function G and develop the self-consistent approximation, let us consider the simplest model of the wave equation in a randomly inhomogeneous medium:

$$\nabla^2\varphi + [\nu - \gamma\rho(\mathbf{x})]\,\varphi = -f. \tag{1}$$

Both parameters, ν and γ, in equation (1) are normalized to the dimension of the squared wave number. For scalar models of electromagnetic or elastic waves, $\nu = (\omega/s)^2$, where ω is the frequency and s is the velocity of the corresponding waves in the medium; for spin waves, $\nu = (\omega - \omega_0)/(\tilde{g}\alpha M_0)$, where ω_0 is the ferromagnetic resonance (FMR) frequency in a homogeneous medium, $\tilde{g}$ is the gyromagnetic ratio, α is the exchange parameter, M_0 is the saturation magnetization, f is an external source exciting the corresponding waves in the medium, γ is the root-mean-square fluctuation of the corresponding inhomogeneities; examples of such inhomogeneities are discussed for spin [30], elastic [31], and electromagnetic [32–34] waves, and $\rho(\mathbf{x})$ is a centered ($< \rho(\mathbf{x}) >= 0$) and normalized ($< \rho^2(\mathbf{x}) >= 1$) random coordinate function. Angle brackets denote averaging over the ensemble of realizations of the corresponding random function.

The equation for the unaveraged Green's function $\tilde{G}$ corresponding to equation (1) has the form:

$$\nabla^2\tilde{G}(\mathbf{x},\mathbf{x}_0) + [\nu - \gamma\rho(\mathbf{x})]\,\tilde{G}(\mathbf{x},\mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0). \tag{2}$$

Let us represent the unaveraged Green's function $\tilde{G}$ as

$$\tilde{G} = g + G', \tag{3}$$

where g is the initial Green's function for a homogeneous medium ($\gamma = 0$), and G' is the correction caused by the inhomogeneous fluctuation parameter.

Substituting expression (3) into (2), we obtain the equation for the initial Green's function g:

$$\nabla^2 g(\mathbf{x},\mathbf{x}_0) + \nu g(\mathbf{x},\mathbf{x}_0) = \delta(\mathbf{x} - \mathbf{x}_0). \tag{4}$$

and for G':

$$\nabla^2 G'(\mathbf{x},\mathbf{x}_0) + \nu G'(\mathbf{x},\mathbf{x}_0) = \gamma\rho(\mathbf{x})\tilde{G}(\mathbf{x},\mathbf{x}_0). \tag{5}$$

According to general rules, the formal solution of equation (5) can be represented as an integral of the product of the unperturbed Green's function and the right-hand side of these equations [32]. It then follows that the generating integral equation for the series of the unaveraged Green's function $\tilde{G}$ has the form

$$\tilde{G}(\mathbf{x},\mathbf{x}_0) = g(\mathbf{x},\mathbf{x}_0) + \gamma \int \tilde{G}(\mathbf{x},\mathbf{x}')\rho(\mathbf{x}')g(\mathbf{x}',\mathbf{x}_0)\,d\mathbf{x}'. \tag{6}$$

In the integral term of the Green's function equation (6), the order of g and $\tilde{G}$ is changed compared with the standard form, where they appear in the reverse sequence [22]. This modification does not violate the equivalence of the equations, because iterative expansion leads to an identical series for the Green's function. The form of equation (6) used here provides convenient ordering of vertex functions when constructing the self-consistent approximation. Functional transformations are applied to equation (6) in accordance with the approach described in [12, 33–36].

Here averaging over the ensemble of realizations of the random function $\rho(\mathbf{x})$ is carried out using Wick's theorem for Gaussian random fields. As a result, we obtain the following system of equations:

$$G(\mathbf{x},\mathbf{x}_0) = g(\mathbf{x},\mathbf{x}_0) + \int\int G(\mathbf{x},\mathbf{x}')\Sigma(\mathbf{x}',\mathbf{x}'')g(\mathbf{x}'',\mathbf{x}_0)\,d\mathbf{x}'\,d\mathbf{x}'', \tag{7}$$

$$\Sigma(\mathbf{x}',\mathbf{x}'') = \gamma^2\int\int \Gamma(\mathbf{x}',\mathbf{x}_1;\mathbf{x}_2)K(\mathbf{x}_2,\mathbf{x}'')G(\mathbf{x}_1,\mathbf{x}'')\,d\mathbf{x}_1\,d\mathbf{x}_2, \tag{8}$$

$$\Gamma(\mathbf{x}',\mathbf{x}_1;\mathbf{x}_2) \approx \delta(\mathbf{x}'-\mathbf{x}_2)\delta(\mathbf{x}'-\mathbf{x}_1) + \gamma^2\int\int\int\int K(\mathbf{x}_1,\mathbf{x}_4)\Gamma(\mathbf{x}',\mathbf{x}_3;\mathbf{x}_4)G(\mathbf{x}_3,\mathbf{x}_5) \times \Gamma(\mathbf{x}_5,\mathbf{x}_6;\mathbf{x}_2)G(\mathbf{x}_6,\mathbf{x}_1)\,d\mathbf{x}_3\,d\mathbf{x}_4\,d\mathbf{x}_5\,d\mathbf{x}_6, \tag{9}$$

where G is the averaged Green's function, Σ is the self-energy, Γ is the vertex function, and K is the correlation function of the form $K(\mathbf{x}',\mathbf{x}'') = K(\mathbf{r}) = \langle\rho(\mathbf{x}'')\rho(\mathbf{x}''+\mathbf{r})\rangle$, where $\mathbf{r} = \mathbf{x}' - \mathbf{x}''$. Equation (7) is the Dyson equation mentioned above. In the general case, equation (9) has the form of an infinite series [22, 34], but here we retain terms only up to order γ^2.

Next, it is convenient to move to $\mathbf{k}$-space (see [22]) and write the system of three equations (7)–(9) in the form:

$$G_{\mathbf{k}} = g_{\mathbf{k}} + g_{\mathbf{k}}\Sigma_{\mathbf{k}}G_{\mathbf{k}}, \tag{10}$$

$$\Sigma_{\mathbf{k}} = \gamma^2(2\pi)^{-\tau}\int S_{\mathbf{k}-\mathbf{k}_1}G_{\mathbf{k}_1}\Gamma_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1}\,d\mathbf{k}_1, \tag{11}$$

$$\Gamma_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1} \approx 1 + \gamma^2(2\pi)^{-\tau}\int S_{\mathbf{k}-\mathbf{q}}G_{\mathbf{q}}G_{\mathbf{q}-\mathbf{k}+\mathbf{k}_1}\Gamma_{\mathbf{q},\mathbf{k}-\mathbf{q}}\Gamma_{\mathbf{q}-\mathbf{k}+\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1}\,d\mathbf{q}. \tag{12}$$

where the initial Green's function $g_{\mathbf{k}}$ is obtained from equation (4) and has the form:

$$g_{\mathbf{k}} = \frac{1}{\nu - k^2}, \tag{13}$$

$S(\mathbf{k})$ is the normalized spectral density, related by the Fourier transform to the normalized correlation function

$$K(\mathbf{r}) = (2\pi)^{-\tau}\int S_{\mathbf{k}}e^{i\mathbf{kr}}d\mathbf{k},\ S_{\mathbf{k}} = \int K(\mathbf{r})e^{-i\mathbf{kr}}d\mathbf{r}, \tag{14}$$

$\mathbf{r} = \mathbf{x}' - \mathbf{x}''$, and τ is the space dimensionality.

Let us transform the system of equations (10)–(12): substitute expression (12) into equation (11), and then equate the obtained result to the exact relation (11). As a result, we obtain the approximate equation

$$\int S_{\mathbf{k}-\mathbf{k}_1}G_{\mathbf{k}_1}\{\Gamma_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1} - \left[1 + \gamma^2(2\pi)^{-\tau}\,\Gamma_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1}\int S_{\mathbf{k}_1-\mathbf{k}_2}G_{\mathbf{k}_2}G_{\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2}\Gamma_{\mathbf{k}_2,\mathbf{k}_1-\mathbf{k}_2}\,d\mathbf{k}_2\right]\Big\}\,d\mathbf{k}_1 \approx 0. \tag{15}$$

In equation (15), the integrand with respect to $\mathbf{k}_1$ contains the difference of two functions depending on the same parameter $\mathbf{k}$. Since the difference of the corresponding integrals must vanish for any values of $\mathbf{k}$, and the factor $S_{\mathbf{k}-\mathbf{k}_1}G_{\mathbf{k}_1} \neq 0$, equation (15) implies that the expression in braces is equal to zero. This makes it possible to obtain a self-consistent equation for the vertex function $\Gamma_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1}$ in the form (17). To close it, an additional equation for the averaged Green's function G is required; it is obtained by substituting (11) into the Dyson equation (10). Thus,

the SCA for the vertex function reduces to a system of two nonlinear integral equations for $G_{\mathbf{k}}$ and $\Gamma_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1}$:

$$G_{\mathbf{k}} = \frac{1}{g_{\mathbf{k}}^{-1} - \gamma^2(2\pi)^{-\tau}\int S_{\mathbf{k}-\mathbf{k}_1} G_{\mathbf{k}_1}\Gamma_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1}\, d\mathbf{k}_1}, \tag{16}$$

$$\Gamma_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1} \approx \frac{1}{1 - \gamma^2(2\pi)^{-\tau}\int S_{\mathbf{k}_1-\mathbf{k}_2} G_{\mathbf{k}_1} G_{\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2}\Gamma_{\mathbf{k}_2,\mathbf{k}_1-\mathbf{k}_2}\, d\mathbf{k}_2}. \tag{17}$$

The first equation (16) is exact, whereas the second one (17) is an approximate self-consistency equation for the Γ function. In addition to all diagrams with nonintersecting correlation lines, SCA-V takes into account a significant number of diagrams with intersecting correlation lines [22].

Neglecting in the denominator of equation (16) the term proportional to γ^2, we obtain the approximate solution of this equation

$$\Gamma_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1} \approx 1 \tag{18}$$

and by substituting this solution into equation (16), we obtain the standard SCA in the form of a single closed equation for the Green's function $G_{\mathbf{k}}$, which is widely used in different areas of physics [1, 2, 6–16]

$$G_{\mathbf{k}} \approx \frac{1}{g_{\mathbf{k}}^{-1} - \gamma^2(2\pi)^{-\tau}\int G_{\mathbf{k}_1} S_{\mathbf{k}-\mathbf{k}_1} d\mathbf{k}_1}. \tag{19}$$

This approximation is of lower order than the SCA for the Γ function; it corresponds to an infinite Green's function series containing only diagrams with nonintersecting correlation lines. It was proposed by Migdal [6] in the study of electron-phonon interaction and by Kraichnan [7] in the study of the turbulence problem. This approximation is also often referred to in the literature (see, for example, [2, 12]) as the self-consistent Born approximation, in contrast to the first Born approximation of perturbation theory.

The first-order perturbation approximation follows from equation (19) by replacing the sought Green's function $G_{\mathbf{k}_1}$ on the right-hand side with the initial Green's function $g_{\mathbf{k}_1}$:

$$G_{\mathbf{k}} \approx \frac{1}{g_{\mathbf{k}}^{-1} - \gamma^2(2\pi)^{-\tau}\int g_{\mathbf{k}_1} S_{\mathbf{k}-\mathbf{k}_1} d\mathbf{k}_1}. \tag{20}$$

This lowest-level approximation is called the Bourret/Born approximation [1–5].

The system of equations (16) and (17) is used as follows. At the first stage, expressions given by equations (18) and (20) are substituted. In equation (20), the wave vector $\mathbf{k}$ is replaced by $\mathbf{k}_1$ and $\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2$, respectively. The obtained expressions are then substituted under the integral into the right-hand sides of equations (16) and (17), which defines the first step of the iterative solution procedure. Then, successive iteration of equations (16) and (17) leads to the formation of infinite branched continued fractions. Successive iterations and numerical evaluation of the integrals ensure convergence, as a result of which the Green's function $G_{\mathbf{k}}$ acquires a smooth limiting behavior.

1.2. Analysis of SCA-V

Next, we analyze the system of equations (16) and (17), illustrate its advantages over other approximations, and compare the results with exact numerical simulation of equation (1). Here we restrict ourselves to one-dimensional inhomogeneities. Then in equations (16) and (17), $\tau = 1$, and the vector $\mathbf{k}$ has one component: $k = k_x$. The correlation properties of the random function $\rho(x)$ are modeled by an exponential correlation function:

$$K(r) = e^{-k_c r}, \quad S_k = \frac{2k_c}{k_c^2 + k^2}, \tag{21}$$

where $r = |x' - x''|$, and k_c is the inhomogeneity correlation wave number ($r_c = k_c^{-1}$ is the inhomogeneity correlation radius).

In the limiting case $k_c = 0$, we have $S_{k-k_1} \to (2\pi)\delta(k - k_1)$. In this case, the number of diagrams in the Green's function expansion can be found analytically. Using the rules for expansion into a binomial series [37], we obtain the following representation for the Green's function:

$$G = g \sum_{n=0}^{\infty} N_n z^n, \quad z = \gamma^2 g^2. \tag{22}$$

The coefficient N_n is the number of diagrams in each n-th order of the Green's function expansion

$$G; \quad N_n = (2n-1)!!, \tag{23}$$

$$G_{SCA-G}; \quad N_n = \frac{(2n)!}{n!(n+1)!} \text{ — Catalan numbers}, \tag{24}$$

$$G_{SCA-V}; \quad N_n = \sum_{m=0}^{\infty} \frac{2^{m+1}(2m)!}{3^{2m+1}(m+1)!(m-n)!n!}\,. \tag{25}$$

$$G_{Lad}; \quad N_1 = 1,\ N_2 = 3,\ N_3 = 12,\ N_4 = 55, \ldots\,. \tag{26}$$

The general formula for the coefficients N_n of the exact Green's function, equation (23), is well known [2]. The general formulas for the coefficients N_n in SCA-G and SCA-V, equations (24) and (25), respectively, were derived in [22]. The coefficients N_n of the ladder approximation [38], equation (26), for $k_c = 0$ were obtained in [29].

Table 1 lists the values of N_n up to sixth order. It is seen that for $n > 1$, the SCA-V approximation includes more diagrams at each order than SCA-G and the ladder approximation. However, the number of diagrams included in SCA-V is still insufficient to obtain the exact Green's function. Figure 1*a* compares the approximations and shows that SCA-V reproduces the shape, width, and amplitude of the exact resonance curve much more accurately. The ladder approximation is difficult to apply in practice because it is not self-consistent; therefore, it is not shown in Fig. 1*a*. Comparison of Fig. 1 and Tab. 1 shows that the more diagrams are included at each order, the closer the approximate curves are to the exact Gaussian dependence (Fig. 1*a*). Note that the exact Gaussian curve is obtained only for $k_c = 0$. For $k_c \neq 0$, approximate methods are used, since exact accounting for an infinite number of diagrams is practically impossible; numerical simulation is used as an alternative to obtain the target Green's function. At the same time, computational complexity compared to SCA-V increases substantially with the problem dimensionality τ: calculations in the two-dimensional case $\tau = 2$ already require significant computational resources, while in the three-dimensional case $\tau = 3$ they are even more resource-intensive.

Table 1. Number of diagrams in each n-th order of the Green's function expansion for the first six orders n, taken into account in the standard (G_{SCA-G}), ladder (G_{Lad}), and new (G_{SCA-V}) approximations, and in the exact expression for the Green's function (G)

G_{SCA-G}	1	2	5	14	42	132
G_{Lad}	1	3	12	55	273	1428
G_{SCA-V}	1	3	13	67	381	2307
G	1	3	15	105	945	10395

For $k_c \neq 0$, the normalized spectral density S is given by equation (21); in this case, the fourth and higher orders of the Green's function expansion G in the SCA-V approximation contain, in

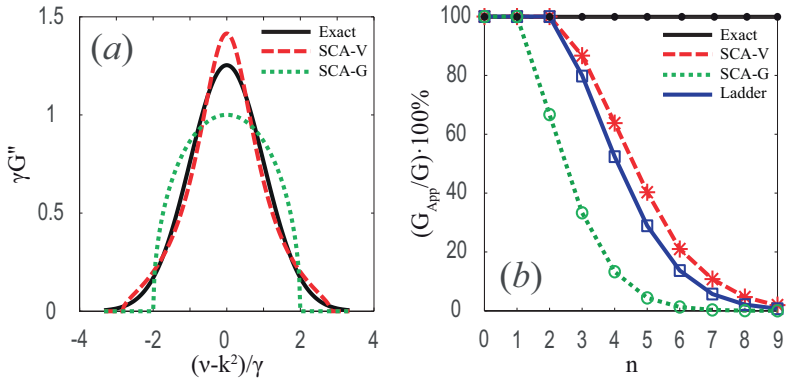


Fig. 1. (a) Imaginary parts of the Green's functions at $k_c = 0$. (b) Relative fractions of the exact number of diagrams (in percent) in each n-th term of the Green's function expansion

addition to correct diagrams, a small number of defective diagrams [22]. It is difficult to estimate the influence of such diagrams in general form. Therefore, comparing results obtained within SCA-V with numerical simulation results (transfer-matrix method) is of particular importance for assessing the accuracy of the proposed method. The numerical simulation procedure is described in [29]. Here we show the result in Fig. 2.

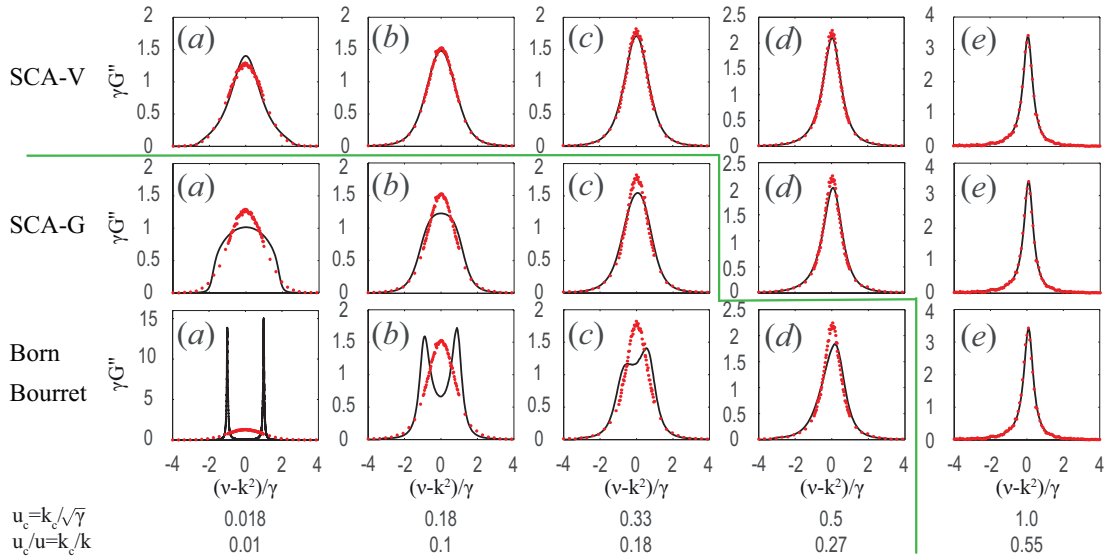


Fig. 2. Imaginary part of the Green's function calculated in different approximations (black curves) and its numerical simulation (red dots)

Figure 2 shows the imaginary part of the Green's function $G''(\omega)$ for SCA-V (top row), SCA-G (middle row), and the Bourret (Born) approximation (bottom row), as well as numerical simulation results (red dots); calculations are performed at fixed wave number $u = k/\sqrt{\gamma} = 1.8$. It is shown that within SCA-V, results practically coincide with the numerical experiment over the entire studied range of dimensionless correlation wave number u_c. In SCA-G, at small $u_c < 0.33$, noticeable discrepancies arise: the resonance peak acquires a dome-like shape and becomes broader than the exact one. For SCA-V at small k_c (large $r_c = k_c^{-1}$), maximum peak

broadening is observed due to stronger scattering; as k_c increases (and r_c decreases), the peak narrows. In addition, under these conditions the number of iterative substitutions in the system of equations (16) and (17), required to reach convergence and obtain a smooth dependence, decreases.

In the Bourret (Born) approximation, a physically incorrect solution with two high resonance peaks is obtained over most of the interval; these peaks merge into one as u_c increases. This approximation is included to illustrate that at $u_c \geqslant 1$ (Fig. 2*e*), results of Bourret (Born), SCA-G, and SCA-V are nearly identical to each other and to numerical simulation.

However, at small k_c, SCA-V has significant advantages. In a number of cases, it also outperforms numerical methods due to higher computational efficiency and the possibility of analyzing three-dimensional problems.

In this section, the system of equations (16) and (17) for SCA-V is obtained. Advantages of this approach in the number of accounted diagrams compared with SCA-G and the ladder approximation are shown, and agreement with exact numerical simulation is demonstrated. Next, we turn to other models and focus on applying SCA-V to resonance phenomena in stochastically inhomogeneous media without going into detailed mathematical derivations. The goal is to show that SCA-V is a universal method that accounts for multiple wave scattering by inhomogeneities and allows resonance peaks (high-frequency susceptibilities) to be studied in inhomogeneous media.

2. Generalization of SCA-V to a system of coupled wave fields

In this section, the self-consistent approximation for the vertex function is generalized to a system of two interacting wave fields of different physical nature. As a model example, spin and elastic waves in a magnetic medium are considered.

We consider a coupled system of two scalar equations for the resonant circular projections of magnetization m and elastic displacements u, in which only the dimensionless magnetoelastic coupling parameter $\varepsilon(x)$ is inhomogeneous [23, 24]:

$$\alpha\left(\nabla^2 + \nu_m\right) m - \varepsilon(x) M_0 \frac{\partial u}{\partial z} = -h, \tag{27}$$

$$\mu\left(\nabla^2 + \nu_u\right) u + M_0 \frac{\partial}{\partial z}\left(\varepsilon(x) m\right) = -f, \tag{28}$$

where μ is the elastic constant, M_0 is the constant magnetization along the external static magnetic field H_0 (the amplitude of transverse circular projections satisfies $m \ll M_0$), h is the external alternating magnetic field perpendicular to H_0, and f is the external body force. The following notation is introduced:

$$\nu_m = \frac{\omega - \omega_0}{\alpha \tilde{g} M_0}, \quad \nu_u = \frac{\omega^2}{v_u^2}, \tag{29}$$

where $v_u = \sqrt{\mu/p}$ is the elastic-wave velocity, and p is the medium density.

Let us represent the magnetoelastic parameter $\varepsilon(x)$ as:

$$\varepsilon(x) = \varepsilon + \Delta\varepsilon\rho(x), \tag{30}$$

where ε is the mean value of this parameter, and $\Delta\varepsilon$ is its root-mean-square fluctuation.

Further calculations for the system of equations (27) and (28) are carried out by analogy with the case of a single wave field (1), yielding in $\mathbf{k}$-space a system of two coupled self-consistent

equations for the matrix Green's function $\hat{G}$ and the vertex function $\hat{\Gamma}$, which is structurally analogous to the SCA-V system for a single wave field (16) and (17):

$$\hat{G}_k = \frac{\hat{E}}{\hat{g}_k^{-1} - \gamma^2 (2\pi)^{-\tau} \hat{E}_k^{(1)} \int S_{k-k_1} \hat{J} \hat{X}_{k_1} \hat{\Gamma}_{k_1,k-k_1} \hat{J} dk_1 \hat{E}_k^{(2)}}, \tag{31}$$

$$\hat{\Gamma}_{k_1,k-k_1} \approx \frac{\hat{E}}{\hat{E} - \gamma^2 (2\pi)^{-\tau} \int S_{k_1-k_2} \hat{\Gamma}_{k_2,k_1-k_2} \hat{J} \hat{X}_{k_2} \hat{J} \hat{X}_{k-k_1+k_2} dk_2}. \tag{32}$$

Here

$$\hat{X}_{k_1} = \hat{E}_k^{(2)} \hat{G}_k \hat{E}_k^{(1)}, \quad \hat{J} = \begin{bmatrix} 0 & \sqrt{\dfrac{\alpha}{\mu}} \\ \sqrt{\dfrac{\mu}{\alpha}} & 0 \end{bmatrix}, \quad \gamma = \frac{\Delta\varepsilon}{\sqrt{\alpha\mu}} M_0. \tag{33}$$

$$\hat{g}_k^{-1} = \begin{bmatrix} \nu_m - k^2 & -i\dfrac{\varepsilon}{\mu} M_0 k_z \\ i\dfrac{\varepsilon}{\alpha} M_0 k_z & \nu_u - k^2 \end{bmatrix}, \quad \hat{g}_k = \begin{bmatrix} g_{mm}(k) & i g_{mu}(k) \\ -i g_{um}(k) & g_{uu}(k) \end{bmatrix}, \tag{34}$$

$$\hat{E}_k^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & -ik_z \end{bmatrix}, \quad \hat{E}_k^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & ik_z \end{bmatrix}, \tag{35}$$

where $\hat{E}$ is the identity matrix, and S is the spectral density defined by equation (21). Note that when expanding the vertex function (32) in a series in γ^2, we obtain repeating products of matrices $\hat{J}\hat{X}$ in each term. The same repetition of matrix products is obtained for the sought Green's function (31) by repeatedly substituting into equations (31) and (32) and expanding the resulting infinite branched continued fraction in a series in γ^2.

Let us represent the matrix Green's function $\hat{G}_k$, similarly to the initial matrix Green's function $\hat{g}_k$, as:

$$\hat{G}_k = \begin{bmatrix} G_{mm}(k) & i G_{mu}(k) \\ -i G_{um}(k) & G_{uu}(k) \end{bmatrix}. \tag{36}$$

The amplitudes m and u are expressed through the averaged Green's functions as follows:

$$m(k) = -\frac{1}{\alpha} \left[G_{mm}(k) h(k) + i G_{mu}(k) f(k) \right], \tag{37}$$

$$u(k) = -\frac{1}{\mu} \left[G_{uu}(k) f(k) - i G_{um}(k) h(k) \right]. \tag{38}$$

As inhomogeneity increases, ε decreases from its initial value ε_0 to zero, while $\Delta\varepsilon$ increases from zero to some maximum value ε', close to ε_0. For simplicity, we set $\varepsilon' = \varepsilon_0$ and study the change of the Green's function elements at the crossing-resonance point $k = k_r$ as $\Delta\varepsilon$ increases and ε decreases while preserving the sum of squares of these quantities:

$$\varepsilon^2 + (\Delta\varepsilon)^2 = \varepsilon_0^2. \tag{39}$$

Of course, experimental samples in which intermediate states are created by changing composition or processing method between the two limiting cases—homogeneous ($\varepsilon = \varepsilon_0$, $\Delta\varepsilon = 0$) and fully stochasticized ($\varepsilon = 0$, $\Delta\varepsilon = \varepsilon_0$)—are not required to obey relation (39). This relation is introduced only for convenient, ordered consideration of all intermediate states with different ratios $(\varepsilon/\varepsilon_0)^2$ and $(\Delta\varepsilon/\varepsilon_0)^2$. At the crossing point of unperturbed dispersion laws ($k = k_r$, $\omega = \omega_r$), a gap appears in the magnetoelastic wave spectrum, whose width Λ is proportional to the coupling parameter in a homogeneous medium ε_0:

$$\Lambda = \frac{\varepsilon_0 M}{\sqrt{\mu}} \sqrt{2\omega_M \omega_r}, \tag{40}$$

where $\omega_M = \tilde{g}M$. The value Λ also determines the distance between the two maxima in the frequency dependence of the imaginary part of the Green's function for both spin and elastic waves in the magnetoelastic resonance region of a homogeneous medium. In units of Λ, we express the dimensionless frequency in Fig. 3.

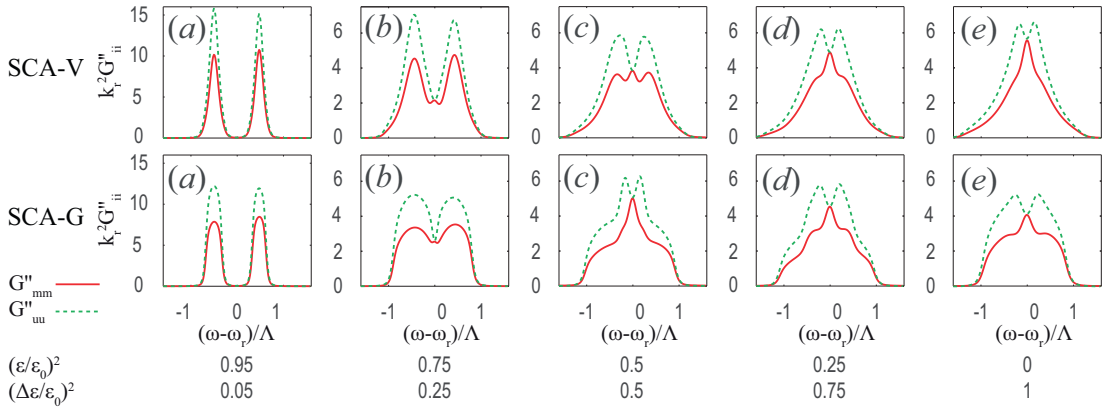


Fig. 3. Imaginary parts of the diagonal elements of the matrix Green's function for spin $G''_{mm}(\omega)$ and elastic $G''_{uu}(\omega)$ waves at $k_c/k_r = 0.01$ and different values of $(\varepsilon/\varepsilon_0)^2$ and $(\Delta\varepsilon/\varepsilon_0)^2$

The main result of applying SCA-V is a substantial improvement in resonance peak shape of high-frequency susceptibility (upper row in Fig. 3). It is shown that the dome-shaped form of magnetoelastic resonance peaks obtained within SCA-G by substituting $\hat{\Gamma} = \hat{E}$ instead of equation (32), as well as the appearance of resonance curves with three maxima (not related to fine structure resonance in the gap center) and bends on the peak slopes (lower row in Fig. 3), are due to the limitations of the SCA-G approximation. At the same time, qualitative regularities revealed within SCA-G are preserved when using SCA-V. In particular, the effects of broadening and convergence of magnetoelastic resonance peaks are reproduced, as well as their merging into one broad peak with increasing root-mean-square fluctuation $\Delta\varepsilon$ and decreasing mean value of the coupling parameter ε. Transition to SCA-V leads only to quantitative changes in these results. Of special interest is the effect of the emergence of spectral fine structure (see [23,24]). It appears as a narrow resonance peak in the Green's function of spin waves (dashed green curve in Fig. 3) and a corresponding narrow dip in the Green's function of elastic waves (solid red curve in Fig. 3). This effect should occur in the interaction of any wave fields of different physical nature. Its magnitude is proportional to the ratio of wave velocities at the crossing-resonance point. In the case of magnetoelastic interaction, this ratio is comparatively small, so the amplitude of fine structure peaks is small. However, for systems in which fields with substantially different velocities interact (for example, polaritons or systems with electron-phonon interaction), fine structure peaks can reach much larger values [39]. As k_c increases, the spectral fine structure is gradually suppressed, and resonance peaks narrow (see [23]).

Thus, magnetoelastic resonance in a stochastically inhomogeneous medium is a clear example of a physical system in which accounting for the matrix vertex function is fundamentally necessary. The matrix form of the self-consistent approximation for the vertex function provides a consistent description of wave hybridization, formation of spectral fine structure, and correct shape of resonance lines. This matrix method will be applied to the problem of ferromagnetic resonance in a thin film in Subsection 4.2 for a single wave field.

3. Ferromagnetic resonance in inhomogeneous thin films

3.1. Effect of stochastic magnetic structure on ferromagnetic resonance

This subsection presents results whose conceptual foundations were formulated by V. A. Ignatchenko in a series of papers [40–42]; however, their analysis and further development are carried out here for inhomogeneous thin magnetic films with perpendicular magnetization orientation under conditions close to magnetic saturation. The key idea is that inhomogeneity in the orientation of the uniaxial magnetic anisotropy axis $\mathbf{l}(\mathbf{x})$ affects FMR in two ways: directly, through the action of $\mathbf{l}(\mathbf{x})$ on dynamic variables $\mathbf{m}(\mathbf{x},t)$, and indirectly, by generating in the sample a stochastic magnetic structure (SMS) $\mathbf{M}(\mathbf{x})$, also called "magnetization ripple", which then affects the dynamic variables ($\mathbf{l}(\mathbf{x}) \to \mathbf{M}(\mathbf{x}) \to \mathbf{m}(\mathbf{x},t)$).

Let us consider the Landau–Lifshitz equation for the magnetization vector $\mathbf{M}$ with a phenomenological damping term in Gilbert form:

$$\frac{\partial \mathbf{M}}{\partial t} = -\tilde{g}[\mathbf{M} \times \mathbf{H}^e] + \frac{\xi}{M_0}\left[\mathbf{M} \times \frac{\partial \mathbf{M}}{\partial t}\right], \tag{41}$$

where ξ is a dimensionless damping parameter, and $\mathbf{H}^e$ is the effective magnetic field defined by

$$\mathbf{H}^e = \alpha \frac{\partial^2 \mathbf{M}}{\partial x_i^2} + \beta_0(\mathbf{nM})\mathbf{n} + \beta(\mathbf{lM})\mathbf{l} + \mathbf{H} + \mathbf{H}^{(d)}, \tag{42}$$

where $\mathbf{H}$ is the external magnetic field, represented below as $\mathbf{H} = H_0\mathbf{n} + \mathbf{h}$; here the direction of the static magnetic field $\mathbf{H}_0$ coincides with the anisotropy axis $\mathbf{n}$ and is perpendicular to the film plane, $\mathbf{h}$ is the alternating magnetic field in the film plane, $\mathbf{H}^{(d)}$ is the magnetodipole field, and β_0 is the homogeneous (induced) uniaxial magnetic anisotropy. For $\beta_0 > 0$, the axis $\mathbf{n}$ oriented along the z axis is easy, whereas for $\beta_0 < 0$ it becomes hard ("easy plane" anisotropy); see [43] for definitions of "easy axis" and "easy plane". Here β is the uniaxial local magnetic anisotropy constant whose easy axis direction is a random function of coordinates: $\mathbf{l} = \mathbf{l}(\mathbf{x})$. The origin of β_0 and the direction of anisotropy unit vector $\mathbf{n}$ are determined by a combination of mechanisms, including surface roughness, thin-film anisotropy formed during deposition, and overall induced anisotropy of the entire film arising from thermal treatment in a static magnetic field or mechanical processing, which form a preferred anisotropy axis direction, among others. We consider waves in a magnetic film unbounded in the xy plane, assuming no pinning of oscillations at the film surface.

Spatial fluctuations of the anisotropy axis lead to an orientationally inhomogeneous equilibrium magnetization state: a stochastic magnetic structure (SMS) appears ("magnetization ripple"). We use a version of SMS theory developed in [42, 44] on the basis of methods from the correlation theory of random functions. In this work, this approach is applied to a perpendicularly magnetized film.

Let us represent the magnetization vector $\mathbf{M}(\mathbf{x},t)$ as

$$\mathbf{M}(\mathbf{x},t) \approx \mathbf{M}(\mathbf{x}) + \mathbf{m}(\mathbf{x},t), \tag{43}$$

where the vector $\mathbf{M}(\mathbf{x})$ describes SMS, and $\mathbf{m}(\mathbf{x},t)$ describes dynamic deviations of $\mathbf{M}(\mathbf{x},t)$ from this structure. Consider magnetic fields H_0 close to film saturation (for exchange-noninteracting crystallites [45]) and directed perpendicular to the film surface:

$$H_0 \geqslant H_A, \tag{44}$$

where

$$H_A = (4\pi - \beta_0 + \sqrt{D^{+-}/2}\,\beta)M_0, \tag{45}$$

$M_0 = |\mathbf{M}|$, and $D^{+-} = 2/15$; this coefficient is obtained for uniaxial anisotropy.

Equation (45) is written for a perpendicularly magnetized inhomogeneous film; in the homogeneous-medium limit ($\beta = 0$), it reduces to $H_A = (4\pi - \beta_0)M_0$ (see [28, 43, 46]). Under these conditions, the orientation of all random vectors $\mathbf{M}(\mathbf{x})$ is concentrated within a cone of small polar angles θ around the z axis. As H_0 increases, this cone narrows, and the average film magnetization $\langle M_z(\mathbf{x})\rangle$ tends to M_0 as $H_0 \to \infty$. Let us represent the SMS vector $\mathbf{M}(\mathbf{x})$ as

$$\mathbf{M}(\mathbf{x}) \approx M_0\mathbf{n} + \boldsymbol{\mu}(\mathbf{x}). \tag{46}$$

Both vectors $\boldsymbol{\mu}(\mathbf{x})$ and $\mathbf{m}(\mathbf{x}, t)$, describing static and dynamic deviations of the vector $\mathbf{M}(\mathbf{x}, t)$ from the z axis, are assumed small,

$$|\boldsymbol{\mu}| \ll M_0, \quad |\mathbf{m}| \ll M_0, \tag{47}$$

but unlike [26], the z projections of these vectors must now be taken into account alongside the x and y projections. Therefore, the z projection of vectors $\mathbf{M}(\mathbf{x}, t)$ has the form

$$M_z(\mathbf{x}, t) \approx M_0 + \mu_z(\mathbf{x}) + m_z(\mathbf{x}, t), \tag{48}$$

where

$$\mu_z(\mathbf{x}) \approx -\frac{1}{2M_0}\left(\mu_x^2(\mathbf{x}) + \mu_y^2(\mathbf{x})\right), \tag{49}$$

$$m_z(\mathbf{x}, t) = -\frac{1}{M_0}\Big(\mu_x(\mathbf{x})m_x(\mathbf{x}, t) + \mu_y(\mathbf{x})m_y(\mathbf{x}, t)\Big). \tag{50}$$

A schematic view of one stochastic realization of vectors $\mathbf{M}(\mathbf{x}, t)$, $\mathbf{M}(\mathbf{x})$, $\boldsymbol{\mu}(\mathbf{x})$, $\mathbf{m}(\mathbf{x}, t)$, and their projections is shown in Fig. 4. Here expression (48) is obtained from the condition of preserving the magnetization vector magnitude $M_0 = |\mathbf{M}|$. Near the field $H_0 = H_A$ under the condition $H_A \gg h$, equation (48) is applicable, and SMS $\mathbf{M}(\mathbf{x})$, including dynamic projections m_j, is not destroyed (see [27]). In the strong-field limit $H_0 \gg H_A$, we have $\mu_x \to 0$ and $\mu_y \to 0$ (see below), hence $M_z = M_0$; this case was considered in [26].

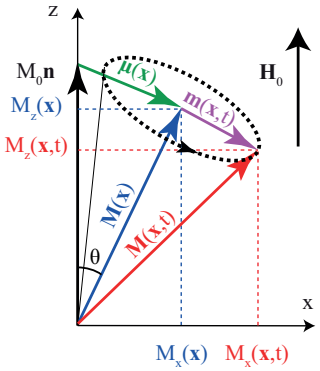


Fig. 4. A schematic view of the components of the magnetization vector $\mathbf{M}(\mathbf{x}, t)$ is shown for azimuthal angle $\varphi = 0$ in the xz plane

The magnetodipole field $\mathbf{H}^{(d)}$ satisfies Maxwell's equations in the magnetostatic approximation. For a thin magnetic film, an analytical solution of this problem is given in [40, 47]. In the case of homogeneous magnetization, the demagnetizing field has the form $H_0^d = -4\pi M_0$, whereas for static inhomogeneities $\boldsymbol{\mu}(\mathbf{x})$, the components of the magnetodipole field in $\mathbf{k}$-space are written as follows:

$$H_{x,y}^d(\mathbf{k}) = -4\pi\Big(1 - V_{\mathbf{k}}\Big)\frac{k_{x,y}}{k^2}(\mathbf{k}\boldsymbol{\mu}_{\mathbf{k}}), \tag{51}$$

$$H_z^d(\mathbf{k}) = 4\pi V_{\mathbf{k}} \mu_z(\mathbf{k}), \ \ (\mu_z < 0), \tag{52}$$

where

$$V_{\mathbf{k}} = \frac{1 - e^{-kd}}{kd}, \quad k = \sqrt{k_x^2 + k_y^2}, \tag{53}$$

d is the film thickness. For the Fourier harmonic $\mathbf{m}_{\mathbf{k}}$ of dynamic magnetization $\mathbf{m}(\mathbf{x}, \omega)$, the magnetodipole fields $\mathbf{h}_{\mathbf{k}}^d(\omega)$ (used below) are determined by expressions analogous to (51)-(53) with substitutions $\mathbf{H}_{\mathbf{k}}^d \to \mathbf{h}_{\mathbf{k}}^d$, $\mu_{x,y}(\mathbf{k}) \to m_{x,y}(\mathbf{k})$, and $\mu_z(\mathbf{k}) \to -m_z(\mathbf{k})$. Limiting cases of coefficients $V_{\mathbf{k}}$ and $1 - V_{\mathbf{k}}$ for small and large values of kd are considered in [26]. For compactness of notation, where convenient and not contradictory, functions depending on vector $\mathbf{k}$ are written as indices: $\mathbf{H}^d(\mathbf{k}) = \mathbf{H}_{\mathbf{k}}^d$.

3.1.1. Stochastic magnetic structure

The equilibrium state of the magnetic system in the form of SMS is described by

$$[\mathbf{M} \times \mathbf{H}^e] = 0, \tag{54}$$

which is the solution of a variational minimization problem for the Lagrangian under the constraint of preserving the magnitude of the vector $\mathbf{M}(\mathbf{x})$ (see, for example, [43, 48]).

Substituting equations (42) and (43) into equation (54), we represent the solution in $\mathbf{k}$-space as a system of equations for projections μ_x and μ_y in complex form $\mu^{\pm} = \mu_x \pm i\mu_y$, retaining only first-order terms in β:

$$\begin{cases} \mu_{\mathbf{k}}^+ = \dfrac{\beta M_0}{\Delta_{\mathbf{k}}^0} \left(a_{\mathbf{k}} \rho_{\mathbf{k}}^+ + b_{\mathbf{k}}^+ \rho_{\mathbf{k}}^- \right), \\ \mu_{\mathbf{k}}^- = \dfrac{\beta M_0}{\Delta_{\mathbf{k}}^0} \left(a_{\mathbf{k}} \rho_{\mathbf{k}}^- - b_{\mathbf{k}}^- \rho_{\mathbf{k}}^+ \right), \end{cases} \tag{55}$$

where $\rho^{\pm} = \rho_{xz} \pm i\rho_{yz}$, $k^{\pm} = k_x \pm ik_y$, and coefficients $a_{\mathbf{k}}$ and $b_{\mathbf{k}}^{\pm}$ are

$$a_{\mathbf{k}} = \frac{H_r}{M_0} + \alpha k^2 + 2\pi(1 - V_{\mathbf{k}}), \tag{56}$$

$$b_{\mathbf{k}}^{\pm} = 2\pi(1 - V_{\mathbf{k}}) \frac{(k^{\pm})^2}{k^2}, \tag{57}$$

and the determinant $\Delta_{\mathbf{k}}^0$ of the system is

$$\Delta_{\mathbf{k}}^0 = \left(\alpha k^2 + \frac{H_r}{M_0} \right) \left(\alpha k^2 + \frac{H_r}{M_0} + 4\pi(1 - V_{\mathbf{k}}) \right), \tag{58}$$

$$H_r = H_0 - (4\pi - \beta_0) M_0. \tag{59}$$

In expression (55), the following dimensionless random coordinate functions are introduced:

$$\rho_{jn}(\mathbf{x}) = l_j(\mathbf{x}) l_n(\mathbf{x}), \quad \rho_{jj}(\mathbf{x}) = l_j^2(\mathbf{x}). \tag{60}$$

indices $j \neq n$ and j, n run over x, y, z.

The representation for μ_z in complex form follows from equation (49) in $\mathbf{k}$-space:

$$\mu_z(\mathbf{k}) = -\frac{1}{2(2\pi)^{\tau} M_0} \int \mu_{\mathbf{k}-\mathbf{k}_1}^+ \mu_{\mathbf{k}_1}^- \, d\mathbf{k}_1. \tag{61}$$

The correlation properties of SMS variables $\mu_z(\mathbf{k})$ depend on the correlators of random functions $\rho_{\mathbf{k}}^{\pm}$ defined by

$$\left\langle \rho_{\mathbf{k}-\mathbf{k}_1}^+ \rho_{\mathbf{k}_1}^- \right\rangle = D^{+-} S_{\mathbf{k}_1} \delta(\mathbf{k}), \tag{62}$$

where the variance $D^{+-} = 2/15$ is well known for uniaxial anisotropy [44, 49].

Substituting expression (61) into the relation $M_z(\mathbf{k}) = M_0\delta(\mathbf{k}) + \mu_z(\mathbf{k})$, obtained from equation (48) by eliminating m_z, gives

$$\langle M_{z,\mathbf{k}} \rangle = (M_0 + \bar{\mu})\,\delta(\mathbf{k}) = \mathcal{M}_z(H_0)\delta(\mathbf{k}). \tag{63}$$

where

$$\bar{\mu} = -\frac{\beta^2 D^{+-} M_0}{2(2\pi)^\tau} \int \frac{S_{\mathbf{k}_1}}{\Delta^0_{\mathbf{k}_1}} d\mathbf{k}_1. \tag{64}$$

For magnetic media, the law of approach to saturation $\mathcal{M}_z(H_0)$ was considered in [44, 45, 50–52], while the influence of SMS on the dispersion law $\omega(\mathbf{k})$ was considered in [40]. Equation (64) is further studied numerically in the magnetic field range satisfying condition (44). In the region $H_0 < H_A$, features associated with the nonmonotonicity of function $\mathcal{M}_z(H_0)$ are possible, as noted earlier in [45, 50, 51].

3.1.2. Dynamic equations

Based on the Landau–Lifshitz equation in Gilbert form (41), an equation is obtained for the resonant circular projection $m = m_x + im_y$ in $\mathbf{k}$-space, represented as [27]

$$\left[\omega_0 + \omega^\alpha_{\mathbf{k}} + \omega^d_{\mathbf{k}} - \omega(1 - i\xi)\right] m_{\mathbf{k}} = g\mathcal{M}_z(H_0) h\delta(\mathbf{k}) - (2\pi)^{-\tau}\beta\omega_M \int \Phi_{\mathbf{k}-\mathbf{k}_1} m_{\mathbf{k}_1} d\mathbf{k}_1, \tag{65}$$

where $\omega^\alpha_{\mathbf{k}}$ and $\omega^d_{\mathbf{k}}$ describe contributions of exchange and magnetodipole interactions, respectively, to the frequency spectrum $\omega(\mathbf{k})$:

$$\omega_0 = gH_r, \quad \omega^\alpha_{\mathbf{k}} = \alpha\omega_M k^2, \quad \omega^d_{\mathbf{k}} = 2\pi(1 - V_{\mathbf{k}})\omega_M. \tag{66}$$

Let us represent the random function $\Phi_{\mathbf{k}-\mathbf{k}_1}$ in the integral term of equation (65) as

$$\Phi_{\mathbf{k}-\mathbf{k}_1} = R_{\mathbf{k}-\mathbf{k}_1} + Q_{\mathbf{k}-\mathbf{k}_1}. \tag{67}$$

Here $R_{\mathbf{k}-\mathbf{k}_1}$ describes the direct influence of local-anisotropy inhomogeneities $\mathbf{l}(\mathbf{x})$ on the dynamic variable $m(\mathbf{x}, \omega)$, considered earlier in [26], while $Q_{\mathbf{k}-\mathbf{k}_1}$ describes indirect interaction through SMS $\mathbf{M}(\mathbf{x})$. Explicit expressions for functions $R_{\mathbf{k}-\mathbf{k}_1}$ and $Q_{\mathbf{k}-\mathbf{k}_1}$ are given in [27] and also below. Let us preliminarily note that in the strong external magnetic field limit $H_0 \gg H_A$, equation (65) yields $\mathcal{M}_z(H_0) \to M_0$ and $Q \to 0$, since $Q \propto \mu^+\mu^-$ [27], so only $R_{\mathbf{k}-\mathbf{k}_1}$ remains. In this case, the considered equation transforms into equation (28) obtained earlier in [26]. The specificity of magnetic system excitation in a thin film is that the high-frequency magnetic field $\mathbf{h}$ depends, to good accuracy, only on frequency ω, since its wavelength is much larger than the film thickness d. In the absence of inhomogeneities, dynamic magnetization $\mathbf{m}(\mathbf{k}, \omega)$ is excited by this field synchronously and in phase at all points of the film. In equation (65), this corresponds to representing the driving field as $h\delta(\mathbf{k})$.

The methods used to study the influence of inhomogeneities on dynamic variables $\mathbf{m}(\mathbf{x}, \omega)$ require introducing a two-point Green's function. For a randomly inhomogeneous medium, before averaging over inhomogeneities, the two-point Green's function must depend on coordinates $\mathbf{x}$ and $\mathbf{x}_0$ separately, i.e., it is a function of two variables $\tilde{G}(\mathbf{x}, \mathbf{x}_0)$. Translational symmetry, in which the function takes the form $G(\mathbf{x} - \mathbf{x}_0)$, is restored on average only after averaging the Green's function over inhomogeneities. This is also true for the Green's function in $\mathbf{k}$-space related to $\tilde{G}(\mathbf{x}, \mathbf{x}_0)$ by the Fourier transform

$$\tilde{G}(\mathbf{x}, \mathbf{x}_0) = (2\pi)^{-2\tau} \iint \tilde{G}_{\mathbf{k},\mathbf{k}_0} e^{i(\mathbf{k}\mathbf{x} + \mathbf{k}_0\mathbf{x}_0)} d\mathbf{k} d\mathbf{k}_0, \tag{68}$$

$$\tilde{G}_{\mathbf{k},\mathbf{k}_0} = \iint \tilde{G}(\mathbf{x},\mathbf{x}_0) e^{-i(\mathbf{k}\mathbf{x}+\mathbf{k}_0\mathbf{x}_0)} d\mathbf{x} d\mathbf{x}_0. \tag{69}$$

The function $\tilde{G}_{\mathbf{k},\mathbf{k}_0}$ is related to the complex amplitude $m_{\mathbf{k}}$ by

$$\frac{g\mathcal{M}_z(H_0)h}{(2\pi)^\tau} \iint \tilde{G}_{\mathbf{k},\mathbf{k}_0} e^{i\mathbf{k}_0\mathbf{x}_0} d\mathbf{k}_0 d\mathbf{x}_0 = m_{\mathbf{k}}. \tag{70}$$

Substituting equation (70) into (65), we obtain the stochastic equation for $\tilde{G}_{\mathbf{k},\mathbf{k}_0}$:

$$\Big[\omega_0 + \omega^\alpha_{\mathbf{k}} + \omega^d_{\mathbf{k}} - \omega\,(1 - i\xi)\Big]\tilde{G}_{\mathbf{k},\mathbf{k}_0} = (2\pi)^\tau \delta(\mathbf{k}+\mathbf{k}_0) - (2\pi)^{-\tau}\beta\omega_M \int \Phi_{\mathbf{k}-\mathbf{k}_1}\tilde{G}_{\mathbf{k}_1,\mathbf{k}_0} d\mathbf{k}_1. \tag{71}$$

Let us average stochastic equation (71) over the ensemble of random realizations $\rho_{\mathbf{k}}$. For the Green's function $\tilde{G}_{\mathbf{k},\mathbf{k}_0}$, we have representation $\left\langle \tilde{G}_{\mathbf{k},\mathbf{k}_0} \right\rangle = (2\pi)^\tau \delta(\mathbf{k}+\mathbf{k}_0) G_{\mathbf{k}}$, with $G_{-\mathbf{k}} \equiv G_{\mathbf{k}}$. Let us represent $\Phi_{\mathbf{k}-\mathbf{k}_1}$ as a sum of its mean value and centered random part: $\Phi = \langle\Phi\rangle + \overset{\circ}{\Phi}$, where $\langle\Phi\rangle = \langle Q\rangle$, $\overset{\circ}{\Phi} = R$, and $\overset{\circ}{\Phi}$ has zero mean. According to equation (44) of [26], $\langle R\rangle = 0$. After averaging equation (71), multiplying by $(2\pi)^{-\tau}$, and integrating over $\mathbf{k}_0$, we finally obtain

$$G_{\mathbf{k}} = g_{\mathbf{k}} - \frac{\beta\omega_M}{(2\pi)^{2\tau}} g_{\mathbf{k}} \iint \left\langle R_{\mathbf{k}-\mathbf{k}_1}\tilde{G}_{\mathbf{k}_1,\mathbf{k}_0} \right\rangle d\mathbf{k}_1 d\mathbf{k}_0, \tag{72}$$

where

$$g_{\mathbf{k}} = \Big[\omega_0 + \omega^\alpha_{\mathbf{k}} + \omega^d_{\mathbf{k}} + \omega^\mu_{\mathbf{k}} - \omega\,(1 - i\xi)\Big]^{-1} \tag{73}$$

is the initial Green's function renormalized by interaction with SMS, and $\omega^\mu_{\mathbf{k}}$ is the renormalization frequency equal to

$$\omega^\mu_{\mathbf{k}} = \beta\omega_M \langle Q_{\mathbf{k}}\rangle, \tag{74}$$

$$\langle Q_{\mathbf{k}}\rangle = \frac{D^{+-}\beta}{(2\pi)^\tau} \int \frac{(2a_{\mathbf{k}_2} - F_{\mathbf{k},\mathbf{k}_2}/2)\, S_{\mathbf{k}_2}}{\Delta^0_{\mathbf{k}_2}} d\mathbf{k}_2, \tag{75}$$

$$F_{\mathbf{k},\mathbf{k}_2} = \alpha\left[2(\mathbf{k}\mathbf{k}_2) - k_2^2\right] - \frac{H_r}{M_0} + 2\beta_0 + 2\pi\Big[(1 - V_{\mathbf{k}}) + 2\,(1 - V_{\mathbf{k}-\mathbf{k}_2})\Big]. \tag{76}$$

Thus, accounting for both direct and indirect effects of inhomogeneities of the magnetic anisotropy axis $\mathbf{l}(\mathbf{x})$ on dynamic variables $m(\mathbf{x},t)$ leads to the following results:
1) The direct action is described by the integral correlator of the Green's function $\tilde{G}_{\mathbf{k}_1,\mathbf{k}_0}$ and random function $R_{\mathbf{k}-\mathbf{k}_1}$ (see [26]).
2) The indirect interaction through SMS is described by the exact analytical expression (71), which renormalizes the wave spectrum of dynamic variables $m(\mathbf{x},t)$.

The standard form of equation (72) allows one to use all major existing approximations to calculate the averaged Green's function $G_{\mathbf{k}}$. We use the SCA-V approximation. The system of equations has the form (16)–(17), where the inhomogeneity parameter is defined as $\gamma^2 = \beta^2 D_R \omega_M^2$, with $D_R = 1/5$ [26].

The normalized spectral density $S_{\mathbf{k}}$ is related to the normalized correlation function of inhomogeneities $K(\mathbf{r})$ by equations (14). For 2D inhomogeneities, we choose the normalized correlation function $K(\mathbf{r})$ in the form

$$K(\mathbf{r}) = e^{-k_c|\mathbf{r}|}. \tag{77}$$

The spectral density corresponding to this correlation function has the form:

$$S(\mathbf{k}) = \frac{2\pi k_c}{\left(k_c^2 + k^2\right)^{3/2}}. \tag{78}$$

The high-frequency complex susceptibility $\chi = \chi' - i\chi''$, measured experimentally and averaged over the sample volume, is defined by [25–27]

$$\chi = \frac{\langle m(\mathbf{x})\rangle}{h} = \frac{1}{h(2\pi)^\tau}\int \langle m_{\mathbf{k}}\rangle\, e^{i\mathbf{k}\mathbf{x}} d\mathbf{k}. \tag{79}$$

Averaging equation (70) and then substituting the result into (79), we obtain

$$\chi = \frac{g\,\mathcal{M}_z(H_0)}{(2\pi)^\tau}\iint G_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}_0)} d\mathbf{x}_0 d\mathbf{k}. \tag{80}$$

After integration over $d\mathbf{x}_0$ and $d\mathbf{k}$, we obtain

$$\chi = g\mathcal{M}_z(H_0)\; G_{\mathbf{k}}|_{\mathbf{k}=0}\,. \tag{81}$$

Before substituting Green's function (16) into equation (81), it is necessary to numerically solve the system of self-consistent equations (16) and (17). The solution is obtained by successive approximations until convergence is reached. As a result, the sought Green's function (16) is determined and then substituted into equation (81). From the resulting expression, the imaginary part of high-frequency susceptibility χ'' is extracted; calculation results for it are shown in Figs. 5–7.

3.1.3. FMR in a thin magnetic film

The problem is considered under application of a static magnetic field $H_0 \gg H_A$; then $\omega^{\mu}_{\mathbf{k}} \to 0$ and $\mathcal{M}_z(H_0) \to M_0$ in equations (72) and (81), with $H_0 \geqslant H_A$. At $H_0 \gg H_A$, magnetization across the whole sample precesses around $\mathbf{H}_0$, and its direction coincides with the direction of effective field $\mathbf{H}^e$. For large fields H_0, the FMR spectrum is shown in Fig. 5. As H_0 decreases to H_A (45), the role of random anisotropy axis $\mathbf{l}(\mathbf{x})$ increases, which leads to formation of SMS: the sample remains single-domain, but a small spatial "magnetization ripple" appears. Random local anisotropy forms an inhomogeneous field $\mathbf{H}^e$, so precession occurs no longer around a homogeneous external field but in an inhomogeneous effective field $\mathbf{H}^e$. This significantly changes FMR dynamic characteristics (Fig. 7).

The problem is formulated under constraints on anisotropy parameters $|\beta_0| < 4\pi$ and $\beta \ll 4\pi$, ensuring positivity of field H_A in expression (45). Similar constraints on anisotropy parameters were considered earlier in [28].

Figure 5 presents imaginary parts of susceptibility χ'' calculated for 2D anisotropy inhomogeneities at strong magnetic fields $H_0 \gg H_A$, with intrinsic damping $\xi = 0.005$, and for various values of normalized correlation wave number $k_c d$. It is shown that the first perturbation-theory approximation (Bourret approximation in random-medium theory and Born approximation in condensed matter physics) is inapplicable in the range $0 < k_c d < 0.2$ (Fig. 5*a–e*), i.e., for relatively large inhomogeneities with ratio $r_c/d > 5$. The self-consistent approximation for the Green's function (SCA-G), defined by equation (19) and obtained under condition (18), satisfactorily describes resonance-peak height, but is unsatisfactory in reproducing line shape in the region $0 < k_c d < 0.02$ (Fig. 5*a–c*). At larger values of $k_c d$, in the interval $0.02 < k_c d < 0.2$, this approximation reproduces line shape acceptably but overestimates its amplitude. Only at $k_c d > 0.3$ does it correctly describe all main parameters of the ferromagnetic resonance line. At the same time, the self-consistent approximation for the vertex function (SCA-V) provides good agreement for all parameters of the Green's function (and, accordingly, the FMR line) over the entire considered range of $k_c d$.

Let us discuss the origin of FMR peak asymmetry at $H_0 > H_A$. In a homogeneous thin film magnetized perpendicular to the plane, excitation by field $\mathbf{h}$ ($|\mathbf{h}| \ll H_0$, $\mathbf{h} \perp \mathbf{H}_0$) produces in-phase precession $\mathbf{M}(t)$ at frequency ω_0, and $\chi''(\omega)$ has a Lorentzian peak with width $2\omega\xi$

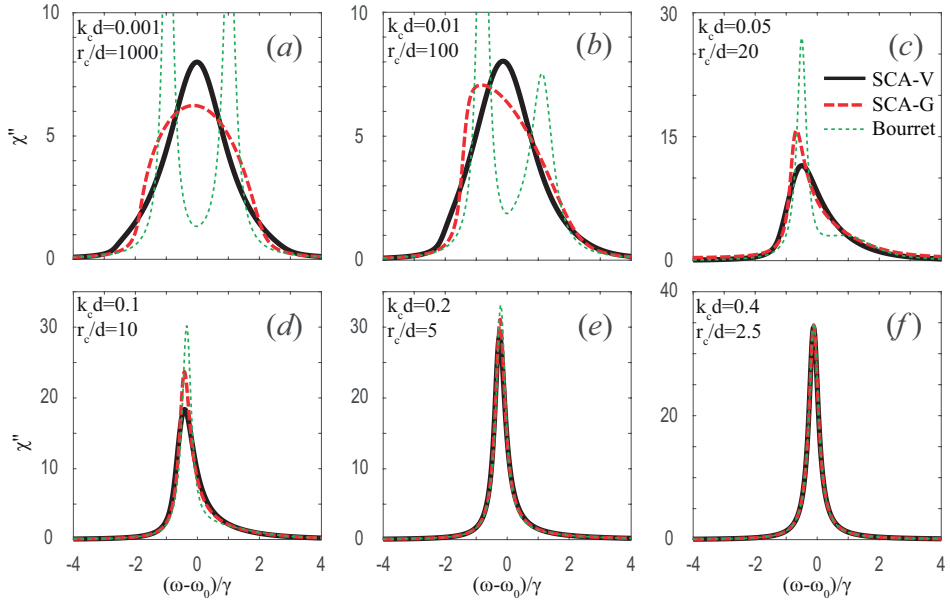


Fig. 5. Imaginary part of susceptibility χ'' versus frequency for 2D anisotropy inhomogeneities at different $k_c d$ (indicated in panels). Comparison of approximations: SCA-V (solid), SCA-G (dashed), Bourret (dotted). Parameters: $\xi = 0.005$, $\gamma/\omega_M = 0.15$

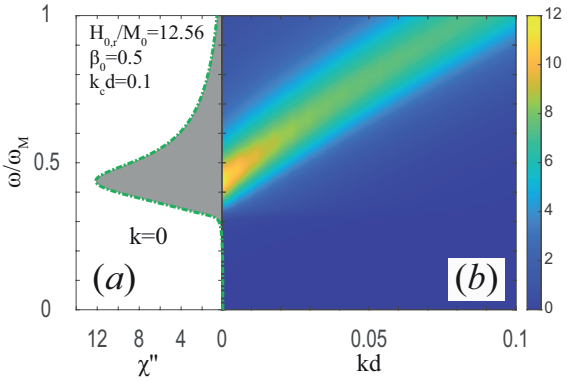


Fig. 6. Imaginary part of susceptibility χ'' in the FMR region: (a) section at $k = 0$ (FMR line); (b) dependence on normalized wave number kd and frequency ω/ω_M

($\xi \sim 10^{-4}$–10^{-2}). In an inhomogeneous film with fluctuating anisotropy axis $\mathbf{l}(\mathbf{x})$, precession becomes spatially inhomogeneous, leading to scattering and excitation of spin waves. Therefore, in equation (81) for FMR, $k = 0$ is taken, but integrations over $k_1, \ldots, k_n$ in the integrals are retained; these form asymmetry of $\chi''(\omega)$ (Fig. 6); the region $k \neq 0$ is not observed experimentally. Consider resonance near ω_0: for $\omega < \omega_0$, the left branch rises sharply, while for $\omega > \omega_0$, the right branch decays more smoothly, forming a tail due to the spin-wave spectrum along the film plane. In thin films, this effect is enhanced due to dominant magnetodipole contribution and linear-in-k dispersion, leading to more pronounced asymmetry [26].

In Fig. 7, columns correspond to different values of homogeneous magnetic anisotropy β_0 at the same value of normalized correlation wave number $k_c d = 0.05$. Figure 7a–a" presents results for the easy-axis case of homogeneous anisotropy $\beta_0 = 0.5$, Fig. 7b–b" for $\beta_0 = 0$, and

Fig. 7*c–c"* for the hard-axis (easy-plane) case $\beta_0 = -0.5$. Figure 7 shows that as static magnetic field H_0 decreases to values on the order of H_A, the effect of SMS on FMR susceptibility χ'' increases substantially, corresponding to an indirect interaction mechanism. This effect appears even for relatively weak changes in homogeneous magnetic anisotropy β_0 and leads to a noticeable change in the dependence of FMR frequency on static magnetic field H_0. For $\beta_0 = 0.5$, as field H_0 decreases to H_A, the FMR frequency either decreases linearly or shifts to lower frequencies as parameter $k_c d$ decreases. At the same time, for $\beta_0 = 0$ and $\beta_0 = -0.5$, the FMR frequency shifts to higher frequencies, and the magnitude of this shift decreases as $k_c d$ increases [27]. The obtained results for a thin magnetic film can be used to determine the sign of homogeneous magnetic anisotropy ($\beta_0 > 0$ or $\beta_0 < 0$) in the parameter range $4\pi > |\beta_0|$.

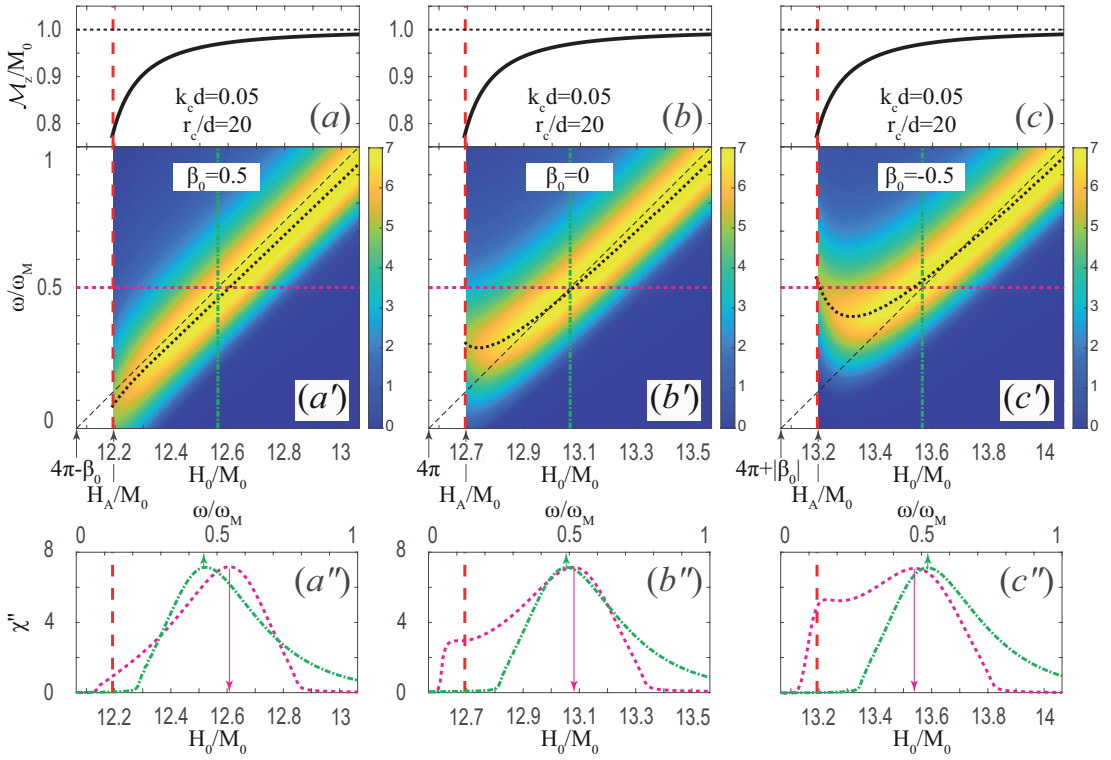


Fig. 7. Dependence of magnetization $\mathcal{M}_z$ and susceptibility χ'' on field H_0 and frequency ω for different β_0: (*a–a"*) 0.5, (*b–b"*) 0, (*c–c"*) −0.5. Top row: $\mathcal{M}_z(H_0)$, middle row: $\chi''(\omega, H_0)$, bottom row: sections $\chi''(\omega)$ (green dash-dotted curve) and $\chi''(H_0)$ (pink dotted curve). Black dotted curves show positions of maxima (*a'–c'*). Parameters: $\xi = 0.01$; α and d as in Fig. 5

It should be noted that the use of SCA-V made it possible to calculate susceptibility χ with SMS taken into account in a thin film, confirming the universality of this approach. Below, another case of ferromagnetic resonance in an inhomogeneous thin film is considered, for which the matrix form of SCA-V is used.

3.2. Effect of 2D inhomogeneities of magnetic anisotropy magnitude and tilt angle of field H_0 on ferromagnetic resonance

In this subsection, the results of [53–56] served as motivation for theoretical investigation. Works [53, 54] showed that inhomogeneities of anisotropy magnitude and anisotropy axis orientation affect FMR linewidth differently: the former can decrease it, while the latter increase

it. Under strong magnetic fields $\mathbf{H}_0$, it was established that anisotropy inhomogeneities in thin films with perpendicular magnetic anisotropy have a characteristic effect on FMR linewidth (see [55, 56]): narrowing of the peak is observed followed by its broadening. To obtain this feature, a theoretical study was carried out in [28]; we use SCA-V in the calculations.

Let us consider a ferromagnet model in which the quantities of uniaxial magnetic anisotropy $\tilde{\beta}_{x,y}(\mathbf{x})$, $\mathbf{x} = \{x, z\}$, are inhomogeneous, with easy axes directed along unit vectors $\mathbf{i}$ and $\mathbf{j}$ as shown in Fig. 8*a*. Let us represent $\tilde{\beta}_{x,y}(\mathbf{x})$ as

$$\tilde{\beta}_{x,y}(\mathbf{x}) = \beta_{x,y} + \Delta\beta_{x,y}\rho(\mathbf{x}), \tag{82}$$

where $\beta_{x,y}$ are mean values of anisotropy parameters, and $\Delta\beta_{x,y}$ are their root-mean-square fluctuations. Representation (82) allows us to consider two separate cases: first, inhomogeneity of anisotropy magnitude along the y axis (unit vector $\mathbf{j}$) ($\tilde{\beta}_y \neq 0$, $\tilde{\beta}_x = 0$), and second, along the x axis (unit vector $\mathbf{i}$) ($\tilde{\beta}_y = 0$, $\tilde{\beta}_x \neq 0$).

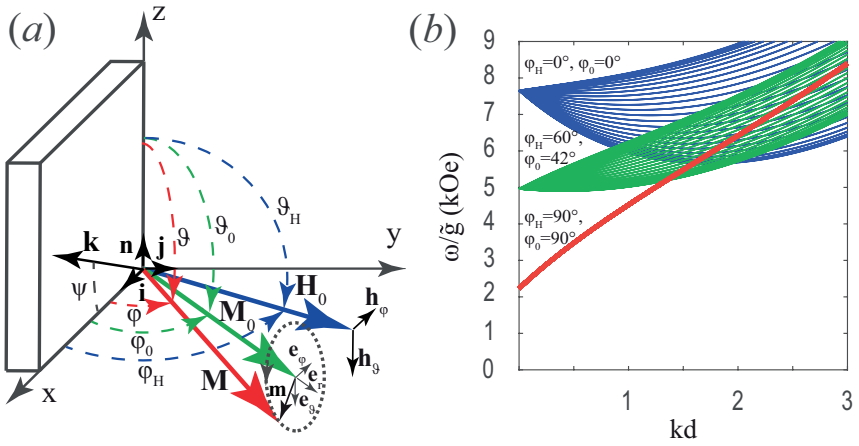


Fig. 8. (*a*) Coordinate system used to describe magnetization $\mathbf{M}$ in a film in external magnetic field $\mathbf{H}_0$. (*b*) Spin-wave dispersion laws $\omega(kd)$ for different field tilt angles φ_H and equilibrium angles φ_0.

For this medium, the equation of motion is the Landau–Lifshitz equation in Gilbert form for the magnetization vector $\mathbf{M} \approx M_0\mathbf{e}_r + m_\theta(\mathbf{x}, t)\mathbf{e}_\theta + m_\varphi(\mathbf{x}, t)\mathbf{e}_\varphi$, where $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\varphi$ are unit vectors of the spherical coordinate system. Let us write it for projections m_φ and m_θ in linear approximation at $\theta_H = \theta_0 = 90°$, assuming $m_\theta, m_\varphi, h \propto \exp(i\omega t)$:

$$\left(\frac{U_{\theta\varphi}}{M_0^2} - \frac{i\omega}{\omega_M}\right) m_\varphi + \left(\frac{U_{\theta\theta}}{M_0^2} + \frac{i\omega\xi}{\omega_M} - \alpha\nabla^2\right) m_\theta + D_{\theta\theta}\rho(\mathbf{x})m_\theta = h_\theta, \tag{83}$$

$$\left(\frac{U_{\varphi\varphi}}{M_0^2} + \frac{i\omega\xi}{\omega_M} - \alpha\nabla^2\right) m_\varphi + \left(\frac{U_{\varphi\theta}}{M_0^2} + \frac{i\omega}{\omega_M}\right) m_\theta + D_{\varphi\varphi}\rho(\mathbf{x})m_\varphi = h_\varphi, \tag{84}$$

where h_θ and h_φ are projections of external alternating magnetic field $\mathbf{h}$, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$, $D_{\theta\theta} = \Delta\beta_y \sin^2\varphi_0 + \Delta\beta_x \cos^2\varphi_0$, $D_{\varphi\varphi} = -\Delta\beta_y \cos 2\varphi_0 + \Delta\beta_x \cos 2\varphi_0$, and functions $U_{\theta\theta}$, $U_{\varphi\varphi}$, and $U_{\theta\varphi} = U_{\varphi\theta}$ are shown below.

The system of equations (83) and (84) is solved within the self-consistent approximation SCA-V, by analogy with the case of a single wave field (1) and the system (27) and (28). This approximation is a system of two coupled self-consistent equations in $\mathbf{k}$-space: an exact equation

for matrix Green's function $\hat{G}_{\mathbf{k}}$ and an approximate equation for matrix vertex function $\hat{\Gamma}_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1}$, of the form

$$\hat{G}_{\mathbf{k}} = \hat{E}\left[\hat{g}_{\mathbf{k}}^{-1} - \frac{1}{(2\pi)^2}\int S_{\mathbf{k}-\mathbf{k}_1}\hat{D}\hat{G}_{\mathbf{k}_1}\hat{\Gamma}_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1}\hat{D}d\mathbf{k}_1\right]^{-1}, \tag{85}$$

$$\hat{\Gamma}_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1} \approx \hat{E}\left[\hat{E} - \frac{1}{(2\pi)^2}\int S_{\mathbf{k}_1-\mathbf{k}_2}\hat{\Gamma}_{\mathbf{k}_2,\mathbf{k}_1-\mathbf{k}_2}\hat{D}\hat{G}_{\mathbf{k}_1}\hat{D}\hat{G}_{\mathbf{k}-\mathbf{k}_1+\mathbf{k}_2}d\mathbf{k}_2\right]^{-1}, \tag{86}$$

where $\hat{g}_{\mathbf{k}} = \omega_M^{-1}\hat{L}_{\mathbf{k}}^{-1}$ is the initial Green's function, S is the normalized spectral density defined by equation (78). A similar structure of a system of two coupled self-consistent equations for matrix functions $\hat{G}_{\mathbf{k}}$ and $\hat{\Gamma}_{\mathbf{k}_1,\mathbf{k}-\mathbf{k}_1}$ was obtained above in (31) and (32) for studying interaction of two coupled wave fields. Here $\hat{L}_{\mathbf{k}}$ and $\hat{D}$ have the form:

$$\hat{L}_k = \begin{bmatrix} \dfrac{U_{\theta\theta}}{M_0^2} + \dfrac{i\omega\xi}{\omega_M} + \alpha k^2 & \dfrac{U_{\theta\varphi}}{M_0^2} - \dfrac{i\omega}{\omega_M} \\ \dfrac{U_{\varphi\theta}}{M_0^2} + \dfrac{i\omega}{\omega_M} & \dfrac{U_{\varphi\varphi}}{M_0^2} + \dfrac{i\omega\xi}{\omega_M} + \alpha k^2 \end{bmatrix}, \ \hat{D} = \begin{bmatrix} D_{\theta\theta} & 0 \\ 0 & D_{\varphi\varphi} \end{bmatrix}, \tag{87}$$

where

$$U_{\theta\theta} = M_0 H_U + 4\pi M_0^2(1 - V_k)\left[\sin^2\psi + \sin^2\varphi_0 - \cos^2\psi\cos^2\varphi_0\right]. \tag{88}$$

$$U_{\varphi\varphi} = M_0 H_U + (4\pi V_k - \beta_y) M_0^2\cos^2\varphi_0 - \beta_x M_0^2\sin^2\varphi_0 + \\ + 4\pi M_0^2(1 - V_k)\left[\sin^2\varphi_0 - \cos^2\psi\cos 2\varphi_0\right], \tag{89}$$

$$U_{\theta\varphi} = U_{\varphi\theta} = 4\pi M_0^2(1 - V_k)\sin\psi\cos\psi\sin\varphi_0, \tag{90}$$

$H_U = H_0\cos(\varphi_0 - \varphi_H) - (4\pi - \beta_y) M_0\sin^2\varphi_0 + \beta_x M_0\cos^2\varphi_0$, $\cos\psi = k_x/k$, $\sin\psi = k_z/k$, and $V_{\mathbf{k}}$ is given by equation (53).

High-frequency susceptibility for ferromagnetic resonance is defined by [25, 26]:

$$\hat{\chi} = \omega_M \hat{G}_{\mathbf{k}}\Big|_{k=0}, \tag{91}$$

where the elements of matrix $\hat{\chi}$ are

$$\hat{\chi} = \begin{bmatrix} \chi_{\theta\theta} & i\chi_{\theta\varphi} \\ -i\chi_{\varphi\theta} & \chi_{\varphi\varphi} \end{bmatrix}, \tag{92}$$

$\chi_{\theta\theta} = \omega_M G_{\theta\theta,\mathbf{k}}\Big|_{k=0}$, $\chi_{\varphi\varphi} = \omega_M G_{\varphi\varphi,\mathbf{k}}\Big|_{k=0}$, $\chi_{\theta\varphi} = -i\omega_M G_{\theta\varphi,\mathbf{k}}\Big|_{k=0}$, and $\chi_{\varphi\theta} = i\omega_M G_{\varphi\theta,\mathbf{k}}\Big|_{k=0}$.

To calculate the matrix Green's function (85) in an inhomogeneous medium, it is first necessary to determine equilibrium angles θ_0 and φ_0. They are found from the solution of the magnetostatic problem [28]. The magnetic film is considered in the configuration shown in Fig. 8*a*. For this configuration, angles θ_H and θ_0 are equal to 90°, while angle φ_H is varied from 0° to 90°. For angles $\varphi_H = 0°$ and $\varphi_H = 90°$, values $\varphi_0 = 0°$ and $\varphi_0 = 90°$ are obtained, respectively [28]; for other values of φ_H, angle φ_0 is determined by

$$\varphi_0 \approx \varphi_H + \frac{B - \sqrt{B^2 + 2A^2}}{2A}, \tag{93}$$

where $A = M_0 [4\pi - \beta_y + \beta_x] \sin(2\varphi_H)$, $B = H_0 + M_0 [4\pi - \beta_y + \beta_x] \cos(2\varphi_H)$. Expression (93) is valid under a strong external magnetic field condition $H_0 > |H_0^c|$,

$$H_0^c = (4\pi - \beta_y + \Delta\beta_y + \beta_x + \Delta\beta_x) M_0. \tag{94}$$

At field values H_0 exceeding (94), the film is characterized by a single-domain state, whereas at $H_0 < |H_0^c|$ a domain magnetic structure may form. In this work, the critical magnetic field value was fixed in all figures as $|H_0^c| < H_0 = 6\,\text{kOe}$. Fluctuations of anisotropy magnitude $\Delta\beta_{x,y}$ must satisfy $\Delta\beta_{x,y} << 4\pi$, under which spatial variations do not lead to formation of an inhomogeneous magnetization state. For analysis of high-frequency susceptibility, value $\Delta\beta_{x,y}/4\pi = 0.125$ was considered. Additional information on characteristic fluctuation magnitudes can be found in [17, 26].

We consider waves in a magnetic film unbounded in the xz plane (Fig. 8*a*) with no pinning of oscillations at its surface. It is also assumed that all considered quantities–magnetic anisotropy inhomogeneities, magnetic inhomogeneities, and oscillations–are uniform across the film thickness. When an external magnetic field $\mathbf{H}_0$ ($H_0 > |H_0^c|$) is applied to a thin inhomogeneous magnetic film, magnetization vector $\mathbf{M}_0$ tends toward the field direction but does not coincide with it, so angles θ_0 and φ_0 are defined. Then, by means of small alternating magnetic fields h_θ, $h_\varphi << H_0$ oriented along unit vectors e_θ and e_φ, magnetization oscillations are excited throughout the film. The excited oscillations represent a superposition of the FMR mode at $k = 0$ and spin (magnetostatic) waves at $k \neq 0$ propagating in the plane of the inhomogeneous film (see Fig. 8*b*). In the long-wavelength region of the spectrum, magnetodipolar interaction dominates [57, 58], whereas in the short-wavelength region the exchange interaction is dominant. In what follows, these spin waves are taken into account within the SCA-V framework.

Next, we numerically calculate elements of the high-frequency susceptibility matrix (92). For this purpose, the system of self-consistent equations (85) and (86) is used, solved numerically by successive approximations until convergence is reached. The main result of [28], computed for the imaginary part of one susceptibility element, $\chi''_{\theta\theta}$, is shown in Fig. 9.

In Fig. 9*a*, the imaginary part of susceptibility $\chi''_{\theta\theta}$ is shown as a function of normalized frequency $\omega/\tilde{g}$ when the field tilt angle φ_H changes from perpendicular to in-plane direction: for anisotropy inhomogeneities along easy axis y ($\Delta\beta_y \neq 0$, $\Delta\beta_x = 0$, thick curves) and without inhomogeneities ($\Delta\beta_y = 0$, $\Delta\beta_x = 0$, thin curves). At $\varphi_H = 90°$, $\varphi_0 = 90°$, the low-frequency peak is maximal in width and asymmetric (steeper decay on the left); as φ_H decreases, width and asymmetry decrease, the peak becomes narrow and symmetric, and its width coincides with the narrow resonance (thin curve) at $\varphi_H \approx 46°$, $\varphi_0 = 30°$; then the peak broadens again and a weak asymmetry of opposite sign appears. In Fig. 9*b*, for inhomogeneities along easy axis x ($\Delta\beta_y = 0$, $\Delta\beta_x \neq 0$, thick curves), the same scenario is observed: at $\varphi_H = 90°$, $\varphi_0 = 90°$, the peak is narrower than in Fig. 9*a* but also asymmetric; as φ_H decreases, it narrows until coinciding with the narrow resonance at $\varphi_H \approx 76°$, $\varphi_0 = 60°$, after which it broadens again with asymmetry of opposite sign and at $\varphi_H = 0°$, $\varphi_0 = 0°$ becomes the broadest. The case with contributions of anisotropies $\beta_y \neq 0$ and $\beta_x \neq 0$ is considered in [28]. Similar linewidth changes are observed experimentally [54–56].

Analysis of the angular dependence of FMR linewidth allows identification of the type of fluctuating anisotropic component. If at angle $\varphi_H = 90°$ the FMR line has increased linewidth, while at angle $\varphi_H = 0°$ it is narrower, then parameter $\Delta\beta_y$ is subject to fluctuations. In the opposite case, fluctuations of parameter $\Delta\beta_x$ dominate.

In the case of 2D inhomogeneities of magnetic anisotropy magnitude, an additional control parameter is the tilt angle of the external static magnetic field relative to the film plane. Varying this angle changes FMR linewidth and asymmetry.

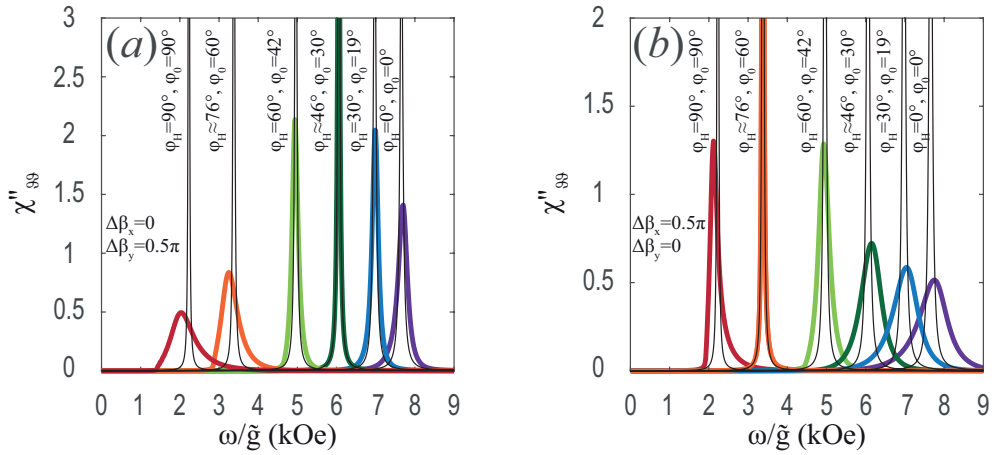


Fig. 9. Imaginary part of susceptibility $\chi''_{\theta\theta}$ versus normalized frequency $\omega/\tilde{g}$ for different angles φ_H and corresponding equilibrium angles φ_0 (indicated in panels). Thick colored curves correspond to anisotropy inhomogeneities: (a) $\Delta\beta_y = 0.5\pi$, $\Delta\beta_x = 0$; (b) $\Delta\beta_y = 0$, $\Delta\beta_x = 0.5\pi$. Thin black curves correspond to the homogeneous case. Parameters: $k_c d = 0.2$, $\xi = 0.005$, $H_0 = 6\,\mathrm{kOe}$, $4\pi M_0 = 3.7\,\mathrm{kOe}$

Conclusion

This review article systematically presents a unified theoretical framework integrating a series of papers devoted to development and application of the self-consistent approximation for analyzing resonance phenomena in stochastically inhomogeneous media. The key result is construction of a self-consistent approach based on self-consistency with respect to the vertex function.

It is shown that the self-consistent approximation for the Green's function (SCA-G), despite its wide applicability, has fundamental limitations due to exclusion of diagrams with intersecting correlation lines. These limitations distort spectral characteristics and become especially significant in media characterized by long-wavelength inhomogeneities.

The developed self-consistent approximation for the vertex function (SCA-V) removes these shortcomings and provides correct limiting transitions to the Bourret/Born and SCA-G approximations. The key methodological element is introducing self-consistency simultaneously for the Green's function and the vertex function. This leads to a system of coupled nonlinear integral equations and allows one to account for an essential class of diagrams with intersecting correlation lines that are inaccessible within SCA-G.

The proposed approach is generalized to systems of coupled wave fields in continuous media, in particular to magnetoelastic waves. For such systems, a matrix representation of the SCA-V approximation is obtained for both the Green's function and the vertex function. It is shown that within this approach, effects of broadening and mutual convergence of magnetoelastic resonance peaks are reproduced correctly. As the distance between resonance peaks decreases in the crossing-resonance region, a spectral fine structure is formed, including resonance-antiresonance features caused by stochastic fluctuations of interaction parameters.

To describe ferromagnetic resonance in inhomogeneous thin films, the SCA-V approximation was used in both scalar and matrix forms when calculating high-frequency susceptibility for various physical situations. The efficiency of this approach is demonstrated for analysis of real systems with spatial inhomogeneities of magnetic anisotropy, taking into account contributions of magnetodipole fields.

It is established that key parameters determining resonance characteristics are the correla-

tion radius of inhomogeneities, the amplitude of their fluctuations, the orientation of the external static magnetic field relative to the film plane, and film thickness, which sets the ratio of exchange- and magnetostatic-wave contributions. It is shown that in thin films, magnetodipole (magnetostatic) waves significantly affect FMR line shape, leading in particular to different asymmetry of the resonance peak. Within SCA-V, propagation of such waves along the film is accounted for correctly, which is reflected in the shape of the high-frequency susceptibility peak. Thus, the SCA-V approximation provides a unified and physically consistent approach to describing the set of effects listed above.

From a methodological viewpoint, the presented cycle of papers demonstrates a consistent transition from scalar problems to matrix systems, from idealized models to realistic geometries, and from qualitative analysis of linewidth broadening to a quantitative theory of resonance peak shape and asymmetry. This makes it possible to regard the self-consistent approximation for the vertex function as a universal analytical tool for studying wave processes in stochastically inhomogeneous media.

Prospects for further research are primarily related to expanding the application domain of the developed approach. Of considerable interest is also tighter integration of the analytical self-consistent approach with numerical simulation and experimental studies, which opens possibilities for direct determination of statistical characteristics of inhomogeneities from measured spectra of resonance responses.

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Самосогласованное приближение для вершинной функции и его применение к волнам в неоднородных средах

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Аннотация. В работе представлен обзор единого теоретического подхода для анализа резонансных явлений в стохастически неоднородных средах на основе самосогласованного приближения для вершинной функции (SCA-V). Показано, что данный подход устраняет ограничения традиционного самосогласованного приближения для функции Грина (SCA-G), известного также как приближение Мигдала, Крейчнана или самосогласованное борновское приближение. Метод обобщён на системы связанных волновых полей, включая магнитоупругие волны, и применён к описанию ферромагнитного резонанса в неоднородных тонких плёнках. Подход обеспечивает последовательное описание уширения резонансных пиков, формирования тонкой структуры в области кроссинг-резонансов и асимметрии линий ФМР, обусловленной магнитостатическими волнами. В целом приближение SCA-V представляет собой универсальный аналитический инструмент для исследования волновых процессов в стохастически неоднородных средах.

Ключевые слова: ферромагнитный резонанс, магнитные пленки, неоднородности магнитной анизотропии, функции Грина, спиновые волны, магнитодипольное поле, самосогласованное приближение.

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On the Boundary Behavior of Potentials of Real Analytic Functions

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Abstract. In this article, we consider potentials of real-analytic densities defined on the boundary of a bounded domain $D \subset \mathbb{R}^n$ and study their boundary behavior. We also obtain estimates for the extensions of these potentials to the complex domain. These estimates can be applied to various multidimensional integral representations (Bochner-Martinelli, Cauchy-Fantappiè) for holomorphic functions in multidimensional complex spaces.

Keywords: simple layer potential, Bochner-Martinelli integral, Cauchy-Fantappiè integral, real analytic functions.

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Introduction

In classical potential theory, formulas are well known that express the boundary values of the potentials of a simple and double layer from various densities and derivatives of them of any order in terms of the potentials of the derivatives of these densities (see, for example, [1, 2]). It follows that for infinitely differentiable densities, the boundary values of the potentials from these densities on the boundary of the domain are also infinitely differentiable up to the boundary of the domain.

In this paper, we consider potentials from real-analytic densities and study their boundary behavior. In addition, we obtain estimates for the continuation of these potentials in the complex domain. This makes it possible to apply them to various multidimensional integral representations for holomorphic functions in a multidimensional complex space.

The conditions for the real analyticity of functions on the boundary of a ball are considered in S. L. Sobolev's fundamental monograph [3, ch. 11]. These conditions for the real analyticity of a function are related to the growth of spherical harmonics, according to which this function decomposes. We extend these statements to arbitrary bounded domains with a real-analytic boundary. In addition, we obtain estimates for the holomorphic continuation of these functions into a multidimensional complex space.

1. Preliminary information

Consider the n-dimensional real space $\mathbb{R}^n$, $n > 1$. The variables in it are the vectors $x = (x_1, \ldots, x_n)$, where x_j are real numbers for all $j = 1, \ldots, n$. We introduce the vector module

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$|x| = \sqrt{|x_1|^2 + \ldots + |x_n|^2}$ and differential forms $dx = dx_1 \wedge \ldots \wedge dx_n$, as well as $dx[k] = dx_1 \wedge \ldots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \ldots \wedge dx_n$. The topology in $\mathbb{R}^n$ is defined by the metric $|x - y|$, $x, y \in \mathbb{R}^n$.

Let D be a bounded domain in $\mathbb{R}^n$ with a real-analytic connected boundary $\partial D = \Gamma$. The orientation of the boundary Γ is induced by the orientation of the domain D. The domain is defined as follows $D = \{x \in \mathbb{R}^n : \rho(x) < 0\}$, where ρ is a real-valued real-analytic function in some neighborhood of the closure of the domain D and $\operatorname{grad}\rho \neq 0$ on Γ.

Recall that the infinitely differentiable function φ is real-analytic at the point

$$x^0 = (x_1^0, \ldots, x_n^0) \in D,$$

if the Taylor series at point x_0 converges to φ in the neighborhood of this point, that is

$$\varphi(x) = \sum_{i_1,\ldots,i_n=0}^{\infty} c_{i_1,\ldots,i_n}(x_1 - x_1^0)^{i_1} \cdots (x_n - x_n^0)^{i_n}. \tag{1}$$

The connectivity of the boundary is necessary in the future for the possibility of solving the Dirichlet problem for harmonic functions (see, for example, [2, ch. 2]).

The coefficients of the series (1) are derivatives of a certain order from the function φ, i.e.

$$c_{i_1,\ldots,i_n} = \frac{1}{i_1! \cdots i_n!} \cdot \partial^J \varphi(x^0) = \frac{1}{i_1! \cdots i_{2n}!} \cdot \frac{\partial^{i_1+\ldots+i_n}\varphi(x^0)}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}},$$

where multiindex $J = (i_1, \ldots, i_n)$.

So it is not difficult to understand (using Stirling's formula) that the convergence of this series in the neighborhood of the point x^0 is equivalent to fulfilling the following condition for the growth of derivatives

$$\left|\partial^J \varphi(x^0)\right| \leqslant C \cdot \frac{(i_1 + \ldots + i_n)!}{R^{i_1+\cdots+i_n}} \tag{2}$$

for some positive constants C, R. The constant C is the maximum value of the function $|\varphi|$ in the neighborhood of the point x^0. This inequality is given in [3] (inequality XI.4.15).

Using (1), (2), we obtain the following estimate for the function $\varphi(x)$

$$\begin{aligned} |\varphi(x)| &\leqslant \sum_{i_1,\ldots,i_n=0}^{\infty} |c_{i_1,\ldots,i_n}| \cdot |x_1 - x_1^0|^{i_1} \cdots |x_n - x_n^0|^{i_n} \leqslant \\ &\leqslant C \sum_{i_1,\ldots,i_n=0}^{\infty} \frac{(i_1 + \ldots + i_n)!}{i_1! \cdots i_n!} \cdot \frac{1}{R^{i_1+\cdots+i_n}} \cdot |x_1 - x_1^0|^{i_1} \cdots |x_n - x_n^0|^{i_n} \leqslant \\ &\leqslant C \sum_{\|J\|=0}^{\infty} \left(\frac{|x_1 - x_1^0|}{R} + \ldots + \frac{|x_n - x_n^0|}{R} \right)^{\|J\|}, \end{aligned}$$

where $\|J\| = i_1 + \ldots + i_n$. The last expression in the previous formula is the sum of a geometric progression that converges at

$$\frac{|x_1 - x_1^0|}{R} + \ldots + \frac{|x_n - x_n^0|}{R} < 1$$

and it does not depend on the type of the function $\varphi(x)$, but depends only on the maximum value of $|\varphi(x)|$ in the neighborhood of the point x^0.

If we give x complex values of w, then we get the following statement.

Lemma 1. *For a real-analytic function $\varphi(x)$, the series(1) converges in some homogeneous polydisc $U(x^0, r) \subset \mathbb{C}^n$ of the form*

$$U(x^0, r) = \{w \in \mathbb{C}^n : |w_1 - x_1^0| < r, \dots, |w_n - x_n^0| < r\}.$$

The radius r of the polydisc does not depend on the type of the function φ, but depends on the constant C in the inequality (2), determined by the maximum value of the function $|\varphi(x)|$ in the neighborhood of the point x^0.

Thus, the function $\varphi(x)$ continues as a holomorphic function from n complex variables $w = (w_1, \dots, w_n)$ to some neighborhood U of the closure of the domain D in $\mathbb{C}^n$.

Using Stirling's formula, the condition (2) can be written more expressively by entering the value

$$\ell(\varphi) = \varlimsup_{\|J\| \to \infty} \sqrt[\|J\|]{\frac{\sup\limits_{\Gamma} |\partial^J \varphi|}{J!}}, \tag{3}$$

where is $J! = i_1! \cdots i_n!$. The condition (2) then means that $\ell(\varphi)$ is finite.

We denote the guiding cosines of the function ρ

$$\rho_k = \frac{1}{|\operatorname{grad} \rho|} \frac{\partial \rho}{\partial x_k}, \quad k = 1, \dots, n. \tag{4}$$

Note that this refers to the usual gradient

$$\operatorname{grad} \rho = \left(\frac{\partial \rho}{\partial x_1}, \dots, \frac{\partial \rho}{\partial x_n} \right).$$

From here

$$|\rho_k| \leqslant 1 \quad \text{in a neighborhood} \quad U, \quad k = 1, \dots, n. \tag{5}$$

Consider the space $\mathcal{A}(\Gamma)$ of real-analytic functions on the boundary of the domain Γ. For further discussion, we will use the definition of the space $\mathcal{A}(\Gamma)$ from Schapira's monograph [4, Ch. 1]: the space $\mathcal{A}(\Gamma)$ is an inductive limit of the spaces of functions $\mathcal{O}(U)$ holomorphic in the neighborhood U of the set Γ in $\mathbb{C}^n$ (each in its neighborhood U), that is

$$\mathcal{A}(\Gamma) = \varinjlim_{U \supset \Gamma} \mathcal{O}(U).$$

It is also noted that every bounded set of $\mathcal{A}(\Gamma)$ is contained and bounded in some space $\mathcal{O}(U)$. So we get that every function $f \in \mathcal{A}(\Gamma)$ is a restriction to Γ of some function h that is holomorphic in the neighborhood of the manifold Γ in $\mathbb{C}^n$.

Let $\varphi \in \mathcal{C}^\infty(\Gamma)$. Define $\ell(\varphi)$ by the formula (3).

Lemma 2. *A function $\varphi \in \mathcal{A}(\Gamma)$ if and only if $\varphi \in \mathcal{C}^\infty(\Gamma)$ and $\ell(\varphi) < \infty$.*

Proof. It immediately follows from the possible decomposition of the function φ into a multiple power series of the form (1) into Γ and the convergence conditions (2) of this series. □

The expression $1/\ell(\varphi)$ is nothing more than the radius of a homogeneous polydisc $U(x^0, r) \subset \mathbb{C}^n$ centered at $x^0 \in \Gamma$ and radius r into which φ holomorphically extends. The union of all such polydiscs $U(x^0, r)$ by $x^0 \in \Gamma$ gives a complex neighborhood $U \subset \mathbb{C}^n$ into which the function φ holomorphically extends.

Proposition 1. *The sequence of functions $\varphi_k \in \mathcal{A}(\Gamma)$ converges to the function $\varphi \in \mathcal{A}(\Gamma)$ as $k \to \infty$ if and only if φ_k converges uniformly with all derivatives to φ on Γ and all numbers $\ell(\varphi_k)$ are uniformly bounded.*

This means exactly that all functions φ_k holomorphically extend into one open set $U \subset \mathbb{C}^n$ and on this set they converge uniformly to φ (due to the topology of the inductive limit).

Consider the space $\mathcal{H}(D)$ of functions f that are harmonic in the domain of D. In him it introduces the topology of uniform convergence on compact subsets from D. Using $\mathcal{H}(\bar{D})$, we denote the space of functions f that are harmonic in some neighborhood of the closure of the domain D (each function f is harmonious in its neighborhood). The topology on $\mathcal{H}(\bar{D})$ is defined as follows. Let

$$D_\varepsilon = \{z : \ \rho(z) < \varepsilon\}, \quad \varepsilon > 0.$$

Then the space

$$\mathcal{H}(\bar{D}) = \liminf_{\varepsilon \to +0} \mathcal{H}(D_\varepsilon).$$

Lemma 3. *The spaces $\mathcal{H}(\bar{D})$ and $\mathcal{A}(\Gamma)$ are topologically isomorphic.*

Proof. Due to the connectivity of the boundary, each function $f \in \mathcal{A}(\Gamma)$ is continued harmonically in D to the function F. Let us apply to this continuation the statement from [5, Ch.1].

Theorem 1. *Let a doubly continuously differentiable surface Σ be defined in the neighborhood of point x^0, and let on one side of Σ in the neighborhood of x^0 (i.e., in some component ω of the set $B(x^0, r) \cap (\mathbb{R}^n \setminus \Sigma)$ for a sufficiently small r), a harmonic function u be defined, and having at each point y of boundaries $\Sigma \cap B(x^0, r)$ limit*

$$\lim_{x \to y} u(x) = U(y),$$

moreover, the function $U(y)$ is doubly continuously differentiable. Then grad u *has a finite limit at each point of the boundary $\Sigma \cap B(x^0, r)$ and can be continuously extended on $\Sigma \cap B(x^0, r)$.*

Therefore, by the theorem 1, the function F is infinitely differentiable up to Γ and since it is real-analytic on Γ, we get that $F \in \mathcal{H}(\bar{D})$. The maximum modulus theorem then shows that the topologies of the spaces $\mathcal{A}(\Gamma)$ and $\mathcal{H}(\bar{D})$ coincide. □

The same arguments apply to the spaces $\mathcal{A}(\Gamma)$ and $\mathcal{H}(\mathbb{C}^n \setminus D)$ (in the latter space, harmonic functions in $\mathbb{C}^n \setminus D$ should be placed, tending to zero at infinity).

2. The simple layer potential

In this section, we will consider the case of the space $\mathbb{R}^n$, $n > 2$. Thus, we include the classical three-dimensional case.

For $n = 2$ on the $\mathbb{R}^2$ plane, harmonic functions are closely related with holomorphic functions of one complex variable (see any textbook on complex analysis, for example, [6]). Therefore, our approach here does not provide anything new.

The Laplace operator Δ has the form

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}.$$

Let $g(x, y)$ be the fundamental solution of the Laplace equation (see, for example, [3, ch. XI], [2, ch. 2]), that is

$$g(x, y) = -\frac{1}{(n-2)s_n} \frac{1}{|x-y|^{n-2}} = -\frac{\Gamma(n/2-1)}{4\pi^{n/2}} \frac{1}{|x-y|^{n-2}}, \quad n > 1,$$

here $s_n = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the unit sphere in $\mathbb{R}^n$, and $\Gamma(\cdot)$ is the Γ-Euler function.

For the function $f \in \mathcal{L}^1(\Gamma)$, we introduce the integral (integral operator) of a simple layer

$$\Phi[f](x) = -\int_\Gamma f(y)g(y,x)\,d\sigma(y) = \frac{1}{(n-2)s_n}\int_\Gamma f(y)\frac{d\sigma(y)}{|y-x|^{n-2}}, \quad x \notin \Gamma,$$

where $d\sigma$ is the surface Lebesgue measure on Γ. Obviously, $\Phi[f](x)$ is a harmonic function outside the boundary of the domain D. The function $\Phi[f](x)$ tends to zero as $|x| \to \infty$.

The boundary properties of the potential of a simple layer for continuous and smooth functions are well studied (see, for example, [1, 7]). The boundary behavior of this potential and its derivatives of various orders is investigated. Thus, it is shown that $\Phi[f]$ is a function of the class $\mathcal{C}^\infty$ up to the boundary of the domain inside and outside for the functions $f \in \mathcal{C}^\infty(\Gamma)$.

The main result of this section is the following statement.

Theorem 2. *Let the bounded domain D have a real-analytic boundary. Then for the function $f \in \mathcal{A}(\Gamma)$, the potential of the simple layer $\Phi[f]$ belongs to the space $\mathcal{H}(\bar{D})$. And if $\ell(f) \leqslant 1/r$ and $\ell(\rho) \leqslant 1/r$, then $\ell(\Phi[f]) \leqslant 1/r$. Moreover, if $|f| \leqslant C$ in some neighborhood $U \subset \mathbb{C}^n$ of the manifold Γ, then $|\Phi[f]| \leqslant C$ in U. The same properties are valid in the complement of the closure of the domain.*

The first part of this theorem was proved earlier in [8, Theorem 5]. Here we give another proof of it, and also prove the boundedness of extending this potential into the complex neighborhood of Γ.

Proof. Spherical polynomials are homogeneous harmonic polynomials Y_k of n variables of some degree k of the form

$$Y_k(x) = \sum_{\|\alpha\|=k} b_\alpha x^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ are multi-indexes, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ are monomials, and $\|\alpha\| = \alpha_1 + \ldots + \alpha_n$.

The theory of homogeneous harmonic polynomials is described in detail in S.L.Sobolev's monograph [3, Chapter XI]. Here we will provide the information we need.

Spherical polynomials of degree k form a linear space $\mathcal{L}_k$ in the space of all polynomials. As shown in [3, chapter XI], its dimension is finite and equal to

$$\sigma(k) = \frac{(n+2k-2)(k+n-3)!}{k!(n-2)!}.$$

Thus, $\sigma(k)$ is a polynomial of degree $n-2$ in k.

The set of all harmonic polynomials is dense in $\mathcal{L}^1(S)$, as well as in $\mathcal{L}^2(S)$ (see, for example, [3, ch. XI, Sec. 3]), so $\bigcup\limits_k \mathcal{L}_k$ is dense in $\mathcal{L}^2(S)$. Recall that the scalar product of two functions $f, h \in \mathcal{L}^2(S)$ in the space $\mathcal{L}^2(S)$ is the integral

$$(f,h) = \int_S f(x)\cdot\bar{h}(x)\,d\sigma(x),$$

where $d\sigma(x)$ is a Lebesgue measure on S.

Moreover, scalar products

$$(Y_k, Y_m) = \int_S Y_k(x)\cdot\overline{Y_m(x)}\,d\sigma(x) = 0,$$

if $k \neq m$. Therefore, we can always choose an orthogonal basis from the spherical polynomials Y_k in the space $\mathcal{L}^2(S)$ (see [3, ch. XI]).

The restriction Y_k onto the sphere S is called the *spherical harmonic of order k*.

We also recall that the Poisson kernel in the ball B has the form (see, for example, [2, Sec. 2.8])

$$P(x,y) = \frac{1}{s_n} \cdot \frac{1-|x|^2}{|y-x|^n}, \quad y \in S, \quad x \in B.$$

The statement is true.

Lemma 4. *The potential of a simple layer from spherical harmonics Y_k of order k is calculated by the formula*

$$\Phi[Y_k](x) = \frac{1}{(n+2k-2)} Y_k(x).$$

Proof. We have

$$\Phi[Y_k](x) = -\int_S Y_k(y) \cdot g(y,x)\, d\sigma(y) = \frac{1}{(n-2)s_n} \int_S Y_k(y) \cdot \frac{1}{|y-x|^{n-2}}\, d\sigma(y) =$$

$$= \frac{1}{(n-2)(1-|x|^2)} \int_S Y_k(y) \cdot P(x,y)(1+|x|^2 - 2 < x,y >)\, d\sigma(y),$$

where $< x,y >= \sum\limits_{m=1}^{n} x_m \cdot y_m$.

To calculate this integral, we need to harmoniously extend the functions $y_m Y_k$ from the sphere S into the ball B. It is easy to verify that the harmonic continuation of $y_m Y_k$ is given by the function

$$f(y) = y_m Y_k(y) + \frac{1-|y|^2}{n+2k-2} \cdot \frac{\partial Y_k}{\partial y_m}.$$

Applying this formula, we obtain that

$$\Phi[Y_k](x) = \frac{1}{(n-2)(1-|x|^2)} \Bigg[(1+|x|^2) Y_k(x) - 2|x|^2 Y_k(x) -$$

$$- \frac{2(1-|x|^2)}{n+2k-2} \sum_{m=1}^{n} x_m \frac{\partial Y_k}{\partial x_m} \Bigg] =$$

$$= \frac{1}{(n-2)(1-|x|^2)} \left((1-|x|^2) Y_k(x) - (1-|x|^2) \frac{2k}{n+2k-2} Y_k(x) \right) =$$

$$= \frac{1}{n+2k-2} Y_k(x).$$

□

Thus, we have found the eigenvalues and eigenfunctions of the operator of the simple layer $\Phi[f]$ in the ball B. All these eigenvalues turned out to be less than 1.

Next, we will need a theorem on the real analyticity of functions on the boundary of the ball B from the monograph by S. L. Sobolev [3] (Theorem XI.3).

Theorem 3. *Let the function $\varphi(x)$ be a real-analytic function on the sphere S and for it on the sphere S a series expansion in spherical harmonics of the form*

$$\varphi(x) = \sum_{k=0}^{\infty} Y_k(x) \tag{6}$$

is valid, then $|Y_k|$ decrease exponentially, i.e. there are constants $K > 0$ and $\eta > 0$ such that inequalities

$$|Y_k| \leqslant Ke^{-\eta k} \tag{7}$$

are valid everywhere on the sphere S.

Conversely, if the function φ is summable on the sphere S and the functions Y_k decrease exponentially (i.e. the condition (7) is fulfilled), then φ is real-analytical on the sphere S. The series (6) will converge uniformly and absolutely in a ball of radius e^{η}.

So if $\ell(\varphi) \leqslant r$, then the lemma 4 and the equality (3) show that $\ell(\Phi[\varphi]) \leqslant r$.

Moreover, from the Theorem 3 and the Lemma 4, we obtain that if

$$|\varphi| \leqslant \sum_{k=0}^{\infty} |Y_k| \leqslant C$$

in some complex neighborhood U of the set $\Gamma \subset \mathbb{C}^n$, then $|\Phi[\varphi]|$ is bounded by the same constant C in the complex neighborhood U. Therefore, the Theorem 2 is fully proved in the case of a ball.

In the future, we will need a local version of the Theorem 3. Let's formulate it.

Theorem 4. *Let the function $\varphi(x) \in \mathcal{L}^2(S)$ and is real-analytic in some neighborhood $W \subset S$ of the point $x^0 \in S$, then the terms of its expansion (6) according to the system of spherical functions Y_k decrease exponentially in W, i.e. there are constants $K > 0$ and $\eta > 0$ such that the condition (7) is satisfied everywhere on W.*

Conversely, if the function φ is summable on the sphere S and the functions Y_k decrease exponentially on W (i.e. the condition (7) is fulfilled), then φ is real-analytical on W. Moreover, this series will uniformly and absolutely converge in the neighborhood $U \subset \mathbb{C}^n$ of the set $W \subset S$.

Proof. The proof follows the scheme of the proof of Theorem XI.3 from [3]. It is based on Lemma XI.2 from [3].

Lemma 5. *Let $u(x)$ and $b_\alpha(x)$, $\|\alpha\| \leqslant m$ be real-analytic in the neighborhood of the origin in $\mathbb{R}^n$ and let the differential operator L have the form*

$$L = \sum_{\|\alpha\| \leqslant m} b_\alpha(x)\partial^\alpha,$$

and

$$f_l(x) = L^l u(x),$$

then there are constants A and K that depend on L and u such that

$$|f_l(x)| \leqslant KA^{ml}(ml)!\,.$$

Hence we get a corollary from [3, ch. XI]. Let us formulate it in a local form (formula (XI.4.20)).

Corollary 1. *For any real-analytic function $\varphi(\theta)$ in the neighborhood $W \subset S$, there is an estimate*

$$|\Delta_\theta^l \varphi(\theta)| \leqslant KA^{2l}(2l)!, \tag{8}$$

here θ is the usual angular coordinates on the sphere S, and Δ_θ is the Laplace operator in these angular coordinates (see, for example, [3, ch. XI, Sec. 1]).

The proof goes on as in Theorem XI.3 of [3], you just need to make sure that instead of the sphere S, the set W is considered. □

Note also that by virtue of the condition (7), the series (6) will uniformly converge on the set

$$W_\eta = \{rx : x \in W, \quad 1 \leqslant r \leqslant e^\eta\}.$$

Let's return to the domain D in a general way. Consider a sufficiently small neighborhood $U(\varepsilon) = U$ of the point $x^0 \in \Gamma$. There is a real-analytic diffeomorphism $x = x(y)$ that maps an open set V to U, and such that a part of the unit sphere $S \cap V$ translates to a part of the hypersurface $\Gamma \cap U$.

Consider the potential of a simple layer on this part of the surface

$$\Phi_\varepsilon[f](u) = c \int\limits_{U\cap\Gamma} \frac{f(x)\, d\sigma(x)}{|x-u|^{n-2}}.$$

Let's replace the variables $x = x(y)$ in it and substitute $u = x(v)$. Then we get

$$\Phi_\varepsilon[f](x(v)) = \Phi^*_\varepsilon[f^*](v) = c \int\limits_{V\cap S} \frac{f^*(y)\, d\sigma^*(y)}{|x(y)-x(v)|^{n-2}},$$

where $f^*(y) = f(x(y))$, and $d\sigma^*(y) = d\sigma(x(y))$. It is clear that $d\sigma^*(y) = \tau(y)\, d\sigma(y)$, where $\tau(y)$ is a positive real-analytic function on the sphere S, and $d\sigma(y)$ is the Lebesgue measure on a sphere.

It is not difficult to get the inequality

$$|x(y)-x(v)|^2 \leqslant T|y-v|^2 = T\sum_{k=1}^{n}(y_k - v_k)^2, \quad y, v \in V.$$

So

$$\frac{|x(y)-x(v)|^2}{\sum\limits_{k=1}^{n}(y_k-v_k)^2} \leqslant T, \quad y, v \in V. \tag{9}$$

Applying an analogue of Riemann's theorem on the elimination of singularities of bounded holomorphic functions (see, for example, [9, clause 2.1.4]), we obtain

$$\frac{|x(y)-x(v)|^2}{\sum\limits_{k=1}^{2n}(y_k-v_k)^2} = \psi(y, v),$$

where $\psi(y, v)$ is a real analytical function in V that does not vanish.

We could not go into the complex space, but apply the results of chapter VI from [10].

Therefore, the potential of a simple layer will take the form

$$\Phi^*_\varepsilon[f^*](v) = c \int\limits_{V\cap S} \frac{f^*(y)\tau(y)\, d\sigma(y)}{\psi^{n-1}(y, v)|y-v|^{2n-2}}.$$

By choosing a sufficiently small neighborhood U and using the inequalities (5), we can assume that

$$\left|\frac{\tau(y)}{\psi^{n-1}(y, v)}\right| \leqslant 1 \quad \text{для} \quad u, v \in V \cap S. \tag{10}$$

Let's continue the function f^* by zero for the rest of the sphere S. We get

$$\Phi^*_\varepsilon[f^*](v) = c \int_S \frac{f^*(y)\, d\sigma^*(y)}{\psi^{n-1}(y, v)|y-v|^{2n-2}}.$$

In fact, this is the potential of a simple layer on a sphere, where the density depends on the parameter v.

Consider an orthonormal basis in the space $\mathcal{L}^2(S)$, consisting of spherical harmonics $Y_{k,l}$. Let's denote it $\mathcal{B} = \{Y_{k,l},\ 0 \leqslant k < \infty,\ 1 \leqslant l \leqslant \sigma(k)\}$. Then any function $h \in \mathcal{L}^2(S)$ can be decomposed into a series according to the basis $\mathcal{B}$ of the form

$$h(y) = \sum_{k,l} b_{k,l} Y_{k,l}(y),$$

converging to it in $\mathcal{L}^2(S)$, here

$$b_{k,l} = \int_S h(y)\bar{Y}_{k,l}\, d\sigma(y).$$

Decompose the function $F(y,v) = \dfrac{f^*(y)\tau(y)}{\psi^{n-1}(y,v)}$ in a series on the basis $\mathcal{B}$ by y, then

$$F(y,v) = \sum_{r,l} b_{k,l}(v) Y_{k,l}(y),$$

where

$$b_{k,l}(v) = \int_S F(y,v)\bar{Y}_{k,l}(y)\, d\sigma(y).$$

It is clear that the functions $b_{k,l}(v)$ are real-analytic in the neighborhood of the set $V \cap S$.

Then, using the lemma 4, we get

$$\Phi^*[f](v) = \sum_{k,l} \frac{1}{2k+n-2} b_{k,l}(v) Y_{k,l}(v).$$

Since the function F is bounded on S, i.e. $|F(y,v)| \leqslant C$ for $y, v \in S$ or even from the neighborhood of S, then

$$|b_{k,l}(v)| \leqslant C \int_S |Y_{k,l}(y)|\, d\sigma(y) \leqslant C_1 \sqrt{\int_S |Y_{k,l}(y)|^2 d\sigma(y)} = C_1.$$

Hence, it is clear that if the conditions of the theorem 4 are fulfilled for the function $F(y,v)$, then they will be fulfilled for the function $\Phi^*[f]$.

Let us now consider the remaining part of the potential of a simple layer over the set $\Gamma \setminus U$, i.e. the integral

$$\Phi^1[f](x) = \int\limits_{\Gamma\setminus U} g(y,x) f(y) d\sigma(y).$$

For him, we have

$$\left|\frac{\partial^\alpha \Phi^1(x^0)}{\alpha!}\right| \leqslant \int\limits_{\Gamma\setminus U(x^0,r)} |f(y)| \left|\frac{\partial^\alpha g(x^0,y)}{\alpha!}\right| d\sigma(y) \leqslant$$

$$\leqslant c_2 \max_\Gamma |f| \frac{(\|\alpha\|+n-2)!}{\alpha!(n-2)!} \frac{1}{r^{\|\alpha\|+n-2}}. \tag{11}$$

Applying Stirling's formula, we get that $\ell(\Phi[f]) \leqslant \ell(f)$.

Moreover, the holomorphic extension of the potential Φ into the complex neighborhood Γ will be limited by a constant determined by the same type of constants for the function $F(u,v)$ and ultimately by constants determined by the functions f, τ, ψ.

Thus, the Theorem 2 proven. □

3. Integral representations of Bochner-Martinelli and Cauchy-Fantappiè

In this section, we will apply the results of the previous paragraph to the study of the boundary behavior of the Bochner–Martinelli and Cauchy–Fantappiè integrals of a certain type from real-analytic functions.

The method of integral representations is one of the main constructive methods in the study of holomorphic functions of many complex variables. The Bochner–Martinelli integral representation plays a particularly important role in multidimensional complex analysis (see, for example, the monographs [6,11–13]). Its kernel is universal (independent of the type of domain) and quite simple (it consists of derivatives of the fundamental solution of the Laplace equation). This kernel has many properties of the Cauchy kernel on the complex plane, with the exception of holomorphicity. The Bochner-Martinelli integral is considered in detail in the monograph [14]. This integral is also closely related to classical potential theory. It is shown in [14, Ch. 1] that it is an analog of the double layer potential. The Bochner-Martinelli integral plays a particularly important role in the analytical continuation of functions of various classes of smoothness (see [14,15]).

Consider the n-dimensional complex space $\mathbb{C}^n$, $n > 1$, of variables $z = (z_1, \dots, z_n)$, $z_j = x_j + ix_{n+j}$, $j = 1, \dots, n$, x_j are real numbers. Thus, we identify the space $\mathbb{C}^n$ with the space $\mathbb{R}^{2n}$.

We introduce the module of the vector $|z| = \sqrt{|z_1|^2 + \dots + |z_n|^2}$ and the differential forms $dz = dz_1 \wedge \dots \wedge dz_n$, $d\bar{z} = d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n$, and also $dz[k] = dz_1 \wedge \dots \wedge dz_{k-1} \wedge dz_{k+1} \wedge \dots \wedge dz_n$.

Consider the Bochner-Martinelli kernel — an external differential form $U(\zeta, z)$ of type $(n, n-1)$ of the form (see, for example, [14, Ch. 1], [15, Ch.1])

$$U(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{n} (-1)^{k-1} \frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^{2n}} \, d\bar{\zeta}[k] \wedge d\zeta.$$

It is a closed differential form with harmonic coefficients. On the complex plane $\mathbb{C}^1$, it coincides with the Cauchy kernel.

The fundamental solution of the Laplace equation $g(\zeta, z)$ in $\mathbb{C}^n$ can be written as

$$g(\zeta, z) = -\frac{(n-2)!}{(2\pi i)^n} \frac{1}{|\zeta - z|^{2n-2}}, \quad n > 1,$$

then

$$U(\zeta, z) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial g}{\partial \zeta_k} \, d\zeta[\bar{k}] \wedge d\zeta.$$

For the function $f \in \mathcal{L}^1(\Gamma)$, we introduce an integral (integral operator) Bochner-Martinelli

$$M[f](z) = \int_{\Gamma} f(\zeta) U(\zeta, z), \quad z \notin \Gamma.$$

It is clear that $M[f]$ is a harmonic function outside of Γ, in particular, it belongs to the spaces $\mathcal{H}(D)$ and $\mathcal{H}(\mathbb{C}^n \setminus \bar{D})$.

Theorem 5. *Let the bounded domain D have a real-analytic boundary. Then for the function $f \in \mathcal{A}(\Gamma)$ we have that $M[f]$ belong to the space $\mathcal{H}(\bar{D})$. Moreover, if $\ell(f) \leqslant 1/r$, $\ell(\rho) \leqslant 1/r$, then $\ell(M[f]) \leqslant 1/r$. The same property holds true in the complement of the closure of the domain. In addition, if $|f| \leqslant C$ in the neighborhood $U \subset \mathcal{C}^{2n}$ of the boundary Γ, then $|M[f]|$ is bounded in U by the same constant C.*

The first part of the Theorem refines the proof of Theorem? 7 from [8].

Proof. Consider the integral Mf. If the domain D is a unit ball B centered at zero, then the homogeneous harmonic polynomials $P_{s,t}$ of degree homogeneity s with respect to z and t with respect to $\bar{z}$ are eigenfunctions of Mf (see [14, Sec. 5]) and

$$M[P_{s,t}] = \frac{n+s-1}{n+s+t-1} P_{s,t} \quad \text{for all} \quad s,t.$$

So that $0 < \dfrac{n+s-1}{n+s+t-1} \leqslant 1$. Therefore, we can completely repeat the reasoning for the potential of a simple layer in a sphere taking the fuctions $P_{s,t}$ with $p+s=k$ as systems of spherical functions Y_k, and obtain the required result.

For an arbitrary bounded domain D with a real-analytic boundary, we represent the Bochner-Martinelli kernel on Γ in the following form (see, for example, [14, Ch. 1])

$$U(\zeta,z)\big|_{\Gamma} = \sum_{k=1}^{n} \frac{\partial g}{\partial \zeta_k}\, \tau_{\bar{k}}\, d\sigma(\zeta), \quad \zeta \in \Gamma, \tag{12}$$

where $\tau_{\bar{k}}$ are "complex" guiding cosines

$$\tau_{\bar{k}} = \frac{1}{|\operatorname{grad}\rho|} \frac{\partial \rho}{\partial \bar{z}_k}, \quad k = 1, \ldots, n.$$

In this case, we mean the "complex" gradient

$$\operatorname{grad}\rho = \left(\frac{\partial \rho}{\partial z_1}, \ldots, \frac{\partial \rho}{\partial z_n} \right).$$

It is clear that $\ell(f)$ does not change during differentiation. And when multiplying the function f by $\rho_{\bar{k}}$, i.e. when considering $\ell(f\rho_{\bar{k}})$, this expression cannot decrease due to the conditions of the theorem and inequalities (5).

We see that the integral Mf consists of derivatives of the potential of a simple layer. To evaluate them in $\mathbb{C}^{2n}$, we can use Cauchy inequalities for holomorphic functions and obtain the same inequality for Mf as for f, reducing (if necessary) the radius of the homogeneous polydisc into which this function extends (see §1 of the article).

□

Consider the differential forms $d\bar{z}[k,l]$, which are obtained from the differential form $d\bar{z}$ by throwing out the differentials $d\bar{z}_k$ and $d\bar{z}_l$.

Let's introduce the functions $a_{k,l}$ — some smooth functions on Γ, $k,l = 1, \ldots, n$.

We will also introduce the kernel

$$W(\zeta,z) = \sum_{k>l} d(a_{k,l}(\zeta) g(\zeta,z)) \wedge d\bar{\zeta}[l,k] \wedge d\zeta.$$

Then we obtain that for holomorphic functions f in the domain D and continuous in the closure of the domain $\bar{D}$, the integral representation

$$f(z) = \int_{\Gamma} f(\zeta)(U(\zeta,z) + W(\zeta,z)), \quad z \in D \tag{13}$$

holds.

Let's denote the integral operator $M+G$ by Q

$$Q[f](z) = \int_{\Gamma} f\big(\zeta)(U(\zeta,z) + W(\zeta,z)\big), \quad z \notin \Gamma. \tag{14}$$

The integral $Q[f](z)$ is a harmonic function outside of Γ.

The integral representation (13) is a special case of the Cauchy–Fantappiè integral representation, so named by J. Leray [16].

Its kernel consists of derivatives of the fundamental Laplace equation, just like the Bochner–Martinelli kernel.

The statement is true.

Theorem 6. *Let the bounded domain D have a real-analytic boundary. Then for the function $f \in \mathcal{A}(\Gamma)$ and the functions $a_{k,l} \in \mathcal{A}(\Gamma)$, $k, l = 1, \dots, n$, we get that $Q[f]$ belongs to the space $\mathcal{H}(\overline{D})$. Moreover, if $\ell(f) \leqslant 1/r$, $\ell(\rho) \leqslant 1/r$, $\ell(a_{k,l}) \leqslant 1/r$, $l = 1, \dots n$, then $\ell(Q[f]) \leqslant 1/r$. The same property holds true in the complement of the closure of the domain. In addition, if $|f| \leqslant C$ in the neighborhood $U \subset \mathbb{C}^{2n}$ of the boundary Γ, then $|Q[f]|$ is bounded in U by the same constant C.*

Proof. For the operator Q, we obtain that the integral contains the derivatives of the fundamental solution of the Laplace equation $g(\zeta, z)$. When differentiating this solution, ℓ does not change. For the product of functions with $\ell \leqslant r$, the value ℓ also does not change.

To evaluate the integrals themselves $Q[f]$, we must restrict the kernel $W(\zeta, z)$ to the boundary of the domain D, then we can assume that the obtained real-analytical functions standing before the derivatives of the potential of a simple layer are bounded by 1.

Applying the Theorem 2, we obtain the necessary statement. □

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О граничном поведении потенциалов от вещественно аналитических функций

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Аннотация. В данной работе мы рассматриваем потенциалы от вещественно аналитических плотностей и изучаем их граничное поведение. Получаем оценки на продолжения этих потенциалов в комплексную область. Это позволяет их применить к различным многомерным интегральным представлениям (Бохнера-Мартинелли, Коши-Фантаппье) для голоморфных функций в многомерном комплексном пространстве.

Ключевые слова: потенциал простого слоя, интеграл Бохнера-Мартинелли, интеграл Коши-Фантаппье, вещественно аналитические функции.

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On Automorphisms of the Matrix Ball in the Space of Rectangular Matrices

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Abstract. The paper considers the matrix ball in the space of rectangular matrices and studies the group of holomorphic automorphisms: explicit formulas are obtained for fractional-linear transformations preserving the ball, and their general structure is proved. Special attention is paid to the construction of an automorphism mapping a given interior point of the ball to the origin. The results generalize the classical theory of automorphisms of the unit ball in $\mathbb{C}^n$ and the ball of square matrices to the case of rectangular matrices.

Keywords: unit disk, matrix scalar product, matrix ball, automorphism.

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1. Introduction and preliminaries

The study of holomorphic automorphisms of domains in multidimensional complex space occupies an important place in modern complex analysis and its applications ([1–6]). Holomorphic automorphisms, i.e., biholomorphic mappings of a domain onto itself, not only describe its internal symmetry but also serve as a powerful analytical tool. Explicit knowledge of the automorphism group allows one to:

construct holomorphic continuations of functions defined on a part of the boundary into the interior of the domain ([7–9]);

derive explicit forms of kernels for integral representations of the Bergman, Cauchy–Szego type, and the Poisson formula that are invariant under the group action. These formulas form the basis for studying spaces of holomorphic functions and solving boundary value problems ([10–13]);

the automorphism group naturally defines invariant metrics (such as the Caratheodory, Kobayashi, and Bergman metrics). Domains with a transitive automorphism group, among which the classical Cartan domains stand out, have a particularly rich structure. The matrix ball is an example of such a domain.

The explicit form of automorphisms for the matrix ball in the space of square matrices was investigated in detail in the works of G. Khudaiberganov and S. Kosbergenov ([9, 12]). Their results formed the basis for the development of analysis in this classical Cartan domain.

The main goal of the present work is to generalize these results to the case of the space of sets of rectangular matrices. We consider a matrix ball defined for a vector composed of rectangular

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matrices. In this paper, we obtain explicit formulas for fractional-linear transformations that preserve such a ball, prove the transitivity of the automorphism group action, and construct an explicit form of the automorphism that maps any interior point to the origin.

The classical theory of automorphisms of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is well known: every holomorphic mapping of the disk onto itself has the form

$$w = e^{i\theta}\frac{z-a}{1-\bar{a}z}, \quad a \in \mathbb{D}. \tag{1.1}$$

In matrix notation, this transformation is represented as a fractional linear

$$w = \frac{az+b}{cz+d},$$

where the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to the group $SU(1,1)$, i.e. satisfies the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

In this paper, we generalize this construction to the case where the variable z is a tuple of rectangular matrices.

2. Matrix ball in the space of rectangular matrices

The term matrix ball was first introduced by G. Khudaiberganov ([14]), who posed several fundamental problems of complex analysis for it: finding the transitive group of automorphisms of the matrix ball, calculating the Bergman and Cauchy-Szego kernels for this domain, and deriving a Carleman formula to recover the values of a holomorphic function within the matrix ball from its values on certain boundary sets of uniqueness. A series of works by S. Kosbergenov, A. Kytmanov, and others ([15–17]) have been devoted to solving these problems.

Let $Z = (Z_1, \dots, Z_n)$ be a vector composed of rectangular matrices Z_j of size $p \times q$ over the field of complex numbers $\mathbb{C}$. Such a set can be identified with an element of the space $\mathbb{C}^{pqn}$. We introduce the matrix "scalar" product on this set:

$$\langle Z, W\rangle = \sum_{j=1}^{n} Z_j W_j^*, \tag{2.1}$$

where W_j^* is the Hermitian conjugate matrix (of size $q \times p$). Note that the result $\langle Z, W\rangle$ is a square matrix of size $p \times p$.

Lemma 2.1. *The product introduced has the following properties:*

1. $\langle Z, Z\rangle \geqslant 0$, *where* $\langle Z, Z\rangle = 0$ *if and only if* $Z = 0$.
2. $\langle Z, W\rangle = \langle W, Z\rangle^*$.
3. $\langle Z^1 + Z^2, W\rangle = \langle Z^1, W\rangle + \langle Z^2, W\rangle$.
4. $\langle \lambda Z, W\rangle = \lambda\langle Z, W\rangle$ *for any square matrix* λ *of size* $p \times p$.
5. *If* $\langle W, W\rangle > 0$, *then the inequality holds*

$$\langle Z, Z\rangle \geqslant \langle Z, W\rangle\langle W, W\rangle^{-1}\langle W, Z\rangle.$$

Proof. Properties 1–4 follow directly from the definition of the matrix "scalar" product.

Let's prove 5. Consider the expression $\langle Z+\lambda W, Z+\lambda W\rangle$, where λ is an arbitrary square matrix of order p. By Property 1, this expression is nonnegative:

$$0 \leqslant \langle Z+\lambda W, Z+\lambda W\rangle.$$

Expanding according to the definition of the matrix scalar product and using the linearity properties 3) and 4), we obtain:

$$\langle Z+\lambda W, Z+\lambda W\rangle = \langle Z,Z\rangle + \lambda\langle W,Z\rangle + \langle Z,W\rangle\lambda^* + \lambda\langle W,W\rangle\lambda^*. \tag{2.2}$$

Let's choose λ in a special way:

$$\lambda = -\langle Z,W\rangle\langle W,W\rangle^{-1}.$$

Such a matrix exists because $\langle W,W\rangle > 0$ and is therefore invertible.

Substitute this λ into (2.2). Let's calculate each term:

1. $\lambda\langle W,Z\rangle = -\langle Z,W\rangle\langle W,W\rangle^{-1}\langle W,Z\rangle$.

2.
$$\langle Z,W\rangle\lambda^* = \langle Z,W\rangle\left(-\langle Z,W\rangle\langle W,W\rangle^{-1}\right)^* = -\langle Z,W\rangle\langle W,W\rangle^{-1}\langle W,Z\rangle,$$
since $\langle W,W\rangle^{-1}$ is Hermitian and $\langle Z,W\rangle^* = \langle W,Z\rangle$.

3.
$$\lambda\langle W,W\rangle\lambda^* = \left(-\langle Z,W\rangle\langle W,W\rangle^{-1}\right)\langle W,W\rangle\left(-\langle Z,W\rangle\langle W,W\rangle^{-1}\right)^* = \\ = \langle Z,W\rangle\langle W,W\rangle^{-1}\langle W,Z\rangle.$$

Substituting these expressions into (2.2), we obtain:

$$0 \leqslant \langle Z,Z\rangle - \langle Z,W\rangle\langle W,W\rangle^{-1}\langle W,Z\rangle - \langle Z,W\rangle\langle W,W\rangle^{-1}\langle W,Z\rangle + \\ + \langle Z,W\rangle\langle W,W\rangle^{-1}\langle W,Z\rangle.$$

After reducing similar terms:

$$0 \leqslant \langle Z,Z\rangle - \langle Z,W\rangle\langle W,W\rangle^{-1}\langle W,Z\rangle.$$

From here we arrive at the desired inequality:

$$\langle Z,Z\rangle \geqslant \langle Z,W\rangle\langle W,W\rangle^{-1}\langle W,Z\rangle.$$

Definition 2.1. *A matrix ball is a domain*

$$\Re_{p,q,n} = \left\{ Z \in \mathbb{C}^{pqn} : I^{(p)} - \langle Z,Z\rangle > 0 \right\},$$

where $I^{(p)}$ is the identity matrix of order p, and the sign > 0 denotes the positive definiteness of the Hermitian matrix.

For $p = q = 1$, we obtain the classical ball in $\mathbb{C}^n$: $\sum |z_j|^2 < 1$. For $n = 1$ — a ball from one matrix $Z \in \mathbb{C}[p \times q]$ with the condition $I^{(p)} - ZZ^* > 0$.

Lemma 2.2. *The matrix ball $\Re_{p,q,n}$ has the following properties:*

1. Boundedness: *There exists a constant $C > 0$ such that $\|Z_j\| \leqslant C$ for all $Z \in \Re_{p,q,n}$, $j = 1,\ldots,n$.*

2. Complete circular domain: *If $Z \in \Re_{p,q,n}$ and $|\alpha| \leqslant 1$, then $\alpha Z \in \Re_{p,q,n}$.*

3. Invariance under unitary transformations*: For any unitary matrices $U \in U(p)$, $V_j \in U(q)$, the transformation $Z_j \mapsto UZ_jV_j$ maps $\Re_{p,q,n}$ to itself.*

4. Convexity*: If $Z, W \in \Re_{p,q,n}$ and $t \in [0,1]$, then $tZ + (1-t)W \in \Re_{p,q,n}$.*

Proof. All properties are proved by direct calculations using Definition 2.1 and the properties of Hermitian matrices. We prove property 4. Let $Z, W \in \Re_{p,q,n}$. This means that

$$I^{(p)} - \langle Z, Z\rangle > 0, \quad I^{(p)} - \langle W, W\rangle > 0.$$

For an arbitrary nonzero vector $\xi \in \mathbb{C}^p$, consider the quadratic form

$$Q(\xi) = \xi^* \left(I^{(p)} - \langle tZ + (1-t)W, tZ + (1-t)W\rangle \right) \xi.$$

We calculate $\langle tZ + (1-t)W, tZ + (1-t)W\rangle$:

$$\langle tZ + (1-t)W, tZ + (1-t)W\rangle = t^2\langle Z, Z\rangle + t(1-t)\langle Z, W\rangle + t(1-t)\langle W, Z\rangle + (1-t)^2\langle W, W\rangle.$$

Hence,

$$Q(\xi) = \|\xi\|^2 - t^2\xi^*\langle Z, Z\rangle\xi - t(1-t)\xi^*\langle Z, W\rangle\xi - t(1-t)\xi^*\langle W, Z\rangle\xi - (1-t)^2\xi^*\langle W, W\rangle\xi.$$

Let's introduce the following notation:

$$a = \sqrt{\xi^*\langle Z, Z\rangle\xi}, \quad b = \sqrt{\xi^*\langle W, W\rangle\xi}.$$

From the condition that Z and W belong to a ball, we have:

$$a^2 < \|\xi\|^2, \quad b^2 < \|\xi\|^2.$$

We estimate the mixed terms using the Cauchy-Bunyakovsky inequality for the matrix scalar product. Property 5 of Lemma 2.1 implies:

$$|\xi^*\langle Z, W\rangle\xi| \leqslant \sqrt{\xi^*\langle Z, Z\rangle\xi} \cdot \sqrt{\xi^*\langle W, W\rangle\xi} = ab.$$

Similarly for $\xi^*\langle W, Z\rangle\xi$.

Thus,

$$Q(\xi) \geqslant \|\xi\|^2 - t^2a^2 - 2t(1-t)ab - (1-t)^2b^2 = \|\xi\|^2 - (ta + (1-t)b)^2.$$

Since $a < \|\xi\|$ and $b < \|\xi\|$, we have:

$$ta + (1-t)b < t\|\xi\| + (1-t)\|\xi\| = \|\xi\|.$$

Therefore,

$$(ta + (1-t)b)^2 < \|\xi\|^2,$$

and hence $Q(\xi) > 0$ for any nonzero ξ.

The positivity of the quadratic form for all $\xi \neq 0$ means that the matrix is positive definite, i.e.

$$I^{(p)} - \langle tZ + (1-t)W, tZ + (1-t)W\rangle > 0,$$

which completes the proof. □

3. Holomorphic automorphisms of the matrix ball

The main idea of the work in studying automorphisms of the matrix ball is to introduce *homogeneous coordinates*. Consider a tuple of matrices

$$\zeta = (\zeta_0, \zeta_1, \dots, \zeta_n),$$

where $\zeta_0 \in \mathbb{C}[p \times p]$ is an invertible square matrix, and $\zeta_j \in \mathbb{C}[p \times q]$ for $j = 1, \dots, n$.

The relationship with the original variables is defined by the relations

$$Z_j = \zeta_0^{-1}\zeta_j, \quad j = 1, \dots, n. \tag{3.1}$$

We introduce a matrix of size $(p + nq) \times (p + nq)$:

$$H = \begin{pmatrix} I^{(p)} & 0 & 0 & \cdots & 0 \\ 0 & -I^{(q)} & 0 & \cdots & 0 \\ 0 & 0 & -I^{(q)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -I^{(q)} \end{pmatrix}, \tag{3.2}$$

where $I^{(p)}$ and $I^{(q)}$ are the identity matrices of order p and q, respectively.

The condition for a point Z to belong to the matrix ball $\Re_{p,q,n}$ in terms of homogeneous coordinates takes the form

$$\zeta H \zeta^* > 0, \tag{3.3}$$

where ζ is regarded as a row matrix of size $p \times (p + nq)$, and the sign > 0 denotes the positive definiteness of a $p \times p$ matrix.

Consider a square matrix A of size $(p + nq) \times (p + nq)$ with the following block structure:

$$A = \begin{pmatrix} A_{00} & A_{01} & A_{02} & \cdots & A_{0n} \\ A_{10} & A_{11} & A_{12} & \cdots & A_{1n} \\ A_{20} & A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n0} & A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}, \tag{3.4}$$

where: $A_{00} \in \mathbb{C}[p \times p]$ is a square matrix,

$A_{0k} \in \mathbb{C}[p \times q]$ for $k = 1, \dots, n$,

$A_{j0} \in \mathbb{C}[q \times p]$ for $j = 1, \dots, n$,

$A_{jk} \in \mathbb{C}[q \times q]$ for $j, k = 1, \dots, n$.

Definition 3.1. *We say that a matrix A defines an* admissible transformation *for the matrix ball $\Re_{p,q,n}$ if the equality holds.*

$$AHA^* = H. \tag{3.5}$$

This condition is a direct matrix generalization of condition (1.2) for the classical case.

Expanding condition (3.5) component by component, we obtain the system of relations:

$$\begin{aligned}
&(1)\quad A_{00}A_{00}^{*}-\sum_{s=1}^{n}A_{0s}A_{0s}^{*}=I^{(p)},\\
&(2)\quad A_{j0}A_{j0}^{*}-\sum_{s=1}^{n}A_{js}A_{js}^{*}=-I^{(q)},\quad j=1,\dots,n,\\
&(3)\quad A_{00}A_{j0}^{*}=\sum_{s=1}^{n}A_{0s}A_{js}^{*},\quad j=1,\dots,n,\\
&(4)\quad A_{j0}A_{k0}^{*}=\sum_{s=1}^{n}A_{js}A_{ks}^{*},\quad j\neq k,\ j,k=1,\dots,n.
\end{aligned}\tag{3.6}$$

These relations are matrix analogues of the conditions for elements of the group $SU(1,n)$ in the scalar case.

We consider a linear transformation via homogeneous coordinates:

$$\omega=\zeta A,\tag{3.7}$$

where $\omega=(\omega_0,\omega_1,\dots,\omega_n)$ with $\omega_0\in\mathbb{C}^{p\times p}$, $\omega_j\in\mathbb{C}^{p\times q}$.

In coordinates:

$$\begin{aligned}
\omega_0&=\zeta_0A_{00}+\sum_{j=1}^{n}\zeta_jA_{j0},\\
\omega_k&=\zeta_0A_{0k}+\sum_{j=1}^{n}\zeta_jA_{jk},\quad k=1,\dots,n.
\end{aligned}\tag{3.8}$$

Using the relation $\zeta_j=\zeta_0Z_j$, we obtain:

$$\begin{aligned}
\omega_0&=\zeta_0\left(A_{00}+\sum_{j=1}^{n}Z_jA_{j0}\right),\\
\omega_k&=\zeta_0\left(A_{0k}+\sum_{j=1}^{n}Z_jA_{jk}\right).
\end{aligned}\tag{3.9}$$

Defining new variables $W_k=\omega_0^{-1}\omega_k$, we arrive at the main result:

Theorem 3.1 (on the form of automorphisms)**.** *Every holomorphic automorphism of the matrix ball $\Re_{p,q,n}$ has the form*

$$W_k=\left(A_{00}+\sum_{j=1}^{n}Z_jA_{j0}\right)^{-1}\left(A_{0k}+\sum_{j=1}^{n}Z_jA_{jk}\right),\quad k=1,\dots,n,\tag{3.10}$$

where the coefficients A_{ij} satisfy conditions (3.6). Conversely, any transformation of the form (3.10) with coefficients satisfying (3.6) is an automorphism of the ball $\Re_{p,q,n}$.

Proof. Necessity follows from the requirement that condition (3.3) be preserved under transformation (3.7), which leads to condition (3.5), equivalent to (3.6). Sufficiency is verified by direct substitution: if conditions (3.6) are satisfied, then transformation (3.10) biholomorphically maps the ball onto itself. □

4. Automorphism that maps a given point to the origin

Theorem 4.1 (on transitivity)**.** *The automorphism group of the matrix ball $\Re_{p,q,n}$ acts transitively. For any point $B = (B_1, \dots, B_n) \in \Re_{p,q,n}$, there exists an automorphism φ_B mapping B to 0.*

Proof. We explicitly construct such an automorphism. Consider a matrix A with the following blocks:

$$\begin{aligned} A_{00} &= R, \\ A_{j0} &= -B_j^* R, \quad j = 1, \dots, n, \\ A_{jk} &= Q_{jk}, \quad j, k = 1, \dots, n, \\ A_{0k} &= -\sum_{s=1}^{n} B_s Q_{sk}, \quad k = 1, \dots, n, \end{aligned} \tag{4.1}$$

where the matrices $R \in \mathbb{C}^{p\times p}$ and $Q = (Q_{jk}) \in \mathbb{C}^{(nq)\times(nq)}$ are determined by the conditions

$$R^* \left(I^{(p)} - \langle B, B \rangle \right) R = I^{(p)}, \quad Q^* \left(I^{(nq)} - B^* B \right) Q = I^{(nq)}. \tag{4.2}$$

Here B^*B is a block matrix with entries $B_j^* B_k$.

Substituting into formula (3.10) yields

$$W_k = R^{-1} \left(I^{(p)} - \sum_{j=1}^{n} Z_j B_j^* \right)^{-1} \sum_{s=1}^{n} (Z_s - B_s) Q_{sk}, \tag{4.3}$$

which exactly maps B to 0. □

Definition 4.1. *A transformation of the form*

$$W_k = \sum_{j=1}^{n} Z_j U_{jk}, \quad k = 1, \dots, n, \tag{4.4}$$

where the block matrix $U = (U_{jk})_{j,k=1}^{n}$ of size $(qn) \times (qn)$ satisfies

$$U^* U = I^{(qn)}, \quad U U^* = I^{(qn)},$$

is called a generalized unitary transformation.

Note that (4.4) is a special case of (3.10) with $A_{00} = I^{(p)}$, $A_{0k} = 0$, $A_{j0} = 0$, $A_{jk} = U_{jk}$. Conditions (3.6) are then satisfied.

We denote by $W = \varphi_B(Z)$ the automorphism mapping B to 0.

Theorem 4.2. *Let ψ be an arbitrary automorphism of the matrix ball $\mathfrak{R}$ and let $B = \psi(0)$. Then there exists a generalized unitary transformation ψ_u such that $\psi = \psi_u \circ \varphi_B^{-1}$.*

Proof. The mapping $\varphi_B \circ \psi$ is an automorphism that fixes the origin 0, since $\varphi_B(\psi(0)) = \varphi_B(B) = 0$. Automorphisms that fix the origin in circular domains are linear, i.e., generalized unitary transformations ψ_u. Consequently, $\varphi_B \circ \psi = \psi_u$, whence $\psi = \varphi_B^{-1} \circ \psi_u = \psi_u \circ \varphi_B^{-1}$ (by the invertibility of φ_B). □

Corollary 4.1. *The following identities hold:*

$$I^{(m)} - \langle \varphi_B(Z), \varphi_B(W)\rangle = R^{-1}(I^{(m)} - \langle Z, B\rangle)^{-1}(I^{(m)} - \langle Z, W\rangle)(I^{(m)} - \langle B, W\rangle)^{-1}R^{*-1}, \quad (4.5)$$

$$\det\big(I^{(m)} - \langle \varphi_B(Z), \varphi_B(W)\rangle\big) = \frac{\det(I^{(m)} - \langle B, B\rangle)\,\det(I^{(m)} - \langle Z, W\rangle)}{\det(I^{(m)} - \langle Z, B\rangle)\,\det(I^{(m)} - \langle B, W\rangle)}. \quad (4.6)$$

Proof of (4.5). Consider the homogeneous coordinates $\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_n)$ and $\tilde{\zeta} = (\tilde{\zeta}_0, \tilde{\zeta}_1, \ldots, \tilde{\zeta}_n)$ corresponding to the points Z and W:

$$\zeta_0 = I^{(p)}, \quad \zeta_j = Z_j, \quad \tilde{\zeta}_0 = I^{(p)}, \quad \tilde{\zeta}_j = W_j.$$

Then

$$\zeta H\zeta^* = I^{(p)} - \langle Z, Z\rangle, \quad \tilde{\zeta}H\tilde{\zeta}^* = I^{(p)} - \langle W, W\rangle, \quad \zeta H\tilde{\zeta}^* = I^{(p)} - \langle Z, W\rangle.$$

From the condition $AHA^* = H$ for the automorphism φ_B, we have:

$$(\zeta A)H(\tilde{\zeta}A)^* = \zeta H\tilde{\zeta}^*. \quad (4.7)$$

Let us write (4.7) in block form. Let $\omega = \zeta A$, $\tilde{\omega} = \tilde{\zeta}A$. Then (4.7) takes the form:

$$\omega_0\tilde{\omega}_0^* - \sum_{j=1}^{n}\omega_j\tilde{\omega}_j^* = \zeta_0\tilde{\zeta}_0^* - \sum_{j=1}^{n}\zeta_j\tilde{\zeta}_j^* = I^{(p)} - \langle Z, W\rangle.$$

The left-hand side can be rewritten as:

$$\omega_0\left(I^{(p)} - \sum_{j=1}^{n}(\omega_0^{-1}\omega_j)(\tilde{\omega}_0^{-1}\tilde{\omega}_j)^*\right)\tilde{\omega}_0^* = \omega_0\left(I^{(p)} - \langle \varphi_B(Z), \varphi_B(W)\rangle\right)\tilde{\omega}_0^*.$$

Thus,

$$\omega_0\left(I^{(p)} - \langle \varphi_B(Z), \varphi_B(W)\rangle\right)\tilde{\omega}_0^* = I^{(p)} - \langle Z, W\rangle. \quad (4.8)$$

Let's find expressions for ω_0 and $\tilde{\omega}_0$. From (4.1) and (3.8) for the automorphism φ_B we have:

$$\omega_0 = \zeta_0 A_{00} + \sum_{j=1}^{n}\zeta_j A_{j0} = I^{(p)} \cdot R + \sum_{j=1}^{n} Z_j(-B_j^* R) =$$

$$= \left(I^{(p)} - \sum_{j=1}^{n} Z_j B_j^*\right) R = \left(I^{(p)} - \langle Z, B\rangle\right) R.$$

Likewise,

$$\tilde{\omega}_0 = \left(I^{(p)} - \langle W, B\rangle\right) R.$$

Note that $\langle W, B\rangle^* = \langle B, W\rangle$, so

$$\tilde{\omega}_0^* = R^*\left(I^{(p)} - \langle B, W\rangle\right).$$

Substituting the found expressions for ω_0 and $\tilde{\omega}_0^*$ into (4.8), we obtain:

$$\left(I^{(p)} - \langle Z, B\rangle\right) R\left(I^{(p)} - \langle \varphi_B(Z), \varphi_B(W)\rangle\right) R^*\left(I^{(p)} - \langle B, W\rangle\right) = I^{(p)} - \langle Z, W\rangle.$$

Multiply on the left by $\left(I^{(p)} - \langle Z, B\rangle\right)^{-1}$ and on the right by $\left(I^{(p)} - \langle B, W\rangle\right)^{-1}$:

$$R\left(I^{(p)} - \langle \varphi_B(Z), \varphi_B(W)\rangle\right) R^* = \left(I^{(p)} - \langle Z, B\rangle\right)^{-1} \left(I^{(p)} - \langle Z, W\rangle\right) \left(I^{(p)} - \langle B, W\rangle\right)^{-1}.$$

Finally, multiply on the left by R^{-1} and on the right by R^{*-1}:

$$I^{(p)} - \langle \varphi_B(Z), \varphi_B(W)\rangle = R^{-1} \left(I^{(p)} - \langle Z, B\rangle\right)^{-1} \left(I^{(p)} - \langle Z, W\rangle\right) \left(I^{(p)} - \langle B, W\rangle\right)^{-1} R^{*-1}.$$

The second identity (4.6) is a direct consequence of the first, obtained by taking determinants of both sides and using the relation $R^*(I^{(m)} - \langle B, B\rangle)R = I^{(m)}$, which implies $|\det R|^2 = \det(I^{(m)} - \langle B, B\rangle)^{-1}$. □

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Об автоморфизмах матричного шара в пространстве прямоугольных матриц

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Аннотация. В работе рассматривается матричный шар в пространстве прямоугольных матриц, изучается группа голоморфных автоморфизмов: получены явные формулы для дробно-линейных преобразований, сохраняющих шар, и доказана их общая структура. Особое внимание уделено построению автоморфизма, переводящего заданную внутреннюю точку шара в начало координат. Результаты обобщают классическую теорию автоморфизмов единичного шара в $\mathbb{C}^n$ и шара из квадратных матриц на случай прямоугольных матриц.

Ключевые слова: единичный круг, матричное скалярное произведение, матричный шар, автоморфизм.

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The Net Subgroups of the Group $GL(3,\mathbb{Q})$, Defining the Structure of Overgroups of a Cyclic Torus

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Abstract. We study nets and net subgroups that define the structure of subgroups of the general linear group $G = GL(3,\mathbb{Q})$ of degree 3 over the field of rational numbers $\mathbb{Q}$, containing a non-split maximal torus $T = T(d)$ associated with a radical (cyclic) extension $\mathbb{Q}(\sqrt[3]{d})$ of degree 3 of the field of rational numbers $\mathbb{Q}$. The calculation of the normalizer $N(\sigma)$ is given of the elementary net subgroup $E(\sigma)$ in the group G.

Keywords: linear groups, net subgroups, non-split maximal torus.

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1. Introduction and preliminaries

This paper continues the work of Z. I. Borevich and the author [1–3] and is devoted to the study of subgroups of the general linear group $G = GL(3,\mathbb{Q})$ that contain a non-split maximal torus $T = T(d)$ associated with a radical (cyclic) extension $K = \mathbb{Q}(\sqrt[3]{d})$ of degree 3 of the field of rational numbers $\mathbb{Q}$. This paper is devoted to the study of net subgroups and their normalizers in the group G, which determine the structure of subgroups containing a non-split maximal torus $T = T(d)$.

The elements of the torus matrices $T = T(d)$ generate some subring $R(d)$ of the field $\mathbb{Q}$. Let R — be an intermediate subring, $R(d) \subseteq R \subseteq \mathbb{Q}$. For a triple of (fractional) ideals A_1, A_2, A_3 of the ring R, related by the relation $dA_3 \subseteq A_1 \subseteq A_2 \subseteq A_3$, we consider the net

$$\sigma = \sigma(A_1, A_2, A_3) = \begin{pmatrix} A_1 & dA_3 & dA_2 \\ A_2 & A_1 & dA_3 \\ A_3 & A_2 & A_1 \end{pmatrix},$$

which is called the net associated with the torus T. The net group $G(\sigma)$ is normalized by the torus T, and therefore the product $TG(\sigma)$ is an intermediate subgroup of the group $G = GL(3,\mathbb{Q})$ containing the torus T (Proposition 3). When studying the structure of intermediate subgroups of the group G containing the torus T, the main role is played by the net $\sigma = \sigma(A_1, A_2, A_3)$ for the case $A_1 = dA_3$. Thus, these net $\sigma = \sigma(dA_3, A_2, A_3)$ are defined by an intermediate ring R, $R(d) \subseteq R \subseteq k$ and a pair of fractional ideals A_2, A_3 of the ring R satisfying the condition $dA_3 \subseteq A_2 \subseteq A_3$. In Section 3 we study nets and net subgroups associated with the torus $T = T(d)$.

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Let us formulate the main result of the work, announced in [4]. Let $d \in \mathbb{Z}$ be an integer not divisible by a cube of any prime, where $d \neq \pm 1$. Consider the net $\sigma = \sigma(dA_3, A_2, A_3)$ and the elementary net subgroup $E(\sigma)$ generated by transvections from the net group $G(\sigma)$. Let $N(\sigma)$ be the normalizer of the subgroup $E(\sigma)$ in the group G. Let, further, q=gcd(b, c), where c and b are determined from the equalities: $d = bc$, $(A_2 : A_3) = cR$, $(dA_3 : A_2) = bR$. Let $B = \dfrac{1}{q}R$ and consider the net

$$\Omega = \begin{pmatrix} R & dB & dR \\ R & R & dB \\ B & R & R \end{pmatrix}.$$

Theorem. *Let $A_1 = dA_3$, $\sigma = \sigma(A_1, A_2, A_3)$ be the net associated with the torus T. The normalizer $N(\sigma)$ of an elementary net subgroup $E(\sigma)$ in the general linear group $G = GL(3,\mathbb{Q})$ coincides with the product of the torus T and the net group $G(\Omega)$: $N(\sigma) = TG(\Omega)$.*

Note that in [5] the normalizer $N(\sigma)$ is calculated for the special case when A_2 and A_3 are integral ideals of the ring R, and $A_2 = A_3$.

The following notations are used in this work:

$d \in \mathbb{Z}$ is an integer free of 3 powers (that is, a product of distinct primes with powers $\leqslant 2$); $k = \mathbb{Q}$ — the field of rational numbers; $x^3 - d$ is an irreducible polynomial over $\mathbb{Q}$; e — the identity matrix of order n; e_{ij} — the matrix in which at position (i, j) there is $1 \in k$, and at the remaining places there are zeros; $t_{ij}(\xi) = e + \xi e_{ij}$ — elementary transvection $\xi \in k^*$, $i \neq j$; transvection is a matrix of the form $(\delta_{ij} + \alpha_i\beta_j)$, where $\alpha_i, \beta_j \in k$, $\sum\limits_{i=1}^{n} \alpha_i\beta_i = 0$ (δ_{ij} — Kronecker symbol). Let $a = (a_{ij})$, $a^{-1} = (a'_{ij})$. Then the formula holds: $(at_{rs}(\xi)a^{-1})_{ij} = \delta_{ij} + a_{ir}\xi a'_{sj}$; $(S)_{ij}$ denotes the element s_{ij} of the matrix $S = (s_{ij})$; s'_{ij} are the elements of the inverse matrix $S^{-1} = (s'_{ij})$.

Let R be a ring, $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$. Recall that an R-module A, $A \subseteq \mathbb{Q}$, is called a fractional ideal of the ring R if $\gamma A \subseteq R$ for some $\gamma \in \mathbb{Q}$. For (fractional) ideals A, B of a ring R, $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$, $(A : B)$ denotes the (fractional) ideal $(A : B) = \{\lambda \in \mathbb{Q} : \lambda B \subseteq A\}$.

For a ring R, $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$, and nonzero integers $a, b \in \mathbb{Z}$, we have

$$aR \cap bR = mR, \quad \frac{1}{a}R \cap \frac{1}{b}R = \frac{1}{q}R,$$

where q=gcd(a, b), $m = \dfrac{ab}{q}$=lcm(a, b).

Let us also recall that the table of additive subgroups $\sigma = (\sigma_{ij})$ of the ring R is called a net [6, 7] if the inclusions $\sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij}$ hold for all i, r, j. For the net $\sigma = (\sigma_{ij})$, $M(\sigma) = \{a = (a_{ij});\ a_{ij} \in \sigma_{ij}\}$ denotes the matrix ring.

2. Cyclic torus and ring associated with a cyclic torus

Consider the radical (cyclic) extension of degree 3 $K = \mathbb{Q}(\sqrt[3]{d})$ of the field $\mathbb{Q}$. There is a regular embedding of the multiplicative group K^* of the field K into the group $GL_{\mathbb{Q}}(K) = Aut_{\mathbb{Q}}(K)$ of all $\mathbb{Q}$-linear invertible mappings of the field K

$$K^* \longrightarrow Aut_{\mathbb{Q}}(K), \quad \alpha \dashrightarrow \widehat{\alpha},$$

where $\widehat{\alpha}(x) = \alpha x$, $x \in K$. The image of K^* under this embedding is denoted by $T = T(K)$ and is called the non-split maximal torus, corresponding to the extension $\mathbb{Q}(\sqrt[3]{d})/\mathbb{Q}$. Let $\theta = \sqrt[3]{d}$ and

fix a basis 1, θ, θ^2 of the field $K = \mathbb{Q}(\sqrt[3]{d})$ over $\mathbb{Q}$. Each element $\alpha = x_1 + x_2\theta + x_3\theta^2 \in K$ in the chosen basis (with the specified embedding) corresponds to a matrix

$$C(x) = \begin{pmatrix} x_1 & dx_3 & dx_2 \\ x_2 & x_1 & dx_3 \\ x_3 & x_2 & x_1 \end{pmatrix}, \ x = (x_1, x_2, x_3) \in \mathbb{Q}^3.$$

Thus, in the chosen basis, the torus $T = T(d) = T(K)$ is defined as the matrix group

$$T = T(d) = \{C(x) : x \in \mathbb{Q}^3 \setminus \bar{0}\}.$$

Each matrix $C = C(x) = (c_{ij})$ has an associated inverse matrix $C^{-1} = C(y) = (c'_{ij})$, $y = (y_1, y_2, y_3) \in k^n$, where $y_i = \dfrac{C_{1i}}{|\, C\,(x)\,|}$, and C_{1i} is the algebraic complement of element c_{1i} of matrix $C = C(x)$. In this paper, we consider the unital subring $R_0 = R(d)$ [1–3, 8] of $\mathbb{Q}$, generated by 1 and the elements x_iy_j, dx_ry_s

$$R_0 = R(d) = \langle 1,\ x_1y_1,\ x_1y_2,\ x_1y_3,\ x_2y_1,\ x_2y_2,\ dx_2y_3,\ dx_3y_2,\ dx_3y_3 : \ \ x \in \mathbb{Q}^3 \setminus \bar{0}\rangle.$$

The ring $R_0 = R(d)$ defined in this way is called the ring associated with the cyclic torus $T = T(d)$. Throughout this article, R is an intermediate subring, $R_0 \subseteq R \subseteq \mathbb{Q}$. Recall that the ring R is a principal ideal domain, and every fractional ideal of R is its principal fractional ideal.

3. Net and net group associated with the torus

Let R be an intermediate subring, $R_0 \subseteq R \subseteq \mathbb{Q}$. Further, let A_1, A_2, A_3 be fractional ideals of R, and

$$dA_3 \subseteq A_1 \subseteq A_2 \subseteq A_3. \tag{1}$$

By $\sigma = (\sigma_{ij}) = \sigma(A_1, A_2, A_3)$ we denote a table that has the form

$$\sigma = \sigma(A_1, A_2, A_3) = \begin{pmatrix} A_1 & dA_3 & dA_2 \\ A_2 & A_1 & dA_3 \\ A_3 & A_2 & A_1 \end{pmatrix}. \tag{2}$$

Proposition 1. *If the table $\sigma = \sigma(A_1, A_2, A_3)$ of form (2) of fractional ideals A_1, A_2, A_3 of the ring R, $R_0 \subseteq R \subseteq \mathbb{Q}$ satisfying condition (1) is a net, then the ideals A_1, A_2, dA_3 are integral ideals of the ring R. More precisely, the inclusion $dA_3 \subseteq A_1 \subseteq A_2 \subseteq R$. holds.*

Proof. By virtue of (1), it is sufficient to prove the inclusion $A_2 \subseteq R$. Let $\alpha, \beta \in A_2 = \sigma_{21} = \sigma_{32}$. Then $\alpha^2, \alpha\beta \in \sigma_{32}\sigma_{21} \subseteq \sigma_{31} = A_3$, hence $d\alpha^2, d\alpha\beta \in \sigma_{12} = \sigma_{23} = dA_3$. That's why

$$d\alpha^2 d\alpha\beta = d^2\alpha^3\beta \in \sigma_{12}\sigma_{23} \subseteq \sigma_{13} = dA_2 \Longrightarrow d^2\alpha^3\beta \subseteq dA_2.$$

Thus, $d\alpha^3 A_2 \subseteq A_2$, so (here we take into account that A_2 is a principal fractional ideal, $A_2 = tR$, $t \in \mathbb{Q}$) we have the inclusion $d\alpha^3 \in R$, whence (d is free of 3 powers) $\alpha \in R$. □

Definition. *A net $\sigma = \sigma(A_1, A_2, A_3)$ of the form (2), defined by a set of fractional ideals A_1, A_2, A_3 (where $A_1, A_2 \subseteq R$ are integer ideals) of an intermediate ring R, $R_0 \subseteq R \subseteq \mathbb{Q}$, with condition (1), is called a net associated with the torus $T = T(d)$, and the net group $G(\sigma)$ we call the net group associated with the torus $T = T(d)$. The subgroup $E(\sigma)$ generated by all (common) transvections of the net group $G(\sigma)$ is called the elementary net subgroup associated with the torus T.*

In [6, 7] the nets $\sigma = (\sigma_{ij})$ of additive subgroups σ_{ij} of a commutative ring R are defined and, in the special case when all σ_{ij} are ideals of a commutative ring R, a formula for the net group $G(\sigma)$ is given. Namely, it is proved that $G(\sigma) = (e + M(\sigma)) \cap GL(n, R)$ [6]. The following proposition presents a net group formula for nets of additive subgroups.

Proposition 2. *Let R be a subring with 1 of the commutative ring Λ, $R \subseteq \Lambda$. Let, further, $\sigma = (\sigma_{ij})$ be a net of additive subgroups of order n over the ring Λ, where σ_{ij} are R-modules and $\sigma_{ii} \subseteq R$, $i = 1, \ldots, n$, $G(\sigma)$ is a net group over the ring Λ. Then*

$$G(\sigma) = \{b \in e + M(\sigma) : \ det\ b \in R^*\}.$$

The last proposition covers all nets considered in this paper, namely, when $\Lambda = \mathbb{Q}$ is the field of fractions of R and $\sigma = (\sigma_{ij})$ is the net of fractional ideals σ_{ij} of $\mathbb{Q}$ of order n, with $\sigma_{ii} \subseteq R$.

The following proposition follows from [8, Theorem 2.7.7].

Proposition 3. *If $\sigma = (\sigma_{ij})$ is the net associated with the torus T, then*

(1) *the torus T normalizes the net group $G(\sigma)$. Therefore, $TG(\sigma)$ is an intermediate subgroup containing the torus T;*

(2) *the torus T normalizes the elementary net group $E(\sigma)$. Therefore, $TE(\sigma)$ is an intermediate subgroup containing the torus T.*

Remark 1. *It can be shown that the group $TE(\sigma)$ is generated by the torus T and the root subgroups $t_{ij}(\sigma_{ij})$, $1 \leqslant i \neq j \leqslant 3$.*

To study the structure of intermediate subgroups of the group G containing the torus T, the networks $\sigma = \sigma(A_1, A_2, A_3)$ play an important role for the case $A_1 = dA_3$.

The following lemma is easily verified given that the integer $d \in \mathbb{Z}$ is free of 3 powers.

Lemma 1. *Let A_2, A_3 be fractional ideals of the ring R, $\mathbb{Z} \subseteq \mathbb{R} \subseteq \mathbb{Q}$, and let $dA_3 \subseteq A_2 \subseteq A_3$. Let, further, $(A_2 : A_3) = cR$, $(dA_3 : A_2) = bR$ $(b, c \in \mathbb{Z})$, q=gcd (b, c), $b = b_1 q$, $c = c_1 q$. Then $d = bc$, and q is either equal to 1 or is a product of distinct primes. Next, gcd (b_1, c_1)=gcd (b_1, q)=gcd $(c_1, q) = 1$.*

3. Normalizer of the elementary net subgroup $E(\sigma)$ in the group $GL(3, \mathbb{Q})$

This section is devoted to the proof of a theorem that gives a calculation of the normalizer $N(\sigma) = N_G(E(\sigma))$ of the elementary network group $E(\sigma)$ in the group $G = GL(3, \mathbb{Q})$ for the network $\sigma = \sigma(dA_3, A_2, A_3)$ associated with the torus T.

Recall that R is an intermediate subring, $R_0 \subseteq R \subseteq \mathbb{Q}$, A_2, A_3 are fractional ideals of R (with A_2, A_3, dA_3 being integral ideals of R), related by the relation

$$dA_3 \subseteq A_2 \subseteq A_3. \tag{3}$$

Remark 2. *By Lemma 1, throughout the article we assume that $d = bc$, $(A_2 : A_3) = cR$, $(dA_3 : A_2) = bR$, q=gcd (b, c), $b = b_1 q$, $c = c_1 q$, gcd (b_1, c_1)=gcd (b_1, q)=gcd $(c_1, q) = 1$, and either $q = 1$ or q is a product of distinct primes.*

Recall that the net σ has the form

$$\sigma = \sigma(dA_3, A_2, A_3) = \begin{pmatrix} dA_3 & dA_3 & dA_2 \\ A_2 & dA_3 & dA_3 \\ A_3 & A_2 & dA_3 \end{pmatrix}. \tag{4}$$

Let $B = \frac{1}{q}R$ (where q=gcd (b,c)), and define the net

$$\Omega = (\Omega_{ij}) = \begin{pmatrix} R & dB & dR \\ R & R & dB \\ B & R & R \end{pmatrix} = \begin{pmatrix} R & \frac{bc}{q}R & dR \\ R & R & \frac{bc}{q}R \\ \frac{1}{q}R & R & R \end{pmatrix}.$$

Lemma 2. *For any i, r, j, the following inclusions are true:*

$$\sigma_{ir}\Omega_{rj} \subseteq \sigma_{ij}, \quad \Omega_{ir}\sigma_{rj} \subseteq \sigma_{ij}.$$

In particular, the matrix ring $M(\sigma)$ is a two-sided ideal of the matrix ring $M(\Omega)$.

Proof. Let us prove the first inclusion (the second is proved similarly)

$$\sigma_{ir}\Omega_{rj} \subseteq \sigma_{ij}. \tag{5}$$

Note (see (4)) that the columns of the net $\sigma = \sigma(dA_3, A_2, A_3)$ increase from right to left by virtue of (3), that is

$$\sigma_{i3} \subseteq \sigma_{i2} \subseteq \sigma_{i1}, \quad i = 1, 2, 3. \tag{6}$$

Let us prove inclusion (5). Let us consider the possibilities for pairs (r, j).

If $(r, j) \in \{(11), (2, 2), (3, 3), (2, 1), (3, 2)\}$, then $\Omega_{rj} = R$ and inclusion (5) is obvious by virtue of (6). If $(r, j) = (1, 3)$, then $\Omega_{13} = dR$, inclusion (6) takes the form $d\sigma_{i1} \subseteq \sigma_{i3}$, which is obvious (see (4) and (3)). If $(r, j) = (3, 1)$, then $\Omega_{31} = B = \frac{1}{q}R$, inclusion (5) takes the form $\sigma_{i3} \subseteq q\sigma_{i1}$, which is obvious from (4), from the fact that $A_2 = cA_3$ and q divides c, q divides d, qc divides d. If $(r, j) = (1, 2)$, then $\Omega_{12} = dB = \frac{d}{q}R$, inclusion (5) takes the form $d\sigma_{i1} \subseteq q\sigma_{i2}$, which is obvious from (4), from the fact that $A_2 = cA_3$ and q divides c, qc divides d. If $(r, j) = (2, 3)$, then $\Omega_{23} = dB = \frac{d}{q}R$, inclusion (5) takes the form $d\sigma_{i2} \subseteq q\sigma_{i3}$, which is obvious from (4), since $A_2 = cA_3$ and q divides d, q divides c, qc divides d. □

Proposition 4. *The net group $G(\Omega)$ is contained in the normalizer of the net subgroup $G(\sigma)$ in the group G, $G(\Omega) \leqslant N_G(G(\sigma))$. In particular, $G(\Omega) \leqslant N(\sigma)$.*

Proof. Let $g = e + h \in G(\Omega)$, $h \in M(\Omega)$, $b = e + a \in G(\sigma)$, $a \in M(\sigma)$. Then according to Proposition 2 for the network group $G(\Omega)$ we have $g^{-1} = e + h_1$, $h_1 \in M(\Omega)$. From here, from Lemma 2, we have

$$gbg^{-1} = e + gag^{-1} = e + (e + h)a(e + h_1) = e + a + ha + ah_1 + hah_1.$$

Therefore, from Lemma 2 we have $gbg^{-1} \in e + M(\sigma)$. □

Proposition 5. *The net group $G(\Omega)$ is normalized by the torus $T = T(d)$, and therefore the product $TG(\Omega)$ is a group. Furthermore,*

$$TG(\Omega) \leqslant N_G(G(\sigma)), \quad TG(\Omega) \leqslant N(\sigma),$$

where $N(\sigma)$ is the normalizer of the subgroup $E(\sigma)$ in the group $G = GL(3, \mathbb{Q})$.

Proof. By definition, the net Ω is a net associated with the torus T. Therefore, according to Proposition 3, the torus T normalizes the net group $G(\Omega)$, and therefore the product $TG(\Omega)$ is a group and for the same reason $T \leqslant N_G(G(\sigma))$ and $T \leqslant N(\sigma)$. It remains to refer to Proposition 4. □

To prove the theorem according to Proposition 5, we need to prove the inverse inclusion $N(\sigma) \subseteq TG(\Omega)$. This is the subject of the rest of the article.

Lemma 3. *Let $g=(g_{ij})\in G=GL(3,\mathbb{Q})$. Then for some matrices $C(x),C(y)\in T$, we have*

$$(gC(x))_{i1}=\delta_{i1},\ (C(y)g)_{1i}=\delta_{1i},\ i=1,2,3.$$

Proof. Let $g^{-1}=(g'_{ij})$. To prove the proposition, it is sufficient to take as $C(x)$ a matrix whose first column coincides with the first column of the inverse matrix g^{-1}, and to take as $C(y)$ a matrix whose first row coincides with the first row of the matrix g^{-1}:

$$C(x)=\begin{pmatrix} g'_{11} & dg'_{31} & dg'_{21}\\ g'_{21} & g'_{11} & dg'_{31}\\ g'_{31} & g'_{21} & g'_{11}\end{pmatrix},\quad C(y)=\begin{pmatrix} g'_{11} & g'_{12} & g'_{13}\\ \frac{1}{d}g'_{13} & g'_{11} & g'_{12}\\ \frac{1}{d}g'_{12} & \frac{1}{d}g'_{13} & g'_{11}\end{pmatrix}.$$

□

Throughout the rest of this section, we assume that $a=(a_{ij})\in N(\sigma)$. According to Proposition 5, the normalizer $N(\sigma)$ contains the torus $T=T(d)$, and therefore, by virtue of Lemma 3, we can assume that

$$a=\begin{pmatrix} 1 & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ 0 & a_{32} & a_{33}\end{pmatrix},\quad a^{-1}=\begin{pmatrix} 1 & a'_{12} & a'_{13}\\ 0 & a'_{22} & a'_{23}\\ 0 & a'_{32} & a'_{33}\end{pmatrix},\quad \Delta=det\ a. \tag{7}$$

We will show (Proposition 10) that $a\in G(\Omega)$.

Proposition 6. *The following inclusions are valid*

$$a_{22},a_{33}\in R,\quad a_{23}\in cR,\quad a_{32}\in\frac{1}{c}R,\quad a_{12}\in bR,\quad a_{13}\in dR,\quad \Delta\in R. \tag{8}$$

$$a'_{22},a'_{33}\in R,\quad a'_{23}\in cR\quad a'_{32}\in\frac{1}{c}R,\quad a'_{12}\in bR,\quad a'_{13}\in dR,\quad \Delta^{-1}\in R. \tag{9}$$

Proof. Let $\alpha\in\sigma_{21}=A_2=cR$. Then $t_{21}(\alpha)\in E(\sigma)$, and therefore $at_{21}(\alpha)a^{-1}\in G(\sigma)$. Because

$$(at_{21}(\alpha)a^{-1})_{11}-1=\alpha a_{12},\ \ (at_{21}(\alpha)a^{-1})_{21}=\alpha a_{22},\ \ (at_{21}(\alpha)a^{-1})_{31}=\alpha a_{32},$$

then $\alpha a_{12}\in\sigma_{11}=dA_3,\ \alpha a_{22}\in\sigma_{21}=A_2,\ \alpha a_{32}\in\sigma_{31}=A_3$. From here

$$a_{12}\in(dA_3:A_2)=bR,\ \ a_{22}\in(A_2:A_2)=R,\ \ a_{32}\in(A_3:A_2)=\frac{1}{c}R.$$

Let $\beta\in\sigma_{31}=A_3=R$. Then $t_{31}(\beta)\in E(\sigma)$, and therefore $at_{31}(\beta)a^{-1}\in G(\sigma)$. Because

$$(at_{31}(\beta)a^{-1})_{11}-1=\beta a_{13},\ \ (at_{31}(\beta)a^{-1})_{21}=\beta a_{23},\ \ (at_{31}(\beta)a^{-1})_{31}=\beta a_{33},$$

then $\beta a_{13}\in\sigma_{11}=dA_3,\ \alpha a_{23}\in\sigma_{21}=A_2,\ \alpha a_{33}\in\sigma_{31}=A_3$. From here

$$a_{13}\in(dA_3:A_3)=dR,\ \ a_{23}\in(A_2:A_3)=cR,\ \ a_{33}\in(A_3:A_3)=R.$$

Thus inclusions (8) are proved. Considering similarly the matrices $a^{-1}t_{21}(\alpha)a$ and $a^{-1}t_{31}(\beta)a$, we obtain inclusions (9).

Finally, from (8) and (9) follow the inclusions $\Delta,\Delta^{-1}\in R$. □

Lemma 4. *The following inclusions are valid*

$$a_{12}a'_{32}\in R,\ \ a_{12}a'_{33}\in cR,\ \ a_{32}a'_{32}\in\frac{1}{b}R,\ \ a_{32}a'_{33}\in R, \tag{10}$$

$$a_{23}a'_{22}\in bR,\ \ a_{23}a'_{23}\in bR. \tag{11}$$

Proof. Let us prove formulas (10). Let $\alpha \in dA_3$. Then $t_{23}(\alpha) \in E(\sigma)$. Therefore $at_{23}(\alpha)a^{-1} \in G(\sigma)$, from here

$$(at_{23}(\alpha)a^{-1})_{12} = \alpha a_{12}a'_{32} \in \sigma_{12} = dA_3, \quad (at_{23}(\alpha)a^{-1})_{13} = \alpha a_{12}a'_{33} \in \sigma_{12} = dA_2.$$

Therefore

$$a_{12}a'_{32} \in (dA_3 : dA_3) = R, \quad a_{12}a'_{33} \in (dA_2 : dA_3) = cR.$$

Next,

$$(at_{23}(\alpha)a^{-1})_{32} = \alpha a_{32}a'_{32} \in \sigma_{32} = A_2, \quad (at_{23}(\alpha)a^{-1})_{33} - 1 = \alpha a_{32}a'_{33} \in \sigma_{33} = dA_3.$$

Therefore

$$a_{32}a'_{32} \in (A_2 : dA_3) = \frac{1}{b}R, \quad a_{32}a'_{33} \in (dA_3 : dA_3) = R.$$

Let us prove formulas (11). Let $\beta \in A_2$. Then $t_{32}(\beta) \in E(\sigma)$, therefore $at_{32}(\beta)a^{-1} \in G(\sigma)$, from here

$$(at_{32}(\beta)a^{-1})_{22} - 1 = \beta a_{23}a'_{22} \in \sigma_{22} = dA_3, \quad (at_{32}(\beta)a^{-1})_{23} = \beta a_{23}a'_{23} \in \sigma_{23} = dA_3,$$

therefore

$$a_{23}a'_{22} \in (dA_3 : A_2) = bR, \quad a_{23}a'_{23} \in (dA_3 : A_2) = bR. \qquad \square$$

Proposition 7. *The following inclusion is valid* $a_{12} \in \Omega_{12} = \dfrac{d}{q}R = \dfrac{bc}{q}R$.

Proof. From (10) $a_{12}a'_{33} \in cR;\ a_{12}a'_{32} \in R$. Next from (9) we have $a'_{22} \in R,\ a'_{23} \in cR$. Therefore

$$a_{12}a'_{33}a'_{22} \in cR; \quad a_{12}a'_{32}a'_{23} \in cR.$$

Therefore (according to (8) and (9) $\Delta \in R^*$),

$$a_{12}\Delta^{-1} = a_{12}(a'_{33}a'_{22} - a'_{32}a'_{23}) \in cR \Longrightarrow a_{12} \in cR.$$

Next, from (8) we have $a_{12} \in bR$. Therefore $a_{12} \in bR \cap cR = \dfrac{bc}{q}R = \dfrac{d}{q}R$ (recall that q=gcd(b,c)). $\square$

Proposition 8. *The following inclusion is valid* $a_{32}, a'_{32} \in R$.

Proof. Let us prove the inclusion $a_{32} \in R$ (the inclusion $a'_{32} \in R$ is proved symmetrically). According to (8) и (9)

$$a_{32}, a'_{32} \in \frac{1}{c}R \Longrightarrow a_{32}a'_{32} \in \frac{1}{c^2}R.$$

On the other hand, from (10) we have $a_{32}a'_{32} \in \dfrac{1}{b}R$. Therefore

$$a_{32}a'_{32} \in \frac{1}{b}R \cap \frac{1}{c^2}R.$$

Next, from (9) we have $a'_{23} \in cR$. Therefore $a_{32}a'_{32}a'_{23} \in \dfrac{c}{b}R \cap \dfrac{1}{c}R$. It is not difficult to see that

$$\frac{c}{b}R \cap \frac{1}{c}R = \frac{1}{bc}\left[c^2R \cap bR\right] = \frac{1}{bc}\left[\frac{c^2b}{q}R\right] = \frac{c}{q}R \subseteq R$$

(according to Remark 2 we have gcd(c^2, b)=q). Therefore $a_{32}a'_{32}a'_{23} \in R$.

On the other hand, it follows from (9) that $a'_{22} \in R$, and from (10) we have the inclusion $a_{32}a'_{33} \in R$. Therefore $a_{32}a'_{33}a'_{22} \in R$. From here $\left(a'_{22}a'_{33} - a'_{32}a'_{23} = \dfrac{1}{\Delta}\right)$,

$$-a_{32}a'_{32}a'_{23} + a_{32}a'_{22}a'_{33} = a_{32} \cdot \frac{1}{\Delta} \in R \Longrightarrow a_{32} \in R. \qquad \square$$

Proposition 9. *The following inclusion is valid* $a_{23} \in \Omega_{23} = \dfrac{bc}{q}R$.

Proof. Let's show the inclusion $a_{23} \in bR$. According to proposition 8 we have $a_{32}, a'_{32} \in R$. Further according to (11)

$$a_{23}a'_{23} \in bR \Longrightarrow a_{23}a'_{23}a'_{32} \in bR.$$

Further according to (11) (and considering that according to (9) we have $a'_{33} \in R$)

$$a_{23}a'_{22} \in bR \Longrightarrow a_{23}a'_{22}a'_{33} \in bR.$$

Therefore

$$a_{23}a'_{22}a'_{33} - a_{23}a'_{23}a'_{32} \in bR \Longrightarrow a_{23} \cdot \frac{1}{\Delta} \in bR \Longrightarrow a_{23} \in bR.$$

So, we have shown that $a_{23} \in bR$. On the other hand, according to (8) we have $a_{23} \in cR$. Therefore

$$a_{23} \in bR \cap cR = \frac{bc}{q}R.$$

□

The following Proposition follows from Propositions 5–8.

Proposition 10. *If* $a = (a_{ij}) \in N(\sigma)$ *is a matrix of the form (7), then* $a \in G(\Omega)$.

Proof of the theorem. The inclusion $N(\sigma) \supseteq TG(\Omega)$ follows from Proposition 5. Let us prove the reverse inclusion. Let $g \in N(\sigma)$. Then according to Lemma 3 for some matrix $C(x) \in T$ we have $(gC(x))_{i1} = \delta_{i1}$. Let $a = gC(x)$. Since $T \subseteq N(\sigma)$, then $a \in N(\sigma)$, and the matrix a has the form (7). But then according to Proposition 10 we have $a \in G(\Omega)$. Hence $a = gC(x) \in G(\Omega)$. Therefore, $g = aC^{-1}(x) \in G(\Omega)T = TG(\Omega)$ (recall that according to Proposition 5, the torus T normalizes the group $G(\Omega)$). Thus, $g \in TG(\Omega)$. The theorem is proved. □

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Сетевые подгруппы группы $GL(3,\mathbb{Q})$, определяющие структуру надгрупп циклического тора

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Аннотация. Исследуются сети и сетевые подгруппы, определяющие структуру подгрупп полной линейной группы $G = GL(3,\mathbb{Q})$ степени 3 над полем рациональных чисел $\mathbb{Q}$, содержащих нерасщепимый максимальный тор $T = T(d)$, связанный радикальным (циклическим) расширением $\mathbb{Q}(\sqrt[3]{d})$ степени 3 поля рациональных чисел $\mathbb{Q}$. Дается вычисление нормализатора $N(\sigma)$ элементарной сетевой подгруппы $E(\sigma)$ в группе G.

Ключевые слова: линейные группы, сетевые подгруппы, нерасщепимый максимальный тор.

EDN: MOJXCH
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Qualitative Properties of Solutions of Parabolic Heat Equation in Nondivergent Form with Two Nonlinear Source Terms

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Abstract. Properties of generalized solutions of parabolic heat conduction equation in nondivergent form with double nonlinear source terms are studied in this paper. The model incorporates nonlinear diffusion with variable density and nonlinear source functions. For cases where classical solutions do not exist the concept of generalized solutions in the sense of integral identities is employed. By applying an appropriate change of variables and the method of self-similar solutions the original nonlinear parabolic equation is reduced to an ordinary differential equation. In the slow diffusion regime, the global existence of solutions is established using the comparison principle. The Fujita-type critical exponent is determined. The obtained results have significant theoretical importance for the numerical analysis of nonlinear models arising in heat transfer, chemical processes, and biological systems.

Keywords: global solution, nondivergent equation, diffusion, asymptotic solution.

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Introduction

Nonlinear parabolic equations, in particular doubly nonlinear diffusion equations in both divergent and nondivergent forms, are actively studied in recent years. Fundamental problems such as blow-up phenomena, global existence, and asymptotic behavior of solutions of nonlinear parabolic equations were systematically studied [1,2]. It provides solid theoretical foundation for subsequent studies.

Issues related to the computational modeling of nonlinear diffusion processes were examined, and correspondence between theoretical results and numerical experiments was demonstrated [2]. Furthermore, the Cauchy problem for doubly nonlinear parabolic equations with variable density was studied in detail, including the existence and qualitative properties of generalized solutions [3]. These studies make a significant contribution to the development of the theory of generalized solutions for equations in nondivergent form [1–4].

The problem of Fujita-type critical exponents for divergent parabolic equations was investigated by Andreucci and Tedeev [5], where threshold conditions separating global existence from blow-up behavior were established. The asymptotic behavior of solutions to doubly nonlinear parabolic equations in nondivergent form was analyzed by Jin and Yin [7]. Zhou and co-authors [8] studied boundary value problems for nondivergent diffusion equations with gradient terms and obtained estimates and qualitative properties of solutions.

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The Cauchy problem for nondivergent diffusion equations, including strong and weak solutions as well as doubly nonlinear equations with source terms were thoroughly studied [**?**]. The existing literature indicates that the study of global solutions, their estimates, and critical exponents for doubly nonlinear parabolic equations in nondivergent and divergent form remains an open and relevant problem. The present paper continues this line of studies and contributes new theoretical results to this field.

1. Problem statement

Consider in $Q = \left\{(x,t):\ \ x \in R^N,\, t>0\right\}$ the following Cauchy problem

$$|x|^{\alpha_1}\frac{\partial u}{\partial t} = u^{\beta}\frac{\partial}{\partial x}\left(|x|^{\alpha_2}u^{m-1}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + |x|^{\alpha_1}u^{q} \tag{1}$$

$$u(x,t)|_{t=0} = u_0(x) \geqslant 0,\ x \in R. \tag{2}$$

Here parameters of this equation satisfy the following inequalities $\beta < 1, m \geqslant 1, p \geqslant \max\{2, \alpha_2-\alpha_1\}, q > 1, \alpha_1 < 0, \alpha_2 < 0$. Initial function $u_0(x)$ is assumed to be continuous, nonnegative and bounded. It is well known that equation (1) is degenerate, and classical solutions may fail to exist in the cases $u(x,t) = 0$ or $|u_x| = 0$. Therefore, solutions of the Cauchy problem (1)–(2) are understood in the sense of generalized solutions defined via integral identities. In particular, it is required that

$$0 \leqslant u, \quad |x|^{\alpha_2}u^{m-1}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x} \in C\left(R\times(t,\,+\infty)\right)$$

and equation (1) is satisfied in the integral sense. Depending on the values of parameters, problem (1)–(2) arises in the mathematical modeling of heat transfer, nonlinear diffusion, chemical reactions, and biological processes. [**?**].

2. Research methodology

First, the following transformation in equation (1) is introduced

$$y = u^{1-\beta}. \tag{3}$$

Under this transformation the Cauchy problem (1)–(2) can be rewritten in the form

$$(1-\beta)L(y) = y^{\frac{\beta}{1-\beta}}\left[|x|^{\alpha_1}y_t - \frac{\partial}{\partial x}\left(|x|^{\alpha_2}y^{\frac{m}{1-\beta}-1}\left|\frac{\partial y^{\frac{1}{1-\beta}}}{\partial x}\right|^{p-2}\frac{\partial y}{\partial x}\right) + (1-\beta)|x|^{\alpha_1}y^{\frac{q-\beta}{1-\beta}}\right] = 0 \tag{4}$$

with the initial condition

$$y|_{t=0} = y_0(x) = [u_0(x)]^{1-\beta},\ x \in R. \tag{5}$$

It is known that if condition $\beta(1-\beta) > 0$ holds then the function is trivial solution of equation (4). In this study, we focus on the existence and qualitative properties of nontrivial solutions. For this purpose, the method of nonlinear separation of variables is employed. Solutions of problem (4)–(5) are sought in the self-similar form

$$y(x,t) = t^{-n_1}g(\eta),\ \ \eta = \frac{|x|}{t^{n_2}}, \tag{6}$$

where $n_1 = \dfrac{1-\beta}{q-1}$, $n_2 = \dfrac{p+q-(\beta+m+2)}{(q-1)(p-\alpha_2+\alpha_1)}$.
Substituting representation (6) into equation (4), the original nonlinear parabolic equation is reduced to the following ordinary differential equation

$$\left(\eta^{\alpha_2} g^{\frac{m+\beta-1}{1-\beta}} \left| \left(g^{\frac{1}{1-\beta}}\right)' \right|^{p-2} g' \right)' + n_2 \eta^{1+\alpha_1} g' + n_1 \eta^{\alpha_1} g + (1-\beta)\, \eta^{\alpha_1} g^{\frac{q-\beta}{1-\beta}} = 0. \tag{7}$$

For the resulting self-similar equation (7), nonnegative, nontrivial solutions are sought that are physically meaningful. These solutions are required to satisfy the following boundary conditions

$$g'(0) = g(d) = 0, \;\; d \in R_+. \tag{8}$$

3. Main results

Assume that condition

$$\frac{m+p-2}{1-\beta} - 1 > 0$$

is satisfied. This case is interpreted as the slow diffusion regime. Under this assumption, the global existence of generalized solutions of equation (1) is established by means of the comparison principle. Since problem (4)–(5) is equivalent to the original Cauchy problem (1)–(2) all subsequent analysis can be carried out for problem (4)–(5). As a consequence of the comparison principle, an upper comparison solution to problem (4)–(5) is constructed in the following self-similar form

$$y_+(x,t) = t^{-n_1} \overline{g}(\eta), \tag{9}$$

where

$$\overline{g}(\eta) = A\left(a - \eta^{\gamma_1}\right)_+^{\gamma_2}. \tag{10}$$

Here, $\gamma_1 = \dfrac{p-\alpha_2+\alpha_1}{p-1}$, $\gamma_2 = \dfrac{(p-1)(1-\beta)}{m-p+\beta+1}$, $a \geqslant 0$, and constant A is defined as

$$A = \left[\left| n_2 \left(\frac{1}{1-\beta} \right)^{2-p} \right|^{1/(p-1)} / (\gamma_1 \gamma_2) \right]^{\gamma_2}, \;\; \gamma_1 \neq 0, \, \gamma_2 \neq 0. \tag{11}$$

The obtained self-similar upper solution plays a key role in obtaining global existence results and qualitative estimates for generalized solutions in the slow diffusion regime.

Theorem 3.1. *Suppose that at least one of the following conditions holds*

$$\alpha_1 < -1, \;\; \mu_1 > \mu_2, \;\; yoki \alpha_1 = -1, \;\; \mu_1 > 1,$$

and assume that inequality

$$y(x,\, t) \leqslant y_+(x,\, t), \;\; x \in R,$$

is satisfied. Then Cauchy problem (4)–(5) admits a global generalized solution in domain Q and this solution satisfies the following estimate

$$y(x,\, t) \leqslant y_+(x,\, t) = t^{-n_1} \overline{g}(\eta). \tag{12}$$

Here,

$$\mu_1 = \frac{q-\beta}{1-\beta}, \;\; \mu_2 = \frac{m+p-2}{1-\beta} + \frac{p-\alpha_2+\alpha_1}{\alpha_1+1}$$

and μ_2 is referred to as the Fujita-type critical exponent.

Proof. Comparison function is defined by equation (10). Substituting this function into equation (4), one can obtain the following estimate

$$\begin{aligned}(1-\beta)Ly = -\, y_+^{\beta/(1-\beta)}\Big[\Big(\eta^{\alpha_2}\overline{g}^{\frac{m}{1-\beta}-1}\big|(\overline{g}^{1/(1-\beta)})'\big|^{p-2}\overline{g}'\Big)' + n_2\eta^{1+\alpha_1}\overline{g}' + \\ + n_1\eta^{\alpha_1}\overline{g} + (1-\beta)\eta^{\alpha_1}\overline{g}^{\mu_1}\Big] \leqslant 0.\end{aligned} \tag{13}$$

Taking into account that function $y_+(x,\,t)$ is nonnegative and that function $\overline{g}(\eta)$ satisfies equation

$$(\eta^{\alpha_2}\overline{g}^{m/(1-\beta)}\Big|(\overline{g}^{1/(1-\beta)})'\Big|^{p-2}\overline{g}')' + n_2\eta^{1+\alpha_1}\overline{g}' + n_2\left(\alpha_1+1\right)\eta^{\alpha_1}\overline{g} = 0$$

inequality (13) can be reduced to the following form

$$\eta^{\alpha_1}\overline{g}\cdot\left[n_1 - n_2\left(\alpha_1+1\right) + (1-\beta)\,\overline{g}^{\mu_1-1}\right] \geqslant 0. \tag{14}$$

Since functions η^{α_1} and $\overline{g}$ are nonnegative and condition $\beta < 1$ holds inequality (14) is satisfied provided that

$$n_1 \geqslant n_2\left(\alpha_1+1\right).$$

If $\alpha_1 + 1 = 0$ then

$$0 \leqslant n_1 = \frac{1}{\mu_1 - 1}$$

and the above inequality is automatically fulfilled. The derived inequalities are consistent with the assumptions of the theorem. Hence,

$$y(x,\,t) \leqslant y_+(x,\;t)$$

and the theorem is proved. □

4. Asymptotic behavior of solutions

In this section, we state a theorem describing the asymptotic properties of solutions obtained for problems (7)–(8) and provide its proof.

Theorem 4.1. *Assume that conditions $\gamma_2 > 0$ and $\mu_1 \geqslant 1 - 1/\gamma_2$ are satisfied. Then the solutions of problem (7)-(8) with compact support admit the following asymptotic representation as $\eta \to a_-^{\frac{1}{\gamma_1}}$*

$$g(\eta) = c_1 A\left(a - \eta^{\gamma_1}\right)_+^{\gamma_2}\left(1 + o(1)\right). \tag{15}$$

Here, c_1 is a constant determined as a solution of the corresponding nonlinear algebraic equation.

Proof. To study the asymptotic behavior of compactly supported solutions of problem (7)–(8), solution of self-similar equation (7) is sought in the following form

$$g(\eta) = \overline{g}(\eta)\, w(\xi),\;\; \xi = -\ln\left(a - \eta^{\gamma_1}\right). \tag{16}$$

Here $\overline{g}(\eta) = A\left(a - \eta^{\gamma_1}\right)_+^{\gamma_2}$, $a = const > 0$, and $w(\xi)$ is an auxiliary function to be determined. It is clear that as $\eta \to a_-^{\frac{1}{\gamma_1}}$ then $\xi \to +\infty$. Substituting (16) into equation (7), one can obtain the following differential equation for function $w(\xi)$

$$\begin{aligned}&\frac{d}{d\xi}\left(K_- w\right) + \left(\frac{\alpha_1+1}{\gamma_1}a_0(\xi) - \gamma_2\right)K_- w + a_1(\xi)\left(\frac{dw}{d\xi} - \gamma_2 w\right) + \\ &+ a_2(\xi)\, w + a_3(\xi)\, w^{\mu_1} = 0\end{aligned} \tag{17}$$

Here, coefficient functions are defined as

$$a_0(\xi)=\frac{e^{-\xi}}{a-e^{-\xi}},\ \ a_1(\xi)=n_2\gamma_1 b,\ \ \ b=\gamma_1^{-p}\left(\frac{1}{1-\beta}\right)^{2-p}A^{\frac{1-(m+p+\beta)}{1-\beta}},\ \ \ a_2(\xi)=n_1 b a_0(\xi),$$

$$a_3(\xi)=(1-\beta)\,bA^{q-1}\frac{e^{(\gamma_2(1-\mu_1)-1)\xi}}{a-e^{-\xi}},\ \ \ K_{\pm}w=w^{\frac{m+\beta(p-1)-1}{1-\beta}}\left|\frac{dw}{d\xi}\pm\gamma_2 w\right|^{p-2}\left(\frac{dw}{d\xi}\pm\gamma_2 w\right).$$

This transformation reduces the asymptotic analysis of the original nonlinear equation to the study of the qualitative behavior of solutions of equation (17) as $\xi\to+\infty$. This completes the proof setup.

Assume that $\eta\in[\eta_0,\eta_1)$, where $0<\eta_0<\eta_1$, and $\eta_1=a^{\frac{1}{\gamma_1}}$. Then function $\xi(\eta)$ has the following properties

$$\eta\in[\eta_0,\eta),\quad \xi'(\eta)>0,\ \xi_0=\xi(\eta_0),\ \lim_{\eta\to\eta_1}\xi(\eta)=+\infty$$

Let us introduce the following notation

$$v(\xi)=K_-(w)=w^{\frac{m+\beta(p-1)-1}{1-\beta}}\left|\frac{dw}{d\xi}-\gamma_2 w\right|^{p-2}\left(\frac{dw}{d\eta}-\gamma_2 w\right). \tag{18}$$

Then equation (17) can be written in the following form

$$v'(\xi)=-\left(\frac{\alpha_1+1}{\gamma_1}a_0(\xi)-\gamma_2\right)v(\xi)-a_1(\xi)\left(\frac{dw}{d\xi}-\gamma_2 w\right)-a_2(\xi)\,w-a_3(\xi)\,w^{\mu_1}. \tag{19}$$

Now, consider the following function

$$z(\xi)=-\left(\frac{(\alpha_1+1)\,a_0(\xi)}{\gamma_1}-\gamma_2\right)\lambda-a_1(\xi)\left(\frac{dw}{d\xi}-\gamma_2 w\right)-a_2(\xi)\,w-a_3(\xi)\,w^{\mu_1}. \tag{20}$$

where $\lambda\in R$.

Let us assume that inequality $\gamma_2>0$ is satisfied, and assume that $\xi\to\infty$. Then

$$\lim_{\xi\to\infty}a_0(\xi)=\lim_{\xi\to\infty}a_2(\xi),\ \lim_{\xi\to\infty}a_1(\xi)=n_2\gamma_1 b,$$

$$\overset{0}{a_3}=\lim_{\xi\to\infty}a_3(\xi)=\begin{cases}\infty, & \gamma_2(1-\mu_1)>1\\ 0, & \gamma_2(1-\mu_1)<1\\ (\beta-1)\,bA^{q-1}a^{(\alpha_1-\alpha_2)/\gamma_1-1}, & \gamma_2(1-\mu_1)=1\end{cases}$$

Taking these limits into account, it follows that for any fixed value of parameter λ function $z(\xi)$ defined by (20) has finite value as $\xi\to+\infty$. Consequently, there exists an interval $[\xi,+\infty)\subset[\xi_0,+\infty)$ such that function $z(\xi)$ preserves its sign in this interval:

$$z(\xi)>0\quad\text{or}\quad z(\xi)<0. \tag{21}$$

Assume that function $z(\xi)$ has a finite nonzero limit as $\xi\to+\infty$. If inequalities (21) hold for sufficiently large values of ξ then the following situations may arise. Since function $z(\xi)$ oscillates around the straight line $\overline{z}=\lambda$ then the function must intersect this straight line infinitely many times in the interval $[\xi,\infty)$. However, this is impossible. Therefore, inequality (21) must be satisfied as $\xi\to+\infty$. As a result, according to equation (20), function $z(\xi)$ can intersect the straight line $\overline{z}=\lambda$ in the interval $[\xi_i,\infty)$, $(i=1,2)$ at only one point. Thus, it is necessary that $z(\xi)\to0$ as $\xi\to+\infty$. Now, consider function $z(\xi)$ defined by (20). If the limit $z(\xi)\to0$ exists

then the limit of its derivative $w'(\xi)$ also exists, and this limit is equal to zero.
Thus, as $\xi \to +\infty$ we have

$$v(\xi) = w^{\frac{m+\beta(p-1)-1}{1-\beta}} \left| \frac{dw}{d\xi} - \gamma_2 w \right|^{p-2} \left(\frac{dw}{d\eta} - \gamma_2 w \right) = -\gamma_2^{p-1} w_0^{\frac{m+p-2}{1-\beta}} + o(1),$$

$$w_0 = \lim_{\xi \to \infty} w(\xi). \tag{22}$$

Taking the limit in (19), it follows that derivative of function $v(\xi)$ has a limit as $\xi \to +\infty$, and this limit is equal to zero. From equations (17)–(22), one can obtain the following result

$$\gamma_2^p w_0^{\frac{m+p-2}{1-\beta}-1} - n_2 b \gamma_1 \gamma_2 + \overset{0}{a}_3 w_0^{\mu_1 - 1} = 0. \tag{23}$$

The theorem is proved. □

Result. If inequality $\dfrac{m+p-2}{1-\beta} - 1 > 0$ holds then the generalized solution of problem (1)–(2) admits the following asymptotic behavior as $|x| \to a^{\frac{1}{\gamma_1}} t^{n_2}$

$$u_A(x,t) = (c_1 A)^{1-\beta} t^{-n_1(1-\beta)} \left(a - \left(|x| t^{-n_2} \right)^{\frac{p-\alpha_2+\alpha_1}{p-1}} \right)_+^{\frac{(p-1)(1-\beta)}{\frac{m+p-2}{1-\beta}-1}} (1 + o(1)).$$

Here c_1 is the solution of algebraic equation (23).
If

$$\frac{m+p-2}{1-\beta} - 1 < 0$$

then this case is referred to as the fast diffusion regime. Assume that equation (7) satisfies the following condition

$$g'(0) = g(\infty) = 0. \tag{24}$$

For the fast diffusion regime, consider function

$$y_-(x,t) = t^{-n_1} h(\eta), \quad h(\eta) = D(d + \eta^{\gamma_1})^{\gamma_2}$$

where D, $d > 0$ and n_1, γ_1 γ_2 are constants defined above.

Theorem 4.2. *Let*

$$\alpha_1 > -1, \quad \mu_1 > \mu_2, \quad \text{and} \quad \frac{D^{\frac{m+p-2}{1-\beta}-1} |\gamma_1 \gamma_2|^p}{(1-\beta)^{p-2}} = n_2, \quad \eta \geqslant \max\left\{ \left(A_1^{1/|\gamma_2|} - d \right)^{1/\gamma_1}, 0 \right\}$$

Assume also that initial condition

$$y(x,0) \geqslant y_-(x,0), \quad x \in R,$$

is satisfied. Then problem (4)–(5) admits a global solution in domain Q. Moreover, the following estimate holds for this solution

$$y(x,t) \geqslant y_-(x,t) = t^{-n_1} h(\eta). \tag{25}$$

Here, $A_1 = \dfrac{(1-\beta)(\mu_1 - 1)(p - \alpha_2 + \alpha_1) D^{\mu_1 - 1}}{(\alpha_1 + 1)(\mu_1 - \mu_2)}$.

The proof of this theorem is carried out by the method used in the proof of Theorem 1.

Theorem 4.3. *If $\gamma_2 < 0$ then the decreasing solutions of problems (7),(24) admit the following asymptotic representation*

$$g(\eta) = c_2 D(d + \eta^{\gamma_1})^{\gamma_2} (1 + o(1)) \tag{26}$$

as $\eta \to \infty$. Here c_2 is a constant determined by the solution of the corresponding nonlinear algebraic equation.

Proof. In order to analyze the asymptotic behavior of solutions of problems (7), (24), consider decreasing solution of equation (7) as $\eta \to \infty$ in the following form

$$g(\eta) = h(\eta) \cdot z(\xi), \quad \xi = \ln(a + \eta^{\gamma_1}), \tag{27}$$

Here $h(\eta) = D(d + \eta^{\gamma_1})^{\gamma_2}$, $d > 0$, and $z(\xi)$ is function to be determined.
Substituting representation (27) into equation (7), one can obtain the following equation for function $z(\eta)$

$$\frac{d}{d\xi}(K_+ z) + \left(\frac{(\alpha_1 + 1)}{\gamma_1} d_0(\xi) + \gamma_2\right) K_+ z + d_1(\xi)\left(\frac{dz}{d\xi} + \gamma_2 z\right) + d_2(\xi) z + d_3(\xi) z^{\mu_1} = 0. \tag{28}$$

Here, $d_0(\xi) = \dfrac{1}{1 - de^{-\xi}}$, $d_1(\xi) = n_2\gamma_1 d_4$, $d_4 = \gamma_1^{-p}\Big(\dfrac{1}{1-\beta}\Big)^{2-p} D^{1-\frac{m-(p-2)}{1-\beta}}$, $d_2(\xi) = n_1 d_4 d_0(\xi)$,
$d_3(\xi) = (1-\beta) d_4 D^{q-1} \dfrac{e^{\gamma_2(\mu_1-1)\xi}}{1 - de^{-\xi}}$, $K_+ z = z^{\frac{m+\beta(p-1)-1}{1-\beta}} \left|\dfrac{dz}{d\xi} + \gamma_2 z\right|^{p-2}\Big(\dfrac{dz}{d\xi} + \gamma_2 z\Big)$. Based on these considerations, Theorem 4 is proved in the same manner as Theorem 2. Then, the following equation is obtained

$$(\gamma_2 + (\alpha_1 + 1)/\gamma_1)\gamma_2^{p-1} z_0^{\frac{m+p-2}{1-\beta}-1} + \overset{0}{d}_3 z_0^{\mu_1 - 1} + (n_1 + n_2\gamma_1\gamma_2) d_4 = 0. \tag{29}$$

Here,

$$\overset{0}{d}_3 = \lim_{\xi\to\infty} d_3(\xi) = \begin{cases} \infty, & \gamma_2(\mu_1 - 1) > 0, \\ 0, & \gamma_2(\mu_1 - 1) < 0, \\ (1-\beta) d_4 D^{q-1}, & \gamma_2(\mu_1 - 1) = 0. \end{cases}$$

□

Result. If inequality $\dfrac{m+p-2}{1-\beta} - 1 < 0$ holds then the generalized solution of problem (1)–(2) admits the following asymptotic behavior

$$u_A(x, t) \approx (c_2 D)^{1-\beta} t^{-n_1(1-\beta)} \left(d + \left(|x| t^{-n_2}\right)^{\frac{p-\alpha_2+\alpha_1}{p-1}}\right)^{(p-1)(1-\beta)/(\frac{m+p-2}{1-\beta}-1)} (1 + o(1))$$

as $\eta = |x| t^{-n_2} \to \infty$. Here, c_2- is the solution of algebraic equation (29).

5. Critical case

The case $\dfrac{m+p-2}{1-\beta} - 1 = 0$ is called the critical case. Let us establish the global existence of generalized solutions of equation (4) in this situation. Using the comparison principle, solution $y_k(x, t)$ is sought in the form

$$y_k(x, t) = t^{-n_1} K(\eta), \tag{30}$$

$$K(\eta) = C_0 \exp(-A_1 \eta^{\gamma_1}),\ C_0 > 0,\ A_1 = \left|\frac{n_2}{(1-\beta)^{2-p}}\right|^{\frac{1}{p-1}} / \gamma_1 \tag{31}$$

Theorem 5.1. *Let* $\dfrac{m+p-2}{1-\beta} - 1 = 0$. *Assume that one of the following conditions holds:* $\alpha_1 < -N,\ \mu_1 > \mu_2$ *or* $\alpha_1 = -N,\ \mu_1 > 1$ *and*

$$y(x, 0) \leqslant y_k(x, 0), \quad x \in R^N.$$

Then problem (19), (20) admits a global solution in domain Q *, and the following estimate holds*

$$y(x, t) \leqslant y_k(x, t) = t^{-n_1} K(\eta), \quad x \in R^N,$$

The proof of the theorem is similar to the proof of Theorem 1.

Conclusion

The Cauchy problem for heat conduction equation in nondivergent parabolic form with double nonlinear source terms is studied in this paper. In the absence of classical solutions, the notion of generalized solution is introduced, and the problem is studied in the sense of integral identities. By applying appropriate transformations of variable and the method of self-similar solutions, the problem is reduced to a simpler form. This allows the detailed analysis of qualitative properties of the equation. In the slow diffusion regime, the global existence of solutions is established using the comparison principle. Moreover, the Fujita-type critical exponent is identified which determines the threshold conditions that separate global-in-time existence from blow-up behavior. The obtained results contribute to the theory of nonlinear parabolic equations of nondivergent type and provide a theoretical basis for applications in heat transfer, chemical processes, and biological modeling.

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Качественные свойства решений параболического уравнения теплопроводности в недивергентной форме с двумя нелинейными источниками

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Аннотация. В данной работе исследуются свойства обобщенных решений параболического уравнения теплопроводности в недивергентной форме с двойными нелинейными источниками. Модель включает нелинейную диффузию с переменной плотностью и нелинейными функциями источников, а для случаев, когда классические решения не существуют, используется концепция обобщенных решений в смысле интегральных тождеств. С помощью соответствующей замены переменных и метода автомодельных решений исходное нелинейное параболическое уравнение сводится к обыкновенному дифференциальному уравнению. В режиме медленной диффузии глобальное существование решений устанавливается с использованием принципа сравнения, и определяется критический показатель типа Фудзиты. Полученные результаты имеют важное теоретическое значение для численного анализа нелинейных моделей, возникающих в теплопередаче, химических процессах и биологических системах.

Ключевые слова: глобальное решение, недивергентное уравнение, диффузия, асимптотическое решение.

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Application of Intertwining Operators to Integrating Differential Equations

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Abstract. We develop an operator approach to the integration of linear differential equations based on intertwining relations between differential operators. Conditions for the existence of intertwining operators are obtained, and it is shown that, in low-order cases, the problem reduces to Riccati-type equations. The method is applied to linear partial differential equations, which makes it possible to construct their solutions. The linear Klein–Gordon equation is presented as an illustrative example.

Keywords: intertwining operators, factorization, partial differential equations, general solutions.

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Introduction

Linear differential equations with variable coefficients arise in many problems of mathematical physics, including wave theory, quantum mechanics, spectral theory, and vibration theory. Of particular interest are constructive methods that establish connections between different differential operators, thereby allowing one to derive new equations and their solutions from already known ones. Such methods include group analysis [1], factorization of differential operators [2,3], Darboux transformations [4], as well as various operator and algebraic approaches that are widely used, in particular, in supersymmetric quantum mechanics and in the theory of exactly solvable models [5, 6].

In the present paper, we employ an operator approach in which a linear differential equation is identified with the action of a differential operator. The central object of the approach is the intertwining relation

$$MT = TL,$$

which connects two operators L and M by means of a differential operator T. This relation defines a map from solutions of the equation $Lu = 0$ to solutions of the equation $Mv = 0$, and thus induces a natural equivalence relation on the set of linear differential operators. We show that the problem of constructing first-order intertwining operators leads to Riccati-type equations, which can be reduced to linear differential equations by a standard substitution.

The operator approach also proves to be effective for linear partial differential equations. The use of commuting operators makes it possible to transfer results obtained for ordinary differential operators to the Klein–Gordon equation

$$u_{tt} = u_{xx} + V(x)u,$$

and to construct equations with new potentials together with their solutions. Such constructions arise naturally in problems of mathematical physics related to wave processes [6–8]. It should be noted that Euler was the first to propose such an approach and to apply it to problems

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in acoustics [9]. At present, this approach is commonly referred to as the Darboux method. Mention should also be made of the largely forgotten article by Moutard on the transformation and integration of linear ordinary differential equations [10].

1. Intertwining operators

Let $\mathcal{F}$ be a commutative ring equipped with a derivation $\partial : \mathcal{F} \to \mathcal{F}$. We denote by $\mathcal{F}[\partial]$ the ring of differential operators of the form

$$L = \sum_{i=0}^{n} a_i \partial^i, \qquad a_i \in \mathcal{F}.$$

If $a_n \neq 0$, then the integer n is called the order of the operator L. Multiplication in the ring of operators is defined by the rule

$$\partial f = f' + f\partial, \qquad f' = \partial(f).$$

The ring $\mathcal{F}[\partial]$ is neither commutative nor a unique factorization domain.

We recall the notion of an intertwining operator. An operator $T \in \mathcal{F}[\partial]$ is said to intertwine an operator L with an operator M if

$$MT = TL. \tag{1.1}$$

The operator T is called an intertwining operator, and equality (1.1) is called the intertwining condition.

From now on, let $\mathcal{F}$ be the ring of germs of infinitely differentiable functions at a point $x_0 \in \mathbb{R}$.

Lemma 1. *Let L be an operator of the form*

$$L = a_n\partial^n + \cdots + a_1\partial + a_0.$$

Then there exist an operator $T = \partial + s$, where $s \in \mathcal{F}$, and an operator M of order n such that the intertwining condition (1.1) *is satisfied.*

Proof. Let us represent the operator M in the form

$$M = b_n\partial^n + b_{n-1}\partial^{n-1} + \cdots + b_0, \qquad b_i \in \mathcal{F},$$

where $b_n(x_0) \neq 0$. We expand the left-hand and right-hand sides of the intertwining condition.

The left-hand side of (1.1) can be written as

$$(b_n\partial^n + \cdots + b_0)(\partial + s) = b_n\partial^{n+1} + c_n\partial^n + \cdots + c_0,$$

where

$$c_m = b_{m-1} + \sum_{k=m}^{n} b_k \binom{k}{m} s^{(k-m)}, \qquad m > 0.$$

The coefficient c_0 is given by

$$c_0 = b_n s^{(n)} + b_{n-1}s^{(n-1)} + \cdots + b_0 s.$$

The right-hand side of (1.1) has the form

$$a_n\partial^{n+1} + (a_n' + sa_n)\partial^n + \cdots + (a_1' + sa_1)\partial + a_0' + sa_0.$$

By comparing the coefficients, we obtain

$$b_n = a_n,$$

and the remaining coefficients $b_{n-1}, \dots, b_0$ are determined uniquely.

For s, we obtain an ordinary differential equation

$$a_n s^{(n)} + b_{n-1} s^{(n-1)} + \cdots + b_0 s = a_0' + s a_0.$$

Given n initial conditions, the solution of this equation depends smoothly on these data.

Let us consider two examples. Suppose that L and M are second-order operators:

$$L = a_2\partial^2 + a_1\partial + a_0, \qquad M = b_2\partial^2 + b_1\partial + b_0.$$

From the intertwining condition (1.1), as described in Lemma 1, we find the coefficients

$$b_2 = a_2, \qquad b_1 = a_2' + a_1, \qquad b_0 = a_0 + a_1' - s a_2' - 2a_2 s'.$$

The differential equation for s takes the form

$$a_2 s'' - 2s s' a_2' - a_2' s^2 + s' a_2' + s' a_1 + s a_1' - a_0' = 0.$$

Integrating this equation, we obtain a Riccati equation

$$a_2(s' - s^2) + a_1 s - a_0 = \lambda, \qquad \lambda \in \mathbb{R}.$$

The substitution $s = -(\ln h)'$ reduces it to the linear equation

$$a_2 h'' + a_1 h' + (a_0 + \lambda) h = 0. \tag{1.2}$$

Now consider a pair of third-order operators:

$$L = a_3\partial^3 + a_2\partial^2 + a_1\partial + a_0, \qquad M = b_3\partial^3 + b_2\partial^2 + b_1\partial + b_0.$$

The intertwining condition yields the coefficients

$$b_3 = a_3, \qquad b_2 = a_2 + a_3',$$
$$b_1 = a_1 + a_2' - s a_3' - 3a_3 s',$$
$$b_0 = a_0 + a_1' + 3s' s a_3 + a_3' s^2 - 2s' a_3' - 2s a_2' - 3a_3 s'' - a_2' s,$$

and leads to a third-order differential equation for s. This equation admits a first integral and can be reduced to a second-order equation

$$a_3(s'' + s^3) - 3a_3 s' s'' - a_2 s^2 + s' a_2 + s a_1 - a_0 = \lambda, \qquad \lambda \in \mathbb{R}.$$

The substitution $s = -(\ln h)'$ reduces it to the linear equation

$$a_3 h''' + a_2 h'' + a_1 h' + (a_0 + \lambda) h = 0.$$

Thus, the equation for s can be regarded as a second-order Riccati-type equation. □

Lemma 2. *Let there be given a linear differential equation of order n*

$$L(u) = a_n u^{(n)} + \cdots + a_1 u' + a_0 u = 0, \qquad a_i \in \mathcal{F}, \tag{1.3}$$

and let $h \in \mathcal{F}$ satisfy the equation

$$Lh = \lambda h, \qquad \lambda \in \mathbb{R}, \tag{1.4}$$

with $h(x_0) \neq 0$. Then the transformation

$$w = u' - \frac{h'}{h} u$$

maps solutions of equation (1.3) *to solutions of some linear differential equation of order n for the function w.*

Proof. By assumption, $h \in \mathcal{F}$ satisfies equation (1.4). Let $u = hv$, where $v \in \mathcal{F}$. Then v satisfies the equation

$$a_n h v^{(n)} + (n a_n h' + a_{n-1} h)\, v^{(n-1)} + \cdots + \Big(a_n h^{(n)} + \cdots + a_1 h' + a_0 h\Big)\, v = 0.$$

Using (1.4), this equation can be rewritten in the form

$$a_n v^{(n)} + \cdots + \lambda v = 0.$$

Differentiating this equation and setting $q = v'$, we obtain an nth-order equation

$$a_n q^{(n)} + \tilde{a}_{n-1} q^{(n-1)} + \cdots + \tilde{a}_0 q = 0, \qquad \tilde{a}_i \in \mathcal{F}.$$

Evidently, this equation is satisfied by the function

$$q = \Big(\frac{u}{h}\Big)'.$$

Thus, the function $w = qh = u' - (\ln h)' u$ also satisfies a linear differential equation of order n.

Let us briefly consider the factorization of a second-order operator

$$L = F\partial^2 + G\partial + H, \qquad F, G, H \in \mathcal{F}.$$

We seek to represent L in the form

$$L = L_2 L_1, \tag{1.5}$$

where $L_1 = \partial + s$, $L_2 = F\partial + p$, and $s, p \in \mathcal{F}$. By comparing the coefficients, we obtain two relations:

$$p = G - sF, \qquad H = Fs' + sp.$$

Thus, L factorizes if

$$L_2 = F\partial + (G - sF) \tag{1.6}$$

and s satisfies the Riccati equation

$$F(s' - s^2) + sG - H = 0.$$

The substitution $s = -(\ln h)'$ reduces this equation to the linear equation

$$Fh'' + Gh' + Hh = 0. \tag{1.7}$$

We observe that equation (1.7) is obtained from (1.2) for $\lambda = 0$. Factorization is used to construct operator intertwinings and transformations of differential equations [6].

As shown above, the operator $T = \partial + s$ intertwines the operators

$$L = F\partial^2 + G\partial + H, \qquad M = F\partial^2 + G_1\partial + H_1,$$

if $G_1 = G + F'$, $H_1 = H + G' - sF' - 2s'F$, and s satisfies the equation

$$F(s' - s^2) + Gs - H = \lambda, \qquad \lambda \in \mathbb{R}. \tag{1.8}$$

A direct calculation shows that the operator

$$T^c = F\partial + (G - sF)$$

intertwines M with L. Formally, T^c coincides with (1.6). However, in the present case s satisfies (1.8), whereas in the factorization case s satisfies this equation only for $\lambda = 0$. □

We say that two operators L and M are equivalent if they are intertwined. It is easy to see that intertwining defines an equivalence relation on the set of operators. This leads to a broader notion of equivalence for differential equations than the one usually adopted [11].

2. Linear partial differential equations

Let us now consider the ring F_2 of germs of smooth functions of two variables at a point $(x_0, y_0) \in \mathbb{R}^2$. Let F_x and F_y denote the rings of germs of smooth functions at the points x_0 and y_0, respectively. Two derivations, ∂_x and ∂_y, act on F_2. The corresponding ring of differential operators is denoted by $F_2[\partial_x, \partial_y]$. The intertwining of operators $L, M \in F_2[\partial_x, \partial_y]$ is defined by formula (1.1), where $T \in F_2[\partial_x, \partial_y]$.

If $L_1 \in F_x[\partial_x]$ and $L_2 \in F_y[\partial_y]$, then these operators commute. Suppose that L_1 is intertwined with $M \in F_x[\partial_x]$ by means of an operator $T \in F_x[\partial_x]$, that is,

$$MT = TL_1.$$

Since $L_2T = TL_2$, it follows that

$$(M + L_2)T = T(L_1 + L_2).$$

Consequently, if $u \in F_2$ satisfies the equation

$$(L_1 + L_2)u = 0,$$

then $v = Tu$ is a solution of the equation $(M + L_2)v = 0$.

Consider the Klein–Gordon equation

$$u_{tt} = u_{xx} + V(x)u, \qquad V(x) \in F_x. \tag{2.9}$$

It corresponds to the operator $\partial_t^2 - \partial_x^2 - V(x)$. As shown above, the operator $L_1 = \partial_x^2 + V(x)$ is intertwined with the operator $M = \partial_x^2 + V(x) + 2(\ln h)''$ by means of $T = \partial_x - (\ln h)'$, provided that h satisfies the equation

$$h'' + (V(x) + \lambda)h = 0. \tag{2.10}$$

Thus, the operator $\partial_t^2 - \partial_x^2 - V(x)$ is intertwined with $\partial_t^2 - \partial_x^2 - (V(x) + 2(\ln h)'')$. This allows us to construct solutions of the equation

$$v_{tt} = v_{xx} + (V(x) + 2(\ln h)'')v, \tag{2.11}$$

provided that solutions of equations (2.9) and (2.10) are known.

Let us present several examples. Consider the wave equation

$$u_{tt} = u_{xx},$$

which has the general solution $u_0 = X(t+x) + Y(x-t)$, where X and Y are arbitrary functions. Since $V(x) = 0$ in this case, equation (2.10) takes the form

$$h'' + \lambda h = 0. \tag{2.12}$$

For $\lambda = 0$, equation (2.12) has the solution

$$h = c_0 + c_1 x, \qquad c_0, c_1 \in \mathbb{R}.$$

In this case, equation (2.11) becomes

$$v_{tt} = v_{xx} - \frac{2}{(x+b)^2}v, \qquad b = \frac{c_0}{c_1}. \tag{2.13}$$

Accordingly, a solution is given by

$$v = \partial_x(u_0) - \frac{u_0}{x+b}.$$

If $\lambda = -k^2$, then a solution of equation (2.12) is

$$h = c_0 \sinh(kx) + c_1 \cosh(kx), \qquad c_0, c_1 \in \mathbb{R}.$$

The corresponding equation (2.11) takes the form

$$v_{tt} = v_{xx} + \frac{2k^2(c_1^2 - c_0^2)}{[c_0 \sinh(kx) + c_1 \cosh(kx)]^2}\, v,$$

and a solution is given by

$$v = \partial_x(u_0) - k\frac{c_0 \cosh(kx) + c_1 \sinh(kx)}{c_0 \sinh(kx) + c_1 \cosh(kx)}\, u_0.$$

For $\lambda = k^2$, the corresponding equation for v and its solution are obtained from the above formulas by replacing hyperbolic functions with trigonometric ones.

Let us now consider equation (2.13) together with the equation for h:

$$h'' + \left(\lambda - \frac{2}{(x+b)^2}\right) h = 0. \tag{2.14}$$

If $\lambda = 0$, then a solution of equation (2.14) is

$$h = c_1(x+b)^2 + \frac{c_2}{x+b}, \qquad c_1, c_2 \in \mathbb{R}.$$

For $\lambda < 0$, we obtain the solution

$$h = c_1 e^{mx}\left(m^2 - \frac{m}{x+b}\right) + c_2 e^{-mx}\left(m^2 + \frac{m}{x+b}\right),$$

where $m^2 = -\lambda$. This allows us to construct solutions of the equation

$$w_{tt} = w_{xx} - 2\left(\frac{1}{(x+b)^2} - (\ln h)''\right) w.$$

The equation

$$v_{tt} = v_{xx} - \frac{2}{\sinh^2 x} v$$

and the corresponding equation for h,

$$h'' - \left(\frac{2}{\sinh^2 x} + A^2\right) h = 0, \qquad A \in \mathbb{R},$$

can be treated in a similar manner. For $A = 2$, a solution of the latter equation is

$$c_1(\cosh(2x) - 1) + c_2 \frac{\sinh(2x)(\cosh(2x) - 2)}{\cosh(2x) - 1}, \qquad c_1, c_2 \in \mathbb{R}.$$

For an arbitrary $A > 0$, the explicit solution formula becomes rather cumbersome.

Remark. This procedure can be continued as long as equation (2.10) can be solved. Using Crum's formula, one can construct solutions of equations for h from known solutions of equation (2.12) corresponding to different values $\lambda_1, \ldots, \lambda_n$.

Separation of variables combined with the operator approach can also be used to construct solutions of equation (2.9). As an example, consider the equation

$$u_{tt} = u_{xx} + \left(-\frac{x^2}{4} + k\right) u, \qquad k \in \mathbb{R}. \tag{2.15}$$

We seek a solution in the form $u = T(t)X(x)$. The functions T and X satisfy the equations

$$T'' + \lambda T = 0, \qquad X'' + \left(k - \lambda - \frac{x^2}{4}\right) X = 0.$$

The equation for X is usually referred to as the Weber equation, and its solutions are well known [13]. In particular, if $k - \lambda = n + \frac{1}{2}$, where n is a nonnegative integer, one of its solutions has the form

$$X = (-1)^n e^{\frac{x^2}{4}} \frac{d^n}{dx^n} \left(e^{-\frac{x^2}{2}}\right).$$

Subsequently, one can apply the operator approach to equation (2.15) using this known solution of the Weber equation or directly employ Crum's determinant formula.

The equation

$$u_{tt} = u_{xx} - \big(k + n(n+1)\wp(x)\big)u$$

can be investigated in a similar manner, where $\wp$ is the Weierstrass elliptic function, n is a positive integer, and $k \in \mathbb{R}$. After separation of variables, we arrive at the Lamé equation, whose solutions can be found in the classical literature [13].

Conclusion

In this paper, we developed an operator approach to linear differential equations based on the intertwining relation $MT = TL$. We showed that the construction of a first-order intertwining operator of the form $T = \partial + s$ is essentially reduced to the solution of a Riccati-type equation for the coefficient s. This observation provides a unifying and structurally transparent framework for Darboux-type transformations.

From an algebraic perspective, the obtained results clarify the structure of factorizations and intertwinings in the ring $\mathcal{F}[\partial]$. From a constructive viewpoint, they provide an effective tool for generating new exactly solvable operators and potentials arising in mathematical physics. Possible directions for further research include the extension of the present approach to higher-order intertwining operators and the investigation of its applications to integrable systems and nonlinear partial differential equations. This approach carries over to the case of many variables.

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Применение сплетающих операторов к интегрированию дифференциальных уравнений

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Аннотация. Разрабатывается операторный подход к интегрированию линейных дифференциальных уравнений, основанный на сплетающих соотношениях между дифференциальными операторами. Получены условия существования операторов сплетения, и показано, что эта задача сводится к уравнениям типа Риккати. Метод применяется к линейным дифференциальным уравнениям в частных производных, что позволяет построить их решения. В качестве иллюстративного примера представлено линейное уравнение Клейна–Гордона.

Ключевые слова: сплетающие операторы, факторизация, дифференциальные уравнения в частных производных, общие решения.

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Problems of Identification of the Unknown Coefficient in the Elliptic Equation

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Abstract. In this paper we study the solvability of inverse problems for an elliptic equation of the form

$$u_{tt} + u_{xx} + \mu(t)u = f(x,t) + q(t)h(x,t),$$

where the coefficient $q(t)$ is unknown. The existence and uniqueness theorems of the regular solutions are proved. The investigation methods are based on the transition from the original inverse problem to a nonlocal problem with conditions of the type of Samarski–Ionkin conditions.

Keywords: elliptic equation, unknown coefficient, boundary condition, existence, uniqueness of solution, Samarski–Ionkin conditions.

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Introduction

In this paper, the authors call the identification problem the problem of determining, along with the solution of the corresponding differential equation, some unknown coefficient that determines the right-hand side (free term) of the equation. Such problems are also called linear inverse coefficient problems in the literature.

In this paper we will study inverse problems in the following formulation: we need to find a solution $u(x,t)$ to the elliptic equation and an unknown multiplier $q(t)$ defining the free term $F(x,t)$ of the form $F(x,t) = f(x,t) + q(t)h(x,t)$ with known functions $f(x,t)$ and $h(x,t)$. Usually in such problems additional conditions (called overdetermined conditions) have to be given together with the boundary conditions which are natural to the class of elliptic equations. In this paper we will use boundary overdetermination conditions, i.e. conditions that set additional boundary data compared to the natural ones.

Free-term identification problems, or linear inverse problems, for elliptic equations have of course been studied before — in [1–4]. But they were studied in different formulations and in different function spaces compared to the present work. Let us note an essential difference of the present paper from the works of its predecessors — all constructions and reasoning in it will be based on the transition from the studied inverse problem to a nonlocal problem of the type of

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the Samarski-Ionkin problem, and on the use of recent results on the solvability of this nonlocal problem (see [5,6]). This was not previously observed in the work of the predecessors.

The essence of the approach used in this paper and the results obtained are presented for model equations with the Laplace operator in the upper part. For more general equations, for problems with different boundary conditions than those used below, the results obtained in this paper are easily transferable - this is discussed at the end of the paper.

Let us specify that the aim of the paper is to prove the existence and uniqueness of regular solutions of both the direct nonlocal boundary value problems arising in the course of the proof and the original inverse problems (regular solutions are solutions in which all S. L. Sobolev generalised derivatives are included in the corresponding equation).

The necessary information on the properties of functions with generalised derivatives can be found in the monographs [7–9].

1. Statement of the problems

Let Q be a cylinder $(0,1) \times (0,T)$ of finite height T of the plane of variables x, t, $f(x,t)$, $h(x,t)$ are functions defined in $\overline{Q}$, the function $\mu(t)$ is on the segment $[0,T]$. We consider two inverse problems.

Inverse Problem I: *Find functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation*

$$u_{tt} + u_{xx} + \mu(t)u = f(x,t) + q(t)h(x,t), \tag{1}$$

provided that $u(x,t)$ satisfies the conditions

$$u(x,0) = u(x,T) = 0, \quad x \in (0,1), \tag{2}$$

$$u(0,t) = u(1,t) = u_x(1,t) = 0, \quad t \in (0,T). \tag{3}$$

Inverse Problem II: *Find functions $u(x,t)$ and $q(t)$ connected in the cylinder Q by the equation (1), provided that $u(x,t)$ satisfies (2) and also the condition*

$$u_x(0,t) + \alpha u(0,t) = u(1,t) = u_x(1,t) = 0, \quad t \in (0,T), \quad (\alpha = const). \tag{4}$$

In the inverse problems I and II, one of the conditions at $x = 1$ can be treated as a condition of the direct (ordinary) problem for the elliptic equation (1) with a known free term, the remaining condition is the boundary overdetermination condition.

2. Auxiliary problems and their solvability

Let us perform some formal constructions. We assume that

$$h(x,t) \in C(\overline{Q}), \quad h_{xx}(x,t) \in C(\overline{Q}), \quad f(x,t) \in L_2(Q), \quad f_{xx}(x,t) \in L_2(Q); \tag{5}$$

and

$$h(1,t) \neq 0 \qquad \text{at} \quad t \in [0,T]. \tag{6}$$

Setting $x = 1$ in equation (1) and using the condition $u(1,t) = 0$ and (6), we find the function $q(t)$:

$$q(t) = \frac{u_{xx}(1,t) - f(1,t)}{h(1,t)}. \tag{7}$$

Let's introduce the notations

$$h_1(x,t) = \frac{h(x,t)}{h(1,t)}, \qquad F_1(x,t) = f(x,t) - f(1,t)h_1(x,t).$$

Let us substitute into the equation (1) the function $q(t)$ calculated by the formula (7). Then we differentiate the equation twice with respect to the variable x. Assuming $v(x,t) = u_{xx}(x,t)$, we obtain that the function $v(x,t)$ in the rectangle Q satisfies the equation

$$v_{tt} + v_{xx} + \mu(t)v = F_{1xx}(x,t) + h_{1xx}(x,t)v(1,t). \tag{8}$$

Let us write out the boundary conditions for the function $v(x,t)$.

If we set $x = 0$ in equation (1) and use the condition $u(0,t) = 0$, we get the boundary condition

$$v(0,t) = F_1(0,t) + h_1(0,t)v(1,t), \quad t \in (0,T). \tag{9}$$

Differentiating equation (1) with respect to x, setting $x = 1$, and taking into account the boundary condition $u_x(1,t) = 0$, we have the boundary condition

$$v_x(1,t) = F_{1x}(1,t) + h_{1x}(1,t)v(1,t), \quad t \in (0,T). \tag{10}$$

Finally, it is obvious that the function $v(x,t)$ satisfies the condition

$$v(x,0) = v(x,T) = 0, \quad x \in (0,1). \tag{11}$$

The equation and boundary conditions obtained for the function $v(x,t)$ mean that this function can be treated as a solution of the boundary value problem (8)–(11).

The boundary value problem (8)–(11) is the first auxiliary problem. The solution $v(x,t)$ of the problem (8)–(11) is used to find the solution $u(x,t)$, $q(t)$ to Inverse Problem I.

Let us construct another auxiliary problem, which can also be used to find the solution of the inverse problem I.

Let the condition (6) be fulfilled instead of the condition (6)

$$h_x(1,t) \neq 0 \qquad \text{at} \quad t \in [0,T]. \tag{12}$$

Differentiating equation (1) with respect to x, setting $x = 1$, and taking into account the boundary condition $u_x(1,t) = 0$, we can represent function $q(t)$ as

$$q(t) = \frac{u_{xxx}(1,t) - f_x(1,t)}{h_x(1,t)}. \tag{13}$$

Let's introduce the notations

$$h_2(x,t) = \frac{h(x,t)}{h_x(1,t)}, \quad F_2(x,t) = f(x,t) - h_2(x,t)f_x(1,t).$$

Let us substitute into the equation (1) the function $q(t)$ calculated by the formula (13). Differentiating the equation (1) twice with respect to x and denoting $v(x,t) = u_{xx}(x,t)$, we see that $v(x,t)$ satsfies the equation

$$v_{tt} + v_{xx} + \mu(t)v = F_{2xx}(x,t) + h_{2xx}(x,t)v_x(1,t), \quad (t,x) \in Q. \tag{14}$$

Using the equality $u(0,t) = u(1,t) = 0$ and (2) it is easy to show that the function $v(x,t)$ satisfies the boundary conditions

$$v(0,t) = F_2(0,t) + h_2(0,t)v_x(1,t), \quad t \in (0,T), \tag{15}$$

$$v(1,t) = F_2(1,t) + h_2(1,t)v_x(1,t), \quad t \in (0,T). \tag{16}$$

$$v(x,0) = v(x,T) = 0, \quad x \in (0,1). \tag{17}$$

Hence, the function $v(x,t)$ can be treated as a solution of the boundary value problem (14)–(17).

Using the technique described above we can construct auxiliary problems for the inverse problem II. If the overdetermination condition (6) is fulfilled, the inverse problem II can be reduced to the direct problem

$$v_{tt} + v_{xx} + \mu(t)v = F_{1xx}(x,t) + h_{1xx}(x,t)v(1,t), \quad (x,t) \in Q, \tag{18}$$

$$v_x(0,t) + \alpha v(0,t) = F_{1x}(0,t) + \alpha F_1(0,t) + (h_{1x}(0,t) + \alpha h_1(0,t))v(1,t), \quad t \in (0,T), \tag{19}$$

$$v_x(1,t) = F_{1x}(1,t) + h_{1x}(1,t)v(1,t), \quad t \in (0,T). \tag{20}$$

$$v(x,0) = v(x,T) = 0, \quad x \in (0,1). \tag{21}$$

If the overdetermination condition (12) is fulfilled, the inverse problem II can be reduced to the direct problem

$$v_{tt} + v_{xx} + \mu(t)v = F_{2xx}(x,t) + h_{2xx}(x,t)v_x(1,t), \quad (x,t) \in Q, \tag{22}$$

$$v_x(0,t) + \alpha v(0,t) = F_{2x}(0,t) + \alpha F_2(0,t) + (h_{2x}(0,t) + \alpha h_2(0,t))v_x(1,t), \quad t \in (0,T), \tag{23}$$

$$v(1,t) = F_2(1,t) + h_2(1,t)v_x(1,t), \quad t \in (0,T). \tag{24}$$

$$v(x,0) = v(x,T) = 0, \quad x \in (0,1). \tag{25}$$

In the next paragraph it will be shown that by solving $v(x,t)$ of the boundary value problems (8)–(11) or (14)–(17) we can find the solution of the inverse problem I, and that by solving $v(x,t)$ of the boundary value problems (18)–(21) or (22)–(25) a solution of the inverse problem II can be found.

Note that, in the authors' opinion, these problems have an independent value. Indeed, each of the constructed problems is a nonlocal boundary value problem for a loaded [10, 11] elliptic equation. Such problems have not been studied before.

Note also the following. All the boundary value problems presented above for the function $v(x,t)$ are nonlocal. In the first of these problems, the nonlocal condition (9) is a generalised condition of Ionkin [12, 13] or the Samarskii–Ionkin condition [14]. In the fourth problem, the condition (23) can also be called a generalised Ionkin condition, but for the derivative $v_x(x,t)$. The nonlocal conditions (15) and (19) for the function $v(x,t)$ in the second and third problems are not generalised Ionkin conditions, but still belong to the class of general nonlocal conditions of Samarskii [14].

As mentioned above, the existence of regular solutions of the nonlocal problems (8)–(11), (14)–(17), (18)–(21) and (22)–(25) guarantees the existence of regular solutions of the original inverse problems I and II.

We prove the solvability of the above nonlocal problems under some simplifying assumptions that do not affect the essence of the results, but allow us to avoid cumbersome calculations and conditions.

We consider the function $\varphi(t) = \dfrac{1}{2} - \dfrac{h_1^2(0,t)}{2} - h_{1x}(1,t)$. Let's represent it as $\varphi(t) = \varphi_+(t) - \varphi_-(t)$, where $\varphi_+(t) = \max\{\varphi(t), 0\}$, $\varphi_-(t) = \varphi_+(t) - \varphi(t)$.

Let $\mu_0 = -\max\limits_{t\in[0;T]} \mu(t)$, $\mu_0 > 0$.

We assume that

$$\varphi_-(t) \leqslant M_1,\ t \in [0,T], \qquad |h_{1xx}(x,t)| \leqslant M_2, \quad (t,x) \in \overline{Q},$$

where M_1, M_2 are constants.

Theorem 1. *Assume that $h(x,t) \in C^2(\overline{Q})$, $f(x,t) \in L_2(Q)$, $f_{xx}(x,t) \in L_2(Q)$, $\mu(t) \in C([0,T])$ and (6) be satisfied. Let the condition*

$$F_1(0,t) = F_{1x}(1,t) = 0, \quad t \in [0,T], \tag{26}$$

$$\mu_0 > \frac{7}{4} + 3M_1 + M_1^2 + \frac{1}{2}M_2^2 \tag{27}$$

holds. Then the problem (8)–(11) has a solution in the space $W_2^2(Q)$ and it is unique.

Proof. In order to prove the solvability of the problem (8)–(11), we use the method of continuation on the parameter [15] and the regularisation method. Consider the boundary value problem

$$v_{tt} + v_{xx} + \mu(t)v - \varepsilon v_{ttxx} = F_{1xx}(x,t) + \lambda h_{1xx}(x,t)v(1,t), \quad (x,t) \in Q, \tag{28}$$

$$v(0,t) = h_1(0,t)v(1,t), \quad t \in (0,T), \tag{29}$$

$$v_x(1,t) = h_{1x}(1,t)v(1,t), \quad t \in (0,T), \tag{30}$$

$$v(x,0) = v(x,T) = 0, \quad x \in (0,1), \tag{31}$$

where λ is an arbitrary number from the segment $[0;1]$, ε — an arbitrary sufficiently small number.

Let us define a Banach space: $V = \{v(x,t) : v(x,t) \in W_2^2(Q), \quad v_{ttx} \in L_2(Q), v_{ttxx} \in L_2(Q)\}$ with the norm $||v||_V = \left(||v||_{W_2^2(Q)} + ||v_{ttx}||_{L_2(Q)} + ||v_{ttxx}||_{L_2(Q)}\right)^{\frac{1}{2}}$.

The solvability of problem (28)–(31) for $\lambda = 0$ and fixed ε is easily proven using the methods of [16, 17]. Next, we show that the estimate

$$||v||_V \leqslant R_0||F_{1xx}||_{L_2(\Omega)} \tag{32}$$

holds for all possible solutions of problem (28)–(31). In (32) the constant R_0 is determined by the functions $\mu(t)$, $h(x,t)$, $\varphi(t)$ and the domain Q.

When obtaining further estimates, we will use the inequality ([6])

$$\psi^2(1) \leqslant \delta^2 \int_0^1 x\psi'^2(x)\,dx + \left(2 + \frac{1}{\delta^2}\right)\int_0^1 x\psi^2(x)\,dx, \tag{33}$$

where $\psi(x) \in W_2^1([0,1])$, δ is an arbitrary positive number.

We multiply the equation (28) by the function $-xv(x,t)$ and integrate over the domain Q. Using the integration by parts formula, taking into account the introduced notation and boundary conditions (29)–(31), we obtain the equality

$$\int_Q xv_t^2 dxdt + \int_Q xv_x^2 dxdt - \int_Q \mu xv^2 dxdt + \varepsilon \int_Q xv_{xt}^2 dxdt + \int_0^T \varphi_+(t)v^2(1,t)\,dt + \\ +\varepsilon \int_0^T \gamma(t)\,dt + \varepsilon \int_0^T \varphi(t)v_t^2(1,t)\,dt = \int_0^T \varphi_-(t)v^2(1,t)\,dt - \int_Q (F_{1xx} + \lambda h_{1xx} v(1,t))xv\,dxdt,$$

where $\gamma(t) = -(h_{1xt}(1,t) + 2h_1(0,t)h_{1t}(0,t))\,v(1,t)\,v_t(1,t) - h_{1x}^2(1,t)v^2(1,t)$.

Taking into account condition (27) and the possibility of choosing ε sufficiently small, and using Young's inequalities along with (33), we arrive at the following estimate

$$\int_Q xv_t^2 dxdt + \int_Q xv_x^2 dxdt + \int_Q xv^2 dxdt + \varepsilon \int_Q xv_{xt}^2 dxdt + \\ + \int_0^T \varphi_+(t)(v^2(1,t) + \varepsilon v_t^2(1,t)\,dt \leqslant C_1 \int_Q F_{1xx}^2\,dxdt. \tag{34}$$

Here and below C_j, $j = 1, 2, \ldots$, are constants independent of ε, λ. C_j, $j = 1, 2, \ldots$, are determined by the functions $\mu(t)$, $h(x,t)$ and the domain Q.

Inequality (34) guarantees that the following estimate

$$\int_Q v^2 dx \leqslant C_2 \int_Q F_{1xx}^2\,dxdt, \tag{35}$$

holds.

We multiply the equation (28) by the function $xv_{tt}(x,t)$ and integrate over the domain Q. Using the estimate (34) and the technique of its derivation, we obtain that the solution $v(x,t)$ of the boundary value problem (28)–(31) satisfies the inequality

$$\int_Q xv_{tt}^2 dxdt + \int_Q xv_{tx}^2 dxdt + \varepsilon \int_Q xv_{xtt}^2 dxdt + \leqslant C_3 \int_Q F_{1xx}^2\,dxdt. \tag{36}$$

To derive the third a priori estimate, we multiply equation (28) by $-xv_{ttxx}(x,t)$ and integrate over Q. Taking into account the estimates (34) and (36), we obtain the inequality

$$\int_Q xv_{ttx}^2 dxdt + \int_Q xv_{txx}^2 dxdt + \varepsilon \int_Q xv_{ttxx}^2 dxdt + \leqslant C_4 \int_Q F_{1xx}^2\,dxdt. \tag{37}$$

Multiply equation (28) by $v_{xx}(x,t)$, integrate over Q, and then use the integration by parts formula on the left-hand side of the resulting equality. The result of the integration is as follows:

$$\int_Q v_{tx}^2 dxdt + \int_0^T (v_{tt}(1,t)v_x(1,t) - v_{tt}(0,t)v_x(0,t))\,dt + \int_0^T \mu(t)(v(1,t)v_x(1,t) - \\ - v(0,t)v_x(0,t))\,dt + \int_Q v_{xx}^2\,dxdt - \int_Q \mu v_x^2 dxdt + \varepsilon \int_Q v_{xxt}^2 dxdt = \\ = \int_Q (F_{1xx} + \lambda h_{1xx} v(1,t))v_{xx}\,dxdt. \tag{38}$$

To obtain an estimate from equality (38), we use the inequality

$$\int_0^T v_x^2(0,t)\,dt \leqslant \delta \int_Q v_{xx}^2\,dxdt + C(\delta) \int_Q v^2\,dxdt,$$

where $\delta > 0$ is an arbitrary constant that can be taken small enough, and $C(\delta)$ is a constant that depends on $\delta > 0$ and the size of Q.

Using Young's inequality for (38) and taking into account the estimate (36), we find that the solution $v(x,t)$ of the boundary value problem (28)–(31) satisfies the inequality

$$\int_Q v_{tx}^2 dxdt + \int_Q v_{xx}^2 dxdt + \int_Q v_x^2 dxdt + \varepsilon \int_Q v_{xxt}^2 dxdt \leqslant C_5 \int_Q F_{1xx}^2 \, dxdt. \tag{39}$$

Using the estimate (39), it is easy to get the estimate

$$\int_Q v_{tt}^2 dxdt + \int_Q v_t^2 dxdt \leqslant C_6 \int_Q F_{1xx}^2 \, dxdt. \tag{40}$$

The following estimate

$$\varepsilon \int_Q v_{ttxx}^2 dxdt \leqslant C_7 \int_Q F_{1xx}^2 \, dxdt. \tag{41}$$

is guaranteed by the inequalities (39)–(40).

From inequalities (39), (40), (41) follows the estimate (32). This means that for any fixed ε the boundary value problem (28)–(31) is solvable in the space V for all $\lambda \in [0,1]$.

Estimates (39), (40), (41) are sufficient to carry out the limit in problem (28)–(31) for $\varepsilon \to 0$. Let $\{\varepsilon_m\}_{m=1}^{\infty}$ be a sequence of positive numbers such that $\varepsilon_m \to 0$ as $m \to \infty$.

Denote by $v_m(x,t) \in W_2^2(Q)$ the solution to the boundary value problem (28)–(31) for $\lambda = 1$ and $\varepsilon = \varepsilon_m$, $m = 1, 2, \ldots$. $W_2^2(Q)$ is the reflexivity of a Hilbert space. Then from the functional sequence $\{v_m(x,t)\}_{m=1}^{\infty}$ one can extract a subsequence $\{v_{m_k}(x,t)\}_{k=1}^{\infty}$ that converges weakly in $W_2^2(Q)$ to some function $v(x,t) \in W_2^2(Q)$. The convergence $\varepsilon_{m_k} v_{ttxx} \to 0$ as $k \to \infty$ weakly in $L_2(Q)$ holds. The limit function $v(x,t)$ belongs to the space $\in W_2^2(Q)$ and it is a solution to the boundary value problems (8)–(11).

The estimates (39),(40) allow us to state that the solution to the problem (8)–(11) in space $W_2^2(Q)$ is unique. The theorem is proved. □

Let's proceed to the study of the solvability of the problem (14)–(17).

Consider the function $\psi(t) = \dfrac{1}{2} - \dfrac{h_2^2(0,t)}{2h_2^2(1,t)} - \dfrac{1}{h_2(1,t)}$, $h_2(1,t) \neq 0$. Represent it as $\psi(t) = \psi_+(t) - \psi_-(t)$, where $\psi_+(t) = \max\limits_{t\in[0;T]} \{\psi(t), 0\}$, $\psi_-(t) = \psi_+(t) - \psi(t)$.

We assume that

$$\psi_-(t) \leqslant M_3, \quad t \in [0,T], \qquad \left| \frac{h_{2xx}(x,t)}{h_2(1,t)} \right| \leqslant M_4, \quad (t,x) \in \overline{Q},$$

where M_3, M_4 are positive constants.

Theorem 2. *Assume that $h(x,t) \in C^2(\overline{Q})$, $f(x,t) \in L_2(Q)$, $f_{xx}(x,t) \in L_2(Q)$, $\mu(t) \in C([0,T])$ and (12) be satisfied. Let the condition*

$$h_2(1,t) \neq 0, \qquad F_2(0,t) = F_2(1,t) = 0, \qquad t \in [0,T], \tag{42}$$

$$\mu_0 > \frac{7}{4} + 3M_3 + M_3^2 + \frac{1}{2} M_4^2 \tag{43}$$

holds. Then the problem (14)–(17) has a solution in the space $W_2^2(Q)$ and it is unique.

Proof. Taking into account the conditions (42), we transform the boundary conditions (15), (16) to the form

$$v(0,t) = h_2(0,t)h_2^{-1}(1,t)v(1,t), \quad v_x(1,t) = h_2^{-1}(1,t)v(1,t). \tag{44}$$

The obtained conditions (44) are similar to the conditions of the boundary value problem, the solvability of which is established in Theorem 1.

We replace the multiplier $v_x(1,t)$ with the expression $h_2^{-1}(1,t)v(1,t)$ on the right-hand side of equation (14). As a result, equation (14) takes the form

$$v_{tt} + v_{xx} + \mu(t)v = F_{2xx}(x,t) + h_2^{-1}(1,t)h_{2xx}(x,t)v(1,t). \tag{45}$$

Thus, problem (14)–(17) is reduced to problem (45), (44), (17). The proof of the solvability of problem (45), (44), (17) repeats the proof of the solvability of problem (8)–(11).

The theorem is proved. □

Next, we investigate the solvability of the boundary value problem (22)–(25).

Theorem 3. *Assume that* $h(x,t) \in C^2(\overline{Q})$, $f(x,t) \in L_2(Q)$, $f_{xx}(x,t) \in L_2(Q)$, $\mu(t) \in C([0,T])$. *Let the conditon (6) and*

$$F_{1x}(0,t) + \alpha F_1(0,t) = 0, \quad F_{1x}(1,t) = 0, \quad t \in [0,T], \tag{46}$$

$$\mu_0 > \frac{1}{2}, \quad \alpha < 0, \quad h_{1x}(1,t) < 0, \quad t \in [0;T], \tag{47}$$

$$\frac{2M_2^2}{2\mu_0 - 1} < \frac{(h_{1x}(0,t) + \alpha h_1(0,t))^2}{4\alpha} - h_{1x}(1,t) \tag{48}$$

hold. Then the problem (18)–(21) has a solution $v(x,t)$ *in the space* $W_2^2(Q)$ *and it is unique.*

Proof. In order to prove the solvability of the problem (18)–(21), we use the method of continuation on the parameter [15]. Taking into account conditions (46), we write the problem (18)–(21) with parameter λ, $\lambda \in [0;1]$.

$$v_{tt} + v_{xx} + \mu(t)v = F_{1xx}(x,t) + \lambda h_{1xx}(x,t)v(1,t), \quad (x,t) \in Q, \tag{49}$$

$$v_x(0,t) + \alpha v(0,t) = (h_{1x}(0,t) + \alpha h_1(0,t))v(1,t), \quad t \in (0,T), \tag{50}$$

$$v_x(1,t) = h_{1x}(1,t)v(1,t), \quad t \in (0,T). \tag{51}$$

$$v(x,0) = v(x,T) = 0, \quad x \in (0,1). \tag{52}$$

The solvability of problem (49)–(52) for $\lambda = 0$ is easy to prove using the methods of [6]. Next, we will show that for all possible solutions of problem (49)–(52), the following estimate holds:

$$||v||_{W_2^2(\Omega)} \leqslant R_1 ||F_{1xx}||_{L_2(\Omega)}, \tag{53}$$

where the constant R_1 is determined by the functions $\mu(t)$, $h(x,t)$ and the domain Q.

We multiply equation (49) by $-v(x,t)$ and integrate over the domain Q. Applying the integration by parts formula, the Cauchy–Bunyakovsky inequality, and Young's inequality, we arrive at the inequality

$$\int_Q (v_t^2(x,t) + v_x^2(x,t))\,dxdt + \left(\mu_0 - \frac{1}{2} - \frac{M_2^2}{2\delta^2}\right)\int_Q v^2(x,t)\,dxdt +$$
$$+ \int_0^T (-\alpha v^2(0,t) + (h_{1x}(0,t) + \alpha h_1(0,t))\,v(1,t)\,v(0,t) - (h_{1x}(1,t) + 2\delta^2)v^2(1,t))dt \leqslant$$
$$\leqslant \frac{1}{2}\int_Q F_{1xx}^2(x,t)\,dxdt.$$

Conditions (47), (48) guarantee the existence of δ^2 such that the inequality $\mu_0 - \frac{1}{2} - \frac{M_2^2}{2\delta^2} > 0$ is satisfied, and the quadratic form $-\alpha v^2(0,t) + (h_{1x}(0,t) + \alpha h_1(0,t))\, v(1,t)\, v(0,t) - (h_{1x}(1,t) + 2\delta^2)v^2(1,t)$ with respect to $v(0,t)$, $v(1,t)$ is positive definite. Then the last inequality yields the estimate

$$\int_Q (v_t^2(x,t) + v_x^2(x,t) + v^2(x,t))\, dxdt \leqslant C_8 ||F_{1xx}||^2_{L_2(Q)}. \tag{54}$$

We multiply equation (49) by $v_{xx}(x,t)$ and integrate over the domain Q. Using the integration by parts formula, Young's inequality, and taking into account the estimate (49), we get the inequality

$$\int_Q (v_{xx}^2(x,t) + v_{xt}^2(x,t))\, dxdt \leqslant C_9 ||F_{1xx}||^2_{L_2(Q)}. \tag{55}$$

From equation (49), based on inequalities (55) and (54), we obtain an estimate for $v_{tt}(x,t)$

$$\int_Q v_{tt}^2(x,t)\, dxdt \leqslant C_{10} ||F_{1xx}||^2_{L_2(Q)}.$$

Thus, estimates (55), (54), and the last inequality guarantee that for any solution $v(x,t)$ of problem (49)–(52), estimate (53) holds, i.e., the conditions of the theorem of the continuation method with respect to parameter [15] are satisfied. And problem (49)–(52) has a solution $v(x,t) \in W_2^2(Q)$ for all values of $\lambda \in [0,1]$. For $\lambda = 1$, problem (49)–(52) coincides with problem (18)–(21). The uniqueness of the solution to the problem (8)–(11) in space $W_2^2(Q)$ follows from the inequality (53).

The theorem is proved. □

Next, we investigate the solvability of the boundary value problem (22)–(25).

We will study this problem under the assumption that $h_2(1,t) \neq 0$. We assume that

$$\left| \frac{h_{2xx}(x,t)}{h_2(1,t)} \right| \leqslant M_5, \quad (x,t) \in Q.$$

Theorem 4. *Assume that $h(x,t) \in C^2(\overline{Q})$, $f(x,t) \in L_2(Q)$, $f_{xx}(x,t) \in L_2(Q)$, $\mu(t) \in C([0,T])$. Let the condition (12) and*

$$F_{2x}(0,t) + \alpha F_2(0,t) = 0, \quad F_2(1,t) = 0, \quad t \in [0,T], \tag{56}$$

$$\mu_0 > \frac{1}{2}, \quad \alpha < 0, \quad h_2(1,t) < 0, \quad t \in [0;T], \tag{57}$$

$$\frac{2M_5^2}{2\mu_0 - 1} < \frac{(h_{2x}(0,t) + \alpha h_2(0,t))^2}{4\alpha h_2^2(1,t)} - h_2^{-1}(1,t) \tag{58}$$

hold.

Then the problem (22)–(25) has a solution $v(x,t)$ in the space $W_2^2(Q)$ and it is unique.

Proof. Taking into account (56), we transform the boundary conditions (23), (24) to the form

$$v_x(0,t) + \alpha v(0,t) = (h_{2x}(0,t) + \alpha h_2(0,t)) h_2^{-1}(1,t)\, v(1,t), \quad v_x(1,t) = h_2^{-1}(1,t)\, v(1,t). \tag{59}$$

The obtained conditions (59) are similar to the conditions of the boundary value problem, the solvability of which is established in Theorem 3. Substituting the expression $h_2^{-1}(1,t)v(1,t)$ instead of the multiplier $v_x(1,t)$ into the right-hand side of equation (22), we write the equation

$$v_{tt} + v_{xx} + \mu(t)v = F_{2xx}(x,t) + h_2^{-1}(1,t) h_{2xx}(x,t) v(1,t). \tag{60}$$

Thus, problem (22)–(25) is reduced to problem (18)–(21), the proof of solvability of which is given in Theorem 3.

The theorem is proved. □

2. Solvability of the inverse problems

Let us investigate the solvability of the inverse problem I (1)–(3) in the case where the overdetermination condition is $u(1,t)=0$.

Theorem 5. *Suppose the fulfillment of the condition*

$$f(0,t)-f(1,t)\frac{h(0,t)}{h(1,t)}\equiv 0,\quad f_x(1,t)-f(1,t)\frac{h_x(1,t)}{h(1,t)}\equiv 0,$$

and also conditons (5), (6) and (27). Then Inverse Problem (1)–(3) has a solution $\{u(x,t),q(t)\}$ *such that* $u(x,t)\in W_2^2(Q)$, $q(t)\in L_2(0;T)$.

Proof. Let us define the function $u(x,t)$ as a solution to the problem

$$u_{xx}(x,t)=v(x,t),\quad (x,t)\in Q,$$
$$u(0,t)=u_x(1,t)=0,\quad t\in[0,T],$$

where $v(x,t)$ is the solutuin of the problem (8)–(11). We define the function $q(t)$:

$$q(t)=\frac{v(1,t)-f(1,t)}{h(1,t)}. \tag{61}$$

To demonstrate that the function $u(x,t)$ meets the overdetermination condition $u(1,t)=0$, we substitute the function $q(t)$, defined by formula (61), into equation (1), and set $x=1$. Let $\beta(t)=u(1,t)$. Then $\beta(t)$ satisfies the equation $\beta''+\alpha\beta=0$ and the boundary conditions $\beta(0)=\beta(1)=0$. Evidently, $\beta(t)=0$. Consequently, it has been proven that the pair of functions $u(x,t),q(t)$ is a solution to the inverse problem I (1)–(3).

The theorem is proved. □

Let us investigate the solvability of the inverse problem I (1)–(3) in the case where the overdetermination condition is $u_x(1,t)=0$.

Theorem 6. *Suppose the fulfillment of the conditions*

$$f(0,t)-f_x(1,t)\frac{h(0,t)}{h_x(1,t)}\equiv 0,\quad f(1,t)-f_x(1,t)\frac{h(1,t)}{h_x(1,t)}\equiv 0,$$

and also conditons (5), (12) and (43). Then Inverse Problem (1)–(3) has a solution $\{u(x,t),q(t)\}$ *such that* $u(x,t)\in W_2^2(Q)$, $q(t)\in L_2(0;T)$.

Proof. Let us define the function $u(x,t)$ as a solution to the problem

$$u_{xx}(x,t)=v(x,t),\quad (x,t)\in Q,$$
$$u(0,t)=u(1,t)=0,\quad t\in[0,T],$$

where $v(x,t)$ is the solution of the problem (14)–(17). The function $q(t)$ is defined by the formula

$$q(t)=\frac{v_x(1,t)-f_x(1,t)}{h_x(1,t)}. \tag{62}$$

Obviously, $u(x,t) \in W_2^2(Q)$, $q(t) \in L_2(0;T)$, and the functions $u(x,t)$, $q(t)$ satisfy equation (1) in Q. In order to show that the overdetermination condition $u_x(1,t) = 0$ is satisfied for the function $u(x,t)$, we substitute the function $q(t)$ defined by formula (62) into equation (1), differentiate it with respect to the variable x and fix $x = 1$. Let $y(t) = u_x(1,t)$. Then $y(t)$ satisfies the equation $y'' + \alpha y = 0$ and the boundary conditions $y(0) = y(1) = 0$. Obviously, $y(t) = 0$. Thus, it is proved that the pair of functions $\{u(x,t), q(t)\}$ is a solution to the inverse problem I (1)–(3). The theorem is proved. □

Let us return to Inverse Problem II (1), (2), (4) in the case where the overdetermination condition is $u(1,t) = 0$. The solution to this inverse problem will be determined by the function $v(x,t)$, which satisfies the auxiliary problem (18)–(21). The solvability of the problem (18)–(21) is established in Theorem 3. Consider a pair of functions $\{u(x,t), q(t)\}$, where $u(x,t)$ is the solution to the problem

$$u_{xx}(x,t) = v(x,t), \quad (x,t) \in Q,$$
$$u_x(0,t) + \alpha u(0,t) = u_x(1,t) = 0, \quad t \in [0,T],$$

and the function $q(t)$ is defined by the formula $q(t) = \dfrac{v(1,t) - f(1,t)}{h(1,t)}$. The following theorem holds

Theorem 7. *Suppose the fulfillment of the condition*

$$f_x(0,t) - f(1,t)\frac{h_x(0,t)}{h(1,t)} + \alpha(f(0,t) - f(1,t)\frac{h(0,t)}{h(1,t)}) \equiv 0, \quad f(1,t) - f_x(1,t)\frac{h_x(1,t)}{h(1,t)} \equiv 0,$$

and also conditons (5), (6), (47) and (48). Then Inverse Problem II (1), (2), (4) has a solution $\{u(x,t), q(t)\}$ *such that* $u(x,t) \in W_2^2(Q)$, $q(t) \in L_2(0;T)$.

Next, we formulate a theorem on the existence of a solution to the inverse problem II in the case where the overdetermination condition is $u_x(1,t) = 0$. In this case, the solution to the inverse problem $\{u(x,t), q(t)\}$ will be determined through the solution to the auxiliary problem (22)–(23). Specifically, the function $u(x,t)$ is the solution to the problem

$$u_{xx}(x,t) = v(x,t), \quad (x,t) \in Q,$$
$$u_x(0,t) + \alpha u(0,t) = u(1,t) = 0, \quad t \in [0,T].$$

And the unknown coefficient $q(t)$ is determined by the formula $q(t) = \dfrac{v(1,t) - f_x(1,t)}{h_x(1,t)}$.

Theorem 8. *Suppose the fulfillment of the conditions*

$$f_x(0,t) - f_x(1,t)\frac{h_x(0,t)}{h_x(1,t)} + \alpha(f(0,t) - f_x(1,t)\frac{h(0,t)}{h_x(1,t)}) \equiv 0, \quad f(1,t) - f_x(1,t)\frac{h(1,t)}{h_x(1,t)} \equiv 0,$$

and also conditons (5), (12), (57) and (58) . Then Inverse Problem II (1), (2), (4) has a solution $\{u(x,t), q(t)\}$ *such that* $u(x,t) \in W_2^2(Q)$, $q(t) \in L_2(0;T)$.

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Задачи идентификации неизвестного коэффициента в эллиптическом уравнении

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Аннотация. В работе исследуется разрешимость обратных задач для эллиптического уравнения вида

$$u_{tt} + u_{xx} + \mu(t)u = f(x,t) + q(t)h(x,t),$$

в котором неизвестен коэффициент $q(t)$. Доказываются теоремы существования и единственности регулярных решений. Методы исследования основаны на переходе от исходной обратной задачи к нелокальной задаче с условиями типа условий Самарского–Ионкина. В качестве дополнительных условий, необходимость которых обусловлена наличием неизвестной величины $q(t)$, используются условия граничного переопределения.

Ключевые слова: эллиптические уравнения, неизвестный коэффициет, существование, единственность решения, условия Самарского–Ионкина.

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Isospectral and Partially Isospectral Dirac Operators and Their Applications

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Abstract. This paper proposes an algorithm for constructing isospectral and partially isospectral Dirac operators on a finite interval. This algorithm is applied to find solutions of mixed problems posed for systems of hyperbolic partial differential equations with variable coefficients.

Keywords: eigenvalues, normalization constants, inverse problems, Fredholm integral equations of the second kind, isospectral and partially-isospectral Dirac operators.

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1. Introduction and preliminaries

The system of differential equations of the form

$$\left\{\sigma_1 \frac{1}{i}\frac{d}{dx} + \sigma_2 p(x) + \sigma_3 q(x)\right\} y = \left\{B\frac{d}{dx} + \Omega(x)\right\} y = \lambda y, \quad y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} \tag{1}$$

is referred to as Dirac system in canonical form.

Here,

$$\sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},$$

where $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices, $p(x), q(x)$ are given scalar functions, $y(x)$ is the unknown function, and λ is the spectral parameter.

The Dirac operator defined by differential expression (1) is widely used in various fields of mathematics and physics. However, the one-dimensional Dirac operator plays an important role as it arises in the study of simple physical models that admit separation of variables. Furthermore, the one-dimensional Dirac operator is of great importance in the theory of integrable systems as it plays a fundamental role in the integration of the nonlinear Schrödinger equation (NLS) and the modified Korteweg-de Vries equation (mKdV) by means of the inverse spectral method.

The physical properties of equation (1) are described in more detail in [30].

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Definition 1. *Boundary value problems for systems of Dirac equations*

$$D^0 y \equiv By' = \lambda y, \quad 0 < x < \pi, \ y = (y_1(x), \ y_2(x))^T, \tag{2}$$

$$y_2(0) = 0, \quad y_2(\pi) = 0 \tag{3}$$

and

$$D(p(x), q(x), \alpha, \beta)y \equiv By' + Q(x)y = \lambda y, \quad 0 < x < \pi, \ y = (y_1(x), \ y_2(x))^T, \tag{4}$$

$$y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0, \quad y_1(\pi)\cos\beta + y_2(\pi)\sin\beta = 0, \ \alpha, \beta \in R \tag{5}$$

are called isospectral if they have the same eigenvalues, that is, $\sigma(D(p(x), q(x), \alpha, \beta) = \sigma(D^0) = \{n, n \in Z\}$.

Definition 2. *Boundary value problems* (2)–(3) *and* (4)–(5) *are called partially isospectral if eigenvalues satisfy conditions* $\lambda_0 \neq \lambda_0^0$; $\lambda_n = \lambda_n^0$ *for* $n \in Z\backslash\{0\}$.

Here

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},$$

and also $p(x)$, $q(x) \in C^1[0, \pi]$ *are real continuously differentiable functions,* λ *is a complex parameter,* $\lambda_n^0 = n, n \in Z$ *are eigenvalues of boundary value problem* (2)–(3).

This work is devoted to the inverse spectral problems that describe all boundary value problems of a system of Dirac differential equations on a finite interval with the same spectrum. Such boundary value problems are called isospectral and were studied in [1–5]. It was shown that Dirac operator on a finite interval is uniquely determined by its eigenvalues and sequence of normalizing constants [6]. It was also obtained necessary and sufficient conditions for the reconstruction of boundary value problems. M. G. Gasymov and B. M. Levitan (see [7]) established necessary and sufficient conditions for the reconstruction of the Dirac operator on the half-line from spectral functions. In constructing isospectral Dirac operators on a finite interval with a prescribed spectrum $\sigma(D(p(x), q(x), \alpha, \beta)) = \{\lambda_n, n \in Z\}$ the Gasymov–Levitan method is used (see [7]). This method is based on the reconstruction of coefficients of the Dirac operator from spectral data by means of a Fredholm integral equation of the second kind with a parameter.

In this paper, an algorithm for constructing isospectral and partially isospectral one-dimensional Dirac operators on a finite interval is proposed. This algorithm is used to construct solutions of mixed problems for systems of partial differential equations of hyperbolic type with variable coefficients.

It should be noted that isospectral Sturm-Liouville and Dirac operators on a finite interval were studied in [8–17]. Partially isospectral Sturm–Liouville and Dirac operators on a finite interval were studied in [18–22]. The theory of inverse spectral problems for the Sturm–Liouville operator and their applications is presented in more detail in [23–29].

The main result of this work is an algorithm for reconstructing a family of Dirac operators $D = D(p(x), q(x), \alpha, \beta)$ and $D = D(a)$ on a finite interval. Spectra of these operators satisfy the following conditions

$$\sigma(D(p(x), q(x), \alpha, \beta)) = \{\lambda_n = n, n \in Z\}$$

and

$$\sigma(D(a)) = \{\lambda_0 = a, \ |a| < 1, \ \lambda_n = n, \ n \in Z\backslash\{0\}\}.$$

2. On the inverse problem for the Dirac operator on a finite interval

Let us denote the solution of equation (4) that satisfies initial conditions

$$\varphi_1(0,\lambda)=\sin\alpha,\quad \varphi_2(0,\lambda)=-\cos\alpha. \tag{6}$$

by $\varphi(x,\lambda)=(\varphi_1(x,\lambda),\varphi_2(x,\lambda))^T$ It is known that solution $\varphi(x,\lambda)$ of problem (4) with conditions (6) exists [25]. It is unique, and for each fixed $x\in[0,\pi]$ is an entire vector-function with respect to λ. In addition, the following integral representation holds

$$\varphi(x,\lambda)=\begin{pmatrix}\sin(\lambda x+\alpha)\\ -\cos(\lambda x+\alpha)\end{pmatrix}+\int_0^x K(x,t)\begin{pmatrix}\sin(\lambda t+\alpha)\\ -\cos(\lambda t+\alpha)\end{pmatrix}dt, \tag{7}$$

where matrix function $K(x,t)=\|K_{ij}(x,t)\|_{i,j=1,2}$ is a solution of problems

$$BK'_x(x,t)+K'_t(x,t)B=-Q(x)K(x,t), \tag{8}$$

$$BK(x,x)-K(x,x)B=-Q(x), \tag{9}$$

$$K_{12}(x,0)=K_{22}(x,0)=0. \tag{10}$$

It is obvious that vector function $\varphi(x,\lambda)=(\varphi_1(x,\lambda),\varphi_2(x,\lambda))^T$ satisfies the boundary condition $\varphi_1(0,\lambda)=\sin\alpha$ for any λ. Therefore, eigenvalues λ_n, $n\in Z$ of problem (4) with conditions (5) are the roots of the following equation

$$\Delta(\lambda)\equiv y_1(\pi)\cos\beta+y_2(\pi)\sin\beta=0. \tag{11}$$

Then, $\varphi(x,\lambda_n)=(\varphi_1(x,\lambda_n),\varphi_2(x,\lambda_n))^T$, $n\in Z$ is an eigenvector function of problem (4), (5). Let us define

$$\alpha_n=\int_0^\pi[\varphi_1^2(x,\lambda_n)+\varphi_2^2(x,\lambda_n)]dx,\ \ n\in Z. \tag{12}$$

Numbers α_n, $n\in Z$ are called the normalization numbers of boundary value problem (4)–(5). The set of numbers $\{\lambda_n,\alpha_n\}_{n=-\infty}^{\infty}$ are called the spectral data of problem (4)–(5).

Theorem 1 ([6]). *If $p(x)$ and $q(x)$ have derivatives of order k in $L^2[0,\pi]$ then the following equalities hold*

$$\lambda_n=n+\frac{\beta-\alpha}{\pi}+\frac{\beta_1}{n}+\cdots+\frac{\beta_{k-1}}{n^{k-1}}+\frac{\beta_{k,n}}{n^k}, \tag{13}$$

$$\alpha_n=\pi+\frac{c_1}{n}+\cdots+\frac{c_{k-1}}{n^{k-1}}+\frac{c_{k,n}}{n^k} \tag{14}$$

for the spectral data $\{\lambda_n,\alpha_n\}_{n=-\infty}^{\infty}$ of problem (4)-(5), where series

$$\sum_{n=-\infty}^{\infty}|\beta_{k,n}|^2<\infty,\qquad \sum_{n=-\infty}^{\infty}|c_{k,n}|^2<\infty$$

converge.

It should be noted that eigenvector functions $\varphi(x,\lambda_n)=(\varphi_1(x,\lambda_n),\varphi_2(x,\lambda_n))^T$, $n\in Z$ of problem (4)–(5) are orthogonal in the Hilbert space of square-summable two-component vector functions $L_2^2(0,\pi)$. For arbitrary vector functions $f(x)=(f_1(x),f_2(x))^T\in L_2^2[0,\pi]$ one can write

$$f(x)=\sum_{n=-\infty}^{\infty}\varphi(x,\lambda_n)\Big\{\frac{1}{\alpha_n}\int_0^\pi\varphi^T(t,\lambda_n)f(t)dt\Big\}.$$

From here it follows symbolic equality

$$\sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n}\varphi(x,\lambda_n)\varphi^T(t,\lambda_n) = I\delta(x-t), \tag{15}$$

where I is identity matrix, $\delta(x)$ is delta function. In particular, if $p(x) \equiv 0,\ q(x) \equiv 0,\ \alpha = \dfrac{\pi}{2}$, $\beta = \dfrac{\pi}{2}$ then

$$\sum_{n=-\infty}^{\infty} \frac{1}{\pi}\begin{pmatrix} \cos nx \\ \sin nx \end{pmatrix}(\cos nt, \sin nt) = I\delta(x-t). \tag{16}$$

Theorem 2 ([6]). *Coefficients $p(x)$ and $q(x)$ of boundary value problem (4)–(5) are determined uniquely from the spectral data $\{\lambda_n, \alpha_n\}_{n=-\infty}^{\infty}$.*

Lemma 1. *The following equality holds*

$$\sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n}\varphi(x,\lambda_n)(\sin(\lambda_n t+\alpha), -\cos(\lambda_n t+\alpha)) = 0,\ \ 0 < t < x < \pi. \tag{17}$$

Theorem 3 ([6]). *Kernel $K(x,t) = \|K_{ij}(x,t)\|_{i,j=1,2}$ of transformation operator (7) satisfies the linear integral equation*

$$K(x,t) + F(x,t) + \int_0^x K(x,s)F(s,t)ds = 0,\ \ (0 < t \le x < \pi), \tag{18}$$

where

$$F(x,t) = \sum_{n=-\infty}^{\infty}\left\{\frac{1}{\alpha_n}\begin{pmatrix} \sin\left(\lambda_n x + \dfrac{\pi}{2}\right) \\ -\cos\left(\lambda_n x + \dfrac{\pi}{2}\right) \end{pmatrix}\left(\sin\left(\lambda_n t + \frac{\pi}{2}\right), -\cos\left(\lambda_n t + \frac{\pi}{2}\right)\right) - \right.$$
$$\left. -\frac{1}{\pi}\begin{pmatrix} \cos nx \\ \sin nx \end{pmatrix}(\cos nt, \sin nt)\right\}. \tag{19}$$

Theorem 4 ([6]). *In order for sequence of real numbers $\{\lambda_n, \alpha_n\}_{n=-\infty}^{\infty}$, $\alpha_n > 0$, $n \in Z$ to be spectral data of boundary value problem of form (4)–(5) with a matrix function of the form*

$$Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},$$

it is necessary and sufficient that asymptotic formulas (13) and (14) are satisfied for $\lambda_n,\ n \in Z$ and $\alpha_n,\ n \in Z$, where $p(x)$ and $q(x)$ have derivatives of order k in $L^2(0,\pi)$.

Thus, let $\{\lambda_n, \alpha_n\}_{n=-\infty}^{\infty}$, $\alpha_n > 0$, $n \in Z$ satisfy conditions (13) and (14). Let us construct matrix function $F(x,t)$ using formula (19) and consider integral equation (18) with respect to $K(x,t) = \|K_{ij}(x,t)\|_{i,j=1,2}$. Then the following theorem holds.

Theorem 5 ([6]). *Integral equation (18) has a unique solution $K(x,t) = \|K_{ij}(x,t)\|_{i,j=1,2}$ for each fixed $x \in (0,\pi)$.*

Now, solving integral equation (18), one can find matrix function $K(x,t) = \|K_{ij}(x,t)\|_{i,j=1,2}$. Next, let us define vector function $\varphi(x,\lambda) = (\varphi_1(x,\lambda), \varphi_2(x,\lambda))^T$ using formula (7). Then it is easy to prove that vector function $\varphi(x,\lambda) = (\varphi_1(x,\lambda), \varphi_2(x,\lambda))^T$ satisfies the following equation (see [6])

$$B\varphi' + Q(x)\varphi = \lambda\varphi, \quad Q(x) = K(x,x)B - BK(x,x)$$

and initial conditions $\varphi_1(0,\lambda) = \sin\alpha,\ \varphi_2(0,\lambda) = -\cos\alpha$.

3. Algorithm for recovering isospectral and partially isospectral boundary value problems

Theorem 6. *Let a pair of sequences of real numbers* $\{\lambda_n, \alpha_n\}_{n=-\infty}^{\infty}$, $\alpha_n > 0$, $n \in Z$ *satisfy the following conditions*

$$\lambda_n = n, \quad n \in Z, \quad \frac{1}{\alpha_n} = \frac{1}{\pi} + \frac{\gamma_n}{n^2+1}, \quad \sum_{n=-\infty}^{\infty} \frac{\gamma_n}{n^2+1} < \infty, \ \gamma_n > 0, \quad n \in Z. \tag{20}$$

Then this pair is the spectral data of the boundary value problem for the Dirac operator of the form $D(p(x, ..., \gamma_{-n}, ..., \gamma_0, ..., \gamma_n, ...), q(x, ..., \gamma_{-n}, ..., \gamma_0, ..., \gamma_n, ...), \alpha, \beta) = D(..., \gamma_{-n}, ..., \gamma_0, ..., \gamma_n, ...)$.

Proof. It is easy to see that sequence $\left\{\lambda_n, \alpha_n\right\}_{n=-\infty}^{\infty}$, $\alpha_n > 0$, $n \in Z$ defined by formulas (20) satisfies conditions (13)–(14) for $\alpha = \dfrac{\pi}{2}, \beta = \dfrac{\pi}{2}$. Therefore, there exists a Dirac operator of the form $D(\ldots, \gamma_{-n}, ..., \gamma_0, ..., \gamma_n, ...)$. Now, by applying the inverse spectral method, one can determine the coefficients of the Dirac operator.

To do this, matrix function $F(x,t)$ is defined using formulas (19) and (20):

$$F(x,t) = \sum_{n=-\infty}^{\infty} \left\{ \frac{\gamma_n}{n^2+1} \begin{pmatrix} \cos nx \\ \sin nx \end{pmatrix} (\cos nt, \sin nt) \right\}. \tag{21}$$

Substituting (21) into integral equation (18), one can obtain

$$K(x,t) = -\sum_{n=-\infty}^{\infty} \frac{\gamma_n}{n^2+1} \varphi(x,n)(\cos nt, \sin nt), \tag{22}$$

where

$$\varphi(x,n) = \begin{pmatrix} \cos nx \\ \sin nx \end{pmatrix} + \int_0^x K(x,s) \begin{pmatrix} \cos ns \\ \sin ns \end{pmatrix} ds. \tag{23}$$

It is known ([6, 25]) that vector function $\varphi(x,\lambda)$ defined by formula (23) satisfies differential equation (4) and initial conditions $\varphi_1(0,\lambda) = 1$, $\varphi_2(0,\lambda) = 0$, where coefficients are defined as follows

$$\begin{aligned} p(x, \ldots, \gamma_{-n}, \ldots, \gamma_0, \ldots, \gamma_n, \ldots) &= \sum_{n=-\infty}^{\infty} \frac{\gamma_n}{n^2+1} [\varphi_2(x,n) \sin nt - \varphi_1(x,n) \cos nt], \\ q(x, \ldots, \gamma_{-n}, \ldots, \gamma_0, \ldots, \gamma_n, \ldots) &= \sum_{n=-\infty}^{\infty} \frac{\gamma_n}{n^2+1} [\varphi_1(x,n) \sin nt + \varphi_2(x,n) \cos nt]. \end{aligned} \tag{24}$$

Next, by substituting expression (22) into formula (23), one can find

$$\varphi(x,n) = \begin{pmatrix} \cos nx \\ \sin nx \end{pmatrix} - \sum_{p=-\infty}^{\infty} \frac{\gamma_p}{p^2+1} \varphi(x,p) \int_0^x (\cos ps, \sin ps) \begin{pmatrix} \cos ns \\ \sin ns \end{pmatrix} ds. \tag{25}$$

Hence, eigenvector function

$$\varphi(x,n) = \frac{1}{1 + \frac{\gamma_n x}{n^2+1}} \begin{pmatrix} \cos nx \\ \sin nx \end{pmatrix}, \ n \in Z,$$

Is obtained. It corresponds to eigenvalues $\lambda_n = n,\ \ n \in Z$ since

$$\int_0^x (\cos ps, \sin ps) \begin{pmatrix} \cos ns \\ \sin ns \end{pmatrix} ds = \begin{cases} x, & p = n, \\ \dfrac{\sin(p-n)x}{p-n}, & p \neq n. \end{cases}$$

It follows that boundary conditions (5) are satisfied, i.e., $\varphi_1(0, \lambda_n) = 1,\ \varphi_2(\pi, \lambda_n) = 0,\ \lambda_n = n, n \in Z$.

Thus, all coefficients of the family of isospectral Dirac operators of the form $D(..., \gamma_{-n}, ..., \gamma_0, ..., \gamma_n, ...)$ are determined. In particular, if $\gamma_0 > 0,\ \gamma_n = 0,\ n \in Z\backslash\{0\}$ then isospectral boundary value problem follows

$$D(\gamma_0)\, y \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y' + \frac{\gamma_0}{1+\gamma_0 x} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} y = \lambda y, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},\ 0 < x < \pi,\ \gamma_0 > 0$$
$$y_2(0) = 0, \quad y_2(\pi) = 0. \tag{26}$$

All eigenvalues of Dirac operator $D(\gamma_0)$ coincide with the prescribed numbers $\lambda_n = n$, $n \in Z$, and the corresponding eigenvector functions are given by

$$y_n(x) = \begin{pmatrix} y_{1n}(x) \\ y_{2n}(x) \end{pmatrix} = \begin{pmatrix} \cos nx \\ \sin nx \end{pmatrix} - \frac{\gamma_0}{n(1+\gamma_0 x)} \begin{pmatrix} \sin nx \\ 0 \end{pmatrix},\ n \in Z.$$

Next, using isospectral boundary value problem (26), one can construct a family of systems of partial differential equations of the form

$$iu_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_x + \frac{\gamma_0}{1+\gamma_0 x} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} u,\ \ 0 < x < \pi,\ \gamma_0 > 0. \tag{27}$$

The main objective of this paper is to find a solution of equation (27) that satisfies the following conditions

$$u(x,0) = f(x),\ \ f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix},\ \ u_2(\pi, t) = 0,\ \ u(x,t) = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}. \tag{28}$$

Using Theorem 6, one can obtain

Corollary 1. The solution of mixed problem (27), (28) can be written in the form

$$u(x,t) = \sum_{n=-\infty}^{\infty} B_n e^{-int} \begin{pmatrix} \cos nx - \dfrac{\gamma_0 \sin nx}{n(1+\gamma_0 x)} \\ \sin nx \end{pmatrix},$$

where $B_n = \int_0^\pi (y_{1n}(x) f_1(x) + y_{2n}(x) f_2(x)) dx, n \in Z$.

Theorem 7. *Let the sequences of real numbers $\{\lambda_n, \alpha_n\}_{n=-\infty}^{\infty}$, $\alpha_n > 0$, $n \in Z$ satisfy conditions*

$$\lambda_0 = a,\ |a| < 1,\ \ \lambda_n = n,\ \ n \in Z\backslash\{0\},\ \ \alpha_n = \pi,\ \ n \in Z. \tag{29}$$

Then this pair is the spectral data of the boundary value problem for the Dirac operator of the form $D(p(x,a), q(x,a), \alpha, \beta) = D(a)$.

Proof. It is easy to see that sequence $\left\{\lambda_n, \alpha_n\right\}_{n=-\infty}^{\infty}$, $\alpha_n > 0$, $n \in Z$ defined by formulas (20) satisfies conditions (13)–(14) for $\alpha = \dfrac{\pi}{2}$, $\beta = \dfrac{\pi}{2}$. Therefore, there exists a Dirac operator with parameters of the form $D(a)$. By applying the inverse spectral method, coefficients of Dirac operator $D(a)$ are determined.

To do this, matrix function $F(x,t)$ is defined using formulas (19) and (29):

$$F(x,t) = \frac{1}{\pi}\begin{pmatrix} \cos ax \\ \sin ax \end{pmatrix}(\cos at, \sin at) - \frac{1}{\pi}\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1,0). \tag{30}$$

Substituting (30) into integral equation (18), one can obtain

$$K(x,t) = -\frac{1}{\pi}f(x)(\cos at, \sin at) + \frac{1}{\pi}g(x)(1,0), \tag{31}$$

where

$$f(x) = \begin{pmatrix} \cos ax \\ \sin ax \end{pmatrix} + \int_0^x K(x,s)\begin{pmatrix} \cos as \\ \sin as \end{pmatrix} ds, \tag{32}$$

$$g(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^x K(x,s)\begin{pmatrix} 1 \\ 0 \end{pmatrix} ds. \tag{33}$$

By substituting expression (30) into formulas (32) and (33), one can obtain

$$f(x) = \begin{pmatrix} \cos ax \\ \sin ax \end{pmatrix} - \frac{x}{\pi}f(x) + \frac{\sin ax}{\pi a}g(x), \tag{34}$$

$$g(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{\sin ax}{\pi a}f(x) + \frac{x}{\pi}g(x). \tag{35}$$

Hence, one can find

$$\begin{aligned} f(x) &= \frac{a^2\pi}{a^2(\pi-x)^2 + \sin^2 ax}\left((\pi-x)\cos ax + \frac{\sin ax}{a}, (\pi-x)\sin ax\right)^T, \\ g(x) &= \frac{a^2\pi}{a^2(\pi-x)^2 + \sin^2 ax}\left(\pi + x - \frac{\sin 2ax}{2a}, -\frac{\sin^2 ax}{a}\right)^T. \end{aligned} \tag{36}$$

Next, using formulas (31) and (36), one can derive

$$K_{11}(x,x) = \frac{a^2}{a^2(\pi^2-x^2) + \sin^2 ax}\left[\pi + x - (\pi-x)\cos^2 ax - \frac{\sin 2ax}{a}\right], \tag{37}$$

$$K_{12}(x,x) = -\frac{a^2}{a^2(\pi^2-x^2) + \sin^2 ax}\left[\frac{(\pi-x)\sin 2ax}{2} + \frac{\sin^2 ax}{a}\right], \tag{38}$$

$$K_{21}(x,x) = -\frac{a^2}{a^2(\pi^2-x^2) + \sin^2 ax}\left[\frac{(\pi-x)\sin 2ax}{2} + \frac{\sin^2 ax}{a}\right], \tag{39}$$

$$K_{22}(x,x) = -\frac{a^2(\pi-x)\sin^2 ax}{a^2(\pi^2-x^2) + \sin^2 ax}. \tag{40}$$

Finally, using formulas (9), (37), (38), (39) and (40), one can obtain

$$p(x,a) = -(K_{12} + K_{21}) = \frac{a^2(\pi-x)\sin 2ax + 2a\sin^2 ax}{a^2(\pi^2-x^2) + \sin^2 ax},$$

$$q(x,a) = K_{11} - K_{22} = \frac{a^2(\pi + x) - a^2(\pi - x)\cos 2ax - a\sin 2ax}{a^2(\pi^2 - x^2) + \sin^2 ax}.$$

Thus, all coefficients of the family of partially isospectral Dirac operators $D(a)$ are determined:

$$\begin{gathered} D(a)\, y \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y' + \begin{pmatrix} p(x,a) & q(x,a) \\ q(x,a) & -p(x,a) \end{pmatrix} y = \lambda y, \ \ 0 < x < \pi, \ \ |a| < 1, \\ y_2(0) = 0, \ \ y_2(\pi) = 0, \ \ y(x) = (y_1(x), y_2(x))^T. \end{gathered} \tag{41}$$

It can be easily seen that all eigenvalues of boundary value problem (41) coincide with the prescribed numbers $\lambda_0 = a$, $|a| < 1$, $\lambda_n = n$, $n \in Z \backslash \{0\}$. The corresponding eigenvector functions are given by

$$y_n(x) = \begin{pmatrix} \cos nx \\ \sin nx \end{pmatrix} - \frac{a^2 \sin(n-a)x}{(n-a)(a^2(\pi^2 - x^2) + \sin^2 ax)} \begin{pmatrix} (\pi - x)\cos ax + \frac{\sin ax}{a} \\ (\pi - x)\sin ax \end{pmatrix} -$$

$$- \frac{a^2 \sin nx}{n(a^2(\pi^2 - x^2) + \sin^2 ax)} \begin{pmatrix} \pi + x - \frac{\sin 2ax}{2a} \\ -\frac{\sin^2 ax}{a} \end{pmatrix}, \quad n \in Z.$$

Now, let us consider the following mixed problem

$$\begin{gathered} iu_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u_x + \begin{pmatrix} p(x,a) & q(x,a) \\ q(x,a) & -p(x,a) \end{pmatrix} u, \ \ 0 < x < \pi, \ \ |a| < 1, \\ u(x,0) = f(x), \ \ f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \ \ u_2(0,t) = 0, \ \ u_2(\pi,t) = 0, \ \ u(x,t) = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}. \end{gathered} \tag{42}$$

Using Theorem 7, one can obtain

Corollary 2. The solution of mixed problem (42) is as follows

$$u(x,t) = B_0 e^{-iat} y_0(x) + \sum_{n=-\infty, n\neq 0}^{\infty} B_n e^{-int} y_n(x),$$

where

$$B_n = \int_0^{\pi} (y_{1n}(x) f_1(x) + y_{2n}(x) f_2(x)) dx, \ \ n \in Z.$$

Conclusion

Isospectral and partially isospectral Dirac operators with parameters $D(..., \gamma_{-n}, ..., \gamma_0, ..., \gamma_n, ...)$ and $D(a)$ were reconstructed. Eigenvalues of these operators coincide with the prescribed numbers

$$\sigma(D(\dots, \gamma_{-n}, \dots, \gamma_0, \dots, \gamma_n, \dots)) = \{\lambda_n = n, n \in Z\}$$

and

$$\sigma(D(a)) = \{\lambda_0 = a, \ |a| < 1, \ \lambda_n = n, \ n \in Z \backslash \{0\}\},$$

respectively. In addition, using isospectral and partially isospectral Dirac operators, solutions of mixed problems for a system of partial differential equations of hyperbolic type with variable coefficients were obtained.

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Изоспектральные и частично изоспектральные операторы Дирака и их приложения

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Аннотация. В работе предлагается алгоритм построения изоспектральных и частично-изоспектральных операторов Дирака на конечном отрезке. Этот алгоритм применяется к процессу нахождения решений смешанных задач, поставленных для систем дифференциальных уравнений с частными производными гиперболического типа с переменными коэффициентами.

Ключевые слова: собственные значения, нормирующие константы, обратные задачи, интегральные уравнения Фредгольма второго рода, изоспектральные и частично-изоспектральные операторы Дирака.

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Mathematical Model of the Data Network between a Control Center and a Group of UAVs with a Deployment Strategy without Interaction

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Abstract. We consider a mathematical model of a communication network for a group of unmanned aerial vehicles (UAVs) that operate without interaction with each other and with a control center, represented as a finite-source queueing system with a fixed number of sources generating connection requests to a server. Each source, when initiating a communication session with the server, generates requests with a constant intensity. If the server is free, the request begins to be processed. The model assumes that the duration of this period is exponentially distributed. If the server is busy at the moment of request initiation, a transmission delay occurs in the form of a blocked request for an exponentially distributed time interval, after which the source attempts to initiate the request again. The object of study is the number of transmission delays during the establishment of communication between UAVs and the control center, which characterizes the performance not only of the control station but also of the entire network. To analyze the number of blocked requests, methods of asymptotic diffusion analysis, numerical analysis, and simulation modeling of the functioning of the considered model are applied. As a result, limiting, numerical, and simulation characteristics of the proposed model are obtained, and the domain of applicability of the results is established.

Keywords: data network, finite-source queueing system, asymptotical analysis method, simulation.

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Introduction

Currently, unmanned aerial vehicles (UAVs) are used in various fields of human activity [1, 2] and have a significant impact on science, industry, agriculture, security, and logistics systems. In most cases, UAVs operate in groups. Two types of groups can be distinguished: groups in

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which UAVs interact with each other [3, 4, 5], and groups in which UAVs solve a specific task in a network-based non-interactive deployment mode. Each UAV in this network functions as an independent unit, directly connected to the control center, which allows for monitoring the status of each UAV and transmitting necessary commands in real time. This approach creates a structure in which each UAV is part of a single computing system, in which control and feedback are implemented through the exchange of data packets between the UAV and the ground system. This allows not only for monitoring flight parameters but also for promptly transmitting commands to change course, speed, and perform other actions, which is critical to the successful completion of assigned missions. However, one of the main problems with this strategy is its dependence on the ground control station, which serves as the single point in the system where a failure, malfunction, or shutdown of the entire network can occur. A malfunction at the ground station will lead to delays in data transmission or a complete halt in communication between the UAV and the control center. This limitation requires a careful approach to system design, as well as the development of backup mechanisms to minimize the risk of communication loss and increase the reliability of the entire control network. The main features of UAV-based networks include high node mobility, three-dimensional freedom of movement in space, low node density in the network, and others. Taking these factors into account in the design of classes of mobile communication networks leads to their extremely dynamic topology and makes the application of standard routing protocols inefficient. It should be noted that theoretical studies of control networks for unmanned objects are poorly developed. Typically, algorithmic approaches are used [6, 7], as well as simulation methods [8] or graph-based approaches [9, 10].

In this paper, it is proposed to model the communication system between a group of non-interacting UAVs and a ground station using methods of probabilistic analysis and queueing theory. The interaction between the group of unmanned aerial vehicles and the control center is represented as a finite-source queueing system with multiple access. As a result of the study, formulas for calculating the probabilistic characteristics of the proposed model will be obtained. Their practical application will make it possible to scientifically substantiate the selection of data transmission network parameters during UAV operation and will significantly expand the possibilities for solving a number of scientific problems in the field of next-generation communication network (NGN) design.

1. Mathematical model

Consider a communication system with a UAV group as a finite-source queueing system with N sources generating connection requests to a server [11–16]. Each source generates a request with rate of λ/N. A source cannot generate a new request until the end of the successful service of this request. The connection request is transmitted to an free server. The duration of the communication session is distributed exponentially with rate μ. If the server is busy when the connection request is initiated, there is a delay and request move into the orbit. The retrial time

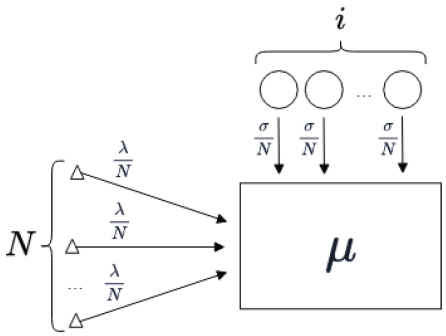


Fig. 1. Mathematical model of a finite-source queuing system with repeated calls

is exponentially distributed with rate σ/N, after which the source attempts to make the request again (Fig.1). Let us denote: $i(t)$ is the number of requests in the orbit; $k(t) = 0,\ 1$ is the state of the server. Here, 0 means that the server is free and 1 means that the server is busy. The goal of the study is obtaining the stationary probability distributions of the number of requests waiting for repeated service (the number of requests in the orbit) and the states of the server. Denote $P(k,i,t) = P\{i(t) = i,\ k(t) = k\}$ the probability that the server is in state k and the number of delays is i requests at time t. We write the differential Kolmogorov system in the steady state

$$\begin{aligned}
&-\left(\lambda + \frac{\sigma-\lambda}{N}i\right)P(0,i) + \mu P(1,i) = 0, \quad i = 0,\dots,(N-1),\\
&-\left(\frac{N-1}{N}\lambda + \mu\right)P(1,0) + \lambda P(0,0) + \frac{1}{N}\sigma P(0,1) = 0,\\
&-\left(\frac{N-1}{N}\lambda + \mu - \frac{\lambda}{N}i\right)P(1,i) + \left(\lambda - \frac{\lambda}{N}i\right)P(0,i) +\\
&+\left(\frac{N-1}{N}\lambda - \frac{\lambda}{N}(i-1)\right)\lambda P(1,i-1) + \frac{i+1}{N}\sigma P(0,i+1) = 0, \quad i = 1,\dots,(N-2),\\
&-\mu P(1,N-1) + \frac{1}{N}\lambda P(0,N-1) + \frac{1}{N}\lambda P(1,N-2) = 0.
\end{aligned} \tag{1}$$

We will rewrite system (1) for the partial characteristic functions. Denoting

$$H(k,u) = \sum_{i=0}^{N-1} e^{jui}P(k,i), \quad j = \sqrt{-1}. \tag{2}$$

We have

$$\begin{aligned}
&-\lambda H(0,u) + j\frac{\sigma-\lambda}{N}\frac{\partial H(0,u)}{\partial u} + \mu H(1,u) = 0,\\
&\lambda H(0,u) + \left(\frac{N-1}{N}\lambda\left(e^{ju}-1\right) - \mu\right)H(1,u) +\\
&\qquad + \left(\lambda - \sigma e^{-ju}\right)\frac{j}{N}\frac{\partial H(0,u)}{\partial u} + \lambda\left(e^{ju}-1\right)\frac{j}{N}\frac{\partial H(1,u)}{\partial u} = 0.
\end{aligned} \tag{3}$$

Let us write one more equation obtained by summing the equations of system (3)

$$\frac{N-1}{N}\lambda H(1,u) + \sigma e^{-ju}\frac{j}{N}\frac{\partial H(0,u)}{\partial u} + \lambda\frac{j}{N}\frac{\partial H(1,u)}{\partial u} = 0. \tag{4}$$

Equation (4) will be called the consistent equation. We propose to get a solution of the system (3)–(4) using the asymptotic analysis method [17, 18] under the limiting condition of an unlimited growing number of sources ($N \to \infty$).

2. Asymptotic analysis method

We will perform the asymptotic analysis method in two stages. The mean value of the asymptotic number of transmission delays and limit probability distribution of the server states are obtained at the first stage. At the second stage, the parameter of asymptotic variance of the number of transmission delays is find. As a result, it will be proved that the limiting probability distribution of the number of requests in the orbit follows a Gaussian distribution with given parameters.

2.1. First-order asymptotics

We introduce notation $\frac{1}{N} = \varepsilon$, where ε is an infinitesimal value (suppose $\varepsilon \to 0$). We rewrite system (3)–(4) in the form

$$\begin{aligned} &-\lambda H(0,u) + j\,(\sigma-\lambda)\,\varepsilon\frac{\partial H(0,u)}{\partial u} + \mu H(1,u) = 0,\\ &\lambda H(0,u) + \left((1-\varepsilon)\lambda\left(e^{ju}-1\right)-\mu\right)H(1,u) + \\ &\qquad + \left(\lambda - \sigma e^{-ju}\right) j\varepsilon\frac{\partial H(0,u)}{\partial u} + \lambda\left(e^{ju}-1\right) j\varepsilon\frac{\partial H(1,u)}{\partial u} = 0,\\ &(1-\varepsilon)\,\lambda H(1,u) + \sigma e^{-ju} j\varepsilon\frac{\partial H(0,u)}{\partial u} + \lambda j\varepsilon\frac{\partial H(1,u)}{\partial u} = 0. \end{aligned} \tag{5}$$

In system (5), we make the following substitutions

$$u = \frac{w}{N} = \varepsilon w, \quad H(k,u) = F_1(k,w,\varepsilon). \tag{6}$$

Condition $N \to \infty$ or $\varepsilon \to 0$ defines the asymptotic regime for a growing number of sources, under which the number of generations to fulfill requests to the server increases without bound. To compensate for this unbounded growth, $u = w\varepsilon$ is substituted, which is equivalent to considering the class of processes $\varepsilon i(t) = i(t)/N$, for which convergence can be expected as $\varepsilon \to 0$. By applying the asymptotic analysis method, in the first stage the limiting characteristic function is constructed, which we will refer to as the first-order asymptotic of the number of transmission delays of requests to the server. We rewrite system (5) taking into account substitutions (6)

$$\begin{aligned} &-\lambda F_1(0,w,\varepsilon) + j\,(\sigma-\lambda)\,\frac{\partial F_1(0,w,\varepsilon)}{\partial w} + \mu F_1(1,w,\varepsilon) = 0,\\ &\lambda F_1(0,w,\varepsilon) + \left((1-\varepsilon)\lambda\left(e^{j\varepsilon w}-1\right)-\mu\right)F_1(1,w,\varepsilon) + \\ &\qquad + \left(\lambda - \sigma e^{-j w\varepsilon}\right) j\frac{\partial F_1(0,w,\varepsilon)}{\partial w} + \lambda\left(e^{jw\varepsilon}-1\right) j\frac{\partial F_1(1,w,\varepsilon)}{\partial w} = 0,\\ &(1-\varepsilon)\,\lambda F_1(1,w,\varepsilon) + \sigma e^{-j\varepsilon w} j\frac{\partial F_1(0,w,\varepsilon)}{\partial w} + \lambda j\frac{\partial F_1(1,w,\varepsilon)}{\partial w} = 0. \end{aligned} \tag{7}$$

Function $F_1(k,w,\varepsilon)$ is the asymptotic characteristic function of the number of requests in the orbit and the state of the server. We will find a solution $F_1(k,w,\varepsilon)$ in the class of functions for which there is a finite limit for the derivative with respect to w as $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0}\frac{\partial F_1(k,w,\varepsilon)}{\partial w} = \frac{\partial F_1(k,w)}{\partial w}.$$

The following theorem is proposed for formulation and proof.

Theorem 1. *Limiting value $F_1(w)$ of solution $F_1(w,\varepsilon)$ of system (7) as $\varepsilon \to 0$ is given by*

$$F_1(w) = \sum_{k=0}^{1} F_1(k,w) = e^{jw\kappa_1}, \tag{8}$$

where the value of κ_1 is determined by the equality

$$\kappa_1 = \frac{\lambda\sigma - 2\lambda^2 - \mu\sigma + \sqrt{\sigma\left(4\lambda^2\mu + \sigma\,(\mu-\lambda)^2\right)}}{2\lambda\,(\sigma-\lambda)}. \tag{9}$$

Probabilities r_0 and r_1 of the server states are determined as follows

$$r_0 = \frac{\mu}{\lambda(1-\kappa_1)+\sigma\kappa_1+\mu}, \quad r_1 = \frac{\lambda(1-\kappa_1)+\sigma\kappa_1}{\lambda(1-\kappa_1)+\sigma\kappa_1+\mu}. \tag{10}$$

Proof. We take the limit transition by $\varepsilon \to 0$ in system (7)

$$\begin{aligned} &-\lambda F_1(0,w) + j\,(\sigma-\lambda)\,\frac{\partial F_1(0,w)}{\partial w} + \mu F_1(1,w) = 0,\\ &\lambda F_1(0,w) - \mu F_1(1,w) + (\lambda-\sigma)\,j\frac{\partial F_1(0,w)}{\partial w} = 0,\\ &\lambda F_1(1,w) + \sigma j\frac{\partial F_1(0,w)}{\partial w} + \lambda j\frac{\partial F_1(1,w)}{\partial w} = 0. \end{aligned} \tag{11}$$

It can be observed that the first and second equations of system (11) are identical with opposite signs; therefore, we will retain only one of them. We write the solution of system (11) in the form $F_1(k,w) = r_k\Phi_1(w)$, where $\Phi_1(w)$ is the characteristic function for the process $i(t)$ as $\varepsilon \to 0$, and r_k, $(k=0,1)$ is the stationary probability distribution of the server states. Next, we substitute the obtained decomposition into system (11)

$$\begin{aligned} &-\lambda r_0 + (\sigma-\lambda)\,r_0 j\frac{\Phi_1'(w)}{\Phi_1(w)} + \mu r_1 = 0,\\ &\lambda r_1 + \sigma r_0 j\frac{\Phi_1'(w)}{\Phi_1(w)} + \lambda r_1 j\frac{\Phi_1'(w)}{\Phi_1(w)} = 0. \end{aligned} \tag{12}$$

Since the ratio $j\dfrac{\Phi_1'(w)}{\Phi_1(w)}$ does not depend on w, the scalar function $\Phi_1(w)$ takes the form $\Phi_1(w) = \exp\{jwx\}$. Here, $j\dfrac{\Phi_1'(w)}{\Phi_1(w)} = -x$. Substituting this value into system (12), we obtain

$$\begin{aligned} -(\lambda(1-x)+\sigma x)\,r_0 + \mu r_1 = 0,\\ \lambda\,(1-x)\,r_1 - \sigma x r_0 = 0. \end{aligned} \tag{13}$$

From the last system, we write down the solutions

$$r_0(x) = \frac{\mu}{\lambda(1-x)+\sigma x+\mu}, \quad r_1(x) = \frac{\lambda(1-x)+\sigma x}{\lambda(1-x)+\sigma x+\mu}. \tag{14}$$

Obtained functions $r_0(x)$ and $r_1(x)$ are substituted into the second equation of system (13), yielding a quadratic equation $\lambda\,(\sigma-\lambda)\,x^2 + \left(2\lambda^2+\mu\sigma-\lambda\sigma\right)x - \lambda^2 = 0$, solving which we obtain two roots

$$x_{1,2} = \frac{\lambda\sigma - 2\lambda^2 - \mu\sigma \pm \sqrt{\sigma\left(4\lambda^2\mu + \sigma\,(\mu-\lambda)^2\right)}}{2\lambda\,(\sigma-\lambda)}. \tag{15}$$

The root x_1 will always lie in the interval $(0,1)$; we denote it by $\kappa_1 = x_1$, which coincides with (9). Substituting the value of κ_1 into the expressions for the functions $r_0(x)$ and $r_1(x)$, we find the probabilities of the server states

$$r_0 = \frac{\mu}{\lambda(1-\kappa_1)+\sigma\kappa_1+\mu}, \quad r_1 = \frac{\lambda(1-\kappa_1)+\sigma\kappa_1}{\lambda(1-\kappa_1)+\sigma\kappa_1+\mu}, \tag{16}$$

which coincides with (10). By determining the probabilities r_k, $k = 0, 1$, and the function $\Phi_1(w)$, we obtain that the limiting solution of system (7) has the form

$$F_1(w) = \sum_{k=0}^{1} F_1(k, w) = \Phi_1(w) = e^{jw\kappa_1},$$

which coincides with (8). The theorem is proved. □

By performing the inverse substitutions from (6), we obtain the limiting characteristic function of the number of transmission delays of requests to the server $H(u) = e^{ju\kappa_1 N}$, which will be referred to as the first-order asymptotic. Clearly, the first-order asymptotic determines mean value $\kappa_1 N$ of the number of requests that waiting for repeated connection to the server. The second stage of the asymptotic analysis method is implemented to construct the limiting probability distribution.

2.2. Second-order asymptotics

We rewrite systems (3) and (4) taking into account the substitution

$$H(k, u) = e^{ju\kappa_1 N} H_2(k, u), \tag{17}$$

yielding

$$\begin{aligned}
&-\left(\lambda\left(1-\kappa_1\right)+\sigma\kappa_1\right)H_2(0,u)+\mu H_2(1,u)+j\frac{\sigma-\lambda}{N}\frac{\partial H_2(0,u)}{\partial u}=0,\\
&\left(\lambda\left(1-\kappa_1\right)+\sigma e^{-ju}\kappa_1\right)H_2(0,u)+\left(\lambda\left(e^{ju}-1\right)\left(\frac{N-1}{N}-\kappa_1\right)-\mu\right)H_2(1,u)+\\
&\qquad+\left(\lambda-\sigma e^{-ju}\right)\frac{j}{N}\frac{\partial H_2(0,u)}{\partial u}+\lambda\left(e^{ju}-1\right)\frac{j}{N}\frac{\partial H_2(1,u)}{\partial u}=0,\\
&\lambda\left(\frac{N-1}{N}-\kappa_1\right)H_2(1,u)-\sigma e^{-ju}\kappa_1 H_2(0,u)+\\
&\qquad+\sigma e^{-ju}\frac{j}{N}\frac{\partial H_2(0,u)}{\partial u}+\lambda\frac{j}{N}\frac{\partial H_2(1,u)}{\partial u}=0.
\end{aligned} \tag{18}$$

We introduce the notation $\dfrac{1}{N} = \varepsilon^2$, then system (18) can be written in the form

$$\begin{aligned}
&-\left(\lambda\left(1-\kappa_1\right)+\sigma\kappa_1\right)H_2(0,u)+\mu H_2(1,u)+j\left(\sigma-\lambda\right)\varepsilon^2\frac{\partial H_2(0,u)}{\partial u}=0,\\
&\left(\lambda\left(1-\kappa_1\right)+\sigma e^{-ju}\kappa_1\right)H_2(0,u)+\left(\lambda\left(e^{ju}-1\right)\left(1-\varepsilon^2-\kappa_1\right)-\mu\right)H_2(1,u)+\\
&\qquad+\left(\lambda-\sigma e^{-ju}\right)j\varepsilon^2\frac{\partial H_2(0,u)}{\partial u}+\lambda\left(e^{ju}-1\right)j\varepsilon^2\frac{\partial H_2(1,u)}{\partial u}=0,\\
&\lambda\left(1-\varepsilon^2-\kappa_1\right)H_2(1,u)-\sigma e^{-ju}\kappa_1 H_2(0,u)+\\
&\qquad+\sigma e^{-ju}j\varepsilon^2\frac{\partial H_2(0,u)}{\partial u}+\lambda j\varepsilon^2\frac{\partial H_2(1,u)}{\partial u}=0.
\end{aligned} \tag{19}$$

And in system (19), we make the following substitutions

$$u = \frac{w}{\sqrt{N}} = \varepsilon w,\ H_2(k, u) = F_2(k, w, \varepsilon). \tag{20}$$

Similarly to the first-order asymptotics, equality $\dfrac{1}{N} = \varepsilon^2$ defines a growing number of sources, under which the number of generations to fulfill requests to the server increases without bound.

To compensate for this unbounded growth, a substitution is performed in which not only the value of process $i(t)$ increases, but also the value of centered process $\{i(t) - \kappa_1 N\}$ increases. To compensate for this growth, the substitution $u = \varepsilon w$ is performed, which is equivalent to considering the process $\left\{\dfrac{i(t) - \kappa_1 N}{\sqrt{N}}\right\} = \varepsilon \{i(t) - \kappa_1 N\}$ for which convergence can be expected as $\varepsilon \to 0$. The result of applying the asymptotic analysis method at the second stage is the limiting characteristic function of the number of transmission delays of requests to the server in the studied finite-source system, which we will refer to as the second-order asymptotic. The first- and second-order asymptotics of the considered asymptotic analysis method are analogous to the law of large numbers and the central limit theorem in probability theory.

We write the system

$$\begin{aligned}
&-\left(\lambda\left(1-\kappa_1\right)+\sigma\kappa_1\right)F_2(0,w,\varepsilon)+\mu F_2(1,w,\varepsilon)+j\left(\sigma-\lambda\right)\varepsilon\frac{\partial F_2(0,w,\varepsilon)}{\partial w}=0,\\
&\left(\lambda\left(1-\kappa_1\right)+\sigma e^{-jw\varepsilon}\kappa_1\right)F_2(0,w,\varepsilon)+\left(\lambda\left(e^{jw\varepsilon}-1\right)\left(1-\varepsilon^2-\kappa_1\right)-\mu\right)F_2(1,w,\varepsilon)+\\
&\qquad+\left(\lambda-\sigma e^{-jw\varepsilon}\right)j\varepsilon\frac{\partial F_2(0,w,\varepsilon)}{\partial w}+\lambda\left(e^{jw\varepsilon}-1\right)j\varepsilon\frac{\partial F_2(1,w,\varepsilon)}{\partial w}=0,\\
&\lambda\left(1-\varepsilon^2-\kappa_1\right)F_2(1,w,\varepsilon)-\sigma e^{-jw\varepsilon}\kappa_1F_2(0,w,\varepsilon)+\\
&\qquad+\sigma e^{-jw\varepsilon}j\varepsilon\frac{\partial F_2(0,w,\varepsilon)}{\partial w}+\lambda j\varepsilon\frac{\partial F_2(1,w,\varepsilon)}{\partial w}=0.
\end{aligned}\tag{21}$$

Function $F_2(k,w,\varepsilon)$ is the asymptotic characteristic function of the number of delayed requests to the server and the state of the server. We will find solution $F_2(k,w,\varepsilon)$ in the class of functions for which there is a finite limit for the derivative with respect to w as $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0}\frac{\partial F_2(k,w,\varepsilon)}{\partial w}=\frac{\partial F_2(k,w)}{\partial w}.$$

Theorem 2. *Limiting value $F_2(w)$ of solution $F_2(w,\varepsilon)$ of system (21) as $\varepsilon \to 0$ is given by*

$$F_2(w)=\sum_{k=0}^{1}F_2(k,w)=e^{\frac{(jw)^2}{2}\kappa_2},\tag{22}$$

where the value of κ_2 is determined by the equality

$$\kappa_2=\frac{\sigma\kappa_1r_0}{\sigma\kappa_1\varphi_0+\sigma r_0+\lambda r_1-\lambda\left(1-\kappa_1\right)\varphi_1}.\tag{23}$$

Stationary probabilities r_k, $k=0,1$, of the server states are calculated according to formula (10), value of κ_1 according to formula (9), and values φ_k, $k=0,1$, are determined by the equalities

$$\varphi_1=\frac{\left(\sigma-\lambda\right)r_0}{\lambda\left(1-\kappa_1\right)+\sigma\kappa_1+\mu},\ \ \varphi_0=-\varphi_1.\tag{24}$$

Proof. We will find the solution of system (21) in the form

$$F_2(k,w,\varepsilon)=\Phi_2(w)\left(r_k+j\varepsilon w f_k\right)+O\left(\varepsilon^2\right),\quad k=0,1,\tag{25}$$

where $\Phi_2(w)$ is the scalar characteristic function of process $\varepsilon\{i(t)-\kappa_1N\}$ as $\varepsilon\to 0$. Here, r_k, $k=0,1$, are the stationary probabilities of the server states. We substitute expansion (25) into

the first two equations of system (21)

$$-\left(\lambda\left(1-\kappa_1\right)+\sigma\kappa_1\right)\left(r_0+j\varepsilon w f_0\right)+\mu\left(r_1+j\varepsilon w f_1\right)+j\left(\sigma-\lambda\right)\varepsilon r_0\frac{\Phi_2^{'}(w)}{\Phi_2(w)}=O\left(\varepsilon^2\right),$$
$$\left(\lambda\left(1-\kappa_1\right)+\sigma\left(1-jw\varepsilon\right)\kappa_1\right)\left(r_0+j\varepsilon w f_0\right)+ \tag{26}$$
$$+\left(\lambda jw\varepsilon\left(1-\kappa_1-\varepsilon^2\right)-\mu\right)\left(r_1+j\varepsilon w f_1\right)+\left(\lambda-\sigma\right)j\varepsilon r_0\frac{\Phi_2^{'}(w)}{\Phi_2(w)}=O\left(\varepsilon^2\right).$$

We transform the last system taking into account (13) and retain only the first equation of the system, since the second equation will later become identical to the first (upon division by $jw\varepsilon$ and in the limit $\varepsilon\to 0$)

$$-\left(\lambda\left(1-\kappa_1\right)+\sigma\kappa_1\right)j\varepsilon w f_0+\mu j\varepsilon w f_1+j\left(\sigma-\lambda\right)\varepsilon r_0\frac{\Phi_2^{'}(w)}{\Phi_2(w)}=O\left(\varepsilon^2\right). \tag{27}$$

Dividing the equation by $jw\varepsilon$ and taking the limit as $\varepsilon\to 0$, we obtain

$$-\left(\lambda\left(1-\kappa_1\right)+\sigma\kappa_1\right)f_0+\mu f_1=\left(\lambda-\sigma\right)r_0\frac{\Phi_2^{'}(w)}{w\Phi_2(w)}. \tag{28}$$

The general solution of the nonhomogeneous system of equations (28) can be written in the form

$$f_k=Cr_k-\varphi_k\frac{\Phi_2^{'}(w)}{w\Phi_2(w)},\quad k=0,1, \tag{29}$$

where C is a constant, probabilities r_k are calculated according to formulas (10), and the terms $\varphi_k\dfrac{\Phi_2^{'}(w)}{w\Phi_2(w)}$ represent a particular solution of nonhomogeneous system (28), satisfying condition $\varphi_0+\varphi_1=0$. Substituting solution (29) into equation (28), we obtain

$$\varphi_1=\frac{(\sigma-\lambda)r_0}{\lambda(1-\kappa_1)+\sigma\kappa_1+\mu},\quad \varphi_0=-\varphi_1. \tag{30}$$

Consider the consistency equation of system (21)

$$\lambda\left(1-\kappa_1\right)F_2(1,w,\varepsilon)-\sigma\left(1-jw\varepsilon\right)\kappa_1F_2(0,w,\varepsilon)+$$
$$+\sigma j\varepsilon\frac{\partial F_2(0,w,\varepsilon)}{\partial w}+\lambda j\varepsilon\frac{\partial F_2(1,w,\varepsilon)}{\partial w}=O\left(\varepsilon^2\right). \tag{31}$$

Substituting solutions (25) and (29) into the last equation, we have

$$\left(\lambda\left(1-\kappa_1\right)\varphi_1-\sigma\kappa_1\varphi_0-\sigma r_0-\lambda r_1\right)\frac{\Phi_2^{'}(w)}{w\Phi_2(w)}=\sigma\kappa_1r_0.$$

Since the ratio $\dfrac{\Phi_2^{'}(w)}{w\Phi_2(w)}$ does not depend on w, scalar function $\Phi_2(w)$ takes the form $\Phi_2(w)=\exp\left\{-\dfrac{w^2}{2}\kappa_2\right\}$. Here, $\dfrac{\Phi_2^{'}(w)}{w\Phi_2(w)}=-\kappa_2$. Substituting this value into the last equation, we obtain

$$\kappa_2=\frac{\sigma\kappa_1r_0}{\sigma\kappa_1\varphi_0+\sigma r_0+\lambda r_1-\lambda(1-\kappa_1)\varphi_1},$$

which coincides with (23). By determining function $\Phi_2(w)$, we find that the limiting solution of system (21) has the form

$$F_2(w)=\sum_{k=0}^{1}F_2(k,w)=\Phi_2(w)=e^{\frac{(jw)^2}{2}\kappa_2},$$

which coincides with (21). The theorem is proved. □

Returning to functions $H(k,u)$, we perform in (17) the substitutions inverse to (20) and obtain the approximation

$$H(u) = \exp\left\{ju\kappa_1 N + \frac{(ju)^2}{2}\kappa_2 N\right\} \tag{32}$$

for the characteristic function of the number of transmission delays of requests. Thus, Theorem 2 shows that the asymptotic characteristic function of the number of transmission delays in the considered finite-source system under the condition of an unlimited growing number of sources is Gaussian with mean $\kappa_1 N$ and variance $\kappa_2 N$, which allows constructing an approximation for the pre-limit distribution. The Gaussian distribution is a continuous probability distribution. For obtaining the discrete probability distribution of the considered process, we use the following formula

$$P_{as}(i) = \frac{F(i+0.5) - F(i-0.5)}{1 - F(-0.5)}, \tag{33}$$

where the numerator represents the probability that the random variable falls within interval $(i-0.5; i+0.5)$, and the denominator ensures normalization, taking into account that the number of requests cannot be negative. Function $F(x)$ is the cumulative normal distribution function with mean $\kappa_1 N$ and variance $\kappa_2 N$.

3. Numerical results

The construction of an approximation of the distribution under study is described by the following algorithm:

1. Set system parameters: N (number of sources), λ (request generation intensity), μ (service rate), σ (retrial rate).
2. Finding parameter κ_1 by equality (9).
3. Finding probabilities r_0 and r_1 via formulas (10).
4. Finding functions φ_0 and φ_1 via formulas (24).
5. Finding parameter κ_2 as a solution to equation (23).
6. A discrete approximation of the probability distribution of the number of requests in the orbit is constructed using formula (33)

We consider simulation model [19] of the finite-source queueing system under study to estimate the accuracy of the obtained approximation. To compare the obtained approximation with the results of the simulation modeling, the Kolmogorov distance metric is used, defined as:

$$\Delta = \max_{0 \leqslant m \leqslant N-1} \left| \sum_{i=0}^{m} (P(i) - P_{\text{as}}(i)) \right|, \tag{34}$$

where $P(i)$ is the pre-limit probability distribution of the number of requests in the orbit obtained via simulation modeling.

Next, graphs of the pre-limit probability distribution of the number of requests in the orbit and the corresponding asymptotic discrete approximation were plotted. The results of the comparison are shown in Fig. 2 for various values of the parameter N, with fixed values of the arrival rate $\lambda = 4$, service rate $\mu = 1.5$, and the rate of repeated requests to the server $\sigma = 12$.

In Tab. 1, each column presents the values of N and the corresponding values of the metric characterizing the accuracy of the approximation. It can be seen that as N increases, the approximation becomes increasingly accurate: the value of Δ decreases, confirming the convergence of the asymptotic approximation to the actual distribution.

Table 1. Kolmogorov distances between methods for varying N

Methods	$N = 25$	$N = 75$	$N = 225$
P_{as} vs P_{sim}	0.112	0.065	0.036

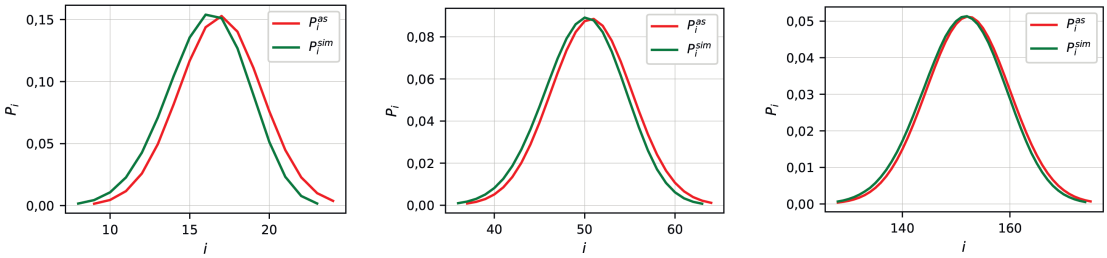


Fig. 2. Comparison of the graphs of the pre-limit probability distribution of the number of requests in the orbit and the asymptotic approximation for $N = 25, 75, 225$

4. Discussion

In this study, a mathematical model of a data transmission network between a flight control center and a group of unmanned aerial vehicles was developed, taking into account a deployment strategy without interaction. We have considered this model in a form of finite-source queueing system with a fixed number of sources generating connection requests to a server. Studying of the model, we have derived steady-state Kolmogorov-equations and applied the asymptotic analysis method to obtain an approximate solution of system (3)–(4). The condition of an unlimited growing number of sources is used for the asymptotic analysis. As the result, we have derived Gaussian approximation (33) of the distribution of the number of requests in the orbit. The obtained approximation should become more precise when the number of sources grows (the asymptotic condition). So, we estimate the approximation precision for different parameters. We have performed the numerical analysis of the obtained results by comparing approximation $P_{as}(i)$ with pre-limit probability distribution $P_{sim}(i)$ obtained via simulation modeling using Kolmogorov distance as an error estimation. Analysis of the results obtained for different values of the parameter N showed that as the number of request-generation sources increases, the Kolmogorov distance decreases, indicating improved accuracy of the approximation. This confirms the validity of the chosen approach. Future studies may be devoted to considering similar models with the general service time. The obtained results can be useful for assessing the performance of distributed communication networks and for designing fault-tolerant control systems.

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Математическая модель сети передачи данных между центром управления полетами и группой БПЛА со стратегией развертывания без взаимодействия

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Аннотация. Мы рассматриваем математическую модель сети связи для группы беспилотных летательных аппаратов (БПЛА), которые функционируют без взаимодействия друг с другом, и центром управления в виде замкнутой системы массового обслуживания с фиксированным числом источников генерации запросов на соединение с сервером. Каждый источник, инициируя сеанс связи с сервером, генерирует запрос с постоянной интенсивностью. Если сервер свободен, то запрос начинает обрабатываться. В модели предполагается, что длительность этого периода распределена по экспоненциальному закону. Если сервер оказывается занят в момент инициирования запроса на связь, то происходит задержка передачи в виде заблокированного запроса в течение экспоненциально распределенного времени, после чего источник вновь делает попытку осуществления запроса. Объектом исследования является число задержек передачи сообщений при установлении связи между БПЛА и центом управления, что характеризует работу не только станции управления, но и работу всей сети. Для анализа числа заблокированных запросов реализуются методы асимптотически диффузионного анализа, численного и имитационного моделирования функционирования рассматриваемой модели. В результате получены предельные, численные и имитационные характеристики предложенной модели, установлена область применимости полученных результатов.

Ключевые слова: сеть передачи данных, замкнутая система массового обслуживания, метод асимптотического анализа, имитационное моделирование.

EDN: USFCZI
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Left-symmetric Structures on Niltriangular Chevalley Algebras

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Abstract. We prove that the algebra $NT(n, \mathbb{K})$ of lower niltriangular $n \times n$ matrices over a field $\mathbb{K}$ is the unique associative exact enveloping algebra of the niltriangular subalgebra $N\Phi(\mathbb{K})$ of the Chevalley algebra of type A_{n-1} that admits a grading compatible with the Cartan grading. We also construct graded left-symmetric structures on the algebras $N\Phi(\mathbb{K})$ of classical types B_n, C_n, and D_n.

Keywords: Lie algebra, Chevalley algebra, niltriangular subalgebra, left-symmetric structure.

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Introduction

Any left-symmetric algebra R has an associated Lie algebra $R^{(-)}$, obtained by equipping R with the commutator bracket $[a, b] \coloneqq ab - ba$. More generally, A. A. Albert [1] calls an arbitrary algebra R *Lie-admissible* if $R^{(-)}$ is a Lie algebra. Following [2–4], we say that R is an *exact enveloping algebra* of a Lie algebra $\mathfrak{L}$ if $R^{(-)} \simeq \mathfrak{L}$.

By the Cartan–Killing classification, every finite-dimensional simple complex Lie algebra $\mathfrak{L}$ corresponds to an indecomposable root system Φ. Choosing a basis Π of simple roots in Φ determines the set of positive roots Φ^+.

The elements e_r $(r \in \Phi)$ together with a suitable basis for Cartan subalgebra of the Lie algebra $\mathfrak{L} = \mathfrak{L}(\Phi, \mathbb{C})$ form a Chevalley basis with integral structure constants. Hence one obtains the Lie algebra $\mathfrak{L}(\Phi, \mathbb{K})$ over an arbitrary field $\mathbb{K}$, called the Chevalley algebra [5]. The subalgebra $N\Phi(\mathbb{K}) \coloneqq \langle e_r \mid r \in \Phi^+ \rangle$ is called the *niltriangular Chevalley algebra*.

The Chevalley basis of the Lie algebra $\mathfrak{L}(\Phi, \mathbb{K})$ of rank n determines on it the *Cartan grading* by the integral root lattice $\mathbb{Z}\Phi \simeq \mathbb{Z}^n$. The homogeneous component of degree zero is the Cartan subalgebra H, whereas all other nonzero homogeneous components are one-dimensional H-invariant subalgebras of $\mathfrak{L}(\Phi, \mathbb{K})$. In particular, all nonzero homogeneous components of the induced grading on $N\Phi(\mathbb{K})$ are one-dimensional.

It is well known that the algebra $NT(n, \mathbb{K})$ of strictly lower triangular $n \times n$ matrices is an exact enveloping algebra of the niltriangular Lie algebra $NA_{n-1}(\mathbb{K})$. In [2], for every indecomposable root system Φ, a family of exact enveloping algebras of $N\Phi(\mathbb{K})$ was constructed; this construction depends on the choice of signs of the Chevalley structure constants. For type A_{n-1}, the uniqueness of $NT(n, \mathbb{K})$ within this family was studied in [3] under an additional condition on ideals, and in [4] under the assumption of associativity.

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In Section 1, we recall basic notions from the theory of graded algebras and prove Proposition 1, which shows that every fine grading on a finite-dimensional nilpotent algebra has width one. This result is a key ingredient in Section 2, where Theorem 2 shows that $NT(n,\mathbb{K})$ is the unique associative exact enveloping algebra of $NA_{n-1}(\mathbb{K})$ that admits a grading compatible with the Cartan grading. Thus, Theorem 2 strengthens the earlier uniqueness results by proving that the existence of such a grading forces any associative exact enveloping algebra of $NA_{n-1}(\mathbb{K})$ to be isomorphic to $NT(n,\mathbb{K})$.

A left-symmetric exact enveloping algebra of a Lie algebra $\mathfrak{L}$ is called a *left-symmetric structure* on $\mathfrak{L}$. It is well known that semisimple Lie algebras over fields of characteristic 0 admit no left-symmetric structures [6]. In [7], J. Milnor posed a question equivalent to the existence problem for left-symmetric structures on solvable Lie algebras. Examples of nilpotent Lie algebras admitting no left-symmetric structures were constructed in [8].

In Section 3, we introduce semidirect sums of left-symmetric algebras, following the corresponding construction for Lie algebras. We show in Theorem 4 that the resulting algebra is again left-symmetric, and deduce in Corollary 4.1 a general construction of left-symmetric structures on semidirect sums of Lie algebras from compatible left-symmetric structures on the two summands.

In Section 4, we apply this construction to the niltriangular Lie algebras $N\Phi(\mathbb{K})$ of types B_n, C_n, and D_n, viewed as semidirect sums of the Lie algebra $NT(n,\mathbb{K})^{(-)}$ and a suitable ideal. Theorem 5 then yields graded left-symmetric structures on $N\Phi(\mathbb{K})$ for all three types, that is, graded left-symmetric exact enveloping algebras of these Lie algebras, realized as semidirect sums of left-symmetric algebras whose two summands are in fact associative.

1. Fine gradings on finite-dimensional nilpotent algebras

Let G be an abelian group. We say that an algebra A is endowed with a *G-grading* Γ if A is represented as a direct sum of subspaces

$$\Gamma\colon A=\bigoplus_{g\in G}A_g$$

such that $A_gA_h\subset A_{g+h}$ for all $g,h\in G$. The set $\operatorname{Supp}\Gamma\coloneqq\{g\in G\mid A_g\neq 0\}$ is called the *support* of the grading Γ.

Gradings of Lie algebras by abelian groups were systematically studied in [9]; see also [10].

An element $a\in A_g$ is said to be *homogeneous of degree g*, and we write $\deg_\Gamma(a)=g$. In particular, 0 is homogeneous of every degree. The subspace A_g is called the *homogeneous component of degree g*. An algebra endowed with a grading Γ is called *graded*.

Every element $a\in A$ admits a unique decomposition $a=\sum\limits_{g\in G}a_g$, where $a_g\in A_g$ are called the *homogeneous components* of a. An ideal $I\subset A$ is called *graded* if it contains the homogeneous components of all its elements, that is,

$$I=\bigoplus_{g\in G}(A_g\cap I).$$

If I is a graded ideal of A, then the quotient A/I inherits the grading ([11, Section 1.3]) given by

$$(A/I)_g\coloneqq(A_g+I)/I=\{a+I\mid a\in A_g\}.$$

Consider two gradings of an algebra A

$$\Gamma_1\colon A=\bigoplus_{g\in G} A_g \quad \text{and} \quad \Gamma_2\colon A=\bigoplus_{h\in H} A_h.$$

We say that Γ_2 is a *refinement* of the grading Γ_1 if there exists a *coarsening homomorphism* $\phi\colon H\to G$ such that $A_h\subset A_{\phi(h)}$ for every $h\in H$. If this inclusion is strict for some $h\in H$, then Γ_2 is called a *proper refinement* of Γ_1.

A grading Γ is said to be *fine* if it admits no proper refinements ([9, Definition 2]). Following [12], we say that a fine grading Γ has *width one* if all its nonzero homogeneous components are one-dimensional.

Lemma 1.1. *Every grading on a finite-dimensional algebra A can be refined to a fine grading.*

Proof. If a grading Γ_1 on A is not fine, then it has a proper refinement Γ_2. Likewise, if Γ_2 is not fine, it has a proper refinement Γ_3, and so on. Since this process cannot continue for more than $\dim A-1$ steps, it terminates after finitely many refinements, yielding a fine grading. □

For any algebra A, besides its one-sided annihilators, one also considers the annihilator

$$\operatorname{Ann}(A)\coloneqq\{\alpha\in A\mid \alpha A=0=A\alpha\}.$$

A nonzero algebra A is said to be *nilpotent of class m* if $A^m\neq 0$ and $A^{m+1}=0$. In particular, $A^m\subset\operatorname{Ann}(A)\neq 0$. We now show that every fine grading on a finite-dimensional nilpotent algebra has width one.

Lemma 1.2. *The annihilator* $\operatorname{Ann}(A)$ *of a graded algebra $A=\bigoplus_{g\in G}A_g$ is a graded ideal of A.*

Proof. By definition, it suffices to show that if $x=\sum\limits_{g\in G}x_g\in\operatorname{Ann}(A)$, then each homogeneous component x_g also belongs to $\operatorname{Ann}(A)$. Fix $h\in G$ and let $y\in A_h$. Since $xA=0$, we have

$$xy=\sum_{g\in G}x_gy=0,$$

where each summand satisfies $x_gy\in A_gA_h\subset A_{g+h}$, and hence lies in a distinct homogeneous component. Since the sum $\bigoplus_{g\in G}A_g$ is direct, it follows from $\sum\limits_{g\in G}x_gy=0$ that $x_gy=0$ for all g. As $y\in A_h$ was arbitrary and $\bigoplus_{h\in G}A_h=A$, we obtain $x_gA=0$ for all g. Similarly, the equality $Ax=0$ implies that $Ax_g=0$ for all g. Hence $x_g\in\operatorname{Ann}(A)$, and therefore

$$\operatorname{Ann}(A)=\bigoplus_{g\in G}(\operatorname{Ann}(A)\cap A_g),$$

that is, $\operatorname{Ann}(A)$ is a graded ideal. □

By a *short exact sequence* ([13, Section 13.3]) we mean a sequence of linear spaces and linear maps

$$0\longrightarrow U\xrightarrow{\alpha}V\xrightarrow{\beta}W\longrightarrow 0$$

such that $\operatorname{Im}\alpha=\ker\beta$, the map α is injective, and the map β is surjective. In particular, these conditions imply that $V=\alpha(U)\oplus S$ for some subspace $S\simeq W$, and hence $V\simeq U\oplus W$. We call such a subspace S of V a *linear section* over W.

Lemma 1.3. *If an algebra A is endowed with a fine grading, then the induced grading on $A/\operatorname{Ann}(A)$ is also fine.*

Proof. Let $I = \operatorname{Ann}(A)$. By Lemma 1.2, the algebra A/I inherits a grading from A:

$$\Gamma_\pi\colon A/I = \bigoplus_{g\in G}(A/I)_g.$$

We claim that the grading Γ_π admits no proper refinements. Suppose otherwise, and let

$$\Gamma'_\pi\colon A/I = \bigoplus_{h\in H}(A/I)_h$$

be a proper refinement of Γ_π. Let $\pi\colon A \twoheadrightarrow A/I$ denote the canonical projection. Consider the exact sequence

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi} A/I \longrightarrow 0$$

and choose a linear section $s\colon A/I \hookrightarrow A$ such that $\pi\circ s = \mathrm{id}_{A/I}$. Then $s(A/I) = \bigoplus_{h\in H} s\big((A/I)_h\big)$, and therefore

$$A = s(A/I) \oplus I = \bigoplus_{h\in H} s\big((A/I)_h\big) \oplus I.$$

The refinement Γ'_π comes with a coarsening homomorphism $\phi\colon H \to G$ such that for every $h \in H$,

$$(A/I)_h \subset (A/I)_{\phi(h)}.$$

We now define a new $G \times H$-grading Γ' on A as follows.

1. If $a \in s((A/I)_h)$, then $\deg_{\Gamma'}(a) = (\phi(h), h)$.
2. If $a \in I$, then $a \in I \cap A_g$ for some $g \in G$, and $\deg_{\Gamma'}(a) = (g, 0)$.

Since $IA = 0 = AI$, the grading Γ' is compatible with the multiplication on A. Thus, it is a refinement of the original grading on A. It remains to show that this refinement is proper.

Indeed, since Γ'_π properly refines Γ_π, there exists a homogeneous component $(A/I)_g$ of Γ_π that splits into several homogeneous components $(A/I)_h$ with $\phi(h) = g$ in Γ'_π. Their images under the section s lie in the component A_g of the original grading Γ, but acquire distinct degrees (g, h) in the grading Γ'. □

Lemma 1.4. *Let a $\mathbb{K}$-algebra A be endowed with a fine G-grading Γ. If a homogeneous component A_g has nonzero intersection with $\operatorname{Ann}(A)$, then $\dim A_g = 1$ and $A_g \subset \operatorname{Ann}(A)$.*

Proof. Let $0 \neq u \in A_g \cap \operatorname{Ann}(A)$. Suppose there exists an element $v \in A_g$ not proportional to u. Then the grading Γ can be refined as follows: set $G' = G \times \mathbb{Z}$ and define a G'-grading Γ' by assigning $\mathbb{K}u$ degree $(g, 1)$, while all other elements of the component A_g retain degree $(g, 0)$; on all other components, we set the second coordinate equal to 0. This contradicts the fineness of Γ. Therefore, $\dim A_g = 1$ and $A_g \subset \operatorname{Ann}(A)$. □

Proposition 1. *Every fine grading on a finite-dimensional nilpotent algebra A has width one.*

Proof. Let

$$\Gamma\colon A = \bigoplus_{g\in G} A_g$$

be a fine G-grading on a finite-dimensional nilpotent algebra A. Consider the quotient algebra $\bar{A} = A/\operatorname{Ann}(A)$. It is again finite-dimensional and nilpotent, and moreover $\dim \bar{A} < \dim A$. By Lemmas 1.2 and 1.3, the grading on A induces a fine grading

$$\Gamma_\pi\colon \bar{A} = \bigoplus_{g\in G} \bar{A}_g$$

on $\bar{A}$, so we may argue by induction on $\dim A$.

By the induction hypothesis applied to $\bar{A}$, we have $\dim \bar{A}_g \leqslant 1$ ($g \in G$). Set $I = \operatorname{Ann}(A)$. The canonical projection $\pi\colon A_g \twoheadrightarrow \bar{A}_g$ has kernel $A_g \cap I = I_g$, and by the isomorphism theorem,

$$\bar{A}_g = (A_g + I)/I \simeq A_g/I_g.$$

For each $g \in G$, considering the exact sequence

$$0 \longrightarrow I_g \xrightarrow{i} A_g \xrightarrow{\pi} \bar{A}_g \longrightarrow 0,$$

we obtain

$$\dim A_g = \dim(\operatorname{Ann}(A) \cap A_g) + \dim \bar{A}_g.$$

To complete the proof, it remains to apply Lemma 1.4. □

Consider two graded algebras

$$\Gamma_1\colon A = \bigoplus_{g\in G} A_g \quad \text{and} \quad \Gamma_2\colon B = \bigoplus_{h\in H} B_h. \tag{1}$$

The gradings Γ_1 and Γ_2 are said to be *equivalent* if there exists an algebra isomorphism $\phi\colon A \to B$ such that for every $g \in G$ there is an $h \in H$ with $\phi(A_g) = B_h$.

2. The uniqueness theorem for type A_n

Fix on the algebra $N\Phi(\mathbb{K})$ the grading Γ_C determined by its root basis $\{e_r \mid r \in \Phi^+\}$:

$$\Gamma_C\colon N\Phi(\mathbb{K}) = \bigoplus_{r\in\Phi^+} \mathbb{K}e_r.$$

Lemma 2.1. *Let R be a nilpotent exact enveloping algebra of the Lie algebra $N\Phi(\mathbb{K})$. If R is endowed with a fine grading Γ such that the induced grading on $R^{(-)}$ is equivalent to the Cartan grading Γ_C, then R can be realized on the vector space $N\Phi(\mathbb{K})$ in such a way that $e_re_s = 0$ whenever $r + s \notin \Phi^+$, and the commutator in $R^{(-)}$ coincides with the commutator in $N\Phi(\mathbb{K})$.*

Proof. By Proposition 1, the fine grading

$$\Gamma\colon R = \bigoplus_{g\in G} R_g \tag{2}$$

on the algebra R has width one and exactly $\dim R$ nonzero one-dimensional components. The grading Γ is also a grading of the Lie algebra $R^{(-)}$, since the multiplication in $R^{(-)}$ is compatible with Γ.

By the assumptions of the lemma, the gradings Γ on $R^{(-)}$ and Γ_C on $N\Phi(\mathbb{K})$ are equivalent. Consequently, there exists a Lie algebra isomorphism $\phi\colon R^{(-)} \to N\Phi(\mathbb{K})$ that sends each nonzero component R_g onto some root component $\mathbb{K}e_r$.

Since $\dim R = \dim N\Phi(\mathbb{K})$, the map ϕ induces a bijection between the support of the grading Γ and the set of positive roots Φ^+. This allows us to reindex the components of R by elements of Φ^+: set $R_r := \phi^{-1}(\mathbb{K}e_r)$. Fix a basis $\{e'_r = \phi^{-1}(e_r) \mid r \in \Phi^+\}$ of R. Clearly, $e'_r \in R_r$ and $R_r = \mathbb{K}e'_r$.

Consider the product $e'_re'_s$ in the algebra R. Since Γ is a group grading and $e'_r \in R_r$, $e'_s \in R_s$, we have $e'_re'_s \in R_{r+s}$. If $r+s \notin \Phi^+$, then $R_{r+s} = 0$, because the support of Γ is identified with Φ^+. Hence $e'_re'_s = 0$.

Now let r, s, and $r+s$ belong to Φ^+. Since ϕ is a Lie algebra isomorphism, in the algebra $N\Phi(\mathbb{K})$ we have $[e_r, e_s] = N_{rs}e_{r+s}$, where N_{rs} are the structure constants of the Chevalley basis. Hence

$$[e'_r, e'_s]_{R^{(-)}} = \phi^{-1}([e_r, e_s]) = \phi^{-1}(N_{rs}e_{r+s}) = N_{rs}e'_{r+s}.$$

Identifying the vector space R with $N\Phi(\mathbb{K})$ via the linear map $e'_r \mapsto e_r$, we obtain the required realization. □

Lemma 2.2. *The algebra $NT(n,\mathbb{K})$ of lower niltriangular $n \times n$ matrices over the field $\mathbb{K}$ is isomorphic to its opposite algebra $NT(n,\mathbb{K})^{(op)}$.*

Proof. Let $J = (j_{ab})$ be the permutation matrix with ones on the antidiagonal, that is, $j_{a,n+1-a} = 1$ and $j_{ab} = 0$ otherwise. Then $J^{-1} = J$. Consider the linear map

$$\varphi\colon NT(n,\mathbb{K}) \to NT(n,\mathbb{K}), \qquad \varphi(X) = JX^tJ^{-1} = JX^tJ,$$

where X^t denotes the transpose of X. If X is lower triangular, then X^t is upper triangular, and conjugation by J sends upper triangular matrices to lower triangular ones. Hence $\varphi(NT(n,\mathbb{K})) = NT(n,\mathbb{K})$. For any $X, Y \in NT(n,\mathbb{K})$, we have

$$\varphi(XY) = J(XY)^tJ = JY^tX^tJ = (JY^tJ)(JX^tJ) = \varphi(Y)\varphi(X),$$

so φ is an antiautomorphism of the algebra $NT(n,\mathbb{K})$. Therefore, the map $X \mapsto \varphi(X)$ defines an isomorphism $NT(n,\mathbb{K}) \simeq NT(n,\mathbb{K})^{(op)}$. □

We shall prove that any associative exact enveloping algebra of the Lie algebra $N\Phi(\mathbb{K})$ of type A_n, where $n \geqslant 2$, is nilpotent. To this end, we need several preliminaries from the theory of noncommutative rings.

Let R be an associative ring. A left R-module M is called *semisimple* if it is a direct sum of a family of simple submodules [14, Theorem 2.4]. The ring R is called *(left) semisimple* if every left R-module is semisimple [14, Theorem 2.5].

We denote by $M_n(D)$ the full $n \times n$ matrix ring over a division ring D.

Theorem 1 ([14, Wedderburn–Artin theorem])**.** *Let R be any left semisimple associative ring. Then $R \simeq \prod_{i=1}^{r} M_{n_i}(D_i)$ for suitable division rings $D_1, \ldots, D_r$ and positive integers $n_1, \ldots, n_r$.*

Next we shall need the extension of scalars functor [13, Section 4.1.7]. Let $\mathbb{K}$ be a subfield of a field $\mathbb{E}$, and let R be a $\mathbb{K}$-algebra. Consider the tensor product of the $\mathbb{K}$-vector spaces $R \otimes_{\mathbb{K}} \mathbb{E}$. Its elements are finite sums of pure tensors $r \otimes a$, where $r \in R$ and $a \in \mathbb{E}$. We endow $R \otimes_{\mathbb{K}} \mathbb{E}$ with the structure of a vector space over $\mathbb{E}$ by setting

$$a(r \otimes b) = r \otimes (ab) \quad (r \in R,\ a, b \in \mathbb{E}).$$

An $\mathbb{E}$-algebra structure on $R \otimes_{\mathbb{K}} \mathbb{E}$ is then obtained by defining multiplication on pure tensors by

$$(r \otimes a)(s \otimes b) = (rs) \otimes (ab) \quad (r, s \in R,\ a, b \in \mathbb{E}),$$

and extending it bilinearly to arbitrary elements of $R \otimes_{\mathbb{K}} \mathbb{E}$ ([15, § 5.4]).

Lemma 2.3. *Let R be an associative $\mathbb{K}$-algebra such that $R^{(-)}$ is nilpotent, and let $\mathbb{E}$ be a field extension of $\mathbb{K}$. Then the Lie algebra $(R \otimes_{\mathbb{K}} \mathbb{E})^{(-)}$ is nilpotent.*

Proof. The Lie bracket in $(R \otimes_{\mathbb{K}} \mathbb{E})^{(-)}$ is given on pure tensors by

$$[r \otimes a, s \otimes b] = [r, s] \otimes (ab),$$

so that $(R \otimes_{\mathbb{K}} \mathbb{E})^{(-)} = R^{(-)} \otimes_{\mathbb{K}} \mathbb{E}$ as Lie algebras over $\mathbb{E}$. If $R^{(-)}$ is nilpotent of class m, that is, $(R^{(-)})^{m+1} = 0$, then

$$\left(R^{(-)} \otimes_{\mathbb{K}} \mathbb{E}\right)^{m+1} = \left(R^{(-)}\right)^{m+1} \otimes_{\mathbb{K}} \mathbb{E} = 0. \qquad \square$$

Lemma 2.4 ([16, Corollary 2.2.11])**.** *Let A be a central simple $\mathbb{K}$-algebra. Then there exists a finite separable field extension $\mathbb{E}/\mathbb{K}$ such that $A \otimes_{\mathbb{K}} \mathbb{E} \simeq M_n(\mathbb{E})$ for some $n \geqslant 1$.*

Lemma 2.5. *Let R be a finite-dimensional semisimple associative $\mathbb{K}$-algebra. If $R^{(-)}$ is nilpotent, then R is commutative.*

Proof. By the Wedderburn–Artin theorem, we have a decomposition

$$R \simeq \prod_{i=1}^{r} M_{n_i}(D_i).$$

Since the coordinate projections $R^{(-)} \to M_{n_i}(D_i)^{(-)}$ are surjective homomorphisms of Lie algebras, the nilpotency of $R^{(-)}$ implies that each Lie algebra $M_{n_i}(D_i)^{(-)}$ is nilpotent.

We claim that $n_i = 1$ for all i. Indeed, if $n_i \geqslant 2$, then $M_{n_i}(D_i)$ contains the standard matrix units e_{pq}, and $[e_{11}, e_{12}] = e_{12}$. Hence $\operatorname{ad} e_{11}$ is not nilpotent, a contradiction. Therefore $n_i = 1$ for all i, and hence

$$R \simeq \prod_{i=1}^{r} D_i.$$

It remains to show that each D_i is commutative. Fix $D = D_i$ and let $F = Z(D)$ be its center. Then D is a central simple algebra over F. By Lemma 2.4, there exists a finite separable field extension E/F such that $D \otimes_F E \simeq M_m(E)$ for some $m \geqslant 1$. By Lemma 2.3, the nilpotency of $D^{(-)}$ implies the nilpotency of $(D \otimes_F E)^{(-)} \simeq M_m(E)^{(-)}$. However, if $m \geqslant 2$, then the Lie algebra $M_m(E)^{(-)}$ is not nilpotent, by the same example with e_{11} and e_{12}. Hence $m = 1$, and therefore $D \otimes_F E \simeq E$. This implies that $D = F$, i.e., D is commutative. Thus all factors D_i are commutative, and therefore R is commutative. $\square$

An associative ring is said to be *(left) artinian* if it satisfies the descending chain condition on left ideals [14, Section 1]. Clearly, every finite-dimensional $\mathbb{K}$-algebra is artinian. For an artinian ring R, the *Jacobson radical* $J(R)$ coincides with the Wedderburn radical, that is, it is the largest nilpotent ideal of R [14, Theorem 4.12].

A ring R is said to be *J-semisimple* if $J(R) = 0$ [14, Definition 4.7]. For any ring R, the quotient ring $R/J(R)$ is J-semisimple [14, Proposition 4.6]. It follows from [14, Theorem 4.14] that if R is artinian, then the quotient ring $R/J(R)$ is semisimple.

Lemma 2.6 ([17, Theorem 1.3.2]). *If R is an artinian ring and $I \neq 0$ is a nonnilpotent right ideal of R then I contains a nonzero idempotent.*

Lemma 2.7. *Let L be a finite-dimensional nilpotent Lie algebra such that $\dim Z(L) = 1$ and $Z(L) \subset [L, L]$. Then every associative exact enveloping algebra of L is nilpotent.*

Proof. Let R be an associative exact enveloping algebra of L, and assume that R is not nilpotent. By Lemma 2.6, there exists a nonzero idempotent $e \in R$.

Denote by L_e and R_e the operators of left and right multiplication by e on R. Since $e^2 = e$, we have $L_e^2 = L_e$, $R_e^2 = R_e$, and $L_e R_e = R_e L_e$. Therefore

$$(\operatorname{ad} e)^3 = (L_e - R_e)^3 = L_e - R_e = \operatorname{ad} e.$$

On the other hand, the Lie algebra $R^{(-)} \simeq L$ is nilpotent, so $\operatorname{ad} e$ is a nilpotent operator. Hence $\operatorname{ad} e = 0$, and therefore e lies in the center of R.

The map $L_e \colon R \to R$ is a nonzero idempotent endomorphism. Therefore

$$R = eR \oplus \ker L_e.$$

Since e is central, both eR and $\ker L_e$ are two-sided ideals of R. Consequently,

$$R^{(-)} = (eR)^{(-)} \oplus (\ker L_e)^{(-)}$$

is a direct sum of Lie ideals. If both summands were nonzero, then, being nilpotent Lie algebras, they would have nonzero centers. Hence $\dim Z(R^{(-)}) \geqslant 2$, contrary to $\dim Z(L) = 1$. Thus $\ker L_e = 0$, and therefore $R = eR$.

Since e is central and $R = eR$, for every $x \in R$ we have $x = ey = ye$ for some $y \in R$, whence $ex = xe = x$. Therefore e is the identity element of R.

The quotient algebra $R/J(R)$ is a finite-dimensional semisimple associative algebra. Since $R^{(-)} \simeq L$ is nilpotent, so is $(R/J(R))^{(-)}$. By Lemma 2.5, the algebra $R/J(R)$ must be commutative, and therefore $[R, R] \subset J(R)$. In particular, $1 \notin [R, R]$.

On the other hand, under the isomorphism $R^{(-)} \simeq L$, the identity element $1 \in R$ corresponds to an element of $Z(L)$. Since $Z(L) \subset [L, L]$, it follows that $1 \in [R, R]$, a contradiction. Therefore R is nilpotent. □

For the root system Φ of type A_n, we use its standard realization [18, Plate 1]:

$$\begin{gathered} A_n = \{\epsilon_i - \epsilon_j \mid 1 \leqslant i \neq j \leqslant n+1\}, \qquad A_n^+ = \{\epsilon_i - \epsilon_j \mid 1 \leqslant j < i \leqslant n+1\}, \\ \Pi(A_n) = \{\alpha_i \coloneqq \epsilon_{i+1} - \epsilon_i \mid 1 \leqslant i \leqslant n\}. \end{gathered} \tag{3}$$

In this realization, it is convenient to identify the basis vectors $e_r = e_{\epsilon_i - \epsilon_j}$ of the algebra $N\Phi(\mathbb{K})$ with the matrix units e_{ij} in $\mathfrak{sl}_{n+1}(\mathbb{K})$ (see, for example, [19, §11.2.1]):

$$e_{\epsilon_i - \epsilon_j} \longleftrightarrow e_{ij} \qquad (1 \leqslant j < i \leqslant n+1). \tag{4}$$

In the realization (4), the center of $N\Phi(\mathbb{K})$ is the one-dimensional subspace $\mathbb{K}e_{n+1,1}$. Moreover, for $n \geqslant 2$ we have $e_{n+1,1} = [e_{n+1,2}, e_{2,1}]$, and therefore Lemma 2.7 immediately yields the following statement.

Corollary 1.1. *Let R be an associative exact enveloping $\mathbb{K}$-algebra of the Lie algebra $N\Phi(\mathbb{K})$ of type A_n ($n \geqslant 2$). Then R is nilpotent.*

Theorem 2. *If an associative exact enveloping algebra R of the Lie algebra $N\Phi(\mathbb{K})$ of type A_n $(n \geqslant 3)$ admits a grading compatible with the Cartan grading, then $R \simeq NT(n+1, \mathbb{K})$.*

Proof. The algebra R may be realized on the vector space $N\Phi(\mathbb{K})$ via the matrix realization (4). In this realization, the commutator of basis elements is given by

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}.$$

By Corollary 1.1, the algebra R is nilpotent. Since R admits a grading compatible with the Cartan grading, Lemma 2.1 applies and gives $e_{ij}e_{kl} = 0$ whenever the sum of the corresponding roots is not a root, while $e_{ij}e_{jk}, e_{jk}e_{ij} \in \mathbb{K}e_{ik}$ otherwise.

For type A_3, this leaves, up to scalar factors, only the following possible nonzero products:

$$e_{32}e_{21},\ e_{21}e_{32} \in \mathbb{K}e_{31}; \qquad e_{43}e_{32},\ e_{32}e_{43} \in \mathbb{K}e_{42};$$

$$e_{42}e_{21},\ e_{21}e_{42} \in \mathbb{K}e_{41}; \qquad e_{43}e_{31},\ e_{31}e_{43} \in \mathbb{K}e_{41}.$$

Hence there exist $a, a', b, b', c, c', d, d' \in \mathbb{K}$ such that

$$e_{32}e_{21} = ae_{31}, \quad e_{21}e_{32} = a'e_{31}; \qquad e_{43}e_{32} = be_{42}, \quad e_{32}e_{43} = b'e_{42};$$

$$e_{42}e_{21} = ce_{41}, \quad e_{21}e_{42} = c'e_{41}; \qquad e_{43}e_{31} = de_{41}, \quad e_{31}e_{43} = d'e_{41}.$$

Since the commutator in $R^{(-)}$ coincides with the matrix commutator, we have

$$a - a' = 1, \qquad b - b' = 1, \qquad c - c' = 1, \qquad d - d' = 1.$$

Associativity is nontrivial only for triples of total degree $\alpha_1+\alpha_2+\alpha_3$ (that is, $\epsilon_4-\epsilon_1$). Taking into account that $e_{43}e_{21} = e_{21}e_{43} = 0$, we obtain the system

$$a'd = 0, \qquad ad' = 0, \qquad bc' = 0, \qquad b'c = 0, \qquad bc = ad.$$

If $d \neq 0$, then $a'd = 0$ implies $a' = 0$ and hence $a = 1$, while $ad' = 0$ gives $d' = 0$ and hence $d = 1$. Then $bc = ad = 1 \neq 0$, so $b, c \neq 0$; the equalities $bc' = 0$ and $b'c = 0$ yield $c' = 0$ and $b' = 0$, and therefore $c = 1$ and $b = 1$. Thus, the multiplication in R coincides with that in $NT(4, \mathbb{K})$. If, on the other hand, $d = 0$, then $d' = -1$, and the same equalities analogously imply $a = 0$, $a' = -1$, $b = 0$, $b' = -1$, $c = 0$, and $c' = -1$. Hence $R \simeq NT(4, \mathbb{K})^{(op)}$, and by Lemma 2.2 we again obtain $R \simeq NT(4, \mathbb{K})$.

For $n > 3$, we may assume by induction that the theorem has been proved for every type A_k, where $3 \leqslant k < n$.

Let Φ_1 and Φ_n be the root subsystems of type A_{n-1} in Φ with bases $\Pi_1 = \Pi \setminus \{\alpha_1\}$ and $\Pi_n = \Pi \setminus \{\alpha_n\}$, respectively. Consider the subalgebras $R(\Phi_1)$ and $R(\Phi_n)$ corresponding to these subsystems. They are associative exact enveloping algebras of type A_{n-1} that admit gradings compatible with the Cartan gradings of the corresponding subsystems. Therefore, $R(\Phi_1) \simeq NT(n, \mathbb{K}) \simeq R(\Phi_n)$.

Taking into account the grading and the prescribed commutators, this means that the multiplication table of basis elements in $R(\Phi_k)$ coincides either with that of $NT(n, \mathbb{K})$ (the standard multiplication), or with that of $NT(n, \mathbb{K})^{(op)}$ (the opposite multiplication).

The intersection $R(\Phi_1) \cap R(\Phi_n)$ corresponds to the subsystem of type A_{n-2} with base $\Pi \setminus \{\alpha_1, \alpha_n\}$. For $n > 3$, this intersection contains elements with nonzero commutator. Therefore,

the multiplication on the intersection cannot be simultaneously standard and opposite. Hence the types of multiplication in $R(\Phi_1)$ and $R(\Phi_n)$ coincide.

Suppose that $R(\Phi_1)$ and $R(\Phi_n)$ carry the standard multiplication. Then all products $e_{ij}e_{kl}$ within $R(\Phi_1)$ and within $R(\Phi_n)$ coincide with the matrix products.

It remains to check the products in which one factor lies in $R(\Phi_1) \setminus R(\Phi_n)$ (and thus involves the index $n+1$), while the other lies in $R(\Phi_n) \setminus R(\Phi_1)$ (and thus involves the index 1).

Consider the products of the form $e_{n+1,k}e_{k,1}$ $(k = 2, \dots, n)$. Using associativity and the fact that the multiplication is standard in both $R(\Phi_1)$ and $R(\Phi_n)$, one shows that their value is independent of k and equal to $X = e_{n+1,n}e_{n,1}$. For example, when $k = 2$,

$$e_{n+1,2}e_{21} = (e_{n+1,n} \dots e_{32})e_{21} = e_{n+1,n}(\dots(e_{32}e_{21})) = e_{n+1,n}e_{n1}.$$

From the commutation relations $[e_{n+1,k}, e_{k1}] = e_{n+1,1}$ $(k = 2, \dots, n)$, it follows that the product $Y = e_{k1}e_{n+1,k}$ also takes the constant value $X - e_{n+1,1}$.

We compute Y for $k = 2$:

$$Y = e_{21}e_{n+1,2} = e_{21}(e_{n+1,3}e_{32}) = (e_{21}e_{n+1,3})e_{32} = 0.$$

Therefore, $X = e_{n+1,1}$. Thus, $e_{n+1,k}e_{k1} = e_{n+1,1}$ for all k, which agrees with the standard multiplication.

If, on the other hand, the multiplication in $R(\Phi_1)$ and $R(\Phi_n)$ is opposite, then similarly we obtain $R \simeq NT(n+1, \mathbb{K})^{(op)} \simeq NT(n+1, \mathbb{K})$. □

Remark 1. If one drops the assumption that R admits a grading compatible with the Cartan grading in Theorem 2, then the algebra R need not be isomorphic to $NT(n+1, \mathbb{K})$.

Indeed, in the algebra $NT(4, \mathbb{K})$, replace the multiplication by a new product $*$ by setting $x * y = xy$ for all pairs of basis elements except for the pair (e_{32}, e_{21}), and additionally defining

$$e_{32} * e_{21} = e_{31} + e_{41}, \qquad e_{21} * e_{32} = e_{41}.$$

Then all commutators of basis elements coincide with those in $NT(4, \mathbb{K})$; hence $(NT(4, \mathbb{K}), *)^{(-)} \simeq N\Phi(\mathbb{K})$ of type A_3. The associativity of $*$ is preserved, since the additional term lies in the annihilator $\mathrm{Ann} = \mathbb{K}e_{41}$.

The resulting associative algebra is not isomorphic to $NT(4, \mathbb{K})$: the dimension of the left annihilator is an invariant. In $NT(4, \mathbb{K})$, we have $\mathrm{Ann}_\ell = \langle e_{21}, e_{31}, e_{41} \rangle$ and $\dim \mathrm{Ann}_\ell = 3$, whereas in $(NT(4, \mathbb{K}), *)$ the element $e_{21} \notin \mathrm{Ann}_\ell$, since $e_{21} * e_{32} = e_{41} \neq 0$, and therefore $\dim \mathrm{Ann}_\ell = 2$.

3. Left-symmetric algebras and their semidirect sums

For each $x \in R$, let $L_x \colon y \mapsto xy$ $(x, y \in R)$ denote the linear operator of left multiplication in the algebra R.

Definition 1. An algebra R over a field $\mathbb{K}$ is called *left-symmetric* or *pre-Lie* if its multiplication satisfies the identity

$$(xy)z - x(yz) = (yx)z - y(xz) \qquad (x, y, z \in R)$$

or, equivalently, the map $L \colon x \mapsto L_x$ defines a representation of $R^{(-)}$ in $\mathfrak{gl}(R)$:

$$[L_x, L_y] = L_{[x,y]} \qquad (x, y \in R).$$

As in the associative case, every left-symmetric algebra is Lie-admissible: its commutator bracket $[x, y] \coloneqq xy - yx$ defines a Lie algebra structure on R.

To construct left-symmetric exact enveloping algebras of the Lie algebras $N\Phi(\mathbb{K})$ of types B_n, C_n, and D_n, we use their decompositions as semidirect sums of the subalgebra $NA_{n-1}(\mathbb{K})$ and a suitable ideal.

Definition 2 ([20, § I.2.11]). The *semidirect sum* $\mathfrak{g} \ltimes_\phi \mathfrak{h}$ of Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is the direct sum of the vector spaces $\mathfrak{g}$ and $\mathfrak{h}$ endowed with the bracket

$$[(x, a), (y, b)] = \big([x, y],\ \phi_x(b) - \phi_y(a) + [a, b]\big),$$

where $\phi\colon \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ is a Lie algebra homomorphism from $\mathfrak{g}$ to the Lie algebra of derivations $\mathrm{Der}(\mathfrak{h})$, and $\phi_x = \phi(x)$.

Motivated by Definition 2, we introduce the notion of a semidirect sum for left-symmetric algebras.

Definition 3. The *semidirect sum* $R \ltimes_\tau I$ of left-symmetric algebras R and I is the direct sum of the vector spaces R and I endowed with the multiplication

$$(x, a)(y, b) = (xy,\ \tau_x(b) + ab),$$

where $\tau\colon R^{(-)} \to \mathrm{Der}(I)$ is a Lie algebra homomorphism from $R^{(-)}$ to the Lie algebra of derivations $\mathrm{Der}(I)$, and $\tau_x = \tau(x)$.

Every derivation of a left-symmetric algebra I is also a derivation of the Lie algebra $I^{(-)}$, that is, $\mathrm{Der}(I) \subset \mathrm{Der}\left(I^{(-)}\right)$. Thus, when considering the semidirect sum of left-symmetric algebras $R \ltimes_\tau I$, one may simultaneously consider the semidirect sum of the associated Lie algebras $R^{(-)}$ and $I^{(-)}$ with the same action τ.

As the next two theorems show, a semidirect sum of associative algebras need not be associative, but it is always left-symmetric.

Theorem 3. *Let R and I be associative algebras, and let $\tau\colon R^{(-)} \to \mathrm{Der}(I)$ be a Lie algebra homomorphism from $R^{(-)}$ to $\mathrm{Der}(I)$. Then the algebra $R \ltimes_\tau I$ is associative if and only if the following conditions hold:*

(A) $\tau_{xy} = \tau_x \tau_y$ *for all* $x, y \in R$;

(B) $\tau_x(I) \subset \mathrm{Ann}_r(I)$ *for all* $x \in R$.

Proof. Assume that conditions (A) and (B) hold. Since R is associative, it suffices to check that the second components coincide in the equality

$$((x, a)(y, b))(z, c) = (x, a)((y, b)(z, c)) \tag{5}$$

for all $x, y, z \in R$ and $a, b, c \in I$. By the definition of the multiplication in $R \ltimes_\tau I$, we have

$$(x, a)(y, b) = (xy,\ \tau_x(b) + ab), \qquad (y, b)(z, c) = (yz,\ \tau_y(c) + bc).$$

Therefore, the second component of the left-hand side is

$$\tau_{xy}(c) + (\tau_x(b) + ab)c = \tau_{xy}(c) + \tau_x(b)c + (ab)c,$$

whereas the second component of the right-hand side is

$$\tau_x(\tau_y(c) + bc) + a(\tau_y(c) + bc) = \tau_x\tau_y(c) + \tau_x(bc) + a\tau_y(c) + a(bc).$$

By (A), we have $\tau_x\tau_y(c) = \tau_{xy}(c)$. Since $\tau_x \in \mathrm{Der}(I)$ and (B) holds, we obtain

$$\tau_x(bc) + a\tau_y(c) = \tau_x(b)c + b\,\tau_x(c) = \tau_x(b)c.$$

Finally, since I is associative, we have $a(bc) = (ab)c$. Hence the second components coincide, which proves that $R \ltimes_\tau I$ is associative.

Conversely, assume that the algebra $R \ltimes_\tau I$ is associative. Substituting the triple $(x,0),(y,0),(0,c)$ into the associativity identity (5), we obtain

$$((x,0)(y,0))(0,c) = (xy,0)(0,c) = (0,\tau_{xy}(c)),$$

$$(x,0)((y,0)(0,c)) = (x,0)(0,\tau_y(c)) = (0,\tau_x\tau_y(c)),$$

whence (A) follows.

Next, substituting the triple $(x,0),(0,b),(0,c)$ into (5), we obtain

$$((x,0)(0,b))(0,c) = (0,\tau_x(b))(0,c) = (0,\tau_x(b)c),$$

$$(x,0)((0,b)(0,c)) = (x,0)(0,bc) = (0,\tau_x(bc)),$$

that is, $\tau_x(bc) = \tau_x(b)c$ for all $b,c \in I$. Since $\tau_x \in \mathrm{Der}(I)$, we have

$$\tau_x(bc) = \tau_x(b)c + b\,\tau_x(c),$$

and therefore $b\,\tau_x(c) = 0$ for all $x \in R$ and $b,c \in I$, which is equivalent to (B). □

Theorem 4. *Let R and I be left-symmetric algebras, and let $\tau\colon R^{(-)} \to \mathrm{Der}(I)$ be a Lie algebra homomorphism from $R^{(-)}$ to $\mathrm{Der}(I)$. Then $R \ltimes_\tau I$ is a left-symmetric algebra. Moreover, its associated Lie algebra is isomorphic to the semidirect sum $R^{(-)} \ltimes_\tau I^{(-)}$, with bracket*

$$[(x,a),(y,b)] = \big([x,y],\ \tau_x(b) - \tau_y(a) + [a,b]\big).$$

Proof. A direct computation shows that the commutator in $(R \ltimes_\tau I)^{(-)}$ is given by

$$[(x,a),(y,b)] = \big([x,y],\ \tau_x(b) - \tau_y(a) + [a,b]\big),$$

and hence $(R \ltimes_\tau I)^{(-)}$ is isomorphic to $R^{(-)} \ltimes_\tau I^{(-)}$.

We prove that $R \ltimes_\tau I$ is left-symmetric. For $(x,a) \in R \ltimes_\tau I$, the operator $L_{(x,a)}$ acts on $R \ltimes_\tau I$ by

$$L_{(x,a)}(y,b) = (xy,\ \tau_x(b) + ab),$$

that is, $L_{(x,a)}|_R = L_x$ and $L_{(x,a)}|_I = \tau_x + L_a$.

Since R and I are left-symmetric, we have $[L_x, L_y] = L_{[x,y]}$ and $[L_a, L_b] = L_{[a,b]}$. Moreover, the fact that τ is a Lie algebra homomorphism from $R^{(-)}$ to $\mathrm{Der}(I)$ implies that $[\tau_x, \tau_y] = \tau_{[x,y]}$,

while $[\tau_x, L_a] = L_{\tau_x(a)}$. Using these identities and restricting the operators to R and I, we obtain

$$\begin{aligned}
[L_{(x,a)}, L_{(y,b)}]\big|_R &= [L_x, L_y] = L_{[x,y]} \\
&= L_{[(x,a),(y,b)]}\big|_R, \\
[L_{(x,a)}, L_{(y,b)}]\big|_I &= [\tau_x + L_a, \tau_y + L_b] \\
&= [\tau_x, \tau_y] + [L_a, \tau_y] + [\tau_x, L_b] + [L_a, L_b] \\
&= \tau_{[x,y]} - L_{\tau_y(a)} + L_{\tau_x(b)} + L_{[a,b]} \\
&= \tau_{[x,y]} + L_{[a,b]+\tau_x(b)-\tau_y(a)} \\
&= L_{([x,y],\ \tau_x(b)-\tau_y(a)+[a,b])}\big|_I \\
&= L_{[(x,a),(y,b)]}\big|_I.
\end{aligned}$$

Since both operators $[L_{(x,a)}, L_{(y,b)}]$ and $L_{[(x,a),(y,b)]}$ preserve the subspaces R and I and coincide on each of them, we obtain

$$[L_{(x,a)}, L_{(y,b)}] = L_{[(x,a),(y,b)]}. \qquad \square$$

Corollary 4.1. *Let $\mathfrak{L} = \mathfrak{l} \ltimes_\phi \mathfrak{h}$ be a semidirect sum of Lie algebras, and let R and I be left-symmetric exact enveloping algebras of $\mathfrak{l}$ and $\mathfrak{h}$, respectively. Suppose further that the Lie algebra homomorphism $\tau\colon R^{(-)} \to \mathrm{Der}(I)$ induces on $I^{(-)}$ the same action of $R^{(-)}$ as ϕ. Then $R \ltimes_\tau I$ is a left-symmetric exact enveloping algebra of the Lie algebra $\mathfrak{L}$.*

Proof. By Theorem 4, the algebra $R \ltimes_\tau I$ is left-symmetric, hence Lie-admissible. Moreover,

$$(R \ltimes_\tau I)^{(-)} \simeq R^{(-)} \ltimes_\tau I^{(-)} \simeq \mathfrak{l} \ltimes_\phi \mathfrak{h} = \mathfrak{L},$$

so $R \ltimes_\tau I$ is an exact enveloping algebra of the Lie algebra $\mathfrak{L}$. $\square$

4. Left-symmetric structures for types B_n, C_n and D_n

We now use Theorem 4 and its Corollary 4.1 to construct nilpotent graded left-symmetric exact enveloping algebras of $N\Phi(\mathbb{K})$ of types B_n, C_n, and D_n.

Let $\Psi = A_{n-1}^+$ be as in (3). The systems of positive roots of types D_n, C_n, and B_n can be written as the unions ([18, Plates I–IV])

$$\begin{aligned}
D_n^+ &= \Psi \cup \{\epsilon_i + \epsilon_j \mid 1 \leqslant j < i \leqslant n\}, \\
C_n^+ &= D_n^+ \cup \{2\epsilon_i \mid 1 \leqslant i \leqslant n\}, \\
B_n^+ &= D_n^+ \cup \{\epsilon_i \mid 1 \leqslant i \leqslant n\}.
\end{aligned}$$

Thus, Φ^+ consists of roots of the form

$$r_{i,mj} = \epsilon_i - m\epsilon_j, \quad 1 \leqslant j \leqslant i \leqslant n,\ m \in \{0, -1, +1\}.$$

We write the set of simple roots of Φ in the form $\Pi(\Phi) = \Pi(\Psi) \cup \{r_\Phi\}$, where

$$r_\Phi := \begin{cases} r_{2,-1}, & \Phi = D_n, \\ r_{1,-1}, & \Phi = C_n, \\ r_{10}, & \Phi = B_n. \end{cases}$$

Clearly, for a root system Φ of rank n, the algebra $N\Psi = \langle e_r \mid r \in \Psi\rangle \simeq NT(n,\mathbb{K})^{(-)}$ is a subalgebra of $N\Phi(\mathbb{K})$.

For roots, we write $s \geqslant r$ if all coefficients in the expansion of $s - r$ with respect to the set of simple roots Π are nonnegative. Define ideals $T(r)$ in $N\Phi(\mathbb{K})$ by

$$T(r) \coloneqq \sum_{s \geqslant r} \mathbb{K} e_s.$$

Denoting by ϕ the adjoint action of $NT(n,\mathbb{K})^{(-)}$ on the ideal $T(r_\Phi)$, we obtain a decomposition of $N\Phi(\mathbb{K})$ as the semidirect sum of Lie algebras

$$N\Phi(\mathbb{K}) \simeq NT(n,\mathbb{K})^{(-)} \ltimes_\phi T(r_\Phi). \tag{6}$$

By Corollary 4.1, in order to construct an exact enveloping algebra of $N\Phi(\mathbb{K})$ of type D_n, C_n, or B_n as a semidirect sum of associative algebras, it suffices to choose an associative multiplication

$$\cdot_\Phi \colon T(r_\Phi) \times T(r_\Phi) \to T(r_\Phi)$$

on the ideal $T(r_\Phi)$ such that $(T(r_\Phi), \cdot_\Phi)^{(-)} = T(r_\Phi)$. Moreover, the adjoint action ϕ of the Lie algebra $NT(n,\mathbb{K})^{(-)}$ on $T(r_\Phi)$ must be compatible with $\cdot_\Phi$: for each $x \in NT(n,\mathbb{K})$, the operator ϕ_x must be a derivation of the associative algebra $\big(T(r_\Phi), \cdot_\Phi\big)$.

For types D_n and C_n, the ideal $T(r_\Phi)$ is abelian, and we may take $\cdot_\Phi$ to be the zero multiplication.

Consider the type B_n. Since the derived subalgebra of the Lie algebra $T(r_{10})$ is contained in its center Z, we have $T(r_{10})^3 = 0$, and the multiplication

$$e_r \cdot_{B_n} e_s \coloneqq \begin{cases} \dfrac{1}{2}[e_r, e_s], & \operatorname{char}\mathbb{K} \neq 2, \\ 0, & \text{otherwise} \end{cases} \qquad \big(e_r, e_s \in T(r_{10})\big)$$

defines an associative exact enveloping algebra $\big(T(r_{10}), \cdot_{B_n}\big)$ of $T(r_{10})$.

Theorem 5. *The semidirect sum of associative algebras*

$$R\Phi(\mathbb{K}) \coloneqq NT(n,\mathbb{K}) \ltimes_\phi \big(T(r_\Phi), \cdot_\Phi\big)$$

is a left-symmetric exact enveloping algebra of $N\Phi(\mathbb{K})$ of type D_n, C_n, or B_n.

Remark 2. Assume that $\operatorname{char}\mathbb{K} \neq 2$ and $n \geqslant 3$. Then the algebra $RB_n(\mathbb{K})$ is nonassociative for any choice of associative multiplication on $T(r_{10})$. Indeed, let $(\cdot)$ be an arbitrary associative multiplication on the vector space $T(r_{10})$ such that $(T(r_{10}), \cdot)^{(-)} = T(r_{10})$.

If the algebra $NT(n,\mathbb{K}) \ltimes_\phi \big(T(r_{10}), \cdot\big)$ were associative, then Theorem 3 would imply that

$$\phi_x\big(T(r_{10})\big) \subset \operatorname{Ann}_r\big(T(r_{10}), \cdot\big) \qquad \text{for all } x \in NT(n,\mathbb{K}).$$

For any $a, b \in \operatorname{Ann}_r\big(T(r_{10}), \cdot\big)$, we have $a \cdot b = b \cdot a = 0$, and hence $[a,b] = 0$. On the other hand, if the classes of $a, b \in T(r_{10})$ in $T(r_{10}) \,/\, Z$ are linearly independent, then $[a,b] \neq 0$. Therefore, the image of the subspace $\operatorname{Ann}_r\big(T(r_{10}), \cdot\big)$ in the quotient $T(r_{10}) \,/\, Z$ has dimension at most 1.

On the other hand, for $n \geqslant 3$, the adjoint action ϕ of the Lie algebra $NT(n,\mathbb{K})$ on $T(r_{10}) \,/\, Z$ is nontrivial and has image of dimension at least 2. Hence there exist $x_1, x_2 \in NT(n,\mathbb{K})$ and $u_1, u_2 \in T(r_{10})$ such that the classes of $\phi_{x_1}(u_1)$ and $\phi_{x_2}(u_2)$ in $T(r_{10}) / Z$ are linearly independent. This contradiction proves the claim.

Remark 3. The algebra $RC_n(\mathbb{K})$ is nonassociative for $n \geqslant 3$. Likewise, the algebra $RD_n(\mathbb{K})$ is nonassociative for $n \geqslant 4$.

Indeed, if one of these algebras were associative, then Theorem 3 would imply condition (A): $\phi_{xy} = \phi_x\phi_y$ for all $x, y \in NT(n, \mathbb{K})$. Take $x = e_{21}$ and $y = e_{32}$, so that $xy = 0$ in $NT(n, \mathbb{K})$, and let $u = e_{2,-1} \in T(r_\Phi)$. Then $\phi_{xy}(u) = 0$, whereas $\phi_x\phi_y(u) \neq 0$, and hence $\phi_{xy} \neq \phi_x\phi_y$. Thus condition (A) fails.

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Левосимметрические структуры на нильтреугольных алгебрах Шевалле

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Аннотация. Доказано, что алгебра $NT(n, \mathbb{K})$ нижних нильтреугольных $n \times n$ матриц над полем $\mathbb{K}$ является единственной ассоциативной точной обертывающей алгеброй нильтреугольной подалгебры $N\Phi(\mathbb{K})$ алгебры Шевалле типа A_{n-1}, допускающей градуировку, согласованную с градуировкой Картана. Для алгебр $N\Phi(\mathbb{K})$ классических типов B_n, C_n и D_n построены градуированные левосимметрические структуры.

Ключевые слова: алгебра Ли, алгебра Шевалле, нильтреугольная подалгебра, левосимметрическая структура.